

THE FORBIDDEN REGION FOR RANDOM ZEROS: APPEARANCE OF QUADRATURE DOMAINS

ALON NISHRY AND ARON WENNMAN

ABSTRACT. Our main discovery is a surprising interplay between quadrature domains on the one hand, and the zero process of the Gaussian Entire Function (GEF) on the other. Specifically, consider the GEF conditioned on the rare *hole event* that there are no zeros in a given large Jordan domain. We show that in the natural scaling limit, a quadrature domain enclosing the hole emerges as a *forbidden region*, where the zero density vanishes. Moreover, we give a description of those holes for which the forbidden region is a disk.

The connecting link between random zeros and potential theory is supplied by a constrained extremal problem for the Zeitouni-Zelditch functional. To solve this problem, we recast it in terms of a seemingly novel obstacle problem, where the solution is forced to be harmonic inside the hole.

CONTENTS

1. Introduction and main results	1
2. Preliminaries and notation	8
3. Perturbation theory for the mixed obstacle problem	18
4. Reformulation of the constrained extremal problem	25
5. Separation of the free boundary from the thin obstacle	31
6. The emergence of quadrature domains	35
7. Global Hölder regularity of the potential	38
8. Quantitative stability under domain perturbations	43
9. The limiting conditional zero distribution	48
10. The inverse problem, disk-like domains and examples	57
Appendix A. Approximation of measures with Fekete points	62
References	68

1. INTRODUCTION AND MAIN RESULTS

1.1. Random zeros and forbidden regions. We are interested in the asymptotic conditional intensity of a stationary point process in the plane, conditional on the rare *hole event* that there are no points in a large region \mathcal{G} . To give some context, we mention briefly two examples where the behavior on the hole event is well-understood. For the homogeneous Poisson process, the spatial independence property shows that the effect of the hole is not felt outside \mathcal{G} . For the Ginibre ensemble, a planar Coulomb gas at critical temperature, the situation is more interesting: the particles accumulate near the boundary of \mathcal{G} in such a way that the

2010 *Mathematics Subject Classification.* 30B20, 35R35, 31A35, 60F10, 30C70.

electrostatic potential generated by the points is asymptotically unchanged outside the hole (a procedure known as *balayage*). Hence, there are no macroscopic effects outside $\bar{\mathcal{G}}$, see Figure 1.1(A). For details we refer to the works by Adhikari and Reddy [2], Armstrong, Serfaty and Zeitouni [6], and Jancovici, Lebowitz and Manificat [40].

Our focus is on the zero process of the Gaussian Entire Function (GEF), introduced by Bogomolny, Bohigas and Labeuf [14], Kostlan [43], and Shub and Smale [68] in the 1990s. For convenience, we consider a fixed domain \mathcal{G} and a zero process with increasing intensity. The GEF is given by the random Taylor series

$$F_L(z) = \sum_{n=0}^{\infty} \xi_n \frac{(Lz)^n}{\sqrt{n!}}, \quad z \in \mathbb{C},$$

where $\xi_n \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ are independent standard complex Gaussian random variables, and its zero set forms an invariant point process with intensity $\pi^{-1}L^2$ with respect to area measure. In fact, the GEF is the only Gaussian analytic function with this property, see [38, § 2.5]. Since its introduction, the GEF has been widely studied, with contributions including [21, 25, 33, 51, 53, 54, 70] to mention a few. For background on the GEF zeros and related models we refer to the monograph [38] by Hough, Krishnapur, Virag and Peres, and the ICM notes [52] by Sodin and Nazarov.

For a bounded plane region \mathcal{G} , we denote the *hole event for random zeros* by

$$\mathcal{H}_L(\mathcal{G}) = \{F_L \text{ is zero-free in the region } \mathcal{G}\}.$$

We let $\mu_L^{\mathbb{C}}$ be the random *empirical measure* obtained by placing a unit point charge at each zero of F_L , and denote by $\mu_{L,\mathcal{G}}^{\mathbb{C}}$ the measure $\mu_L^{\mathbb{C}}$ conditioned on $\mathcal{H}_L(\mathcal{G})$.

We will show in the course of the proof of Theorem 1.6 (c.f. Proposition 4.5) that for a rather general hole \mathcal{G} , the rescaled measure $L^{-2}\mu_{L,\mathcal{G}}^{\mathbb{C}}$ converges to a limiting measure $\mu_{\mathcal{G}}^{\mathbb{C}}$, which splits into a singular part supported on $\partial\mathcal{G}$, and a continuous part supported on the complement of a larger region $\Omega = \Omega(\mathcal{G})$ containing \mathcal{G} . There is always a macroscopic gap $\Omega \setminus \bar{\mathcal{G}}$ between the two components of the limiting measure, where the limiting zero density vanishes. We refer to Ω as the *forbidden region*¹. This phenomenon was suspected to occur for circular holes by Nazarov and Sodin, and it was recently proved by Ghosh and the first-named author in [24]. We find the following problems natural.

Problem 1.1. *Determine the possible shapes of forbidden regions.*

Problem 1.2. *Given a forbidden region, determine which holes give rise to it.*

Regarding the first problem, we will show that if the hole is a smooth Jordan domain, the forbidden region takes the shape of a quadrature domain. Under mild conditions on \mathcal{G} , we solve the inverse problem (Problem 1.2) when the forbidden region is a disk. We start with the inverse problem.

¹This terminology appears in other contexts, e.g. in quantum mechanics and semiclassical analysis [36] and for random polynomials and partial Bergman kernels [66, 60]. These notions are related with ours in that a limiting density of states vanishes, but for entirely different reasons.

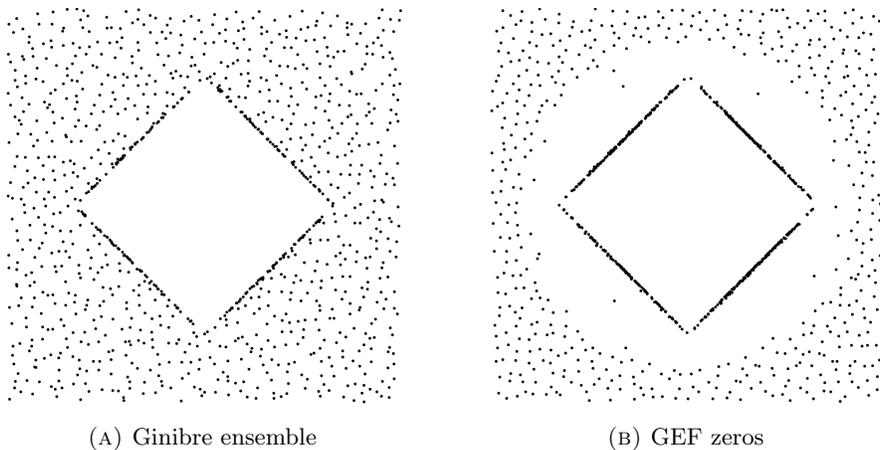


FIGURE 1.1. Illustration of the square hole event.

1.2. The inverse problem and disk-like domains. The connection between the hole \mathcal{G} and the associated forbidden region $\Omega(\mathcal{G})$ appears through a delicate variational problem (see § 1.4 below), and one might initially suspect that little could be established, besides regularity properties of the measure $\mu_{\mathcal{G}}^{\mathbb{C}}$ and (free) boundary $\partial\Omega$. However, in one important special case we have a complete solution to the inverse problem.

Definition 1.3. A Jordan domain \mathcal{G} is said to be *disk-like* with center z_0 and radius r , if the Riemann mapping $\varphi: \mathcal{G} \rightarrow \mathbb{D}$, which maps z_0 to the origin with $\varphi'(z_0) =: r^{-1}$, satisfies the bound

$$|\varphi(z)| \geq \frac{|z-z_0|}{r} \exp\left(-\frac{|z-z_0|^2}{2er^2}\right), \quad z \in \mathcal{G}.$$

One can verify² that the square, the regular pentagon, ellipses up to a critical eccentricity as well as a wide class of more general perturbations of the disk are all disk-like. That \mathcal{G} is disk-like implies that z_0 is a local conformal center with (inner) conformal radius r (see [58, § 6.3] and § 10.2 below). In addition, disk-likeness implies that $\mathcal{G} \subset \mathbb{D}(z_0, \sqrt{e}r)$.

Theorem 1.4. *Let \mathcal{G} be a Jordan domain with piecewise smooth boundary without cusps. Then the forbidden region Ω is the disk $\mathbb{D}(z_0, \sqrt{e}r)$ if and only if \mathcal{G} is disk-like with center z_0 and radius r .*

It follows from the proof (see § 10.2) that the leading order asymptotics for the probability of the hole event $\mathcal{H}_L(\mathcal{G})$ for disk-like \mathcal{G} depends only on the radius r . It is plausible that this result should hold even under less restrictive regularity conditions on the boundary $\partial\mathcal{G}$. For an illustration see Figure 1.1(B).

1.3. Appearance of quadrature domains. A priori, it is not clear if there are any restrictions on the shape of the forbidden region Ω . However, the stability of

²For a graphical illustration for regular n -gons with $n \geq 4$, see [55]. The equilateral triangle is not disk-like.

the circular case suggests a strong rigidity, and this led us to consider the following notion, which is classical in potential theory. We let dA denote area measure.

Definition 1.5. A domain $\Omega = \Omega_\nu$ is said to be a (subharmonic) quadrature domain with respect to a finite measure ν , if Ω contains $\text{supp}(\nu)$ and if for all integrable subharmonic functions u on Ω it holds that

$$\int u(z) d\nu(z) \leq \frac{1}{\pi} \int_{\Omega} u(z) dA(z).$$

For background on quadrature domains, we refer to the surveys [28, 30] by Gustafsson and Putinar, and Gustafsson and Shapiro, respectively. The most notable example of a quadrature domain is a disk. More generally, *classical* quadrature domains correspond to finitely supported positive measures ν , and these may be thought of as potential theoretic *sums* of disks, see [28]. Classical quadrature domains are rather outstanding domains with algebraic boundaries [65, § 5.1], which appear in several areas (see § 1.6 below). Going back to Problem 1.1, they appear as forbidden regions, seemingly out of nowhere.

Theorem 1.6. *Let \mathcal{G} be a Jordan domain with C^2 -smooth boundary. The rescaled conditional empirical measures $L^{-2} \mu_{L,\mathcal{G}}^{\mathbb{C}}$ converge vaguely in distribution as $L \rightarrow \infty$ to the measure*

$$d\mu_{\mathcal{G}}^{\mathbb{C}} = \sum_{\lambda \in \Lambda} \rho_{\lambda} d\omega_{\mathcal{G},\lambda} + \frac{1}{\pi} \chi_{\mathbb{C} \setminus \Omega_{\nu}} dA,$$

where $\Lambda \subset \mathcal{G}$ is a finite set, ρ_{λ} are strictly positive weights, $\omega_{\mathcal{G},\lambda}$ are the harmonic measures supported on $\partial\mathcal{G}$ relative to the points λ , and Ω_{ν} is the (unique) quadrature domain with respect to $\nu = \sum_{\lambda} \rho_{\lambda} \delta_{\lambda}$.

The assumption that $\partial\mathcal{G}$ is C^2 -smooth is made to fix ideas. A stronger version (Theorem 9.1) is proven in § 9 below.

The notion of vague convergence in distribution is described in [41, Ch. 4]. In our case, it amounts to proving that for any continuous compactly supported test-function f , the random variables $L^{-2} \int f d\mu_{L,\mathcal{G}}^{\mathbb{C}}$ converges in probability to $\int f d\mu_{\mathcal{G}}^{\mathbb{C}}$.

Remark 1.7. Let $\nu = \sum_{\lambda} \rho_{\lambda} \delta_{\lambda}$ be a finitely supported positive measure, and denote by Ω the associated subharmonic quadrature domain. Then, for any bounded harmonic function h on Ω , we have the quadrature rule

$$\sum_{\lambda} \rho_{\lambda} h(\lambda) = \frac{1}{\pi} \int_{\Omega} h(z) dA(z).$$

Quadrature domains for *harmonic functions* are those domains Ω who satisfy this equality for all bounded harmonic h . For many measures ν the two notions are equivalent, but in general a harmonic quadrature domain is *not* uniquely determined by the measure ν (see [28, Theorem 4.1 and § 10.4]).

1.4. Characterization of extremal measures. The work [73] of Zeitouni and Zelditch and [24] suggest that the solution to the hole problem can be understood in terms of constrained minimizers of the convex energy functional

$$(1.4.1) \quad I_{\alpha}(\mu) = -\Sigma(\mu) + 2 \sup_{z \in \mathbb{C}} \left(U^{\mu}(z) - \frac{|z|^2}{2\alpha} \right),$$

over the class of probability measures on \mathbb{C} which give no mass to the hole, where $\alpha > 0$ is a large parameter. Here, U^μ and $\Sigma(\mu)$ denote, respectively, the logarithmic potential and the (negative) logarithmic energy

$$U^\mu(z) = \int \log |z - w| d\mu(w), \quad -\Sigma(\mu) = - \int \log |z - w| d\mu(z)d\mu(w).$$

The functional (1.4.1) was originally introduced in [73] and studied in [24] in connection with the circular hole problem $\mathcal{G} = \mathbb{D}$. It may be mentioned that I_α is the Large Deviations Principle (LDP) rate function for the zero process associated to the (Weyl) polynomials obtained by truncating the GEF.

