1 Vad blir allmänna lösningen till ekvationen

gul. $y'' + 5y' + 6y = -x$

$r^2 + 5r + 6 = 0$

rätt svar $(r+2)(r+3) = 0 \Rightarrow r = -2$ el $r = -3$

a

$y_p = ax + b$

$y_p' = a$

$\Rightarrow 5a + 6ax + 6b = -x$

$6a = -1 \Rightarrow a = -\frac{1}{6}$

$-\frac{5}{6} + 6b = 5a + 6b = 0 \Rightarrow b = \frac{5}{36}$

grön: $y'' - y' - 6y = 2x$

b

$$r^2 - r - 6 = 0$$

$$(r-3)(r+2) = 0$$

$$r = 3,\ r = -2$$

$$y_p = ax + b$$

$$y_p' = a$$

$$-a - 6ax - 6b = 2x$$

$$-6a = 2 \Rightarrow a = -\frac{1}{3}$$

$$-a - 6b = 0 \Rightarrow b = \frac{1}{3 \cdot 6} = \frac{1}{18}$$

rosa: $y'' + y' - 6y = x$

$r^2 + r - 6 = 0$

a $(r+3)(r-2) = 0$

$r = -3, \quad r = 2$

$y_p = ax + b$

$y_p' = a$

$a - 6ax - 6b = x$

$-6a = 1 \Rightarrow a = -\frac{1}{6}$

$a - 6b = 0 \Rightarrow b = \frac{a}{6} = -\frac{1}{36}$

2. gul Vad blir $\int \frac{x+1}{x^2-4}\,dx$?

$$\frac{x+1}{(x-2)(x+2)} = \frac{x+1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} = \frac{\frac{3}{4}}{x-2} + \frac{\frac{1}{4}}{x+2}$$

(c)

grön $\int \frac{x}{x^2-2x-3}\,dx$

$$\frac{x}{x^2-2x-3} = \frac{x}{(x-3)(x+1)} = \frac{\frac{3}{4}}{x-3} + \frac{\frac{1}{4}}{x+1}$$

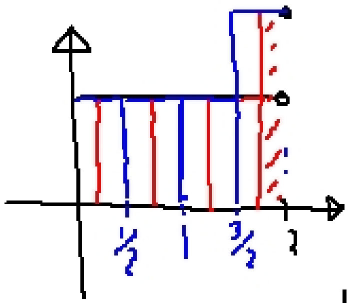
(b)

rosa: $\displaystyle\int \frac{x-2}{x^2+3x-4}\,dx$

$$\frac{x-2}{x^2+3x-4} = \frac{x-2}{(x+4)(x-1)} = \frac{\frac{6}{5}}{x+4} + \frac{-\frac{1}{5}}{x-1}$$

a

Generaliserade integraler



$$f(x) = \begin{cases} 1 & 0 \le x < 2 \\ 2 & x = 2 \end{cases}$$

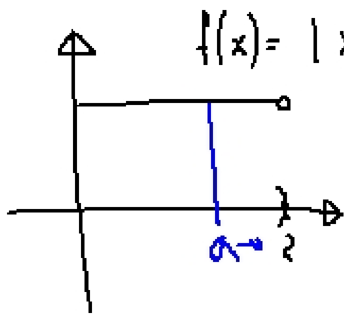
När indelningen blir finare minskar sista stapelns betydelse och arean går mot 2.

Detta indikerar också att integralen för f över det halvöppna intervallet $[0,2)$

borde bli samma som

den över det slutna

intervallet $[0,2]$.



