När är en (2×2)-matris $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ inverterbar?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{a} \Leftrightarrow \begin{pmatrix} 1 & b/a \\ c & d \end{pmatrix} -c \Leftrightarrow \begin{pmatrix} 1 & b/a \\ 0 & d-bc/a \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix} -b/a \Leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Vi har antagit $a \neq 0$ och $d - \frac{bc}{a} \neq 0$. Uttrycket $ad-bc$ kallas matrisens determinant och betecknas: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ eller $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Vi ser att

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ inverterbar} \Leftrightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$$

Motsvarande för (3×3)-matriser

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \frac{1}{x_1} \Leftrightarrow \begin{pmatrix} 1 & y_1/x_1 & z_1/x_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} -x_2 \quad -x_3$$

$$\Leftrightarrow \begin{pmatrix} 1 & y_1/x_1 & z_1/x_1 \\ 0 & y_2 - x_2 y_1/x_1 & z_2 - x_2 z_1/x_1 \\ 0 & y_3 - x_3 y_1/x_1 & z_3 - x_3 z_1/x_1 \end{pmatrix} \frac{x_1}{x_1 y_2 - x_2 y_1}$$

$$\Leftrightarrow \begin{pmatrix} 1 & y_1/x_1 & z_1/x_1 \\ 0 & 1 & \dfrac{x_1 z_2 - x_2 z_1}{x_1 y_2 - x_2 y_1} \\ 0 & y_3 - x_3 y_1/x_1 & z_3 - x_3 z_1/x_1 \end{pmatrix} -\frac{y_1}{x_1} \quad -y_3 + \frac{x_3 y_1}{x_1}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 0 & \dfrac{z_1 y_2 - y_1 z_2}{x_1 y_2 - x_2 y_1} \\ 0 & 1 & \dfrac{x_1 z_2 - x_2 z_1}{x_1 y_2 - x_2 y_1} \\ 0 & 0 & \dfrac{x_1 y_2 z_3 - \boxed{x_3 y_2 z_1} - x_2 y_1 z_3 - x_1 y_3 z_2 + \boxed{x_2 y_3 z_1} + x_3 y_1 z_2}{x_1 (x_1 y_2 - x_2 y_1)} \end{pmatrix}$$

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_1y_3z_2 - x_2y_1z_3 - x_3y_2z_1$$

id , (1 2 3) , (1 3 2) jämna

(2 3) , (1 2) , (1 3) udda

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1y_2 - x_2y_1$$

id jämn , (12) udda.

$$\begin{vmatrix} 1 & 5 \\ 3 & 4 \end{vmatrix} = 4 - 15 = -11$$

$$\begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} = 15 - 4 = 11$$

(5)

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = x_1 y_2 z_3 + y_1 z_2 x_3 + z_1 x_2 y_3 - x_1 z_2 y_3 - y_1 x_2 z_3 - z_1 y_2 x_3$$

$$\det A = \det(A^T)$$

$$\begin{vmatrix} x_0 + x_1 & y_0 + y_1 \\ x_2 & y_2 \end{vmatrix} = \begin{vmatrix} x_0 & y_0 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

$$(x_0 + x_1) y_2 - x_2 (y_0 + y_1) = (x_0 y_2 - x_2 y_0) + (x_1 y_2 - x_2 y_1)$$

$$\begin{vmatrix} kx_1 & ky_1 & kz_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = kx_1 y_2 z_3 + ky_1 z_2 x_3 + kz_1 x_2 y_3 - kx_1 z_2 y_3 - ky_1 x_2 z_3 - kz_1 y_2 x_3$$

$$= k \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$\begin{vmatrix} x_1 & y_1 \\ x_1 & y_1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x_1 + kx_2 & y_1 + ky_2 \\ x_2 & y_2 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

Exempel: Beräkna $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 2 & 0 & 2 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & 1 \\ 2 & 0 \end{matrix}$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -5 \\ 0 & -4 & -4 \end{vmatrix} = -4 \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -5 \\ 0 & 1 & 1 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -3 & -5 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{vmatrix} = -8 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -8 \cdot 1$$

$$1 \cdot 1 \cdot 2 + 2 \cdot 1 \cdot 2 + 3 \cdot 2 \cdot 0 - 1 \cdot 1 \cdot 0 - 2 \cdot 2 \cdot 2 - 3 \cdot 2 \cdot 1$$

Definition: Låt $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

då sätter vi

$$\det A = \sum (-1)^{\sigma} a_{1\sigma(1)} \cdot \ldots \cdot a_{n\sigma(n)}$$

där summan går över alla permutationer.