Here, we study the constrained extremal problem for a general bounded Jordan domain \mathcal{G} . We denote by $\mu_{\alpha, \mathcal{G}}$ the extremal measure obtained as the unique solution to the problem

$$(1.4.2) \quad \text{minimize } I_\alpha(\mu) \quad \text{subject to } \mu \in \mathcal{M}_{\mathcal{G}},$$

where $\mathcal{M}_{\mathcal{G}}$ is the collection of all probability measures on \mathbb{C} with $\mu(\mathcal{G}) = 0$.

If $\partial\mathcal{G}$ is a piecewise smooth curve without cusps, we denote its set of corner points by $\mathcal{E} = \mathcal{E}(\partial\mathcal{G})$. We say that μ has *regular support* in $\bar{\mathcal{G}}$, if $\text{supp}(\mu)$ is contained in a set Λ consisting of a finite number of analytic cross-cuts³ and countably many points, such that in addition $\Lambda \cap \partial\mathcal{G} \subset \mathcal{E}$. We denote by $\text{Bal}(\nu, \mathcal{G}^c)$ the *balayage* of a measure ν to the boundary $\partial\mathcal{G}$. We recall the definition of this potential theoretic notion in § 2.6.

The following result is used to establish our main results for the GEF.

Theorem 1.8. *There exists an absolute constant α_0 , such that if $\mathcal{G} \subset \mathbb{D}$ is a Jordan domain with piecewise C^2 -smooth boundary without cusps, then there exists a positive measure ν with regular support in $\bar{\mathcal{G}}$, such that for all $\alpha \geq \alpha_0$ the extremal measure is given by*

$$d\mu_{\alpha, \mathcal{G}} = \frac{1}{\alpha} d\text{Bal}(\nu, \mathcal{G}^c) + \frac{1}{\pi\alpha} \chi_{\mathbb{D}(0, \sqrt{\alpha}) \setminus \Omega_\nu} dA,$$

where Ω_ν denotes the subharmonic quadrature domain with respect to ν .

The proof is contained in § 6. For comparison with Theorem 1.6, we mention that the balayage of a unit point charge at $\lambda \in \mathcal{G}$ to \mathcal{G}^c is the harmonic measure $\omega_{\mathcal{G}, \lambda}$.

1.5. Connection to an obstacle problem. Theorem 1.8 is obtained by a variational argument in two steps, which we briefly describe.

STEP 1: An implicit obstacle problem. For a *thin obstacle* g in the Sobolev space $H^{\frac{1}{2}}(\partial\mathcal{G})$ (see § 2.3 below for details), the full obstacle function $\phi_\alpha(z) = \frac{1}{2\alpha}|z|^2$ and $D = \mathbb{D}(0, \sqrt{\alpha})$, we consider the class of functions

$$\mathcal{K}_g = \left\{ u \in H^1(D) : u \leq g \text{ on } \partial\mathcal{G}, u \leq \phi_\alpha \text{ on } D, u = \phi_\alpha \text{ on } \partial D \right\}.$$

Whenever \mathcal{K}_g is non-empty, there is a unique solution u_g to the obstacle problem

$$(1.5.1) \quad \inf_{u \in \mathcal{K}_g} \mathcal{D}(u) := \inf_{u \in \mathcal{K}_g} \int_D |\nabla u|^2 dA.$$

³A cross-cut is an arc $\gamma \subset \mathcal{G}$ without loops, which connects two boundary points $z, w \in \partial\mathcal{G}$, thereby splitting \mathcal{G} in two connected components.

We will show that for α large enough, there is *some* thin obstacle function g such that the extremal measure $\mu_{\alpha, \mathcal{G}}$ equals the Riesz measure $\mu_g = \frac{1}{2\pi} \Delta u_g$. In fact, it holds that $I_\alpha(\mu_g) = \mathcal{D}(u_g) + C(\alpha)$ for an explicit constant $C(\alpha)$, and if we denote by \tilde{g} the harmonic extension of g to \mathcal{G} , the extremal problem (1.4.2) reduces to

$$(1.5.2) \quad \text{minimize } \mathcal{D}(u_g) \quad \text{subject to } \tilde{g} \leq \phi_\alpha \text{ on } \mathcal{G} \text{ and } g \in H^{\frac{1}{2}}(\partial\mathcal{G}).$$

Here, the condition that the harmonic extension \tilde{g} lies below the full obstacle ϕ_α ensures that $\mu_g(\mathcal{G}) = 0$.

STEP 2: Finding the optimal thin obstacle. We wish to find the thin obstacle g which solves (1.5.2), and to this end we devise a perturbation argument. We put

$$B_\alpha(\mu) = \sup_{z \in \mathbb{C}} \left(U^\mu(z) - \frac{1}{2\alpha} |z|^2 \right)$$

and define the *interior coincidence set*

$$\mathcal{I} = \left\{ z \in \bar{\mathcal{G}} : U^{\mu_{\alpha, \mathcal{G}}}(z) - \frac{1}{2\alpha} |z|^2 = B_\alpha(\mu_{\alpha, \mathcal{G}}) \right\}.$$

If g is the solution to (1.5.2), then \mathcal{I} is also the coincidence set on $\bar{\mathcal{G}}$ for u_g with the full obstacle ϕ_α . For a harmonic function h on \mathcal{G} which is negative on \mathcal{I} , we denote by $u_\epsilon = u_{g_\epsilon}$ the solutions to the obstacle problem (1.5.1) where g is replaced by $g_\epsilon = g + \epsilon h|_{\partial\mathcal{G}}$ with $\epsilon > 0$. It turns out that for such h , the Riesz measures $\mu_\epsilon := \mu_{g_\epsilon}$ belong to $\mathcal{M}_\mathcal{G}$ for small ϵ . One of the keys to our approach is the variational formula of Corollary 3.7 below, which reads

$$\mathcal{D}(u_\epsilon) = \mathcal{D}(u_0) - 4\pi\epsilon \int_{\partial\mathcal{G}} h \, d\mu_0 + o(\epsilon), \quad \epsilon \rightarrow 0^+.$$

In view of (1.5.2), the integral appearing on the right-hand side must be *negative* for all admissible perturbations h , or else g would not be extremal. Using harmonic interpolation, this inequality yields Theorem 1.8 provided that the set \mathcal{I} is finite. When $\partial\mathcal{G}$ is C^2 -smooth, we show that the potential $U^{\mu_{\alpha, \mathcal{G}}}$ is *non-degenerate*, meaning that

$$(1.5.3) \quad \sup_{z \in \partial\mathcal{G}} \left(U^{\mu_{\alpha, \mathcal{G}}}(z) - \frac{|z|^2}{2\alpha} \right) < B_\alpha(\mu).$$

In particular, the coincidence set \mathcal{I} is separated from $\partial\mathcal{G}$, see Theorem 5.1 below. A classical regularity result of Caffarelli and Rivière [15] implies that under such a separation condition, either \mathcal{I} has positive area (impossible), or it is a finite set.

For a more general Jordan domain \mathcal{G} , we argue by approximation with smooth domains from inside.

Remark 1.9. We stress that the assumption that \mathcal{G} is simply connected is crucial for the conclusion of Theorem 1.6. Indeed, in the theorem of Caffarelli-Rivière, closed curves are also possible components of the coincidence set. Since we work with simply connected \mathcal{G} and a subharmonic full obstacle, the region enclosed by the curve would then be part of the coincidence set as well. Since \mathcal{I} has vanishing area, this cannot happen. If \mathcal{G} is not simply connected, then we cannot draw this conclusion, and the interior coincidence is not finite in general.

1.6. Related work. In one dimension, hole probabilities are also known as *gap probabilities*, and have been studied extensively. We mention in particular work by Majumdar, Nadal, Scardicchio and Vivo [49] on a log-gas with quadratic potential, where the probability of large gaps is computed, and the conditional distribution is described in detail. A closely related problem is the computation of *persistence probabilities* for real stochastic processes, see [7].

In the two-dimensional setting, the hole event has been studied for Coulomb gases, including the Ginibre ensemble. In this case, the LDP functional is the weighted logarithmic energy (see [63]). In the work by Armstrong, Serfaty and Zeitouni [6] the constrained minimization problem for the limiting deficiency and overcrowding events⁴ is studied with variational techniques; in the work [2] by Adhikari and Reddy, a connection to the Ginibre ensemble is drawn and the hole probabilities are computed for a large class of holes. Shirai [67] studies annular holes for Ginibre-type point processes, and describes the distribution of points on the hole event, in particular near the singular part of the limiting measure.

For the random zeros considered in this work, the asymptotics of hole probabilities was first studied by Sodin and Tsirelson [71] who estimated the decay rate of hole probabilities for circular holes. In [56], the exact leading order asymptotics was found, and as mentioned previously the existence of forbidden regions was observed for the first time in [24] in the context of a circular hole. For a more detailed account of related work, see [23] and the references therein.

The emergence of a forbidden region seems unique to random zeros, as it comes about due to the non-local nature of the term B_α of the functional I_α . To our knowledge, similar phenomena have not been discovered elsewhere. To the best of our knowledge, the functional I_α appeared first in work by Zeitouni and Zelditch [73], who established a LDP for zero sets of random polynomials with Gaussian coefficients. This was originally carried out on Riemann surfaces (see also the works by Butez [13] and Zelditch [74]).

The hole event is also interesting for dynamic point processes, for instance the dynamic GEF obtained by letting the random variables $\xi_n = \xi_{n,t}$ undergo an independent Ornstein-Uhlenbeck evolution. It was shown by Hough [37] that the probability that a hole to persist for times $0 \leq t < T$ is exceedingly small (of order $\exp(-Te^{cL^2})$). For other dynamic point processes such as lattice gases, one can answer more delicate questions, such as how the hole appears and how it subsequently disappears, see the article [44] by Krapivsky, Meerson and Sasorov.

For further results pertaining to probabilities of rare events for stationary random processes, we refer to [4, 5, 8, 17, 20, 45, 57].

Recently, questions pertaining to quantitative stability of obstacle problems have been studied. Among recent results we mention work by Serfaty and Serra [64], and Blank and LeCrone [9]. This connects to our developments in § 3.

Quadrature domains appear in several different areas of mathematics. Examples include the study of aggregation models in probability through the Diaconis-Fulton smash sum [18] studied by Levine and Peres [48], fluid mechanics and random matrix theory as explained in [31, §6.2] (see also the work by Mineev-Weinstein,

⁴In the p -deficiency (resp. q -overcrowding) problem, the hole event is replaced by the event that \mathcal{G} contains at most a fraction p with $0 \leq p < 1$ of the expected number of points (resp. at least a fraction $q > 1$ of the expected number).

Wiegmann and Zabrodin [50]), and complex dynamics as illustrated by the work of Lee, Lyubich and Makarov [46, 47].

1.7. Structure of the paper. Section 2 contains preliminary material from geometric function theory, potential theory and the theory of free boundary problems.

The main ideas underlying the proof of Theorem 1.8 were sketched above, and this program is carried out in § 3 to § 6.

In § 7 and § 8 we study regularity for the potential of the extremal measure, and quantitative stability under domain perturbations, respectively. These results are used in § 9 to prove Theorem 1.6 (see 9.2 for an overview of the proof).

In § 10 we obtain Theorem 1.4 as a consequence of a general form of Theorem 1.6 and a *sufficient condition* for a measure μ_0 to be extremal for I_α over \mathcal{M}_g . We also discuss a family of examples of holes for which the forbidden regions are two-point quadrature domains.

Finally, the appendix (a collaboration with S. Ghosh) contains a result on approximation of measures using weighted Fekete points, which is used to obtain lower bounds for the hole probability.

2. PRELIMINARIES AND NOTATION

2.1. Notational conventions. We use *positive* and *negative* in the weak sense, allowing e.g. the value zero for a positive parameter. Similarly, *greater than* and *smaller than* allow for equality, unless otherwise stated.

Unless we indicate otherwise, inequalities between functions in L^p and Sobolev spaces are interpreted in the almost everywhere sense.

For a set $E \subset \mathbb{C}$, the standard symbols E^c , E° and \overline{E} are used to denote complement, interior and closure (as a subset of Euclidean space $\mathbb{C} = \mathbb{R}^2$). Moreover, $|E|$ will be used to denote the Lebesgue measure of E . If it is understood that E is confined to a one-dimensional rectifiable subset Γ , then $|E|$ denotes the arc-length measure of E . When more precision is needed, we express these measures in terms of the length and area elements, which are denoted by ds and dA , respectively.

We will oftentimes deal with asymptotics of quantities depending on a parameter, and in particular deal with inequalities between quantities depending on that parameter. If $(a(t))_{t \in \mathcal{T}}$ and $(b(t))_{t \in \mathcal{T}}$ are quantities depending on the parameter t in an index set \mathcal{T} , then we write

$$a(t) \lesssim b(t)$$

if there exists a constant $C \in \mathbb{R}^+$ such that for all $t \in \mathcal{T}$ we have

$$a(t) \leq Cb(t).$$

The constant C may depend on other parameters. If $a(t) \lesssim b(t)$ and $a(t) \gtrsim b(t)$ hold simultaneously, then we write $a(t) \cong b(t)$. In addition we will use the standard notation $f = O(g)$ and $f = o(g)$.