$\int_0^a f(x)\,dx$ är

definierad som vanligt

då kan vi definiera $\int_0^2 f(x)\,dx$ som

$$\lim_{a\to 2^-} \int_0^a f(x)\,dx$$

Ex: $\displaystyle \int_0^1 x^\alpha \, dx = \left[\begin{array}{ll} \frac{x^{\alpha+1}}{\alpha+1} & \alpha \neq -1 \\ \ln|x| & \alpha = -1 \end{array}\right]_0^1 =$

$$= \lim_{a \to 0^+} \left[\begin{array}{ll} \frac{x^{\alpha+1}}{\alpha+1} & \alpha \neq -1 \\ \ln x & \alpha = -1 \end{array}\right]_a^1 = \lim_{a \to 0^+} \begin{cases} \frac{1}{\alpha+1} - \frac{a^{\alpha+1}}{\alpha+1} \\ \ln 1 - \ln a \end{cases}$$

$$\Rightarrow \lim_{a \to 0^+} \int_a^1 x^\alpha \, dx \qquad = \begin{cases} \frac{1}{\alpha+1} & \alpha + 1 > 0 \\ \pm\infty & \alpha + 1 \leq 0 \end{cases}$$

$$\int_1^\infty x^\alpha \, dx = \lim_{a \to \infty} \int_1^a x^\alpha \, dx = \lim_{a \to \infty} \left[\begin{array}{ll} \frac{x^{\alpha+1}}{\alpha+1} & \alpha \neq -1 \\ \ln x & \alpha = -1 \end{array}\right]_1^a$$

$$= \lim_{a \to \infty} \begin{cases} \frac{a^{\alpha+1}}{\alpha+1} - \frac{1}{\alpha+1} & \alpha \neq -1 \\ \ln a - 0 & \alpha = -1 \end{cases} = \begin{cases} -\frac{1}{\alpha+1} & \alpha + 1 < 0 \\ \infty & \alpha + 1 \geq 0 \end{cases}$$

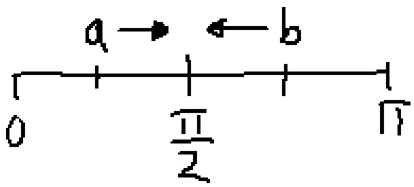
Ex: $\displaystyle\int \frac{\cos x}{\sin x + \cos x}\,dx = \int \frac{1}{1+\tan x}\,dx = \ldots$

$\displaystyle\int_0^{\pi} \frac{\cos x}{\sin x + \cos x}\,dx$ ger problem då tangen ej är definierad då $x = \frac{\pi}{2}$.

$$\int_0^{\pi} \frac{\cos x}{\sin x + \cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\cos x}{\sin x + \cos x}\,dx =$$

$$= \lim_{a \to \frac{\pi}{2}^-} \int_0^{a} \frac{\cos x}{\sin x + \cos x}\,dx + \lim_{b \to \frac{\pi}{2}^+} \int_b^{\pi} \frac{\cos x}{\sin x + \cos x}\,dx =$$

$$= \lim_{a \to \frac{\pi}{2}^-} \int_0^{a} \frac{1}{1+\tan x}\,dx + \lim_{b \to \frac{\pi}{2}^+} \int_b^{\pi} \frac{1}{1+\tan x}\,dx =$$



$$= \text{(enl tidigare räkningar)} =$$

$$= \lim_{a \to \frac{\pi}{2}^-} \left[\frac{1}{2} \ln|1+\tan x| - \frac{1}{4} \ln|1+\tan^2 x| + \frac{x}{2} \right]_0^a +$$

$$+ \lim_{b \to \frac{\pi}{2}^+} \left[\frac{1}{2} \ln|1+\tan x| - \frac{1}{4} \ln|1+\tan^2 x| + \frac{x}{2} \right]_b^\pi =$$

$$= \lim_{a \to \frac{\pi}{2}^-} \frac{1}{2} \ln \left| \frac{1+\tan a}{\sqrt{1+\tan^2 a}} \right| + \frac{a}{2} + \lim_{b \to \frac{\pi}{2}^+} \frac{\pi}{2} -$$

$$- \frac{1}{2} \ln \left| \frac{1+\tan b}{\sqrt{1+\tan^2 b}} \right| - \frac{b}{2} =$$

