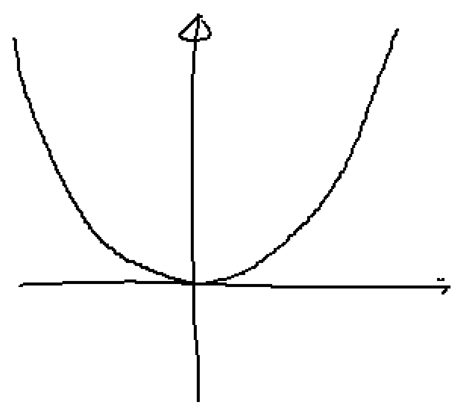
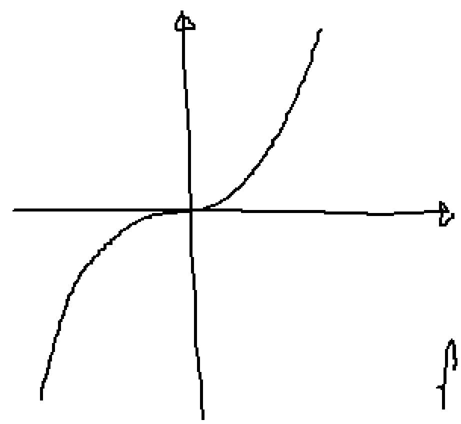


$f(x) = x^2$

$x = 0$

$f'(x) \quad - \quad 0 \quad +$ min för f

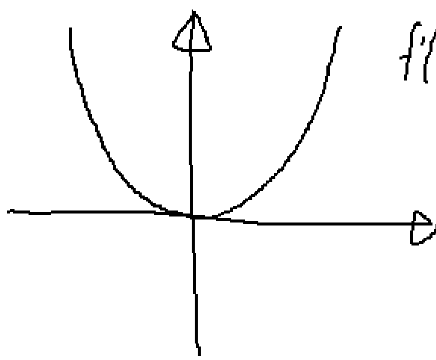
Om lokalt maximum eller minimum så är $f'(x) = 0$.