For a measure μ on \mathbb{C} , we use the notation μ^s and μ^c for the singular and continuous parts in the Lebesgue decomposition with respect to area measure.

2.2. Logarithmic potentials. For a signed measure μ , we recall that the logarithmic potential U^μ of μ is the convolution of μ with the logarithmic kernel

$$U^\mu(z) = \int_{\mathbb{C}} \log |z - w| d\mu(w),$$

where we have chosen the normalization such that

$$\frac{1}{2\pi}\Delta U^\mu = \mu,$$

in the sense of distributions. The logarithmic energy of μ is the quantity

$$-\Sigma(\mu) = - \int \log |z - w| d\mu(z)d\mu(w) = - \int U^\mu(z)d\mu(z)$$

which we consider for compactly supported finite measures μ for which the function $|\log |z - w||$ is integrable with respect to the product measure $\mu \times \mu$ (so that the above computation is justified by Fubini's theorem). These measures are known as measures of finite logarithmic energy. A standard reference for potential theory in the plane is the monograph [63] by Saff and Totik. As a word of caution to the reader, we mention that in [63] the logarithmic potential is defined to have the *opposite* sign compared to what we use here.

2.3. Sobolev spaces. Unless otherwise stated, every domain D considered in this article will be a Lipschitz domain, meaning that at each boundary point z_0 there exists a number $\epsilon > 0$ such that $\partial D \cap \mathbb{D}(z_0, \epsilon)$ is the graph of a Lipschitz function $f : [0, 1] \rightarrow \mathbb{C}$, after applying an appropriate rotation.

For the basic properties and definitions of Sobolev spaces, we refer the reader e.g. to the monograph [1]. Below, we will merely fix notation and recall some properties that are especially important in what follows.

For a (Lipschitz) domain $D \subset \mathbb{C}$ we consider the space $H^1(D)$ of functions $u \in L^2(D)$ whose first order (distributional) partial derivatives lie in $L^2(D)$ as well. If Γ is a closed Lipschitz curve enclosing some domain Ω , we denote the (Dirichlet) trace space of $H^1(\Omega)$ by $H^{\frac{1}{2}}(\Gamma)$. The space $H^{\frac{1}{2}}(\Gamma)$ is itself a Sobolev space with an intrinsic characterization [19]. The dual spaces are defined as usual, and they are denoted by $H^{-1}(D) = (H^1(D))^*$ and $H^{-\frac{1}{2}}(\Gamma) = (H^{\frac{1}{2}}(\Gamma))^*$, respectively. The spaces $H^1(D)$, $H^{\frac{1}{2}}(\Gamma)$ as well as their duals are Hilbert spaces, and as such are in particular weakly sequentially compact. For the precise definitions and further basic properties of the spaces mentioned above, the reader may consult e.g. the monograph [1].

We mention that whenever Γ is a simple closed Lipschitz curve lying inside D and u is a function in $H^1(D)$ with $\Delta u \in L^2(D)$ (in particular, if u is harmonic), then we can also define a Neumann trace of u on Γ . We denote by $\mathcal{N}_\Gamma(u)$ the unique trace function which for smooth functions u satisfies

$$\mathcal{N}_\Gamma(u) = -\left(\partial_{n_{D^+}} + \partial_{n_{D^-}}\right)u,$$

where D^\pm denote the two components of $D \setminus \Gamma$, and $\partial_{n_{D^\pm}}$ denote the normal derivatives in the outward normal direction from inside D^+ and D^- , respectively. By a standard mollification argument, it is readily verified that $\mathcal{N}_\Gamma(u) \in H^{-\frac{1}{2}}(\Gamma)$ whenever $u \in H^1(D)$ with $\Delta u \in L^2(D)$ (cf. Proposition 3.1 below).

2.4. Harmonic measures and the Poisson kernel. For a Lipschitz domain Ω and a point $z_0 \in \Omega$, the harmonic measure $\omega_{\Omega, z_0}(\cdot) = \omega(z_0, \cdot, \Omega)$ is a Borel measure on $\partial\Omega$, with the property that for any $E \subset \partial\Omega$, the measure $\omega_{\Omega, z_0}(E)$ is the probability that a Brownian motion started at z_0 exits Ω through the set E . We will only need some very basic properties of harmonic measure. The first property

is that when Ω is simply connected, the density of harmonic measure with respect to arc length measure ds on $\partial\Omega$ is given by the Poisson kernel:

$$\frac{d\omega_{\Omega, z_0}}{ds} = P_{\Omega}(z_0, \cdot) \quad \text{on } z \in \partial\Omega.$$

This allows us to estimate harmonic measures using conformal invariance and Kellogg's theorem, which we proceed to state (see p. 426 in [26]). For a domain Ω we denote by $C^{k, \beta}(\Omega)$ the space of k times differentiable functions in Ω , whose partial derivatives of order k satisfy a Hölder condition with exponent β on $\bar{\Omega}$.

Theorem 2.1. *Denote by Ω a simply connected domain with $C^{k, \beta}$ -smooth Jordan curve boundary, for some integer $k \geq 1$ and Hölder exponent $0 < \beta < 1$. Denote by φ a conformal mapping of \mathbb{D} onto Ω and by ϕ its inverse. Then φ and ϕ are $C^{k, \beta}$ -smooth on the closed unit disk and the closure of Ω , respectively.*

The statement for the inverse mapping is, strictly speaking, not included in the presentation in [26], but follows from standard methods in view of the non-vanishing of the derivative on the closure $\bar{\mathbb{D}}$.

Another important feature of harmonic measures is the following simple monotonicity property. If Ω is a subset of Ω' and $E \subset \partial\Omega$ is a subset of $E' \subset \partial\Omega'$, then for any $z_0 \in \Omega$, it trivially holds that

$$(2.4.1) \quad \omega_{\Omega, z_0}(E) \leq \omega_{\Omega', z_0}(E').$$

This is most easily seen from the equivalent definition of $z \mapsto \omega(z, E, \Omega)$ as the solution to the Dirichlet problem with boundary data χ_E . We can also argue probabilistically as follows. Any Brownian path which starts at z_0 and exits Ω through E is also a path in Ω' which exits Ω' through E , so in particular it exits through E' .

2.5. Piecewise smooth domains and conformal mappings near corners. A simple closed curve Γ is said to be C^2 -smooth if it admits a C^2 -smooth parameterization $f : [0, 1] \rightarrow \Gamma$ with non-vanishing derivative. If Γ is a simple arc, we say that it is C^2 -smooth if there exists a larger arc $\tilde{\Gamma}$ with a C^2 -smooth (surjective) parameterization $f : [0, 1] \rightarrow \tilde{\Gamma}$ with non-vanishing derivative, such that Γ is contained in the image $f([\epsilon, 1 - \epsilon])$ for some $\epsilon > 0$.

Definition 2.2. We say that a Jordan curve Γ is piecewise smooth if it is made up of finitely many C^2 -smooth arcs $(\Gamma_j)_{j=1}^M$. When two arcs Γ_j and Γ_{j+1} meet (we identify Γ_{M+1} with Γ_1), they form an interior angle $\pi\sigma$, where $\sigma = \sigma(\Gamma_j, \Gamma_{j+1}) \in [0, 2]$. If each angle $\sigma\pi$ lies in the open interval $(0, 2\pi)$ we say that \mathcal{G} is piecewise smooth without cusps.

A domain is said to be piecewise smooth if its boundary curve is.

The following is a small modification of Theorem 3.9 of [59] (cf. Exercise 3.4.1 *loc. cit.*).

Proposition 2.3. *Denote by \mathcal{G} a piecewise smooth simply connected domain without cusps, and let $w_0 \in \partial\mathcal{G}$ a corner point with interior angle $\pi\sigma$. If f denotes a conformal mapping of \mathbb{D} onto \mathcal{G} with $f(\zeta) = w_0$, there exists a non-zero complex constant a such that for any $\epsilon > 0$, we have with $\sigma' = \min\{\sigma, 1\}$ that*

$$f(z) = f(\zeta) + a(z - \zeta)^\sigma + O(|z - \zeta|^{\sigma + \sigma' - \epsilon}), \quad z \in \bar{\mathbb{D}}.$$

and

$$\frac{f'(z)}{(z - \zeta)^{\sigma-1}} = a\sigma + O(|z - \zeta|^{\sigma'-\epsilon}), \quad z \in \overline{\mathbb{D}}.$$

Proof sketch. We explain how the proof of Theorem 3.9 of [59] should be modified to yield this claim (in this proof we use the terminology of that reference). Without loss of generality we may assume that $w_0 = 0$ and $\zeta = 1$.

As explained in the proof of [59, Theorem 3.9], we may localize to a piecewise smooth domain \mathcal{G}_0 containing $\mathcal{G} \cap \mathbb{D}(0, \delta)$, for some small $\delta > 0$. This localization is done so that 0 is the only singular boundary point at $\partial\mathcal{G}_0$, and amounts to constructing a conformal mapping φ of \mathbb{D} onto $f^{-1}(\mathcal{G}_0) \subset \mathbb{D}$ (cf. fig 3.3 in *loc. cit.*) which is $C^{1,1-\epsilon'}$ -smooth for any given $\epsilon' > 0$.

Denote by H the domain resulting from applying the straightening mapping $w^{\frac{1}{\sigma}}$ to \mathcal{G}_0 near 0, with 0 being mapped to the origin. We claim the boundary of H consists of two arcs (corresponding to the two arcs Γ^\pm meeting at w_0) which are both $C^{1,\sigma(1-\epsilon')}$ -smooth for any $\epsilon' > 0$, and that are tangent at 0. Indeed, the mapping straightens the angle at w_0 by construction, and if w denotes the given parameterization of one of the arcs Γ^\pm , then a one-sided parameterization v of ∂H at 0 is obtained by $v(t) = (w(ct^\sigma))^{1/\sigma}$ where $c = 1/|w'(0)|$. Following along the lines of the original proof, we have

$$(2.5.1) \quad v'(t) = c^{\frac{1}{\sigma}} w'(ct^\sigma) u(ct^\sigma), \quad u(s) = \left(\frac{1}{s} w(s)\right)^{\frac{1-\sigma}{\sigma}}.$$

The non-vanishing of $w'(0)$ and hence of $w(s)/s$ for s small shows that the modulus of continuity of the function $u(s)$ is controlled by that of $s^{-1}w(s)$. Here, we recall that the modulus of continuity $\omega(\delta, F)$ of a function F is defined by

$$\omega(\delta, F) = \sup_{|z-w| \leq \delta} |F(z) - F(w)|.$$

In view of the computation

$$\frac{w(s_1)}{s_1} - \frac{w(s_2)}{s_2} = \int_0^1 (w'(s_1 x) - w'(s_2 x)) dx$$

it follows that $\omega(\delta, u) \lesssim \omega(\delta, w')$. It follows from (2.5.1) that $\omega(\delta, v')$ is controlled by $\omega(c\delta^{\sigma'}, w)$ where $\sigma' = \min\{\sigma, 1\}$ (cf. (8) and the equation immediately following it in *loc. cit.*). So, after straightening the corner, the domain is $C^{1,\sigma'(1-\epsilon')}$ -smooth for any $\epsilon' > 0$, where $\sigma' = \min\{\sigma, 1\}$. The claim then follows by invoking Kellogg's theorem (Theorem 2.1, cf. [59, Theorem 3.6]) for the mapping $h : \mathbb{D} \rightarrow H$. Indeed, the mapping f is given in terms of h and the localization mapping φ by $f(z) = (h(\varphi^{-1}(z)))^\sigma$, where $h \circ \varphi^{-1}$ is a $C^{1,\sigma'(1-\epsilon')}$ -smooth for any ϵ' . \square

Remark 2.4. For future reference we record that in the above proof, we found that $f(z) = w_0 + (g(z) - g(1))^\sigma$, where for any $\epsilon > 0$ the mapping g belongs to the space $C^{1,\sigma'(1-\epsilon)}$ with $\sigma' = \min\{\sigma, 1\}$, where $w_0 = f(1)$. Hence both f and f^{-1} are Hölder continuous.

Remark 2.5. Let H_t and H be a family of Jordan domains with C^2 -smooth boundaries, such that H_t approaches H as $t \rightarrow 0$. Suppose ∂H_t has a parameterization w_t , and denote by w the corresponding parameterization of ∂H . Then, by a stability theorem of Warschawski's [76], if $\|w_t'' - w''\|_{L^2[0,1]} \rightarrow 0$ and $w_t \rightarrow w$ uniformly,

then we have

$$\sup_{z \in \mathbb{D}} |h'_t(z) - h'(z)| \rightarrow 0.$$

Assume now that \mathcal{G}_t and \mathcal{G} are piecewise smooth, such that the above condition holds for all the C^2 -smooth boundary subarcs $\Gamma_{t,j}$ and Γ_j . Then apply Warschawski's theorem to the mapping functions h_t and h_0 , and φ_t and φ_0 appearing in the proof of Proposition 2.3 onto the domains H_t and H , obtained by straightening a given angle at \mathcal{G}_t and \mathcal{G} . This shows in particular that the implicit constants in Remark 2.4 may be taken uniform over appropriately convergent sequences of piecewise smooth domains \mathcal{G}_t . Indeed, the functions g_t with $f_t(z) = f_t(1) + (g_t(z) - g_t(1))^{\sigma_t}$ appearing in Remark 2.4 (σ_t being the t -dependent angle) are C^1 with uniform control on their C^1 -norm.

