SF2832 - Mathematical systems theory, Autumn 2016
Exercise session 5 - An example on stabilizability

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Since I did not have time to do this example, but I think it is an illustrative example that connects several parts of the course material, I have written this short note.

Problem: Consider the system

$$\dot{x} = \underbrace{\begin{bmatrix} 13 & 6 & -8 \\ -12 & -5 & 8 \\ 8 & 4 & -5 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{b} u.$$

Find a feedback control $u = Kx$ that stabilize the system.

Solution: We solve this problem in several steps.

Step 1: Is A a stable matrix?

$$\chi_A(\lambda) = \det(\lambda I - A) = \ldots = (\lambda - 3)(\lambda - 1)(\lambda + 1).$$

Hence two eigenvalues are real part > 0 and hence A is not a stable matrix. This disqualifies the feedback $K = 0$.

Step 2: Is the system reachable?

$$\Gamma = \begin{bmatrix} b & Ab & A^2b \end{bmatrix} = \begin{bmatrix} 1 & 5 & 17 \\ 0 & -4 & 16 \\ 1 & 3 & 9 \end{bmatrix}.$$

However $-3v_1 + 4v_2 - v_3 = 0$ which means that Γ is not full rank (can also be seen from $\det(\Gamma) = 0$). Moreover, it can be shown that $\text{rank}(\Gamma) = 2$. Hence the system is not reachable and we cannot for sure use pole assignment to stabilize the system.

Step 3: However, we can still stabilize the system if the uncontrollable part is stable. Recall the Kalman decomposition:

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} u.$$

If the block

$$\begin{bmatrix} A_{33} & A_{34} \\ 0 & A_{44} \end{bmatrix}$$

is stable, we can use control in order to stabilize the first two components, and hence the entire system would be stable. However, since we are not interested in the output in this case, we only need to make a coordinate change that separates the controllable and uncontrollable part.

To this end, we note that

$$\mathrm{Im}(\Gamma) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} \right\}$$

In order to make the transformation we now need a last vector w so that

$$\mathrm{span}\{w\} \oplus \mathrm{Im}(\Gamma) = \mathbb{R}^3.$$

We can for example take

$$w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This gives a transformation

$$T = \begin{bmatrix} 1 & 5 & 0 \\ 0 & -4 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

and hence

$$x = Tz \implies \dot{z} = T^{-1}ATz + T^{-1}Bu = \begin{bmatrix} 0 & -3 & 2 \\ 1 & 4 & -2 \\ 0 & 0 & -1 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u.$$

We note that the uncontrollable part, which is the third component (since $\mathrm{rank}(\Gamma) = 2$) is stable. Hence we can stabilize this system using feedback! In order to do that we consider the system described by the first two components

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u.$$

By construction this is controllable, and hence we can do pole assignment. The ansatz $u = k_1 z_1 + k_2 z_2$ gives the matrix

$$A + bK = \begin{bmatrix} k_1 & -3 + k_2 \\ 1 & 4 \end{bmatrix},$$

which has a characteristic polynomial

$$\chi_{A+bK}(\lambda) = \det(\lambda I - (A + bK)) = \ldots = \lambda^2 + (-4 - k_1)\lambda + (4k_1 - 3k_2 + 3).$$

For a monic polynomial of degree 2, the roots are negative if both coefficients are positive.[1] Hence a sufficient condition for stability is

$$\begin{aligned} 4 + k_1 &< 0 \\ 4k_1 - 3k_2 + 3 &> 0 \end{aligned}$$

[1] To see this, just note that the equation $x^2 + ax + b = 0$ has the solutions $x = -a/2 \pm \sqrt{a^2/4 - b}$.

which is fulfilled for, e.g., $k_1 = -5$ and $k_2 = -6$. A stabilizing control law is thus given by

$$u = \begin{bmatrix} -5 & -6 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -5 & -6 & 0 \end{bmatrix} T^{-1} x.$$