Exempel: Beräkna $\begin{vmatrix} 1 & 4 & 7 & 3 \\ 0 & 3 & 1 & 10 \\ 0 & 3 & 2 & 6 \\ 0 & 0 & 2 & 4 \end{vmatrix}$

$$= 3 \begin{vmatrix} 1 & 4 & 7 & 3 \\ 0 & 1 & \frac{1}{3} & \frac{10}{3} \\ 0 & 3 & 2 & 6 \\ 0 & 0 & 2 & 4 \end{vmatrix} = 3 \begin{vmatrix} 1 & 4 & 7 & 3 \\ 0 & 1 & \frac{1}{3} & \frac{10}{3} \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 2 & 4 \end{vmatrix}$$

$$=3\begin{vmatrix} 1 & 4 & 7 & 3 \\ 0 & 1 & \frac{1}{3} & \frac{10}{3} \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 12 \end{vmatrix} = 3\cdot 1\cdot 1\cdot 1\cdot 12 = 36$$

Sats 6.3: Följande påståenden är ekvivalenta:

i) $\det A \neq 0$

ii) A är inverterbar

iii) $A\mathbb{x} = \mathbb{b}$ har precis en lösn.

iv) $A\mathbb{x} = \mathbb{0}$ har bara lösningen $\mathbb{x} = \mathbb{0}$.

v)

Ex: Låt $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ och lös $A\mathbb{x} = \begin{pmatrix} 10 \\ -5 \end{pmatrix}$.

$$\left(\begin{array}{cc|c} 1 & 2 & 10 \\ 3 & 1 & -5 \end{array}\right) -3 \Leftrightarrow \left(\begin{array}{cc|c} 1 & 2 & 10 \\ 0 & -5 & -35 \end{array}\right) -\tfrac{1}{5} \Leftrightarrow$$

$$\Leftrightarrow \left(\begin{array}{cc|c} 1 & 2 & 10 \\ 0 & 1 & 7 \end{array}\right) -2 \Leftrightarrow \left(\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 7 \end{array}\right)$$

$x_1 = -4$
$x_2 = 7$

$$\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1\cdot 1 - 2\cdot 3 = -5$$

$$\begin{vmatrix} 1 & 10 \\ 3 & -5 \end{vmatrix} = -5 - 30 = -35$$

$$\begin{vmatrix} 10 & 2 \\ -5 & 1 \end{vmatrix} = 10 - (-5)\cdot 2 = 20$$

$$x_1 = \frac{20}{-5}, \quad x_2 = \frac{-35}{-5}$$

$$A^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{pmatrix}$$

$$= -\frac{1}{5} \begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}$$

$$A^{-1} \begin{pmatrix} 10 \\ -5 \end{pmatrix} = -\frac{1}{5} \cdot \begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 10 \\ -5 \end{pmatrix}$$

$$= -\frac{1}{5} \begin{pmatrix} 1\cdot 10 + (-2)(-5) \\ (-3)\cdot 10 + 1\cdot(-5) \end{pmatrix}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = x_1 y_2 z_3 + x_2 y_3 z_1 + x_3 y_1 z_2 - x_1 y_3 z_2 - x_2 y_1 z_3 - x_3 y_2 z_1$$

$$= x_1 (y_2 z_3 - y_3 z_2) + x_2 (y_3 z_1 - y_1 z_3) + x_3 (y_1 z_2 - y_2 z_1) = x_1 \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} - x_2 \begin{vmatrix} y_1 & z_1 \\ y_3 & z_3 \end{vmatrix} + x_3 \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}.$$

