Differentierbara funktioner

①

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Longleftrightarrow$$

$$\begin{cases} f'(x) = \dfrac{f(x+h) - f(x)}{h} - R(x,h) \\ \lim\limits_{h \to 0} R(x,h) = 0 \end{cases}$$

$$\Longleftrightarrow$$

$$\begin{cases} f(x+h) = f(x) + f'(x) \cdot h + R(x,h)h \\ \lim\limits_{h \to 0} R(x,h) = 0 \end{cases}$$

Def: En funktion, $\vec{f}$, av typ $\mathbb{R}^n \to \mathbb{R}^m$ sägs vara differentierbar i punkten $\vec{x}$ om den är definierad i en omgivning till $\vec{x}$ och

$$\vec{f}(\vec{x}+\vec{h}) = \vec{f}(\vec{x}) + A\vec{h} + \vec{R}(\vec{x},\vec{h})|\vec{h}|$$

där A inte beror av $\vec{h}$ och

$$\lim_{\vec{h}\to\vec{0}} \vec{R}(\vec{x},\vec{h}) = \vec{0}.$$

Exempel: Om $n=m=1$ har vi

$$A = f'(x).$$

Exempel: $n=1$ och $m=2$ ③

$$\vec{f}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\begin{pmatrix} x(t+h) \\ y(t+h) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} h + \begin{pmatrix} R_1(t,h) \\ R_2(t,h) \end{pmatrix} |h|$$

och $\lim_{h \to 0} R_1(t,h) = \lim_{h \to 0} R_2(t,h) = 0$

Vi kan räkna komponentvis och ser att

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}.$$

Exempel: $n=2$ och $m=1$.

$$f(x+h, y+k) = f(x,y) + (a,b)\begin{pmatrix} h \\ k \end{pmatrix} + R(\vec{x},\vec{h})|\vec{h}|$$

$$\text{och} \lim_{\vec{h}\to\vec{0}} R(\vec{x},\vec{h}) = 0.$$

$\underline{h=0}$:

$$f(x, y+k) = f(x,y) + bk + R(\vec{x},\vec{h})|k|$$

$\underline{k=0}$:

$$f(x+h,y) = f(x,y) + ah + R(\vec{x},\vec{h})|h|$$

Sätt $g_x(y) = f(x,y)$ x fixt

$$g_x(y+k) = g_x(y) + bk + R(\vec{x},\vec{h})|k|.$$

$$\Rightarrow b = g_x'(y)$$

b kallas f's partialderivata med avseende på y.

På samma sätt får (5)
vi att a är f's partialderivata
med avseende på x.

$$\boxed{a = \frac{\partial f}{\partial x}(x,y),\ b = \frac{\partial f}{\partial y}(x,y)}$$

Vi har så

$$f(x+h, y+k) = f(x,y) + \frac{\partial f}{\partial x}(x,y)h + \frac{\partial f}{\partial y}(x,y)k + R(\vec{x}, \vec{h})|\vec{h}|.$$

~~Sätt~~ Vi byter beteckningar
till
$$(x, y) \to (a, b)$$
$$(x+h, y+k) \to (x, y)$$

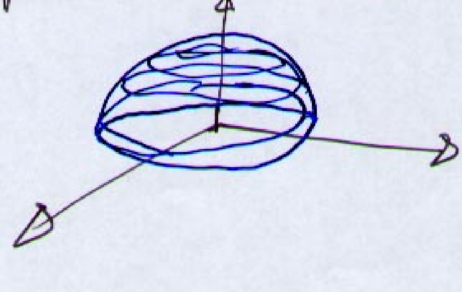
$$f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)\,(x-a) + \frac{\partial f}{\partial y}(a,b)\,(y-b) + R(\vec{a}, \vec{x}-\vec{a})\,|\vec{x}-\vec{a}|$$

$$\lim_{\vec{x}\to\vec{a}} R(\vec{a}, \vec{x}-\vec{a}) = 0$$

Ytan $z = f(x,y)$ approximeras i (a,b) av tangentplanet

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b).$$

Exempel: $z = f(x,y) = \sqrt{1-x^2-y^2}$



Övre halvan av ett klot

Tangentplanet i punkten ⑦

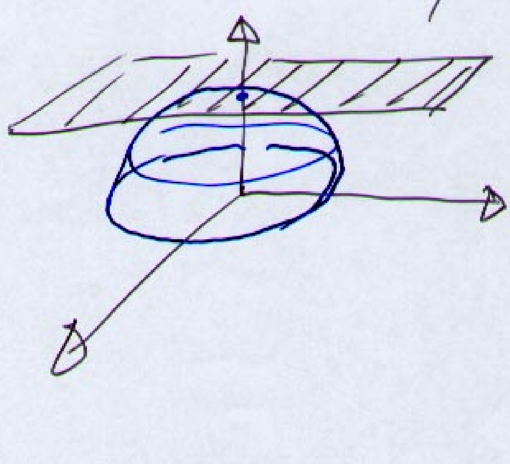
(0,0) blir

$$z = f(0,0) + \frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}\left(\sqrt{1-x^2-y^2}\right) = \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2-y^2}} = \frac{-x}{\sqrt{1-x^2-y^2}}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{-\cancel{2}y}{\cancel{2}\sqrt{1-x^2-y^2}} = \frac{-y}{\sqrt{1-x^2-y^2}}$$