$f(x) = x^3$

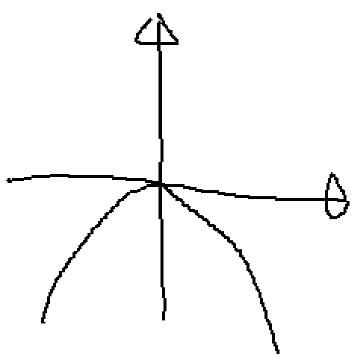
$f'(0) = 0$

men $x=0$ är varken lokalt max eller minimum



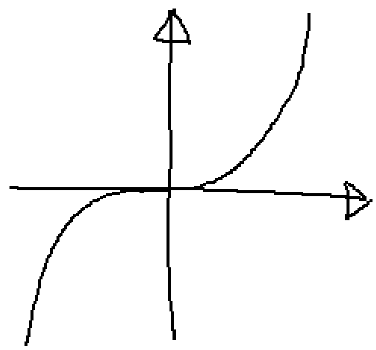
$f(x) = x^2$ $\quad f''(0) = ? > 0$

lok min



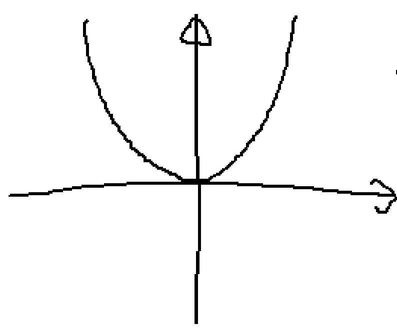
$f(x) = -x^2$ $\quad f''(0) = -2$

lok max



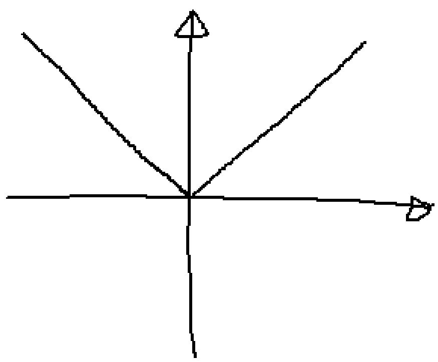
$f(x) = x^3$

$f''(0) = 0$



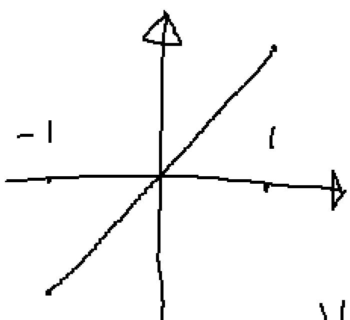
$f(x) = x^4$

$f''(0) = 0$



$f(x) = |x|$

f ej deriverbar i punkten $x = 0$

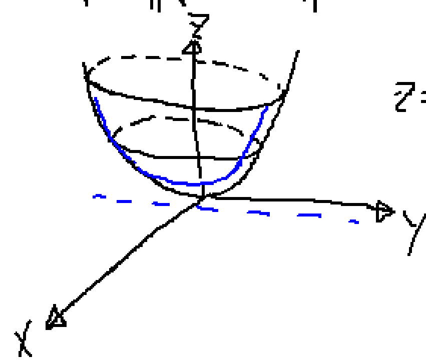


$f(x) = x$

$f'(x) = 1 \neq 0$ för alla x

Vi får min och max i ändpunkterna.

$f: \mathbb{R}^2 \to \mathbb{R}$



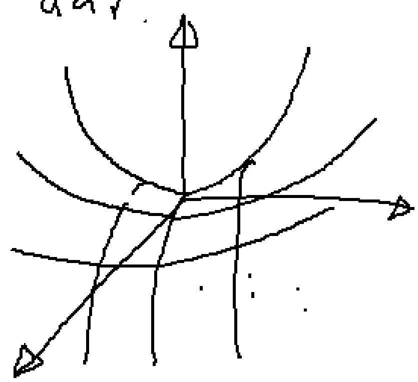
$z = f(x,y) = x^2 + y^2$

Om vi fixerar x är vi tillbaka i en variabelfallet, så lokalt max eller min ger $\frac{\partial f}{\partial y} = 0$.

På samma sätt får vi att $\frac{\partial f}{\partial x} = 0$ i punkten.

Lokalt max eller min medför att $\operatorname{grad} f = (0,0)$.

grad $f(a,b) = (0,0)$ ger inte nödvändigtvis att funktionen har lokalt max eller min där.



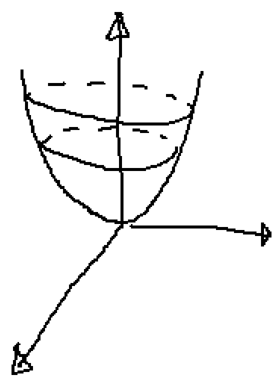
$z = f(x,y) = y^2 - x^2$

$\text{grad} f(0,0) = (0,0)$

men $(0,0)$ varken min eller max

En sadelpunkt.

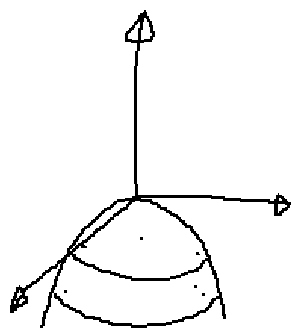
En punkt där grad $f = (0,0)$ kallas kritisk eller stationär punkt



$$z = f(x,y) = x^2 + y^2$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = 2, \quad \frac{\partial^2 f}{\partial y^2}(0,0) = 2$$

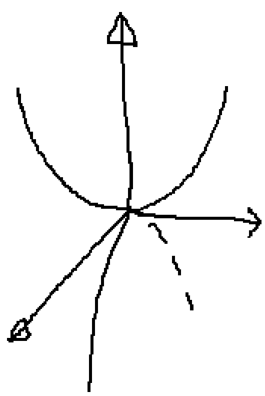
$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$$



$$z = f(x,y) = -x^2 - y^2$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = -2$$

$$\frac{\partial^2 f}{\partial y^2}(0,0) = -2 \qquad \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$$



$$z = f(x,y) = y^2 - x^2$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = -2 \qquad \frac{\partial^2 f}{\partial y^2}(0,0) = 2$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$$

$$f(x+h, y+k) = f(x,y) + \frac{\partial f}{\partial x}(x,y)\,h +$$

$$+ \frac{\partial f}{\partial y}(x,y)\,k + \frac{1}{2}\Big(\frac{\partial^2 f}{\partial x^2}(x,y)\,h^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x,y)\,hk$$

$$+ \frac{\partial^2 f}{\partial y^2}(x,y)\,k^2 \Big) + O(3)$$

Om stationär punkt har vi att $\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = 0$.