$$= x_1 (y_2 z_3 - y_3 z_2) - y_1 (x_2 z_3 - x_3 z_2) + z_1 (x_2 y_3 - x_3 y_2) = x_1 \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} - y_1 \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} + z_1 \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}.$$

$$\begin{array}{ccc} + & - & + \\ \end{array}$$
$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Exempel:

$$\begin{array}{c} \begin{matrix} + & - & + & - \end{matrix} \\ \begin{vmatrix} 1 & 2 & 4 & 1 \\ 1 & 0 & 2 & 1 \\ 3 & 1 & 1 & 0 \\ 3 & 0 & 2 & 0 \end{vmatrix} \end{array} = -1 \cdot \begin{vmatrix} 2 & 4 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{vmatrix}$$

$$+ 0 - 2 \cdot \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & 1 \\ 3 & 0 & 2 \end{vmatrix} =$$

$$= -2 + 6 + (2 + 6 + 0 - 12 - 12 - 0) =$$

$$= 4 + 8 - 24 = -12.$$

$$= -2 \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 3 & 2 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 & 1 \\ 1 & 2 & 1 \\ 3 & 2 & 0 \end{vmatrix} =$$

$$= -2(6 - 3) - (12 + 2 - 6 - 2) =$$

$$= -6 - 6 = -12.$$

$$\det(AB) = \det A \cdot \det B$$

$$\begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 0 \\ 2 & 4 & 2 \end{pmatrix}\begin{pmatrix} 4 & 1 & 0 \\ 2 & 3 & 0 \\ 1 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 15 & 35 & 20 \\ 2 & 3 & 0 \\ 18 & 24 & 8 \end{pmatrix}$$

$$\begin{vmatrix} 15 & 35 & 20 \\ 2 & 3 & 0 \\ 18 & 24 & 8 \end{vmatrix} = -2\begin{vmatrix} 35 & 20 \\ 24 & 8 \end{vmatrix} + 3\begin{vmatrix} 15 & 20 \\ 18 & 8 \end{vmatrix}$$

$$= -2(280 - 480) + 3(120 - 360) = -320$$

$$\begin{vmatrix} 1 & 3 & 5 \\ 0 & 1 & 0 \\ 2 & 4 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 5 \\ 2 & 2 \end{vmatrix} = 2 - 10 = -8$$

$$\begin{vmatrix} 4 & 1 & 0 \\ 2 & 3 & 0 \\ 1 & 5 & 4 \end{vmatrix} = 4 \cdot \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} = 4 \cdot 10 = 40$$

$$4 \cdot \begin{vmatrix} 3 & 0 \\ 5 & 4 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ 5 & 4 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = 48 - 8 + 0$$

Speciellt gäller

$$1 = \det(\underbrace{AA^{-1}}_{E}) = \det A \cdot \det A^{-1}$$

$$\boxed{\det(A^{-1}) = \frac{1}{\det A}}$$

Exempel: $A = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 0 \\ 2 & 4 & 2 \end{pmatrix}$ $\det A = -8$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 4 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} -2 \\ \leftarrow \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & -8 & -2 & 0 & 1 \end{array}\right) \begin{array}{cc} \\ -3 & 2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & -3 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -8 & -2 & 2 & 1 \end{array}\right) \quad -\frac{1}{8}$$

$$\Leftrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & -3 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{8} \end{array}\right) \quad -5$$

$$\Leftrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{4} & -\frac{7}{4} & \frac{5}{8} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{8} \end{array}\right)$$

$$\begin{vmatrix} -\frac{1}{4} & -\frac{7}{4} & \frac{5}{8} \\ 0 & 1 & 0 \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{8} \end{vmatrix} = 1 \cdot \begin{vmatrix} -\frac{1}{4} & \frac{5}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{vmatrix} =$$

$$= \left(-\frac{1}{8}\right)^2 \begin{vmatrix} 2 & -5 \\ -2 & 1 \end{vmatrix} = -\frac{1}{8}$$

