Optimality conditions

- **FoNoc** = First-order Necessary optimality conditions
- **SoNoc** = Second-order Necessary optimality conditions
- **SoSoc** = Second-order Sufficient optimality conditions
- Assume f, g and $h \in \mathcal{C}^2$. Also assume that x^* is a regular point.

λ^* denotes the Lagrangian multiplier vector.

Optimality conditions for unconstrained problems

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x)$$

- **FoNoc:** $\nabla f(x^*) = 0$.
- **SoNoc:** FoNoc and $\nabla^2 f(x^*) \succeq 0$.
- **SoSoc:** SoNoc and $\nabla^2 f(x^*) \succ 0$.

Optimality conditions for linear equality constrained problems

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & Ax = b \end{array}$$

- **FoNoc:** $Ax^* = b$ and $Z^T \nabla f(x^*) = 0$.
- **SoNoc:** FoNoc and $Z^T \nabla^2 f(x^*) Z \succeq 0$.
- **SoSoc:** SoNoc and $Z^T \nabla^2 f(x^*) Z \succ 0$.

where Z denotes a matrix (not unique) whose columns form a base of $\mathcal{N}(A)$.
e.g. let $A = (B \ N)$, then a

$$Z = \begin{pmatrix} -B^{-1}N \\ I \end{pmatrix}.$$

- An alternative of **FoNoc** is: $Ax^* = b$ and $\nabla f(x^*) = A^T \lambda^*$.

Optimality conditions for linear inequality constrained problems

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & Ax \geq b \end{array}$$

- **FoNoc:** $Ax^* \geq b$ and $\nabla f(x^*) = A_A^T \lambda_A^*$, $\lambda_A^* \geq 0$.
- **SoNoc:** FoNoc and $Z_A^T \nabla^2 f(x^*) Z_A \succeq 0$.
- **SoSoc:** SoNoc and $Z_+^T \nabla^2 f(x^*) Z_+ \succ 0$.

where A_A correspond to the active constraints in $Ax^* \geq b$ (i.e. $A_A x^* = b_A$).
where Z_A is a matrix whose columns form a base of $\mathcal{N}(A_A)$.
where Z_+ is a matrix whose columns form a base of $\mathcal{N}(A_+)$.
where A_+ is a matrix of the rows of A_A for which $\lambda_A^* > 0$.

- An alternative of **FoNoc** is: $Ax^* \geq b$, $\nabla f(x^*) = A^T \lambda^*$, $\lambda^* \geq 0$,

and $\lambda_i^* (\sum_{j=1}^n a_{ij} x_i^* - b_i) = 0$ for all i.

Optimality conditions for equality constrained problems

$$\begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & g(x)=0 \end{array}$$

• **FoNoc:** $g(x^*)=0$ and $\nabla f(x^*) = A(x^*)^T\lambda^*$.
• **SoNoc:** FoNoc and $Z(x^*)^T\nabla^2_{xx}\mathcal{L}(x^*,\lambda^*)Z(x^*)\succeq 0$.
• **SoSoc:** SoNoc and $Z(x^*)^T\nabla^2_{xx}\mathcal{L}(x^*,\lambda^*)Z(x^*)\succ 0$.

where

$$A(x) = \begin{pmatrix} \nabla g_1(x)^T \\ \vdots \\ \nabla g_m(x)^T \end{pmatrix}$$

where $\mathcal{L}(x,\lambda) = f(x) - \sum_{i=1}^m \lambda_i g_i(x)$
where $Z(x^*)$ is a matrix whose columns form a base of $\mathcal{N}(A(x^*))$.

Optimality conditions for inequality constrained problems

$$\begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & g(x)\geq 0 \end{array}$$

• **FoNoc:** $g(x^*)\geq 0$ and $\nabla f(x^*) = A_A(x^*)^T\lambda^*_A$, $\lambda^*_A\geq 0$.
• **SoNoc:** FoNoc and $Z_A(x^*)^T\nabla^2_{xx}\mathcal{L}(x^*,\lambda^*)Z_A(x^*)\succeq 0$.
• **SoSoc:** SoNoc and $Z_+(x^*)^T\nabla^2_{xx}\mathcal{L}(x^*,\lambda^*)Z_+(x^*)\succ 0$.

where A_A correspond to the active constraints in $g(x^*)\geq 0$ (i.e. $j: g_j(x^*)=0$).
where $Z_A(x^*)$ is a matrix whose columns form a base of $\mathcal{N}(A_A(x^*))$.
where Z_+ is a matrix whose columns form a base of $\mathcal{N}(A_+)$.
where $A_+(x^*)$ is a matrix of the rows of $A_A(x^*)$ for which $\lambda^*_A > 0$.
• An alternative of **FoNoc** is: $g(x^*)\geq 0$, $\nabla f(x^*) = A(x^*)^T\lambda^*$, $\lambda^*\geq 0$, and $\lambda^*_i g_i(x^*)=0$ for all i.

Optimality conditions for inequality and equality constrained problems

$$\begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & g(x)\geq 0 \\ & h(x)=0 \end{array}$$

• **FoNoc:** $g(x^*)\geq 0$, $h(x^*)=0$ and $\nabla f(x^*) = A(x^*)^T\lambda^*$, $\lambda^*_i\geq 0\ \forall\, i\in\mathcal{I}$, $\lambda^*_i g_i(x^*)=0\ \forall\, i\in\mathcal{I}$, (where $\mathcal{I}$ is the inequalities).

Please contact Axel (aringh@kth.se) if you find an error.