$$\frac{\partial^2 f}{\partial x^2}(x,y)\,h^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x,y)\,hk + \frac{\partial^2 f}{\partial y^2}(x,y)\,k^2 =$$

$$(h,k)\begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}$$

$\mathbb{K}$ (symmetrisk matris)

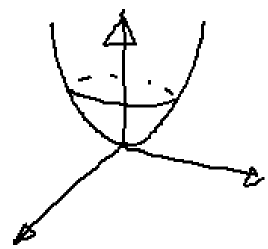
$$\det \mathbb{K} = \frac{\partial^2 f}{\partial x^2}(x,y)\,\frac{\partial^2 f}{\partial y^2}(x,y) - \left(\frac{\partial^2 f}{\partial x \partial y}(x,y)\right)^2$$

$$\| $$

$$\lambda_1 \cdot \lambda_2 = \det \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

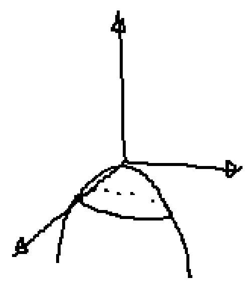
$\det \mathbb{H} > 0$.

$\lambda_1, \lambda_2 > 0$ $\quad \lambda_1 \xi^2 + \lambda_2 \eta^2$



lokalt min

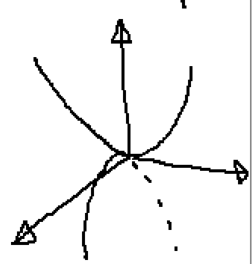
$\lambda_1, \lambda_2 < 0$ $\quad -|\lambda_1| \xi^2 - |\lambda_2| \eta^2$



lokalt max

$\det \mathbb{H} < 0$ $\quad -|\lambda_1| \xi^2 + |\lambda_2| \eta^2$

$\lambda_1 < 0, \lambda_2 > 0$



sadelpunkt

Exempel: $f(x,y) = xy$

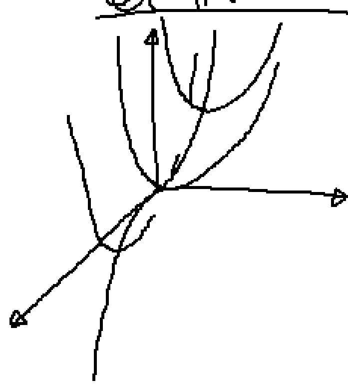
$\operatorname{grad} f(x,y) = (y, x)$

$(x,y) = (0,0)$ stationär punkt

$\frac{\partial^2 f}{\partial x^2}(0,0) = 0, \quad \frac{\partial^2 f}{\partial y^2}(0,0) = 0, \quad \frac{\partial^2 f}{\partial x \partial y}(0,0) = 1$

$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$ sadelpunkt

$\det \mathbb{H} = 0.$

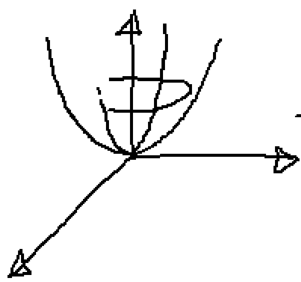


$$z = f(x,y) = y^2 - x^3$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = 0 \qquad \frac{\partial^2 f}{\partial y^2}(0,0) = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\frac{\partial^2 f}{\partial x^2}(0,0)\,\frac{\partial^2 f}{\partial y^2}(0,0) - \left(\frac{\partial^2 f}{\partial x \partial y}(0,0)\right) = 0 \cdot 2 - 0 = 0$$



$$z = f(x,y) = y^2 + x^4$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = 0 \quad \frac{\partial^2 f}{\partial y^2}(0,0) = 2, \quad \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$$

$$\det \mathbb{H} = 0.$$

Exempel: $f(x,y) = x^2 + axy + y^2$

$$\frac{\partial f}{\partial x}(x,y) = 2x + ay, \quad \frac{\partial f}{\partial y}(x,y) = ax + 2y$$