Definition 2.6. A domain \mathcal{G} is said to meet a uniform cone condition, if there exist numbers ϵ and $\theta > 0$, such that for any boundary point $z_0 \in \partial\mathcal{G}$ there exists a $c \in [0, 2\pi)$, such that the planar cone

$$W = e^{ic} \{z \in \mathbb{C} : |\arg(z - z_0)| \leq \theta\},$$

with apex at z_0 locally contains $\partial\mathcal{G}$, in the sense that $\partial\mathcal{G} \cap \mathbb{D}(z_0, \epsilon) \subset W$.

The parameter θ is called the *aperture* of the cone.

2.6. Balayage measures. For a measure $\nu \in H^{-1}(D)$ where D is a bounded Lipschitz domain, we define its balayage $\text{Bal}(\nu, D^c)$ (sweeping) to the complement D^c as the unique measure μ supported on ∂D which satisfies

$$U^\mu(z) = U^\nu(z), \quad z \in D^c.$$

By the maximum principle, it follows that we have the global bound

$$U^\mu(z) \geq U^\nu(z), \quad \text{q.e. } z \in \mathbb{C},$$

and from the equality of the potentials near infinity it follows that μ and ν have the same mass. If μ has finite logarithmic energy and D is a Lipschitz domain, then for $\nu = \text{Bal}(\mu, D^c)$ we have $-\Sigma(\nu) \leq -\Sigma(\mu)$.

Remark 2.7. More generally, if ν is supported on \mathbb{C} , we define the balayage measure as

$$\text{Bal}(\nu, D^c) = \text{Bal}(\chi_D \nu, D^c) + \chi_{D^c} \nu,$$

provided that $\chi_D \nu$ is regular enough.

Let us only say a few words about the existence of Balayage measures. In view of the assumption $\nu \in H^{-1}(\Omega)$, the Dirichlet problem on Ω with boundary datum $f = U^\nu|_\Omega \in H^{\frac{1}{2}}(\partial\Omega)$ admits a solution $V \in H^1(\Omega)$, and the function V_0 which equals V on Ω and U^μ elsewhere has (distributional) Laplacian in the Sobolev space $H^{-1}(\Omega)$, in fact in $H^{-\frac{1}{2}}(\Gamma)$. The balayage measure of ν to Ω^c then equals $\mu = \frac{1}{2\pi} \Delta V_0$. In particular, under these mild conditions, the balayage measure exists. For an introductory account of these matters, we refer to [63, Ch. II].

Let ν be a measure supported on a bounded domain Ω , which we assume to be simply connected. Let $\mu = \text{Bal}(\nu, \Omega^c)$. For $w \in \Omega$, the potential of the Balayage measure $\text{Bal}(\delta_w, \Omega^c)$ is given by

$$U^{\text{Bal}(\delta_w, \Omega^c)}(z) = \begin{cases} \log \frac{|z-w|}{|\phi_w(z)|}, & z \in \Omega \\ \log |z-w|, & z \in \Omega^c, \end{cases}$$

where ϕ_w is the Riemann mapping of Ω onto \mathbb{D} with $\phi_w(w) = 0$. In view of this formula, it is clear that we have the representation

$$(2.6.1) \quad U^\mu(z) = \int \log \frac{|z-w|}{|\phi_w(z)|} d\nu(w), \quad z \in \Omega,$$

see eq. (5.3) on p. 124 in [63]. Similarly, one can deduce the representation formula

$$(2.6.2) \quad \frac{d\text{Bal}(\nu, \Omega^c)}{ds} = \int \frac{d\omega_{\Omega, z}(\cdot)}{ds} d\nu(z)$$

for the density with respect to arc length on $\partial\Omega$, whenever ν is compactly supported on the domain Ω (c.f. Eq. (4.44) on p. 122 in [63]).

2.7. Quadrature domains. We already encountered the notion of (subharmonic⁵) quadrature domains in Definition 1.5. Let ν be a positive atomic measure

$$\nu = \sum_{\lambda \in \mathcal{I}} \omega_\lambda \delta_\lambda$$

with finite support $|\mathcal{I}| < \infty$. Recall that a planar open set Ω (not necessarily connected) is said to be a quadrature domain for subharmonic functions (or simply a ‘subharmonic quadrature domain’) with respect to the measure ν if, for any L^1 -integrable subharmonic function u on Ω we have

$$\sum_{\lambda \in \mathcal{I}} \omega_\lambda u(\lambda) \leq \frac{1}{\pi} \int_{\Omega} u(z) dA(z).$$

The simplest quadrature domain is a disk, as follows from the sub-mean value property for subharmonic functions. The simplest non-trivial example is given by the *Neumann oval*, which we will return to later in § 10.

Given a finitely supported atomic measure ν , there exists a unique subharmonic quadrature domain Ω_ν relative to ν ⁶ (see [62], Theorems 3.4 and 3.5), and it may be constructed by a sweeping out process known as partial balayage (see [62, Ch. 2]). We will need the following characterization of Ω_ν , see Theorem 4.8 in [29]. We denote by $\text{SH}(\mathbb{C})$ the class of subharmonic functions on \mathbb{C} .

Theorem 2.8. *Given a finitely supported positive atomic measure ν , the quadrature domain Ω_ν is the non-coincidence set $\{z \in \mathbb{C} : U(z) < \frac{1}{2}|z|^2\}$ for the subharmonic envelope function*

$$U(z) = \sup \{u(z) : u \in \text{SH}(\mathbb{C}), u(z) \leq \frac{1}{2}|z|^2 - U^\nu(z)\}.$$

In view of Sakai’s regularity theorem (see Theorem 2.17 below), the boundary $\partial\Omega$ is piecewise real-analytic. As mentioned above, Aharonov and Shapiro [3] established the stronger result that the boundary of a subharmonic quadrature domain is algebraic.

Remark 2.9. We use the notion of quadrature domain in the extended sense when the measure ν is supported on union of a finitely many analytic curves and countably many points. The same definition applies, and Theorem 2.8 holds also in this case.

⁵Quadrature domains can also be defined for harmonic or analytic functions instead of subharmonic ones. Here, the notion of quadrature domains refers only to the subharmonic version, which is the most restrictive of the three.

⁶Actually, this holds for much more general measures: ν has to be sufficiently concentrated, which holds, for instance for measures ν of *regular support* as defined before Theorem 1.8.

2.8. Several variants of the obstacle problem. Obstacle problems may be posed in many different ways, which is one of the reasons why they are so useful. The *Dirichlet energy* $\mathcal{D}(u)$ of $u \in H^1(D)$ is defined as

$$\mathcal{D}(u) = \int_D |\nabla u|^2 \, dA.$$

We remind the reader that for f and g in the Sobolev space $H^1(D)$ (or $H^{\frac{1}{2}}(\Gamma)$), the inequality $f \leq g$ is understood as $f(z) \leq g(z)$ a.e. $z \in D$ (resp. $z \in \Gamma$).

Definition 2.10. (a) For a Lipschitz domain D and functions $\psi \in H^1(D)$, and $f \in H^{\frac{1}{2}}(\partial D)$, we denote by $\mathcal{K}_\psi^f = \mathcal{K}_\psi^f(D)$ the class

$$\mathcal{K}_\psi^f = \{u \in H^1(D) : u \leq \psi, u|_{\partial D} = f\}.$$

The minimizer v_0 of $\mathcal{D}(v)$ over $v \in \mathcal{K}_\psi^f$ is said to solve the obstacle problem with full obstacle ψ and boundary datum f .

(b) If moreover Γ denotes a Lipschitz curve whose closure is contained in D , and $g \in H^{\frac{1}{2}}(\Gamma)$, we denote by $\mathcal{K}_{\psi,g}^f = \mathcal{K}_{\psi,g}^f(D)$ the class

$$\mathcal{K}_{\psi,g}^f = \{u \in H^1(D) : u \leq \psi, u|_\Gamma \leq g, u|_{\partial D} = f\}.$$

The minimizer v_0 of $\mathcal{D}(v)$ for $v \in \mathcal{K}_{\psi,g}^f$ is said to solve the mixed obstacle problem with thin obstacle g on Γ , full obstacle ψ on D , and boundary datum f .

When the boundary datum agrees with the full obstacle, we denote the classes simply by \mathcal{K}_ψ and $\mathcal{K}_{\psi,g}$, respectively. To simplify terminology, if \mathcal{K} is any convex and nonempty subset of $H^1(D)$, we speak of the solution to the obstacle problem for \mathcal{K} .

Remark 2.11 (§3.19, 3.21 and in [35]). Whenever the class \mathcal{K} is non-empty, there exists a unique solution to any of the obstacle problems of Definitions 2.10. Moreover, the solution is always subharmonic. In addition, if v_0 denotes the minimizer the Dirichlet energy over the convex class \mathcal{K} , it holds that

$$\int \nabla v_0 \cdot \nabla(v - v_0) \, dA \geq 0, \quad v \in \mathcal{K}.$$

Remark 2.12. If G denotes the harmonic function on $D \setminus \Gamma$ which equals g on Γ and f on ∂D , then the solution of the mixed obstacle problem for $\mathcal{K}_{\psi,g}^f$ agrees with the classical solution for \mathcal{K}_η , where

$$\eta = \min\{G, \psi\}.$$

To see why the two solutions agree, one simply notices that the two classes agree: indeed, for any $u \in \mathcal{K}_{\psi,g}^f$ we have $u \leq G$ by the maximum principle. Moreover, u has the right boundary data, and $u \leq \psi$. Hence, $u \in \mathcal{K}_\eta$. The reverse direction follows since traces respect inequalities among functions in $H^1(D)$.

We will moreover need to consider obstacle problems for *unbounded* domains. In this situation, minimization of Dirichlet energy does not make sense, but we instead rely on an envelope formulation involving subharmonic functions (see e.g. [39] for

background on subharmonic functions). For the definition, we need the following notions. We denote by $\text{SH}_1(\mathbb{C})$ the class of subharmonic functions of growth

$$v(z) = \log |z| + O(1), \quad z \rightarrow \infty.$$

Equivalently, $\text{SH}_1(\mathbb{C})$ is the class of subharmonic functions whose Riesz masses are probability measures with finite logarithmic energy. When working with global obstacle problems on the entire complex plane, we will consider obstacles ψ subject to the growth condition

$$(2.8.1) \quad \liminf_{|z| \rightarrow \infty} \frac{\psi(z)}{\log |z|} > 1.$$

The following definition is central to our work.

Definition 2.13. (a) Let $\psi \in H_{\text{loc}}^1(\mathbb{C})$ be a real-valued function subject to the growth bound (2.8.1). We say that v_0 solves the global obstacle problem with full obstacle ψ if $v_0 \in \text{SH}_1(\mathbb{C})$ with $v_0 \leq \psi$ and

$$v_0(z) = \sup\{v(z) : v \in \text{SH}_1(\mathbb{C}), v \leq \psi\}.$$

(b) A function v_0 is said to solve the global obstacle problem with thin obstacle g in $H^{\frac{1}{2}}(\Gamma)$ and full obstacle $\psi \in H_{\text{loc}}^1(\mathbb{C})$ if $v_0 \in \text{SH}_1(\mathbb{C})$, $v_0 \leq \psi$, $v_0|_{\Gamma} \leq g$ and

$$v_0(z) = \sup\{v(z) : v \in \text{SH}_1(\mathbb{C}), v \leq \psi, \text{ and } v|_{\Gamma} \leq g\}.$$

Remark 2.14. In part (a) of Definitions 2.10 and 2.13, the solution v_0 is known to be as regular as the obstacle ψ , up to order $C^{1,1}$. Provided that ψ is smooth enough, the function v_0 solves the PDE

$$\Delta v_0 = \chi_{\mathcal{S}} \Delta \psi,$$

in the distributional sense for some compact set \mathcal{S} . Since this set is a priori unknown, its boundary is known as a *free boundary*. The set \mathcal{S} is precisely the *coincidence set*

$$(2.8.2) \quad \mathcal{S} = \{z : v_0(z) = \psi(z)\}.$$

In part (b) of the same definitions, the same holds away from Γ . In addition to being supported on \mathcal{S} , the Laplacian Δv_0 will contain a *singular part*, supported on Γ (see Proposition 3.1 below).

Remark 2.15. In the context of any of the above obstacle problems (Definitions 2.10 and 2.13), if the thin obstacle g is replaced by another obstacle \tilde{g} with $\tilde{g} < g$, then the corresponding solution \tilde{v}_0 satisfies $\tilde{v}_0 \leq v_0$. As a consequence, the coincidence set with the fixed full obstacle decreases. Similarly, if the full obstacle ψ is replaced by $\tilde{\psi} < \psi$, then the coincidence set with the fixed thin obstacle g on Γ shrinks.

Remark 2.16. In case of obstacle problems on a bounded domain (Definition 2.10), the solution is in fact known to be given as the upper envelope

$$v_0(z) = \sup\{v(z) : v \in \mathcal{K} \cap \text{SH}(D)\},$$

where $\text{SH}(D)$ denotes the class of subharmonic functions on D (see [42, Theorem 6.2, Ch. II]). This observation puts this problem on the same ground as those in Definition 2.13. When available, the energy minimization point of view is often advantageous, since tools from functional analysis are more readily available.