$$\left(\lim_{a \to \frac{\pi}{2}^-} \ln \left| \frac{1+\tan a}{\sqrt{1+\tan^2 a}} \right| = \begin{bmatrix} t = \tan a \\ a \to \frac{\pi}{2}^- \\ t \to \infty \end{bmatrix} = \right.$$

$$= \lim_{t \to \infty} \ln \left| \frac{1+t}{\sqrt{1+t^2}} \right| = \lim_{t \to \infty} \ln \left| \frac{\cancel{t}\left(\frac{1}{t}+1\right)}{\cancel{t}\sqrt{\frac{1}{t^2}+1}} \right| =$$

$$= \ln 1 = 0$$

$$\lim_{b \to \frac{\pi}{2}^+} \ln \left| \frac{1+\tan b}{\sqrt{1+\tan^2 b}} \right| = \begin{bmatrix} t = \tan b \\ b \to \frac{\pi}{2}^+ \\ t \to -\infty \end{bmatrix} =$$

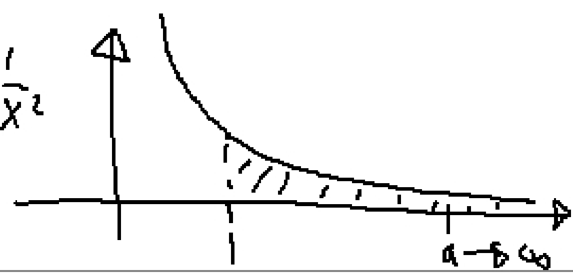
$$\left. = \lim_{t \to -\infty} \ln \left| \frac{1+t}{\sqrt{1+t^2}} \right| = 0 \right)$$

$$= \lim_{a \to \frac{\pi}{2}^-} \frac{a}{2} + \frac{\pi}{2} - \lim_{b \to \frac{\pi}{2}^+} \frac{b}{2} =$$

$$= \frac{\pi}{4} + \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{2}$$

OBS! variabelbytet $t = \tan x$ i ursprungsintegralen hade givet gränserna $t_1 = \tan 0 = 0$, $t_2 = \tan \pi = 0$

$f(x) = \frac{1}{x^2}$



Ex: $\displaystyle\int_1^{\infty} \frac{dx}{1+x^2} = \lim_{a\to\infty} \int_1^{a} \frac{dx}{1+x^2} = \lim_{a\to\infty} \Big[\arctan x\Big]_1^a =$

$\displaystyle = \lim_{a\to\infty} \arctan a - \arctan 1 =$

$\displaystyle = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

Ex $\displaystyle\int_{-1}^{1} \frac{dx}{\sqrt{|x|}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} + \lim_{b\to 0^-} \int_{-1}^{b} \frac{dx}{\sqrt{-x}} =$

$\displaystyle = \lim_{a\to 0^+} \Big[2\sqrt{x}\Big]_a^1 + \lim_{b\to 0^-} \Big[-2\sqrt{-x}\Big]_{-1}^{b} =$

$\displaystyle = \lim_{a\to 0^+} 2 - 2\sqrt{a} + \lim_{b\to 0^-} -2\sqrt{-b} + 2\sqrt{1} = 4$

Ex: $\displaystyle\int_0^1 (\ln x)^2\,dx = \lim_{a\to 0} \int_a^1 1\cdot(\ln x)^2\,dx =$

$\displaystyle= \lim_{a\to 0} \left[x(\ln x)^2\right]_a^1 - \int_a^1 \cancel{x}\, 2\ln x \cdot \frac{1}{\cancel{x}}\,dx$

$\displaystyle= \lim_{a\to 0} \left[x(\ln x)^2\right]_a^1 - 2\int_a^1 1\cdot\ln x\,dx =$

$\displaystyle= \lim_{a\to 0} -a(\ln a)^2 - 2\left(\left[x\ln x\right]_a^1 - \int_a^1 x\,\frac{1}{x}\,dx\right) =$

$\displaystyle= \lim_{a\to 0} -a(\ln a)^2 + 2a\ln a + 2(1-a) =$

$= 0 + 0 + 2 - 0 = 2$