$$z = 1 - 0x - 0y = 1$$



Sats 4.4

a) f av typ $\mathbb{R}^n \to \mathbb{R}$

$$\operatorname{grad} f = A = \left(\frac{\partial f}{\partial x_1}(\vec{x}), \dots, \frac{\partial f}{\partial x_n}(\vec{x})\right)$$

b) f av typ $\mathbb{R}^n \to \mathbb{R}^m$

$$J_f = A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Exempel: $f(x,y) = \sqrt{1-x^2-y^2}$

$$\operatorname{grad} f = \left(\frac{-x}{\sqrt{1-x^2-y^2}}, \frac{-y}{\sqrt{1-x^2-y^2}}\right)$$

Tangentplanet till ytan (9)

$z = f(x,y)$ har normalvektor $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right)$.

Sätt $F(x,y,z) = f(x,y) - z$

då blir $\operatorname{grad} F = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right)$.

Exempel: $f(x,y) = (x^2 + xy, 1 - y^2)$

$$J_f = \begin{pmatrix} \frac{\partial}{\partial x}(x^2+xy) & \frac{\partial}{\partial y}(x^2+xy) \\ \frac{\partial}{\partial x}(1-y^2) & \frac{\partial}{\partial y}(1-y^2) \end{pmatrix} =$$

$$= \begin{pmatrix} 2x+y & x \\ 0 & -2y \end{pmatrix}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

Exempel: $f(x,y) = x^2 + yx$

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 0, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(x) = 1$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x+y) = 1$$

Sats 4.5: Om $\frac{\partial^2 f}{\partial x \partial y}$ och $\frac{\partial^2 f}{\partial y \partial x}$ är kontinuerliga i närheten av en punkt så är

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\begin{cases} \vec{f}(\vec{x}+\vec{h}) = \vec{f}(\vec{x}) + A\vec{h} + \vec{R}(\vec{x},\vec{h})|\vec{h}| \\ \lim_{\vec{h}\to\vec{0}} \vec{R}(\vec{x},\vec{h}) = \vec{0}. \end{cases}$$ 11

När $\vec{h}\to\vec{0}$ får vi

$$\lim_{\vec{h}\to\vec{0}} \vec{f}(\vec{x}+\vec{h}) = \vec{f}(\vec{x})$$

Sats 4.6: $\vec{f}$ differentierbar $\Rightarrow$ $\vec{f}$ kontinuerlig och har partiella derivator

Exempel: $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

$\lim_{x\to 0} f(x,0) = 0$ men

$$\lim_{t\to 0} f(t,t) \overset{\lim}{=} \frac{t^2}{t^2+t^2} = \frac{1}{2}$$

$\Rightarrow$ f ej kontinuerlig i $(0,0)$

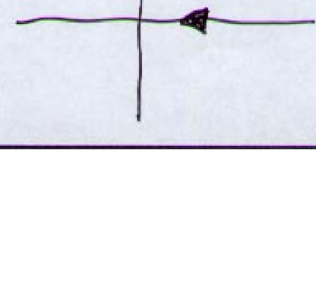
$\Rightarrow$ ej differentierbar i $(0,0)$

men

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$$

$$\left(\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}\left(\frac{xy}{x^2+y^2}\right) = \frac{y(x^2+y^2)-2x^2y}{(x^2+y^2)^2} = \frac{y^3-x^2y}{(x^2+y^2)^2} \right)$$

och $$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0$$



Sats 4.7 Om alla partiella derivator till $\vec{f}$ är kontinuerliga i en omgivning av en punkt, så är $\vec{f}$ differentierbar i omgivningen.

$$\lim_{t \to 0} \frac{\partial f}{\partial x}(0, t) = \lim_{t \to 0} \frac{t^3}{t^4} = \lim_{t \to 0} \frac{1}{t}.$$

Däremot så är funktionen differentierbar utanför origo.

Sats 4.8: Om $\vec{f}$ är av typ $\mathbb{R}^p \to \mathbb{R}^n$ och $\vec{g}$ av typ $\mathbb{R}^n \to \mathbb{R}^p$ och båda dessutom differentierbara så är $\vec{f}(\vec{g}(\vec{x}))$ också differentierbar och

$$J_{\vec{f}\circ\vec{g}}(\vec{x}) = J_{\vec{f}}(\vec{g}(\vec{x}))\, J_{\vec{g}}(\vec{x}).$$

$$\left([\vec{f}(\vec{g}(\vec{x}))]' = \vec{f}\,'(\vec{g}(\vec{x})) \cdot \vec{g}\,'(\vec{x}). \right)$$