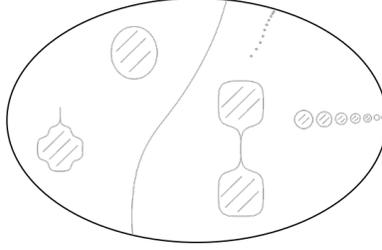


FIGURE 2.1. The free boundary in the obstacle problem with analytic obstacle, showing regular and singular boundary points, and possible degeneracies near the boundary.

The upper envelope u_0 of a family of subharmonic functions may fail to be upper semi-continuous. It is then convenient to define its upper semi-continuous regularization $u_0^*(z)$. It is known that $u_0 = u_0^*$ outside a polar set, i.e. a set of vanishing logarithmic capacity (see [63, pp. 24-25]). In view of the BreLOT–Cartan theorem ([63, Ch. II.2]), the function u_0^* is a subharmonic function on the domain D of u_0 . As such, the F. Riesz Theorem ([63, Ch. II.3]) yields the existence of a measure μ_0 , called the Riesz mass of u_0^* on D and a harmonic function h such that

$$u_0^*(z) = U^{\mu_0}(z) + h(z), \quad z \in D.$$

Abusing the notation slightly, will refer to μ_0 as the Riesz mass of u_0 as well.

2.9. Classical regularity and stability for the obstacle problem. Next, we will need some results pertaining to the regularity of the coincidence set \mathcal{S} for the obstacle problem. The following result was obtained by Sakai, see [61], Theorem 1.1.

Theorem 2.17. *Denote by ψ a strictly subharmonic real-analytic function in \mathbb{D} , and assume that $u_0 \in C^1(\mathbb{D})$ solves*

$$\Delta u_0 = 0 \quad \text{in } \mathbb{D} \setminus \mathcal{S},$$

where \mathcal{S} is the coincidence set $\mathcal{S} = \{z : u_0(z) = \psi(z)\}$, and that $0 \in \partial\mathcal{S}$. Then there exists a positive number $\epsilon > 0$ such that either

- (i) 0 is a regular boundary point of \mathcal{S} in the sense that $\mathcal{S} \cap \mathbb{D}(0, \epsilon)$ is a simply connected domain with non-empty interior, and $\partial\mathcal{S} \cap \mathbb{D}(0, \epsilon)$ is an analytic arc.
- (ii) the set $\mathcal{S} \cap \mathbb{D}(0, \epsilon)$ is open, consisting of either one (I) or two (II) simply connected components, and $\partial\mathcal{S} \cap \mathbb{D}(0, \epsilon)$ consists of two analytic arcs which terminate at the origin in a cusp (case I), or pass through and are tangent at 0 forming a double point (case II).
- (iii) the origin is a degenerate point, in the sense that $\mathcal{S} \cap \mathbb{D}(0, \epsilon)$ is either an isolated point or an analytic arc.

For a schematic illustration of the different free boundary types allowed by Sakai's theorem, see Figure 2.1. In particular, there are only three types of singular boundary points: cusps, double points and sets with non-empty interior. The latter consists of isolated points or a analytic curves, which has to terminate at the boundary and be loop-free (cross-cuts). In addition, accumulation of isolated points or of components of \mathcal{I} with positive area can only occur at the boundary.

In view of Proposition 3.1 below, Sakai's theorem applies for the solution v_0 to the mixed obstacle problem for $\mathcal{K}_{\psi,g}$ for free boundary points away from Γ , wherever ψ is real-analytic with strictly positive Laplacian. In particular, if \mathcal{G} is simply connected and the interior coincidence set $\mathcal{I} = \mathcal{S} \cap \bar{\mathcal{G}}$ has zero area, then $\partial\mathcal{I} = \mathcal{I}$ is a union of isolated points which accumulate only on $\partial\mathcal{G}$, as well as real-analytic curves (cross-cuts) in \mathcal{G} . If we moreover know that $\partial\mathcal{I}$ is separated from $\partial\mathcal{G}$, then \mathcal{I} can only be a finite collection of points. We remark that the finiteness of \mathcal{I} may be deduced from the earlier regularity theorem of Caffarelli and Rivière [15], but we will find the other conclusions of Sakai's theorem useful later on.

Next, we need an elementary approximation property for obstacle problems on bounded domains. The following is Theorem 3.78 of [35]. When formulating results for a general obstacle problem, we agree to tacitly assume that the sets \mathcal{K} involved in their definitions are non-empty.

Proposition 2.18. *Assume that $\psi_t \in H^1(D)$ is a convergent sequence, which approximates the limit $\psi \in H^1(D)$ from below, and denote by v_t the solutions to the obstacle problem for \mathcal{K}_{ψ_t} . Then v_t converges in $H^1(D)$ to the solution v to the obstacle problem for \mathcal{K}_ψ .*

The following stability result will be useful.

Proposition 2.19. *Let $g \in H^{\frac{1}{2}}(\Gamma)$ and assume that $h \in C(\Gamma)$. Let $g_\epsilon = g + \epsilon h$ and let v_ϵ denote the solution to the obstacle problem for $\mathcal{K}_{\psi,g_\epsilon}$, where $\psi \in C^{1,1}(D)$. Then we have the linear L^∞ -stability bound*

$$(2.9.1) \quad \|v_\epsilon - v_0\|_{L^\infty(D)} \lesssim \epsilon \|h\|_\infty, \quad \text{as } \epsilon \rightarrow 0.$$

Proof. We define an auxiliary function H as the solution to the thin obstacle problem on D with thin obstacle $-\|h\|_\infty$ on Γ , vanishing Dirichlet boundary condition on ∂D and trivial full obstacle which equals $+\infty$ throughout D . Notice that H is homogeneous with respect to rescaling of h .

We claim that

$$v_0 + \epsilon H \leq v_\epsilon \leq v_0 - \epsilon H \quad \text{on } D.$$

The lower bound is evident in view of the fact that

$$v_0 + \epsilon H \leq g - \epsilon \|h\|_\infty \leq g + \epsilon h \quad \text{on } \Gamma$$

so that $v_0 + \epsilon H \in \mathcal{K}_{\psi,g_\epsilon} \cap \text{SH}(D)$ in view of the bound $H \leq 0$ on D . Similarly, we have

$$v_\epsilon + \epsilon H \leq g + \epsilon h - \epsilon \|h\|_\infty \leq g \quad \text{on } \Gamma,$$

so $v_\epsilon + \epsilon H \in \mathcal{K}_{\psi,g} \cap \text{SH}(D)$. But then it follows that $v_\epsilon \leq v_0 - \epsilon H$ on D , so we get the two-sided bound

$$-\epsilon \|H\|_\infty \leq v_\epsilon - v_0 \leq \epsilon \|H\|_\infty$$

on D . As the supremum norm of H only depends on the curve Γ and the domain D , and linearly on the number $\|h\|_{L^\infty(\Gamma)}$, this completes the proof. \square

Lastly, we need a restriction property (see p. 61 in [35]).

Proposition 2.20. *Denote by Ω a subdomain of D with Lipschitz boundary, let v_0 denote the solution to the obstacle problem for $\mathcal{K}_\psi^f(D)$ and set $g = v_0|_{\partial\Omega}$. Then $v_0|_\Omega$ solves the obstacle problem for $\mathcal{K}_\psi^g(\Omega)$.*

3. PERTURBATION THEORY FOR THE MIXED OBSTACLE PROBLEM

3.1. Perturbations of the thin obstacle: a preliminary view. The topic of the following section could be of independent interest, and therefore we will work in slightly greater generality than needed for the applications we have in mind. Denote by Γ a simple Lipschitz curve in \mathbb{C} , which lies in a bounded domain D with Lipschitz boundary. Denote by $\psi \in H^1(D)$ a full obstacle function (below we will impose additional regularity on ψ), and let $f \in H^{\frac{1}{2}}(\partial D)$ be a boundary datum with $f \leq \psi|_{\partial D}$. Let moreover $g \in H^{\frac{1}{2}}(\Gamma)$ be a thin obstacle with $g \leq \psi|_{\Gamma}$, where we remind the reader once again that inequalities in Sobolev spaces are understood in the almost everywhere sense, unless specifically stated otherwise.

We recall the notation $\mathcal{D}(u)$ for the Dirichlet energy

$$\mathcal{D}(u) = \int_D |\nabla u|^2 \, dA, \quad u \in H^1(D),$$

where dA denotes planar area measure, and consider the solution v_0 to the obstacle problem for $\mathcal{K}_{\psi,g}^f$, that is

$$(3.1.1) \quad \inf_{u \in \mathcal{K}_{\psi,g}^f} \mathcal{D}(u),$$

$$\text{where } \mathcal{K}_{\psi,g}^f = \{u \in H^1(D) : u|_{\partial D} = f, u \leq \psi \text{ on } D, \text{ and } u|_{\Gamma} \leq g\}.$$

Recall that, in view of [42, Theorem 6.4, Ch II], the solution v_0 is just the upper envelope of the class $\mathcal{K}_{\psi,g}^f \cap \text{SH}(D)$.

Let now $h \in H^{\frac{1}{2}}(\Gamma)$, define for $\epsilon > 0$ a family of thin obstacles g_ϵ on Γ by $g_\epsilon = g + \epsilon h$ and let v_ϵ be the solution to the obstacle problem for $\mathcal{K}_{\psi,g_\epsilon}^f$. We are interested in the stability of this obstacle problem as ϵ varies. In particular, we seek a *variational formula* for the Dirichlet energy of the solution.

Assuming that h and g are sufficiently smooth, and that $g - \psi|_{\Gamma}$ is negative and bounded away from zero, one can prove $C^{1,\beta}$ -stability for v_ϵ valid up to the curve Γ . Moreover, we recall the linear L^∞ -stability with respect to the data on the curve Γ of Proposition 2.19. Using these facts, it is rather straightforward to derive a variational formula for the Dirichlet energy

$$\mathcal{D}(v_\epsilon) = \mathcal{D}(v_0) - 4\pi\epsilon \int_{\Gamma} h \, d\mathcal{N}_{\Gamma}(v_0) + o(\epsilon)$$

as $\epsilon \rightarrow 0$. We omit the details since we will show something more general below. We return to the variational formula after discussing a preliminary result on the regularity of the solution of the extremal problem in (3.1.1).

3.2. Structure of the solution to the mixed obstacle problem. If the obstacle ψ in Definition 2.10 is regular enough, then the Riesz mass μ_0 of the solution splits into two components. One is singular with respect to area measure and lives on the curve Γ , while the other is continuous with respect to area measure and is supported on the set $\mathcal{S} = \{z \in D : v_0(z) = \psi(z)\}$. This is the content of the next result. It is likely well-known to experts, but we include a proof of it for completeness.

Proposition 3.1. *The Riesz mass μ_0 of the solution v_0 to the obstacle problem*

$$\inf_{v \in \mathcal{K}} \mathcal{D}(v), \quad \mathcal{K} = \{v \in H^1(D) : v|_\Gamma \leq g, v \leq \psi, v|_{\partial D} = f\}$$

with thin obstacle $g \in H^{\frac{1}{2}}(\Gamma)$, boundary datum $f \in H^{\frac{1}{2}}(\partial D)$ and full obstacle $\psi \in C^{1,1}(\overline{D})$ on a Lipschitz domain D containing the Lipschitz curve Γ enjoys a decomposition

$$d\mu_0 = d\mu_0^s + \frac{1}{2\pi} \Delta\psi \chi_{\mathcal{S}} dA,$$

where $\mu_0^s \in H^{-\frac{1}{2}}(\Gamma)$, and the solution v_0 belongs to $C^{1,\beta}(D \setminus \Gamma)$ for any $\beta < 1$. In addition, Green's formula

$$(3.2.1) \quad \int_D \nabla u \cdot \nabla v_0 dA = -2\pi \int_\Gamma u d\mu_0^s - \int_{\mathcal{S}} u \Delta\psi dA$$

holds for any $u \in H_0^1(D)$.

Proof. The proof splits into two steps: first we obtain the result under the condition that $g \leq \psi|_\Gamma - \delta$ for some $\delta > 0$, then this condition is relaxed by an approximation argument.

STEP 1. Assume first that $g \leq \psi|_\Gamma - \delta$ for some $\delta > 0$. It is clear that \mathcal{S} is separated from Γ , by upper continuity of $v_0 - \psi$, since otherwise there would exist a sequence of points z_j converging to Γ , along which $u - \psi$ vanishes, which would force $v_0 - \psi$ to vanish on Γ . As a consequence, v_0 is harmonic in a region $V \setminus \Gamma$, where V is a neighbourhood of Γ .

In view of Proposition 2.20 we may think of v_0 as the solution to the classical obstacle problem in each component of $D \setminus \Gamma$ separately, with boundary data $g_0 = v_0|_\Gamma$. Since the coincidence set \mathcal{S} remains bounded away from the fixed boundary Γ , it follows from classical theory, see e.g. [16, Theorem 2.3], that the solution is $C^{1,1}$ smooth away from Γ and that $\Delta v_0 = 0$ outside the coincidence set \mathcal{S}_0 . In fact, the distributional Laplacian away from Γ takes the form $\chi_{D \setminus \Gamma} \mu_0 = \frac{1}{2\pi} \chi_{\mathcal{S}} \Delta\psi$, see [34, Theorem 3.10].

