Serier

Aritmetiska serier

samma Differens

$1+2+3+4+\ldots+99+100 \qquad 1=2-1=3-2=4-3\ldots$

$23+18+13+8+3 \qquad -5=18-23=13-18=\ldots$

$$A = a+(a+d)+(a+2d)+\ldots+(a+nd) = \sum_{k=0}^{n}(a+kd)$$

$$= a(n+1)+d(1+2+\ldots+n) = a(n+1)+d\,\frac{n(n+1)}{2}$$

Alternativt

$a=0, d=1$

$n=100$

$$\begin{array}{ccccccccccc} 1 & + & 2 & + & 3 & + & \ldots & + & 99 & + & 100 \\ 100 & + & 99 & + & 98 & + & \ldots & + & 2 & + & 1 \\ 101 & + & 101 & + & 101 & + & \ldots & + & 101 & + & 101 \end{array}$$

$1+2+3+\ldots+99+100 = \frac{100\cdot 101}{2}$

$101+101+101+\ldots+101+101 = 100\cdot 101$

$$\begin{array}{ccccccccc}
a & + & (a+d) & + & (a+2d) & + & \cdots & + & (a+nd) \\
+ & & + & & & & & & + \\
(a+nd) & + & (a+(n-1)d) & + & \cdots & & & + & a \\
\parallel & & \parallel & & & & & & \parallel \\
2a+nd & + & 2a+nd & + & \cdots & & & + & 2a+nd
\end{array}$$

$$= (n+1)(2a+nd) = 2A \Rightarrow A = \frac{(n+1)(2a+nd)}{2}$$

<u>Geometriska serier</u> samma kvot

$$1+2+4+8+16+32 \qquad 2 = \frac{2}{1} = \frac{4}{2} = \frac{8}{4} = \ldots$$

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{3^{10}} \qquad \frac{1}{3} = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{\frac{1}{27}}{\frac{1}{9}}$$

$$G = a + ak + ak^2 + ak^3 + \dots + ak^n$$

$$= \sum_{j=0}^{n} ak^j$$

Observera att

$$(1-k)G = a + \cancel{ak} + \cancel{ak^2} + \cancel{ak^3} + \dots + \cancel{ak^n} -$$

$$- \cancel{ak} - \cancel{ak^2} - \cancel{ak^3} - \dots \cancel{ak^n} - ak^{n+1} = a - ak^{n+1}$$

$$k \neq 1 \Rightarrow \quad G = a\left(\frac{1-k^{n+1}}{1-k}\right)$$

$$k = 1 \Rightarrow \quad G = \sum_{j=0}^{n} a = a(n+1)$$

Ex: $1+2+4+8+16+32$

$a=1,\ k=2,\ n=5$

$$G = a\,\frac{1-k^{n+1}}{1-k} = 1\cdot\frac{1-2^6}{1-2} = 63$$

Ex: $\frac{1}{3}+\frac{1}{9}+\frac{1}{27} = \frac{1}{3}\cdot\left(\frac{1-\left(\frac{1}{3}\right)^3}{1-\frac{1}{3}}\right) = \frac{26}{2\cdot 27} = \frac{13}{27}$

$a=\frac{1}{3},\ k=\frac{1}{3},\ n=2$

Ex: $x^2+xy+y^2 = x^2\left(\frac{1-\left(\frac{y}{x}\right)^3}{1-\frac{y}{x}}\right) = x^2\left(\frac{\frac{x^3-y^3}{x^3}}{\frac{x-y}{x}}\right) =$

$a=x^2,\ k=\frac{y}{x},\ n=2$

$$= \frac{x^3-y^3}{x-y} \Rightarrow x^3-y^3=(x-y)(x^2+xy+y^2)$$

$k<1$ $$G = a\,\frac{1-k^{n+1}}{1-k} = \frac{a}{1-k} - \underbrace{\frac{a\,k^{n+1}}{1-k}}_{\to 0 \text{ när } n\to\infty}$$

Ex: $$G_n = 1 + \frac{1}{2} + \dots + \frac{1}{2^n} = \sum_{k=0}^{n} \frac{1}{2^k}$$

$$G_n = 1\cdot\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 2 - \frac{1}{2^n}$$

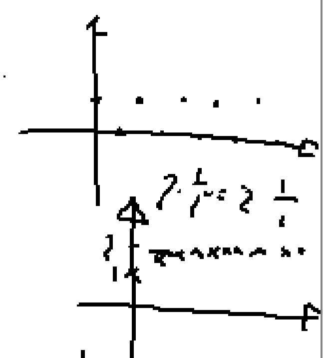
$$\lim_{n\to\infty} G_n = 2 \qquad \sum_{k=0}^{\infty} \frac{1}{2^k} = 2$$

Def. Låt $a_1, a_2, \dots$ vara en oändlig talföljd och sätt $S_N = \sum_{n=1}^{N} a_n$ (partiell/del summa). Serien är konvergent om $\lim_{N\to\infty} S_N$ konvergerar.

