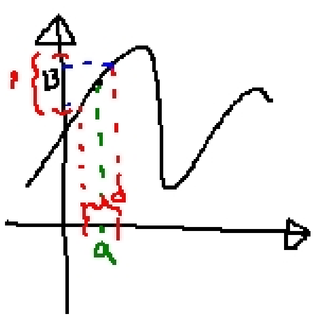
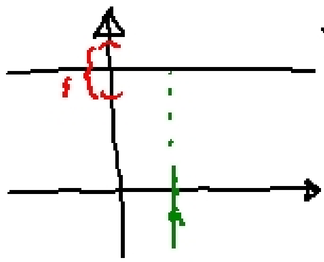
Att funktionen f går mot B när x närmar sig punkten a det bestände vi vara samma sak som att givet $\varepsilon>0$ ska det finnas ett tal $\delta>0$ så att



$$0<|x-a|<\delta \Rightarrow |f(x)-B|<\varepsilon.$$



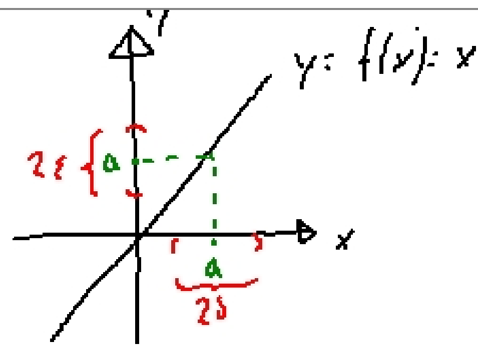
$$\lim_{x \to a} f(x) = C$$

Vi vill visa att givet $\varepsilon > 0$
så finns $\delta > 0$ så att

$|f(x) - c| < \varepsilon$ för alla $0 < |x-a| < \delta$.

Men
$$|f(x) - c| = |c - c| = 0$$

så δ kan väljas godtyckligt.



Förmodan

$$\lim_{x \to a} f(x) = a$$

Givet $\varepsilon > 0$ vill vi hitta $\delta > 0$ så att $|f(x) - a| < \varepsilon$ för alla $0 < |x - a| < \delta$.

Vi har

$$|f(x) - a| = |x - a|$$ så om $\delta = \varepsilon$

får vi

$$|f(x) - a| < \varepsilon \text{ så snart } 0 < |x - a| < \varepsilon = \delta$$

Om $\lim_{x \to a} f(x) = A$ och $\lim_{x \to a} g(x) = B$

så gäller

i) $\lim_{x \to a} \left(f(x) + g(x) \right) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) =$

$= A + B$

$$|f(x) + g(x) - A - B| = |f(x) - A + g(x) - B|$$

$$\leq |f(x) - A| + |g(x) - B|$$

Så om δ väljs så att $|f(x) - A| < \frac{\varepsilon}{2}$

och $|g(x) - B| < \frac{\varepsilon}{2}$ blir $|f(x) + g(x) - A - B| < \varepsilon$

Ex: Vad blir $\lim_{x \to 4} (x+1)$?

$$\lim_{x \to 4} (x+1) = \lim_{x \to 4} x + \lim_{x \to 4} 1 =$$

$$= 4 + 1 = 5.$$

ii) $\lim_{x \to a} f(x) \cdot g(x) = A \cdot B$

Ex: $\lim_{x \to 4} x^2 = \lim_{x \to 4} x \cdot x = 4 \cdot 4 = 16$

"Motivation": $\left| f(x)g(x) - AB \right| = \left| (f(x)-A)g(x) + A(g(x)-B) \right|$

$$= \left| (f(x)-A)(g(x)-B) + B(f(x)-A) + A(g(x)-B) \right|$$

alla termerna går att göra små

Ex. Vad blir $\lim\limits_{x \to 1} x^3 + 2x + 5$?

SVAR: 8

$$\lim_{x \to 1} x^3 + 2x + 5 = \left(\lim_{x \to 1} x\right)^3 + 2\left(\lim_{x \to 1} x\right) + 5$$

$$= 1 + 2 + 5 = 8$$

iii) Om $B \neq 0$ gäller

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{A}{B}$$

Motivation: $\left|\frac{f(x)}{g(x)} - \frac{A}{B}\right| = \left|\frac{f(x)B - Ag(x)}{g(x)B}\right| =$

$$= \left|\frac{(f(x) - A)B + A(B - g(x))}{(g(x) - B)B + B^2}\right|$$

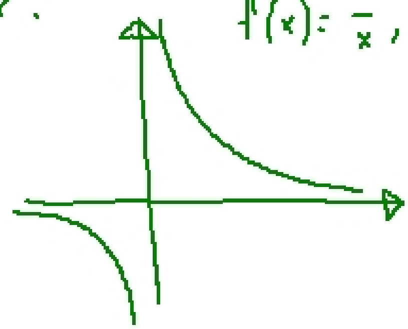
kan göras godtyckligt litet

Ex: Vad blir $\lim_{x \to 2} \frac{x^2-1}{x^3+5}$?