Punkten $(x,y) = (0,0)$ är en stationär punkt.

$$\frac{\partial^2 f}{\partial x^2}(0,0) = 2 \qquad \frac{\partial^2 f}{\partial y^2}(0,0) = 2$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = a$$

$$\frac{\partial^2 f}{\partial x^2}(0,0)\frac{\partial^2 f}{\partial y^2}(0,0) - \left(\frac{\partial^2 f}{\partial x \partial y}(0,0)\right)^2 = 4 - a^2$$

		-2		2	
$4-a^2$	$-$	0	$+$	0	$-$

$|a| > 2 \Rightarrow 4-a^2 < 0 \Rightarrow$ sadelpunkt

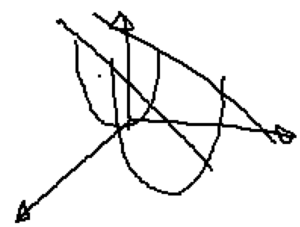
$-2 < a < 2 \Rightarrow 4-a^2 > 0 \Rightarrow$ lokalt max eller min

$$\begin{vmatrix} 2-\lambda & a \\ a & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - a^2 \qquad \begin{aligned} \lambda_1 &= 2+a \\ \lambda_2 &= 2-a \end{aligned}$$

$\lambda_1, \lambda_2 > 0 \Rightarrow$ lokalt minimum.

$a = -2$: $x^2 - 2xy + y^2 = (x-y)^2$

konstant = 0
längs $x = y$



$a = 2$: $x^2 + 2xy + y^2 = (x+y)^2$

konstant = 0
längs linjen $x = -y$.

$\lambda_1, \lambda_2 > 0$ $\quad$ $\lambda_1 \xi^2 + \lambda_2 \eta^2 \geq 0$

och $= 0$ enbart då

$(\xi, \eta) = (0,0)$

positivt definit

$\lambda_1, \lambda_2 < 0$ $\quad$ $-|\lambda_1| \xi^2 - |\lambda_2| \eta^2 \leq 0$

och $= 0$ precis då

$(\xi, \eta) = (0,0)$.

negativt definit

$\lambda_1 > 0, \lambda_2 < 0$: $\lambda_1 \xi^2 - |\lambda_2| \eta^2 > 0$

för vissa (ξ, η) men

< 0 för andra.

indefinit

$\lambda_1 > 0, \lambda_2 = 0$: $\lambda_1 \xi^2 \geq 0$ $= 0$ precis då $\xi = 0$

positivt semidefinit

$\lambda_1 < 0, \lambda_2 = 0$ $-|\lambda_1| \xi^2 \leq 0$ $= 0$ om $\xi = 0$

negativt semidefinit

Exempel: $x^2 + 2xy + y^2 = (x+y)^2 \geq 0$

$(x+y)^2 = 0$ om $x = -y$

positivt semidefinit

Sats: $f(\vec{x}+\vec{h}) = f(\vec{x}) + P_k(\vec{h}) + O(k+1)$

där $f: \mathbb{R}^n \to \mathbb{R}$, P_k är en form av grad k, så gäller

$P_k(\vec{h})$ positivt def $\Rightarrow$ lok min

$P_k(\vec{h})$ neg def $\Rightarrow$ lok max

Exempel:

$f(x) = x^3$

indefinit

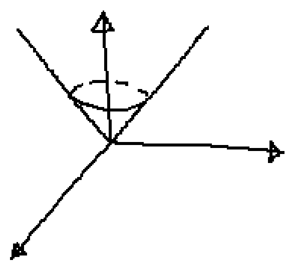
inget extremvärde

$f(x) = x^4$

pos def

$x=0$ lokalt min

Exempel: $z = f(x,y) = \sqrt{x^2+y^2}$



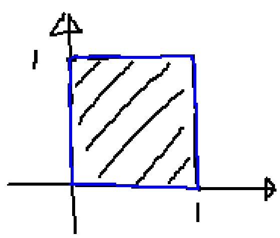
$$\frac{\partial f}{\partial x}(x,y) = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

Derivatorna är ej definierade när $(x,y) = (0,0)$.