We may then define a two-sided Neumann trace on Γ . By a standard convolution argument there exists a sequence $(v_j)_j$ of smooth functions whose gradients are continuous up to the boundary of $D \setminus \Gamma$, such that $v_j \rightarrow v_0$ in $H^1(D)$. In addition we have $\Delta v_j \rightarrow \Delta v_0$ pointwise a.e. and in L^2 . But then by the standard Green's formula we have

$$\int_D \nabla u \cdot \nabla v_j dA = -2\pi \int_\Gamma u d\mu_j^s - \int_D u \Delta v_j dA,$$

where μ_j^s is the Riesz measure of v_j on Γ . From the convergences $v_j \rightarrow v_0$ in $H^1(D)$ and $\Delta v_j \rightarrow \Delta v_0$ in $L^2(D)$ it follows that μ_j^s converges to a bounded functional on $H^{\frac{1}{2}}(\Gamma)$, and we find that the limiting Green's formula for v_0 holds.

In summary, we have found that μ_0 is supported on $\Gamma \cup \mathcal{S}$ and since \mathcal{S} is relatively compact in $D \setminus \Gamma$, the Riesz mass decomposes as

$$d\mu_0 = d\mu_0^s + \frac{1}{2\pi} \Delta\psi \chi_{\mathcal{S}} dA$$

and that Green's formula holds. The singular measure (with respect to dA) μ_0^s is the Neumann trace $\mathcal{N}_\Gamma(v_0)$ and belongs to $H^{-\frac{1}{2}}(\Gamma)$.

STEP 2. We now consider a general thin obstacle g , which does not necessarily lie strictly below the full obstacle. We will approximate g and f from below, and show that we have a strong enough convergence of the corresponding solutions in order to reach the desired conclusions for the limiting object. We let $g_\epsilon = g - \epsilon$ and $f_\epsilon = f - \epsilon$, and denote by v_ϵ the corresponding solutions. For each v_ϵ we may apply the above argument. By the approximation property of Proposition 2.18, we know that $v_\epsilon \rightarrow v_0$ in $H^1(D)$. But then it follows that $\mu_\epsilon \rightarrow \mu_0$. Let \mathcal{S}_ϵ denote the coincidence set for v_ϵ with the full obstacle ψ . The set \mathcal{S}_ϵ increases monotonically as ϵ tends to zero (see Remark 2.15), and in view of the regularity assumption we may conclude that $\mu_\epsilon^c = \frac{1}{2\pi}\chi_{\mathcal{S}_\epsilon}\Delta\psi$ is convergent in L^2 (e.g. by monotone convergence) towards the limit function $\frac{1}{2\pi}\chi_{\mathcal{S}^*}\Delta\psi$, where $\mathcal{S}^* = \cup_\epsilon \mathcal{S}_\epsilon$. But then it also follows from Green's formula for v_ϵ that μ_ϵ^s is convergent as a functional on $H^{\frac{1}{2}}(\Gamma)$ towards $\mu_0^s \in H^{-\frac{1}{2}}(\Gamma)$. In summary, we have the convergence

$$\int_D \nabla u \cdot \nabla v_0 \, dA = - \lim_{\epsilon \rightarrow 0} \left(2\pi \int_\Gamma u \, d\mu_\epsilon^s + \int_{\mathcal{S}_\epsilon} u \, \Delta\psi \, dA \right) = -2\pi \int_\Gamma u \, d\mu_0^s - \int_{\mathcal{S}^*} u \, \Delta\psi \, dA$$

It only remains to conclude that $\mathcal{S}^* = \mathcal{S}$ up to the removal of a null set. But this follows from the fact that $v_0 \in \text{SH}(D)$, that $v_0|_{D \setminus \Gamma}$ is continuous, the linear stability result of Proposition 2.19 together with the stability for the coincidence set under perturbations, see e.g. [9]. Indeed, outside any fixed neighbourhood Γ_δ of Γ , the above results combine to say that $|(\mathcal{S} \setminus \mathcal{S}_\epsilon) \cap \Gamma_\delta^c| = O(\epsilon)$, so that $|(\mathcal{S} \setminus \mathcal{S}^*) \cap \Gamma_\delta^c| = 0$. This completes the proof.

As a consequence of the smoothness of ψ , we saw that the solution v_0 solves the PDE $\Delta v_0 = \Delta\psi\chi_{\mathcal{S}} \in L^\infty$ in $D \setminus \Gamma$. In view of [75, Proposition 1.1] this implies that $v_0 \in W^{2,p}(D \setminus \Gamma)$ for any $p < \infty$, so by the classical Sobolev embedding theorem it follows that $v_0 \in C^{1,\beta}$ for any $\beta < 1$. \square

3.3. A few subtleties. In this section, we verify two seemingly obvious facts, which are slightly subtle due to the low regularity of the function $g \in H^{\frac{1}{2}}(\Gamma)$.

Proposition 3.2. *Let $f \in H^{\frac{1}{2}}(\Gamma)$ and assume that $f \geq 0$ in the sense of traces, i.e. there exists $u \in H^1(D)$ with $u \geq 0$ Lebesgue a.e. in D , with $u|_\Gamma = f$. Then, for any positive measure $\mu \in H^{-\frac{1}{2}}(\Gamma)$ we have $\int f \, d\mu \geq 0$.*

Proof. A standard dilation and mollification trick shows that any non-negative $u \in H^1(D)$ can be approximated by non-negative smooth functions in $H^1(D)$.

We denote by u the non-negative $H_0^1(D)$ -function which is harmonic in $D \setminus \Gamma$ while $u|_\Gamma = f$, and apply this approximation trick to get $u_j \in C^\infty(D)$ with $u_j \rightarrow u$ in $H^1(D)$ and $u_j \geq 0$. Denote by v an element of $H_0^1(D)$ with Riesz mass μ . Then

$$\int_\Gamma f \, d\mu = \frac{1}{2\pi} \int_D \nabla u \cdot \nabla v \, dA = \lim_{j \rightarrow \infty} \frac{1}{2\pi} \int_D \nabla u_j \cdot \nabla v \, dA = \lim_{j \rightarrow \infty} \int_\Gamma u_j \, d\mu \geq 0.$$

This completes the proof. \square

If the obstacles ψ and g are smooth, and u denotes the solution to the obstacle problem for $\mathcal{K}_{\psi,g}$, then $(u - g)d\mathcal{N}_\Gamma = 0$ pointwise on Γ , and $(u - \psi)\Delta u = 0$ on $D \setminus \Gamma$. The latter survives in our setting, but we prefer not to attempt to interpret the former in a pointwise sense. However, we have the following averaged version.

Proposition 3.3. *Assume that v_0 solves the mixed obstacle problem with thin obstacle g on Γ and full obstacle $\psi \in C^{1,1}(\bar{D})$. Then, if μ_0^s denotes the singular part of the Riesz mass of v_0 , we have $\int_\Gamma (v_0 - g)d\mu_0^s = 0$.*

Proof. Denote by η the $H^1(D)$ -obstacle discussed in Remark 2.12

$$\eta = \min\{G, \psi\},$$

where G is the harmonic function in $D \setminus \Gamma$ which equals f on ∂D and g on Γ . On the one hand, $\eta \in \mathcal{K}_\eta = \mathcal{K}_{\psi, g, f}$ so in view of Remark 2.11 we have

$$\int_D \nabla(v_0 - \eta) \cdot \nabla v_0 \, dA \leq 0,$$

by the variational inequalities for minimization of the Dirichlet energy (obtained as usual by noticing that $u_t = (1-t)v_0 + t\eta$ is a competitor for the minimization for any $t \geq 0$, and then expanding the energy and letting $t \rightarrow 0^+$). On the other hand we have $v_0 \leq G$ by the maximum principle, and $v_0 \leq \psi$ by definition, so it holds that $v_0 \leq \eta$, from which we conclude that

$$\int \nabla(v_0 - \eta) \cdot \nabla u \, dA = -2\pi \int (v_0 - \eta) d\mu_0 \geq 0.$$

Hence, $\langle v_0 - g, v_0 \rangle = 2\pi \int_D (v_0 - \eta) d\mu_0 = 0$. By Proposition 3.1, Green's formula applies, so by (3.2.1) we have

$$\begin{aligned} & \int_\Gamma (v_0 - g) d\mu_0^s \\ &= \frac{1}{2\pi} \int \nabla(v_0 - \eta) \cdot \nabla v_0 \, dA - \frac{1}{2\pi} \int_{\mathcal{S}_0} (v_0 - \eta) \Delta \psi \, dA = -\frac{1}{2\pi} \int_{\mathcal{S}_0} (v_0 - \eta) \Delta \psi \, dA \end{aligned}$$

where \mathcal{S}_0 denotes the coincidence set $\{v_0 = \psi\}$. But since $v_0 \leq \eta \leq \psi$ we find that $v_0 = \psi = \eta$ on the support of Δv_0 , so the last integral vanishes and the claim follows. \square

3.4. The variational formula. We begin with a preliminary stability estimate for the singular part of the Riesz mass of v_ϵ .

Lemma 3.4. *In the setting of Proposition 3.1, it holds that*

$$\int_\Gamma h \, d\mathcal{N}_\Gamma(v_0 - v_\epsilon) = o(1) + O(\mathcal{D}(v_\epsilon - v_0)^{\frac{1}{2}}) \quad \text{as } \epsilon \rightarrow 0.$$

Proof. Let us look at the integral

$$\int_D \nabla(v_0 - \psi) \cdot \nabla H \, dA,$$

where H is the harmonic extension to $D \setminus \Gamma$ of the boundary values given by h on Γ and by 0 on ∂D . In terms of the Neumann jump $\mathcal{N}_\Gamma(v_0)$, Green's formula (3.2.1) reads

$$\int_D \nabla(v_0 - \psi) \cdot \nabla H \, dA = - \int_{D \setminus \mathcal{S}} H \Delta \psi - 2\pi \int_\Gamma h \, d\mathcal{N}_\Gamma(v_0).$$

Subtracting the same calculation for v_ϵ , we see that

$$(3.4.1) \quad 2\pi \int_\Gamma h \, d\mathcal{N}_\Gamma(v_0 - v_\epsilon) = \int_D \nabla(v_\epsilon - v_0) \nabla H \, dA - \int_{\mathcal{S}_\epsilon \Delta \mathcal{S}_0} \text{sgn}(v_\epsilon - v_0) H \Delta \psi \, dA,$$

where $A \Delta B$ denotes the symmetric difference $(A \setminus B) \cup (B \setminus A)$. Invoking the Cauchy-Schwarz inequality, we find

$$2\pi \int_\Gamma h \, d\mathcal{N}_\Gamma(v_0 - v_\epsilon) = - \int_{\mathcal{S}_\epsilon \Delta \mathcal{S}_0} \text{sgn}(v_\epsilon - v_0) H \Delta \psi \, dA + O(\mathcal{D}(v_\epsilon - v_0)^{\frac{1}{2}}).$$

It remains to show that the symmetric difference $\mathcal{S}_\epsilon \Delta \mathcal{S}_0$ has $|\mathcal{S}_\epsilon \Delta \mathcal{S}_0| = o(1)$. Indeed, since $H \in L^2$ and $\Delta \psi \in L^\infty$, the claim would then follow by an application of the Cauchy-Schwarz inequality. To study the symmetric difference, we note that \mathcal{S}_ϵ and \mathcal{S}_0 are the coincidence sets for a classical obstacle problem on the domain $D \setminus \Gamma$ where the full obstacle is given by ψ , and the boundary data on Γ given by the functions $v_\epsilon|_\Gamma$ and $v_0|_\Gamma$, respectively, which by Proposition 2.19 differ by $O(\epsilon)$. We wish to deduce the result by invoking a result of Blank and LeCrone (see below), but their theorem requires that the boundary data is continuous. This issue may be circumvented, as follows. First, notice that it is enough to prove that for any given $\eta > 0$, we have

$$(3.4.2) \quad |(\mathcal{S}_\epsilon \Delta \mathcal{S}_0) \cap \{z \in D : d(z, \Gamma) \geq \eta\}| = O(\epsilon)$$

as $\epsilon \rightarrow 0$ with η held fixed. Indeed, if we assume that (3.4.2) holds, it follows that

$$\limsup_{\epsilon \rightarrow 0} |\mathcal{S}_\epsilon \Delta \mathcal{S}_0| \leq |\{z \in D : d(z, \Gamma) \leq \eta\}| \leq C_0 \eta,$$

where C_0 is a uniform constant that depends on Γ . But η was arbitrary, so it follows that $|\mathcal{S}_\epsilon \Delta \mathcal{S}_0| = o(1)$.

To see why the former bound (3.4.2) holds, let Γ_η^\pm , denote two smooth Jordan curves between Γ and the sets

$$D_\eta^\pm := \{z \in D^\pm : d(z, \Gamma) \geq \eta\},$$

respectively, where we recall that D^\pm denote the two components of $D \setminus \Gamma$. We require that the two curves Γ_η^\pm lie at a positive distance from Γ as well as from the corresponding set D_η^\pm . The restrictions of v_ϵ and v_0 to the curves Γ_η^\pm are C^1 -smooth by Proposition 3.1, and by the linear stability bound (2.9.1) the difference $v_0 - v_\epsilon$ is of order $O(\epsilon)$ on D , so in particular on each of the two curves. Hence, the assumptions of Blank and LeCrone's results in [9] are satisfied, so applying their result twice, once for each domain D_η^\pm , we get that (3.4.2) holds. \square

The following stability estimate for the energy $\mathcal{D}(v_\epsilon)$ as $\epsilon \rightarrow 0$ is a key ingredient in what follows.