Exempel: $f : \mathbb{R}^2 \to \mathbb{R}$

$\vec{g} : \mathbb{R} \to \mathbb{R}^2$

(15)

$$f(x,y) = x^2 + y^2$$

$$\vec{g}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$$J_f = \operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x, 2y)$$

$$J_g = \begin{pmatrix} \frac{d}{dt}\cos t \\ \frac{d}{dt}\sin t \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

Om $F_1(t) = f(\vec{g}(t))$ har vi

$$J_{F_1}(t) = (2\cos t, 2\sin t)\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} = 0.$$

Om $F_2(x,y) = \vec{g}(f(x,y))$ får vi

$$J_{F_2}(x,y) = \begin{pmatrix} -\sin(x^2+y^2) \\ \cos(x^2+y^2) \end{pmatrix}(2x, 2y) =$$

$$= \begin{pmatrix} -2x\sin(x^2+y^2) & -2y\sin(x^2+y^2) \\ \cos(x^2+y^2)2x & 2y\cos(x^2+y^2) \end{pmatrix}$$ 16

$$F_1(t) = f(\vec{g}(t)) = \cos^2 t + \sin^2 t = 1$$

$$F_2(x,y) = \begin{pmatrix} \cos(x^2+y^2) \\ \sin(x^2+y^2) \end{pmatrix}$$

Kedje regeln:

$$f: \mathbb{R}^n \to \mathbb{R} \qquad g: \mathbb{R} \to \mathbb{R}^n$$

$$F(t) = f(g(t))$$

$$\operatorname{grad} f = J_f(x_1, \dots, x_n) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

$$J_g(t) = \begin{pmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix}$$

$$J_F(t) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) \begin{pmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix} =$$

$$= \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \ldots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}.$$

Riktningsderivator

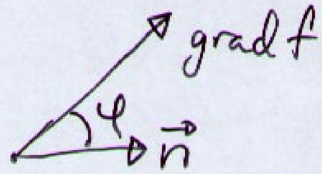
Def 4.5: Om $\vec{n}$ är en enhetsvektor så sägs $(f : \mathbb{R}^n \to \mathbb{R})$

$$f'_{\vec{n}}(\vec{x}) = \lim_{t \to 0} \frac{f(\vec{x} + t\vec{n}) - f(\vec{x})}{t} =$$

vara f:s **riktningsderivata** i punkten $\vec{x}$ i riktning $\vec{n}$.

Sats 4.9:

$$\boxed{f'_{\vec{n}}(\vec{x}) = \operatorname{grad} f(\vec{x}) \cdot \vec{n}}$$



$$\operatorname{grad} f \cdot \vec{n} = |\vec{n}|\,|\operatorname{grad} f| \cos\varphi$$
$$= |\operatorname{grad} f| \cos\varphi$$

$\Rightarrow$ $f'_{\vec{n}}$ är störst i gradientens riktning.

Transformation av diffekv

Exempel: $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$

Vi vill tranformera det till en ekvation med derivator i variablerna u, v där

$$x = u + v$$
$$y = \frac{1}{2}(u^2 + v^2)$$

Vi har att

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = \left(\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

$$\left(\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(u+v) = 1$$

$$\frac{\partial x}{\partial v} = 1$$

$$\frac{\partial y}{\partial u} = u \ , \ \frac{\partial y}{\partial v} = v$$

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ u & v \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} v & -1 \\ -u & 1 \end{pmatrix} \frac{1}{(v-u)}$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{v}{v-u}\frac{\partial z}{\partial u} - \frac{u}{v-u}\frac{\partial z}{\partial v} -$$

$$- \frac{1}{v-u}\frac{\partial z}{\partial u} + \frac{1}{v-u}\frac{\partial z}{\partial v} =$$

$$= \frac{(v-1)}{(v-u)}\frac{\partial z}{\partial u} - \frac{(u-1)}{v-u}\frac{\partial z}{\partial v}$$

$$G(x) = \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy$$

$$\boxed{\begin{aligned} G'(x) = \int_{\alpha(x)}^{\beta(x)} \frac{\partial f}{\partial x}(x,y)\,dy + f(x,\beta(x))\beta'(x) \\ - f(x,\alpha(x))\alpha'(x). \end{aligned}}$$

Bevis: Låt $F(x,u,v) = \int_v^u f(x,y)\,dy$

Så $G(x) = F(x, \beta(x), \alpha(x))$.

Kedjeregeln ger

$$G'(x) = \frac{\partial F}{\partial x} \cdot 1 + \frac{\partial F}{\partial u}\frac{du}{dx} + \frac{\partial F}{\partial v}\frac{dv}{dx}$$

$$= \int_v^u \frac{\partial f}{\partial x}(x,y)\,dy + f(x,u)\frac{du}{dx} - f(x,v)\frac{dv}{dx} = \begin{bmatrix} u = \beta(x) \\ v = \alpha(x) \end{bmatrix}$$

$$= \int_{\alpha(x)}^{\beta(x)} \frac{\partial f}{\partial x}(x,y)\,dy + f(x,\beta(x))\beta'(x) - f(x,\alpha(x))\alpha'(x).$$