V. kan ha extrempunkter även i sådana punkter.

Exempel: $f(x,y) = x^2 + y^2$



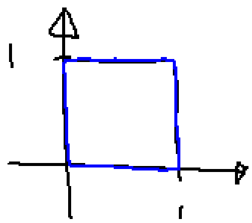
Bestäm max och min för f i området

– Stationära punkter

$\text{grad} f(x,y) = (2x, 2y) \neq (0,0)$

i det inre av området $\Longrightarrow$

inga stationära punkter



Randen består av sträckorna

I1. $x=0, \quad 0 \leq y \leq 1$

I2. $x=1, \quad 0 \leq y \leq 1$

I3. $y=0, \quad 0 \leq x \leq 1$

I4. $y=1, \quad 0 \leq x \leq 1$

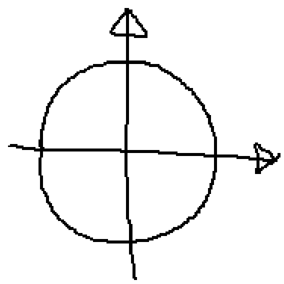
Vi vill maximera f när vi är på I1, I2, I3, I4.

I1: $g(y)=f(0,y)$ ska maximeras då $0 \leq y \leq 1$.

Kolla om $g'(y)=0$ för något $0<y<1$

och sedan $y=0$ och $y=1$.

Exempel:



Området

$\{ x^2 + y^2 \leq 1 \}$

$f(x,y) = x$

Kolla stationära punkter i det inre av området

$$\operatorname{grad} f(x,y) = (1,0) \neq (0,0)$$

Kolla randen: $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$ $\quad 0 \leq t \leq 2\pi$

$$g(t) = f(\cos t, \sin t) = \cos t$$

$$g'(t) = -\sin t = 0 \text{ om } t = \cancel{0}, \pi, 2\cancel{\pi}$$

$t = \pi \Rightarrow f(\cos t, \sin t) = \cos t = -1$

Slutligen kollar vi ändpunkterna

$$\left.\begin{array}{l} t = 0 \\ t = 2\pi \end{array}\right. \Rightarrow f(\cos t, \sin t) = \cos t = 1.$$

Så f antar max i punkten $(1,0)$
och min i punkten $(-1,0)$.

Det kan vara svårt att hitta en parametrisering av randen. Men om randen ges av en funktion

$$g(x,y)=0$$

och

$$\text{grad}\, g(x_0,y_0) \neq 0$$

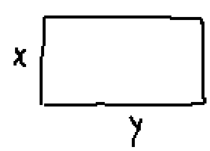
så vet vi att den kan beskrivas nära punkten som en parameterkurva.

Nära randpunkten (x_0, y_0)
ges randen av $\vec{r}(t)$.
Det betyder att vi vill
minimera/maximera $f(\vec{r}(t))$ som funktion
av t. Från envariabel analys följer
att $$0 = \frac{d}{dt} f(\vec{r}(t)) = \operatorname{grad} f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$
Vi kan också beskriva randen
som en nivåkurva $g(x,y) = 0$
så grad g är normal till kurvan.

Det betyder att

$$\operatorname{grad} f(x_0, y_0) = \lambda \operatorname{grad} g(x_0, y_0)$$

Exempel:



$g(x,y) = x + y - 1 = 0$

Bestäm x, y så att rektangeln får maximal area.

$f(x,y) = x \cdot y$

$\operatorname{grad} f(x,y) = (y, x)$

$\operatorname{grad} g(x,y) = (1, 1)$

$g(x,y) = 0$

$\operatorname{grad} f \parallel \operatorname{grad} g$

$\Rightarrow x = y = \frac{1}{2}$

Dvs max när kvadrat.

$f(x,y) = x \qquad g(x,y) = x^2 + y^2 - 1 = 0$

$\operatorname{grad} f = (1,0) \qquad \operatorname{grad} g(x,y) = (2x, 2y)$

$\Rightarrow$ extrempunkt när $y = 0$

$\Rightarrow \quad x^2 - 1 = 0 \quad \Rightarrow \quad x = \pm 1.$