Lemma 3.5. *In the setting of Proposition 3.1, we have the stability*

$$\mathcal{D}(v_\epsilon - v_0) = o(\epsilon)$$

as $\epsilon \rightarrow 0$.

Proof. Recall that for $\epsilon \geq 0$, the function v_ϵ solves the obstacle problem for the class $\mathcal{K}_{\psi, g_\epsilon}^f$, where $g_\epsilon = g + \epsilon h$ and $g, h \in H^{\frac{1}{2}}(\Gamma)$. By Green's formula (3.2.1) applied to the difference $v_\epsilon - v_0$ we have

$$\mathcal{D}(v_\epsilon - v_0) = -2\pi \int_\Gamma (v_\epsilon - v_0) d\mathcal{N}_\Gamma(v_\epsilon - v_0) - \int_{D \setminus \Gamma} (v_\epsilon - v_0) \Delta(v_\epsilon - v_0) dA,$$

where we write $D \setminus \Gamma$ for the domain of integration in the area integral simply to indicate that Green's formula was applied in this domain. We add and subtract ϵh inside the boundary integral, and find

$$\begin{aligned} \mathcal{D}(v_\epsilon - v_0) &= -2\pi \int_\Gamma (v_\epsilon - v_0 - \epsilon h) d\mathcal{N}_\Gamma(v_\epsilon - v_0) - \int_{D \setminus \Gamma} (v_\epsilon - v_0) \Delta(v_\epsilon - v_0) dA \\ &\quad + 2\pi\epsilon \int_\Gamma h d\mathcal{N}_\Gamma(v_0 - v_\epsilon). \end{aligned}$$

We next claim that

$$(3.4.3) \quad - \int_{\Gamma} (v_{\epsilon} - v_0 - \epsilon h) d\mathcal{N}_{\Gamma}(v_{\epsilon} - v_0) \leq 0$$

and that

$$(3.4.4) \quad - \int_{D \setminus \Gamma} (v_{\epsilon} - v_0) \Delta(v_{\epsilon} - v_0) dA \leq 0.$$

We first argue heuristically. On the support of the (positive) measure $\mathcal{N}_{\Gamma}(v_{\epsilon})$, we have $v_{\epsilon} = g + \epsilon h$, while $-v_0 \geq -g$ holds on the entire curve Γ , so that $v_{\epsilon} - v_0 - \epsilon h \geq 0$ holds on $\text{supp}(\mathcal{N}_{\Gamma}(v_{\epsilon}))$. It would follow that

$$(3.4.5) \quad - \int_{\Gamma} (v_{\epsilon} - v_0 - \epsilon h) d\mathcal{N}_{\Gamma}(v_{\epsilon}) \leq 0.$$

Moreover, on the support of $d\mathcal{N}_{\Gamma}(v_0)$, we have $v_{\epsilon} - v_0 - \epsilon h \leq 0$, so that

$$(3.4.6) \quad \int_{\Gamma} (v_{\epsilon} - v_0 - \epsilon h) d\mathcal{N}_{\Gamma}(v_0) \leq 0.$$

Adding these up would give (3.4.3).

Since the function g is possibly very irregular, we prefer not to rely on pointwise equalities on the supports $\text{supp}(\mathcal{N}_{\Gamma}(v_{\epsilon})) \subset \Gamma$. We may justify (3.4.5) and (3.4.6) by reinterpreting the above heuristics in a weak sense, with the aid of § 3.3. Indeed, we illustrate this for the term (3.4.5). We rewrite

$$v_{\epsilon} - v_0 - \epsilon h = (v_{\epsilon} - g - \epsilon h) - (v_0 - g),$$

and notice that in view of Proposition 3.3, we have

$$\int (v_{\epsilon} - g - \epsilon h) d\mathcal{N}_{\Gamma}(v_{\epsilon}) = 0,$$

from which we conclude that

$$- \int (v_{\epsilon} - v_0 - \epsilon h) d\mathcal{N}_{\Gamma}(v_{\epsilon}) = - \int (g - v_0) d\mathcal{N}_{\Gamma}(v_{\epsilon}) \leq 0,$$

where the last inequality uses Proposition 3.2 and the bound $v_0 \leq g$ on Γ (a.e.). As a consequence, the weak interpretation of the initial inequality yields the desired claim.

The claim (3.4.4) is proven similarly, but the argument can be understood in the pointwise sense, in view of the regularity imposed on the full obstacle ψ : outside the union $\mathcal{S}_{\epsilon} \cup \mathcal{S}_0$, both functions are harmonic and so do not contribute to the integral. On the coincidence set \mathcal{S}_{ϵ} (which contains $\text{supp}(\Delta v_{\epsilon})$), we have $v_{\epsilon} - v_0 \geq 0$, so

$$- \int_{D \setminus \Gamma} (v_{\epsilon} - v_0) \Delta v_{\epsilon} dA \leq 0.$$

Similarly, on \mathcal{S}_0 we have $v_{\epsilon} - v_0 \leq 0$, so

$$\int_{D \setminus \Gamma} (v_{\epsilon} - v_0) \Delta v_0 dA \leq 0.$$

Adding up these terms gives (3.4.4). In conclusion, we find that

$$\mathcal{D}(v_{\epsilon} - v_0) \leq 2\pi\epsilon \int_{\Gamma} h d\mathcal{N}_{\Gamma}(v_0 - v_{\epsilon}).$$

In view of Lemma 3.4, we find that

$$\mathcal{D}(v_\epsilon - v_0) \leq C\epsilon \mathcal{D}^{\frac{1}{2}}(v_\epsilon - v_0) + o(\epsilon),$$

which implies that $\mathcal{D}(v_\epsilon - v_0) = o(\epsilon)$, which proved the claim. \square

Remark 3.6. In case it holds that $g < \psi|_\Gamma$, so that \mathcal{S}_ϵ stays bounded away from Γ , we can improve this to an optimal stability bound of the form

$$\mathcal{D}(v_\epsilon - v_0) = O(\epsilon^2).$$

Indeed, the number η appearing in the proof of Lemma 3.4 can be chosen sufficiently small but fixed, that the whole of the symmetric difference $\mathcal{S}_\epsilon \Delta \mathcal{S}_0$ is contained in $D_\eta^+ \cup D_\eta^-$, so we find $|\mathcal{S}_\epsilon \Delta \mathcal{S}_0| = O(\epsilon)$. Inserting this into (3.4.1) and applying the Cauchy-Schwarz inequality, we find that

$$\mathcal{D}(v_\epsilon - v_0) \leq C\epsilon \mathcal{D}^{\frac{1}{2}}(v_\epsilon - v_0) + C'\epsilon^2,$$

which can be solved to yield the claimed conclusion.

We can now deduce our key stability lemma.

Corollary 3.7. *In the setting of Lemma 3.5, we have the variational formula*

$$\mathcal{D}(v_\epsilon) = \mathcal{D}(v_0) - 4\pi\epsilon \int_\Gamma h \, d\mathcal{N}_\Gamma(v_0) + o(\epsilon).$$

Remark 3.8. If in addition we assume that $g_\epsilon - \psi|_\Gamma$ is negative and bounded away from zero for small enough ϵ , the error term may be improved to $O(\epsilon^2)$.

Proof. As observed above, we by Green's formula (3.2.1)

$$\int_D \nabla u \cdot \nabla v_\epsilon \, dA = -2\pi \int_\Gamma u \, d\mathcal{N}_\Gamma(v_\epsilon) - \int_{D \setminus \Gamma} u \Delta v_\epsilon \, dA, \quad u \in H_0^1(D).$$

A computation using this yields that

$$\begin{aligned} \mathcal{D}(v_\epsilon) &= \mathcal{D}(v_0) + 2 \int_D \nabla(v_\epsilon - v_0) \cdot \nabla v_0 \, dA + \mathcal{D}(v_\epsilon - v_0) \\ (3.4.7) \quad &= \mathcal{D}(v_0) - 4\pi \int_\Gamma (v_\epsilon - v_0) \, d\mathcal{N}_\Gamma(v_0) - 2 \int_{D \setminus \Gamma} (v_\epsilon - v_0) \Delta v_0 \, dA + \mathcal{D}(v_\epsilon - v_0). \end{aligned}$$

The last term is $o(\epsilon)$ by the improved stability bound of Lemma 3.5. The main non-trivial contribution to the first variation in (3.4.7) should come from the second term, while the third should be small. We notice that the second term of (3.4.7) can be renormalized with the function h ,

$$-2 \int_\Gamma (v_\epsilon - v_0) \, d\mathcal{N}_\Gamma(v_0) = -4\pi\epsilon \int_\Gamma h \, d\mathcal{N}_\Gamma(v_0) - 4\pi \int_\Gamma ((v_\epsilon - v_0) - \epsilon h) \, d\mathcal{N}_\Gamma(v_0).$$

The key point here is that the last integral on the right-hand side is negative (so that the last term is positive). Indeed, repeating the argument in the proof of Lemma 3.5 we find that where the (positive) measure $\mathcal{N}_\Gamma(v_0)$ is supported, v_0 reaches up to its obstacle g , while $v_\epsilon \leq g + \epsilon h$, and hence $v_\epsilon - v_0 - \epsilon h \leq 0$. Similarly, we have that

$$-4\pi \int_\Gamma ((v_\epsilon - v_0) - \epsilon h) \, d\mathcal{N}_\Gamma(-v_\epsilon) \geq 0,$$

so we obtain a two-sided bound for the integral over Γ

$$\begin{aligned} -4\pi\epsilon \int_{\Gamma} h \, d\mathcal{N}_{\Gamma}(v_0) &\leq -4\pi \int_{\Gamma} (v_{\epsilon} - v_0) \, d\mathcal{N}_{\Gamma}(v_0) \\ &\leq -4\pi\epsilon \int_{\Gamma} h \, d\mathcal{N}_{\Gamma}(v_0) - 4\pi \int_{\Gamma} ((v_{\epsilon} - v_0) - \epsilon h) \, d\mathcal{N}_{\Gamma}(v_0 - v_{\epsilon}) \end{aligned}$$

An analogous comparison shows that where the positive measure Δv_0 is supported, we have $v_{\epsilon} - v_0 \leq 0$. Similarly, on the support of the measure Δv_{ϵ} , we have $v_{\epsilon} - v_0 \geq 0$. Hence, we obtain

$$0 \leq -2 \int_{D \setminus \Gamma} (v_{\epsilon} - v_0) \Delta v_0 \, dA \leq 2 \int_{D \setminus \Gamma} (v_{\epsilon} - v_0) \Delta (v_{\epsilon} - v_0) \, dA.$$

Adding up everything, we see that

$$\begin{aligned} -4\pi\epsilon \int_{\Gamma} h \, d\mathcal{N}_{\Gamma}(v_0) &\leq 2 \int_D \nabla(v_{\epsilon} - v_0) \cdot \nabla v_0 \, dA \leq -4\pi\epsilon \int_{\Gamma} h \, d\mathcal{N}_{\Gamma}(v_0) \\ &\quad - 4\pi \int_{\Gamma} ((v_{\epsilon} - v_0) - \epsilon h) \, d\mathcal{N}_{\Gamma}(v_0 - v_{\epsilon}) - 2 \int_{D \setminus \Gamma} (v_0 - v_{\epsilon}) \Delta (v_{\epsilon} - v_0) \, dA. \end{aligned}$$

By applying Green's formula in reverse we see that we may rewrite the right-hand side as

$$\begin{aligned} &-4\pi \int_{\Gamma} ((v_{\epsilon} - v_0) - \epsilon h) \, d\mathcal{N}_{\Gamma}(v_0 - v_{\epsilon}) - 2 \int_{D \setminus \Gamma} (v_{\epsilon} - v_0) \Delta (v_0 - v_{\epsilon}) \, dA \\ &= 4\pi \int_{\Gamma} (v_{\epsilon} - v_0) \, d\mathcal{N}_{\Gamma}(v_{\epsilon} - v_0) + 2 \int_{D \setminus \Gamma} (v_{\epsilon} - v_0) \Delta (v_{\epsilon} - v_0) \, dA - 4\pi\epsilon \int_{\Gamma} h \, d\mathcal{N}_{\Gamma}(v_{\epsilon} - v_0) \\ &= -2\mathcal{D}(v_{\epsilon} - v_0) - 4\pi\epsilon \int_{\Gamma} h \, d\mathcal{N}_{\Gamma}(v_{\epsilon} - v_0) = o(\epsilon), \end{aligned}$$

where the last step uses Lemma 3.4 and Lemma 3.5. So, we obtain the inequality

$$-4\pi\epsilon \int_{\Gamma} h \, d\mathcal{N}_{\Gamma}(v_0) \leq 2 \int_D \nabla(v_{\epsilon} - v_0) \cdot \nabla v_0 \, dA \leq -4\pi\epsilon \int_{\Gamma} h \, d\mathcal{N}_{\Gamma}(v_0) + o(\epsilon).$$

Using this and Lemma 3.5 in (3.4.7), it we find that

$$\mathcal{D}(v_{\epsilon}) = \mathcal{D}(v_0) - 4\pi\epsilon \int_{\Gamma} h \, d\mathcal{N}_{\Gamma}(u_0) + o(\epsilon).$$

This completes the proof of the claim. \square

4. REFORMULATION OF THE CONSTRAINED EXTREMAL PROBLEM

4.1. Constrained extremal measures solve an obstacle problem. The following result is the starting point for our approach to obtaining Theorem 1.8. It serves as a first step in a two-step variational characterization of the equilibrium measure. We recall briefly the extremal problem under consideration. The functional is given, for $\alpha > 0$, by

$$I_{\alpha}(\mu) = -\Sigma(\mu) + 2B_{\alpha}(\mu),$$

where

$$B_{\alpha}(\mu) = \sup_{z \in \mathbb{C}} (U^{\mu}(z) - \frac{1}{2\alpha}|z|^2).$$

The minimizer of I_α among all $\mu \in \mathcal{M}_G$ - the class of compactly supported probability measures on \mathbb{C} with finite logarithmic energy, which give no mass to \mathcal{G} - is denoted by $\mu_{\alpha, \mathcal{G}}$. In order to keep the notation manageable, we allow ourselves to denote the measure $\mu_{\alpha, \mathcal{G}}$ by μ_0 when there can occur no confusion.