$$\lim_{x \to 2} \frac{x^2-1}{x^3+5} = \frac{\lim_{x \to 2} x^2-1}{\lim_{x \to 2} x^3+5} = \frac{\left(\lim_{x \to 2} x\right)^2-1}{\left(\lim_{x \to 2} x\right)^3+5} =$$

$$= \frac{2^2-1}{2^3+5} = \frac{3}{13}$$

Ex:

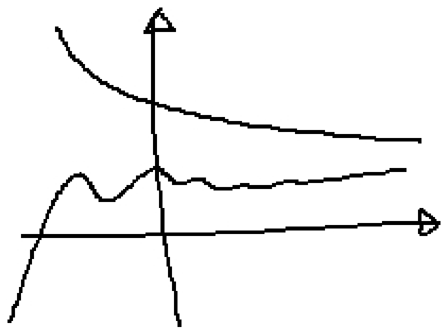


$f(x) = \frac{1}{x}$, $g(x) = 0$ $\quad x > 0 \quad f(x) > g(x)$

$$\lim_{x \to \infty} f(x) \geq \lim_{x \to \infty} g(x) = 0$$

$x < 0 \quad f(x) < g(x)$

$$\lim_{x \to -\infty} f(x) \leq \lim_{x \to -\infty} g(x) = 0$$



Om $f(x) \geq g(x)$

så gäller

$A = \lim_{x \to a} f(x) \geq \lim_{x \to a} g(x) = B$

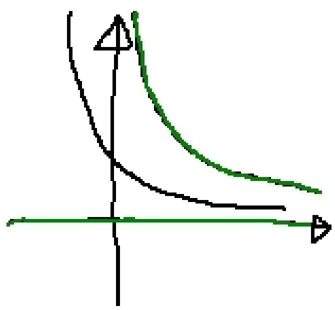
motivation:

$A - B = A - f(x) + f(x) - g(x) + g(x) - B$

$> -2\varepsilon$

ε godtycklig $\Rightarrow A - B \geq 0$

Instängningsprincipen



$f(x) = \frac{1}{x}$, $g(x) = 0$, $h(x) = \frac{1}{1+x}$

$$x > 0 \qquad 0 < \frac{1}{1+x} < \frac{1}{x}$$

$$\lim_{x \to \infty} g(x) = 0, \quad \lim_{x \to \infty} f(x) = 0$$

$$0 = \lim_{x \to \infty} f(x) \geq \lim_{x \to \infty} h(x) \geq \lim_{x \to \infty} g(x) = 0$$

$$\Rightarrow \lim_{x \to \infty} h(x) = 0$$

Ex: $\lim_{x \to 2} \overset{h(x)}{1 + x^2} = 5$, $\lim_{x \to 2} \overset{g(x)}{x^2} = 4$, $\lim_{x \to 4} \overset{f(x)}{1 + x} = 5$

Sätt $f(x) = 1 + x$, $g(x) = x^2$, $h(x) = f \circ g(x) = 1 + x^2$

Om $\lim_{x \to a} g(x) = A$, $\lim_{u \to A} f(u) = B$

så blir $\lim_{x \to a} f \circ g(x) = B$

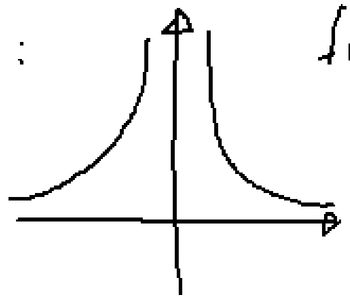
Ex: $f(x) = \frac{1}{x}$, $g(x) = 1 + x$

$f(g(x)) = f \circ g(x) = \frac{1}{1+x}$ $\quad \lim_{x \to \infty} g(x) = \infty$

$f(1+x)$ $\quad \lim_{x \to \infty} f \circ g(x) = \lim_{u \to \infty} f(u) =$

$= \lim_{u \to \infty} \frac{1}{u} = 0$

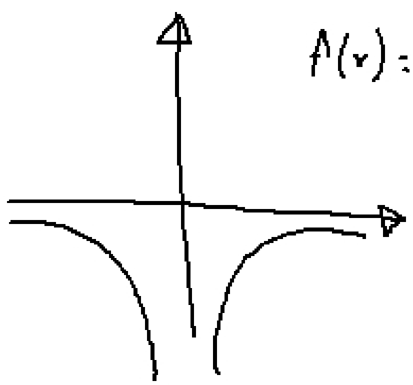
Ex:



$$f(x) = \frac{1}{x^2}$$

$$\lim_{x \to 0} f(x) = \infty$$

$$\lim_{x \to 0} \frac{1}{f(x)} = \lim_{x \to 0} x^2 = 0$$



$$f(x) = -\frac{1}{x^2}$$

$$\lim_{x \to 0} f(x) = -\infty$$

$$\lim_{x \to 0} \frac{1}{f(x)} = \lim_{x \to 0} -x^2 = 0$$