Adjunkter

$$C_{11} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} C_{11} - a_{12} C_{12} + a_{13} C_{13}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{12} C_{12} + a_{22} C_{22} - a_{32} C_{32}$$

$$\operatorname{adj} A = \begin{pmatrix} C_{11} & -C_{21} & C_{31} \\ -C_{12} & C_{22} & -C_{32} \\ C_{13} & -C_{23} & C_{33} \end{pmatrix}$$

$$-a_{11} C_{21} + a_{12} C_{22} - a_{13} C_{23} =$$

$$= -a_{11} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} -$$

$$- a_{13} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

$$\boxed{A \cdot \operatorname{adj} A = \det(A)\, E}$$

$$\boxed{A^{-1} = \frac{1}{\det A} \operatorname{adj} A}$$

Exempel: $A = \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 5 & 7 & 0 & 1 \end{array}\right) \begin{array}{c} -5 \\ \end{array} \Leftrightarrow \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -8 & -5 & 1 \end{array}\right) \begin{array}{c} \\ -\frac{1}{8} \end{array}$$

$$\Leftrightarrow \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{5}{8} & -\frac{1}{8} \end{array}\right) \begin{array}{c} \\ -3 \end{array} \Leftrightarrow \left(\begin{array}{cc|cc} 1 & 0 & -\frac{7}{8} & \frac{3}{8} \\ 0 & 1 & \frac{5}{8} & -\frac{1}{8} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{7}{8} & \frac{3}{8} \\ \frac{5}{8} & -\frac{1}{8} \end{pmatrix}$$

$$\operatorname{adj} A = \begin{pmatrix} 7 & -3 \\ -5 & 1 \end{pmatrix} \quad \begin{pmatrix} C_{11} & -C_{21} \\ -C_{12} & C_{22} \end{pmatrix}$$

$$\det A = 7 - 15 = -8$$

$$\frac{1}{\det A} \operatorname{adj} A = -\frac{1}{8} \begin{pmatrix} 7 & -3 \\ -5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} = 1 \cdot 1$$

$$\operatorname{adj} \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1 \qquad \operatorname{adj} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Lös ekvationssystemet

$$\begin{cases} 5x_1 + x_2 + 2x_3 = 9 \\ 10x_1 + 6x_3 = 16 \\ x_2 + 3x_3 = 5 \end{cases}$$

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 10 & 0 & 6 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\det A = 5\begin{vmatrix} 0 & 6 \\ 1 & 3 \end{vmatrix} - 10\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} =$$

$$= -30 - 10 = -40$$

$$\operatorname{adj} A = \begin{pmatrix} \begin{vmatrix} 0 & 6 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 6 \end{vmatrix} \\ -\begin{vmatrix} 10 & 6 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 5 & 2 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 5 & 2 \\ 10 & 6 \end{vmatrix} \\ \begin{vmatrix} 10 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 5 & 1 \\ 10 & 0 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -6 & -1 & 6 \\ -30 & 15 & -10 \\ 10 & -5 & -10 \end{pmatrix}$$

$A\mathbb{x} = \mathbb{b}$

$A^{-1}A\mathbb{x} = A^{-1}\mathbb{b}$

$\mathbb{x} = A^{-1}\mathbb{b}$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{1}{\det A} \operatorname{adj} A \begin{pmatrix} 9 \\ 16 \\ 5 \end{pmatrix}$$

$$= -\frac{1}{40} \begin{pmatrix} \begin{vmatrix} 9 & 1 & 2 \\ 16 & 0 & 6 \\ 5 & 1 & 3 \end{vmatrix} \\[2ex] \begin{vmatrix} 5 & 9 & 2 \\ 10 & 16 & 6 \\ 0 & 5 & 3 \end{vmatrix} \\[2ex] \begin{vmatrix} 5 & 1 & 9 \\ 10 & 0 & 16 \\ 0 & 1 & 5 \end{vmatrix} \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \qquad \operatorname{adj} A = \begin{pmatrix} C_{11} & -C_{21} & C_{31} \\ -C_{12} & C_{22} & -C_{32} \\ C_{13} & -C_{23} & C_{33} \end{pmatrix}$$

$$\begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 5 & 1 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} \\ \begin{vmatrix} 5 & 1 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 5 & 1 \end{vmatrix} \end{pmatrix}$$