Ex: $1+1+1+\dots+1 = \sum_{k=1}^{N} 1 = S_N$

$\lim_{N\to\infty} S_N = \lim_{N\to\infty} N = \infty$ divergent

Ex: $\sum_{k=0}^{N}(-1)^k = \underbrace{(1-1)}_{=0}+\underbrace{(1-1)}_{=0}+\underbrace{(1-1)}_{=0}+\underbrace{(1-1)}_{=0}+\dots$



Om N udda $\sum_{k=0}^{N}(-1)^k = 0$

N jämnt

$\sum_{k=0}^{N} = (1-1)+(1-1)+(1-1)+\dots+(1-1)+1 = 1$

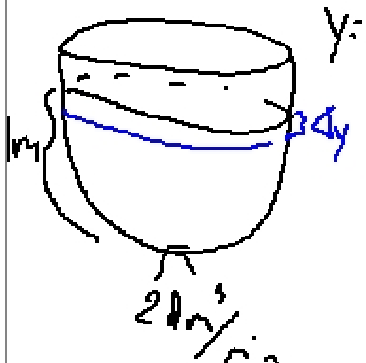
$\Rightarrow \lim_{N\to\infty}\sum_{k=0}^{N}(-1)^k$ existerar ej

serien divergerar

Ex: Om $k > 1$ konvergerar

$$S_N = \sum_{n=1}^{N} \frac{1}{k^n}$$ för

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1}{k}\left(\frac{1-\frac{1}{k^N}}{1-\frac{1}{k}}\right) = \frac{1}{k-1}$$



$y = x^2$ roterad

$x = \sqrt{y}$, $y \geq 0$

$\Delta V = \pi x^2 \Delta y = \pi y \Delta y$

$$\frac{\Delta V}{\Delta t} = 2\ \mathrm{dm}^3/\mathrm{min}$$

Hur lång tid tar det innan tanken är tömd?

$$\pi y \frac{\Delta y}{\Delta t} = 2$$

$$\pi y \frac{\Delta y}{\Delta t} \Delta t = 2 \Delta t$$

Vi låter $\Delta t \to 0$ och summerar över tiden

$$\int_0^{10} \pi y\,dy = \int \pi y \frac{dy}{dt}\,dt = \int_0^{T} 2\,dt = 2T$$

totala tiden

$$\left[\pi \frac{y^2}{2}\right]_0^{10} = \frac{\pi 100}{2} = 50\pi$$

$\Rightarrow$ Tiden det tar att tömma tanken blir 25π min.

(Tenta 2002-01-08) Tal 8 a)

Avgör om serien

$$\sum_{n=0}^{\infty} \frac{2^n}{3^n}$$

är konvergent eller ej.

$$S_N = \sum_{n=0}^{N} \frac{2^n}{3^n} = \sum_{n=0}^{N} \left(\frac{2}{3}\right)^n$$

geometrisk serie

$a = 1,\ k = \frac{2}{3}$

$$= 1 \cdot \frac{\left(1 - \left(\frac{2}{3}\right)^{N+1}\right)}{1 - \frac{2}{3}} = 3 - 2\left(\frac{2}{3}\right)^N \longrightarrow 3$$

Svar: Konvergent

(Tenta 2007-08-15) Tal 1

Bestäm $\displaystyle\int_{-1}^{0} \frac{dx}{x^2+x-2}$

$x^2+x-2=0 \qquad (x+2)(x-1)$

$$\frac{1}{x^2+x-2} = \frac{-1/3}{x+2} + \frac{\frac{1}{3}}{x-1}$$

$$\int_{-1}^{0} \frac{dx}{x^2+x-2} = \int_{-1}^{0} \frac{-\frac{1}{3}}{x+2} + \frac{\frac{1}{3}}{x-1}\,dx = \left[-\frac{1}{3}\ln|x+2| + \frac{1}{3}\ln|x-1|\right]_{-1}^{0}$$

$$= -\frac{1}{3}\ln 2 + \frac{1}{3}\ln 1 + \frac{1}{3}\ln 1 - \frac{1}{3}\ln 2 = -\frac{2}{3}\ln 2$$