Theorem 4.1. *There exists a function $g \in H^{\frac{1}{2}}(\partial\mathcal{G})$ such that $\mu_{\alpha, \mathcal{G}}$ is the Riesz mass of the solution v_0 to the mixed obstacle problem with thin obstacle g on $\partial\mathcal{G}$ and global obstacle $\frac{1}{2\alpha}|z|^2$, in the sense of Definition 2.13(b).*

Proof. To ease notation, set $\mu_0 = \mu_{\alpha, \mathcal{G}}$ and $v_0 = U^{\mu_{\alpha, \mathcal{G}}}$. Define the function w by $w = v_0 - B_\alpha(\mu_0)$, and denote by g its restriction to $\partial\mathcal{G}$. We moreover consider the solution u to the obstacle problem with full obstacle $\frac{1}{2\alpha}|z|^2$ and thin obstacle g , see Definition 2.13, Part (b). It is immediate that the upper semicontinuous regularization u^* of u is a subharmonic function and that its Riesz mass is a probability measure. Moreover, it may happen that $u^* \neq u$ only on a polar subset of Γ . In addition, since u^* satisfies the sub-mean value inequality and $u^*(z) \leq \frac{1}{2\alpha}|z|^2$ a.e., it is easy to see that $\sup_{z \in \mathbb{C}} (u^*(z) - \frac{1}{2\alpha}|z|^2) = 0$. Abusing notation slightly, we write u for the regularized function.

Observe that w is harmonic in the hole \mathcal{G} , so it follows that the set

$$\mathcal{I}' = \left\{ z \in \overline{\mathcal{G}} : w(z) = \frac{|z|^2}{2\alpha} \right\}$$

has zero (planar) Lebesgue measure. As a consequence of the maximum principle we have $u \leq w$ on \mathcal{G} , and we conclude that also the interior coincidence set

$$\mathcal{I} = \left\{ z \in \overline{\mathcal{G}} : u(z) = \frac{|z|^2}{2\alpha} \right\}$$

has zero Lebesgue measure as well. Since u solves the obstacle problem in \mathcal{G} with obstacle $\frac{1}{2\alpha}|z|^2$, it follows (e.g. by Proposition 3.1) that the Riesz measure of u is contained in \mathcal{I} . By Grishin's lemma [27] (see also [69] and [11]), we have $\Delta u \leq \Delta \psi_\alpha$ on \mathcal{I} , so it follows that the Riesz measure η of u gives total mass 0 to \mathcal{G} . Let C_0 be the constant such that $U^\eta = u + C_0$. Such a constant exists, since by Riesz' theorem it holds that $U^\eta = u + h$ for some harmonic function, which has to be constant since the functions $U^\eta(z)$ and $u(z)$ have the same growth at infinity, up to order $O(1)$.

The function w is an admissible subharmonic function for the obstacle problem on \mathbb{C} with thin obstacle g on $\partial\mathcal{G}$ and full obstacle $\frac{1}{2\alpha}|z|^2$, so we have $u \geq w = v_0 - B_\alpha(\mu_0)$, and as a consequence also

$$U^\eta(z) \geq v_0 - B_\alpha(\mu_0) + C_0 = U^{\mu_0} - B_\alpha(\mu_0) + C_0.$$

From this it follows that, using the notation $\Sigma(\mu, \mu') = -\int U^\mu d\mu'$, we have

$$-\Sigma(\eta) = -\Sigma(\eta, \eta) \leq -\Sigma(\mu_0, \eta) - C_0 + B_\alpha(\mu_0) \leq -\Sigma(\mu_0) - 2C_0 + 2B_\alpha(\mu_0).$$

Since $u(z) \leq \frac{1}{2\alpha}|z|^2$ a.e. and since u satisfies the sub-mean value property, it follows that $B_{\alpha, \mathcal{G}}(\eta) \leq C_0$. Adding up the two terms that make up $I_\alpha(\eta)$ we find that the constants $2C_0$ cancel, and that $I_\alpha(\eta) \leq I_\alpha(\mu_0)$. Since the extremal measure μ_0 is the unique minimizer of I_α , the result follows. \square

4.2. Two elementary observations. We begin with an observation which tells us how the potential of the extremal measure $\mu_{\alpha, \mathcal{G}}$ behaves on its support away from the hole.

Proposition 4.2. *Let $z_0 \in \text{supp}(\mu_{\alpha, \mathcal{G}}) \setminus \partial \mathcal{G}$. Then*

$$U^{\mu_{\alpha, \mathcal{G}}}(z_0) - \frac{|z_0|^2}{2\alpha} = B_{\alpha}(\mu_{\alpha, \mathcal{G}}).$$

Proof. Suppose, on the contrary, that we have

$$U^{\mu_{\alpha, \mathcal{G}}}(z_0) - \frac{|z_0|^2}{2\alpha} < B_{\alpha}(\mu_{\alpha, \mathcal{G}}).$$

We may choose a small neighbourhood V of z_0 where the equilibrium measure has positive mass. If V is made small enough, we may then replace $\mu_{\alpha, \mathcal{G}}$ by

$$\eta = \chi_{V^c} \mu_{\alpha, \mathcal{G}} + \text{Bal}(\mu_{\alpha, \mathcal{G}}, V^c)$$

without increasing the value of $B_{\alpha}(\mu_{\alpha, \mathcal{G}})$. Since the energy $-\Sigma$ only decreases under the balayage operation it follows that $I_{\alpha}(\eta) < I_{\alpha}(\mu_{\alpha, \mathcal{G}})$, which is a contradiction. \square

We also have the following simple lemma.

Lemma 4.3. *Assume that \mathcal{G} is contained in the unit disk. For $\alpha > e$, the value $B_{\alpha}(\mu_{\alpha, \mathcal{G}})$ is attained by $U^{\mu_{\alpha, \mathcal{G}}}(z) - \frac{|z|^2}{2\alpha}$ on \mathcal{G} as well as on $\overline{\mathcal{G}}^c$. Moreover, the singular mass is bounded by*

$$\mu_{\alpha, \mathcal{G}}^s(\mathbb{C}) \leq e\alpha^{-1}.$$

Proof. We first show that the measure $\mu_{\alpha, \mathcal{G}}$ is not concentrated on $\partial \mathcal{G}$, meaning that

$$\mu_{\alpha, \mathcal{G}}^s(\mathbb{C}) = \mu_{\alpha, \mathcal{G}}(\partial \mathcal{G}) < 1.$$

In fact, we will find a stronger bound on the total mass of $\mu_{\alpha, \mathcal{G}}^s$. Suppose that $\mu_{\alpha, \mathcal{G}}(\partial \mathcal{G}) = \frac{p}{\alpha}$ for some $p > 1$, so that $\mu_{\alpha, \mathcal{G}}$ is a candidate for the overcrowding problem on \mathbb{D} with parameter p (see [24]). But if $p > e$, we have from the assumption of mass concentration on $\partial \mathcal{G}$, the result of [24] on the optimal values for the overcrowding problem, and lastly from the inclusion $\mathcal{G} \subset \mathbb{D}$ that

$$I_{\alpha}(\mu_{\alpha, \mathcal{G}}) \geq I_{\alpha}(\mu_{\alpha, \mathbb{D}}^p) > I_{\alpha}(\mu_{\alpha, \mathbb{D}}) \geq I_{\alpha}(\mu_{\alpha, \mathcal{G}}),$$

which yields a contradiction.

Since there exist points $z \in \partial \mathcal{G}^c$ which belong to the support of $\mu_{\alpha, \mathcal{G}}$, we find that $B_{\alpha}(\mu_{\alpha, \mathcal{G}})$ is attained outside the closure of \mathcal{G} by Proposition 4.2.

For the other direction, assume that the supremum is not attained on $\overline{\mathcal{G}}$. But then it is not attained on $\overline{\mathcal{G}}_1$, where \mathcal{G}_1 contains a neighbourhood of $\overline{\mathcal{G}}$. We split $\mu_{\alpha, \mathcal{G}}$ into a sum $\mu_1 + \mu_2$, where μ_1 is supported on $\partial \mathcal{G}$ and μ_2 is supported on $\overline{\mathcal{G}}^c$. If μ_1 is non-zero, then we may form the balayage measure $\eta = \text{Bal}(\mu_1, \mathcal{G}_1^c)$ and put

$$\mu_t = (1-t)\mu_1 + t\eta + \mu_2$$

to obtain an admissible measure μ_t for which $-\Sigma(\mu_t) < -\Sigma(\mu_{\alpha, \mathcal{G}})$ while $B_{\alpha}(\mu_t) = B_{\alpha}(\mu_{\alpha, \mathcal{G}})$. This contradicts minimality of $I_{\alpha}(\mu_{\alpha, \mathcal{G}})$. It remains to be proved that μ_1 is non-zero. This is obvious, however, from the obstacle problem. Indeed, if $\mu_1 = 0$ this means that the obstacle on Γ is inactive, which would say that

$$U^{\mu_{\alpha, \mathcal{G}}}(z) = \sup \{u(z) : u \in \text{SH}_1, u(z) \leq \frac{|z|^2}{2\alpha} + C\},$$

for some constant C , the solution of which is the potential of the unconstrained minimization of I_{α} , that is equal to $U^{\alpha^{-1}\chi_{\mathbb{D}(0, \sqrt{\alpha})}}$, which is inadmissible. The conclusion of the lemma follows. \square

4.3. The structure of extremal measures. We next supply a crude description of the structure of equilibrium measures on the hole event. The information gained on the coincidence set is of particular importance for later applications, as it allows to localize the problem of minimizing $I_\alpha(\mu)$ over \mathcal{M}_G to the disk $\mathbb{D}(0, \sqrt{\alpha})$, fix the value of $B_\alpha(\mu_{\alpha, G})$, and replace the logarithmic energy by a Dirichlet integral.

Before we proceed, we need a specific function to compare the solution of an obstacle problem with. Denote by $\nu_{\alpha, r}$ the measure

$$\nu_{\alpha, r} = \frac{1}{\pi\alpha} \chi_{\mathbb{D}(0, \sqrt{\alpha}) \setminus \mathbb{D}(0, \sqrt{r})} dA,$$

and define the function V on $\mathbb{D}(0, \sqrt{\alpha})$ by

$$\begin{aligned} V(z) &= \frac{r}{\alpha} \log |z| + U^{\nu_{\alpha, r}}(z) - c_\alpha \\ (4.3.1) \quad &= \frac{r}{\alpha} \left(U^{\delta_0}(z) - U^{\frac{1}{\pi r} \chi_{\mathbb{D}(0, \sqrt{r})}}(z) \right) + U^{\frac{1}{\pi\alpha} \chi_{\mathbb{D}(0, \sqrt{\alpha})}}(z) - c_\alpha, \end{aligned}$$

where the constant c_α is given by

$$(4.3.2) \quad c_\alpha = \frac{1}{2} (\log \alpha - 1).$$

Proposition 4.4. *The function V meets the bound $V(z) \leq \frac{|z|^2}{2\alpha}$ on $\mathbb{D}(0, \sqrt{\alpha})$, and if $\alpha \geq \alpha_0 := 16e^3$ and $r = \frac{1}{4}\alpha$, the coincidence set for V with ψ_α contains $\mathbb{A}(\frac{1}{2}\sqrt{\alpha}, \sqrt{\alpha})$, and we have $V|_{\mathbb{T}(0, 2)} \leq -\frac{1}{4}$.*

Proof. That the global bound holds and that the coincidence set contains the annulus $\mathbb{A}(\sqrt{r}, \sqrt{\alpha})$ is clear from the fact that the first term in (4.3.1) is negative, and vanishes outside $\mathbb{D}(0, \sqrt{r})$. Provided that $|z|^2 \leq r$ we have

$$V(z) = \frac{r}{2\alpha} \left(\log \frac{|z|^2}{r} + \frac{|z|^2}{r} \right),$$

so if $r > 2$ we find that

$$V(z) = -\frac{r}{2\alpha} \log \left(\frac{r}{4e^{4/r}} \right), \quad |z| = 2.$$

Let $\alpha \geq \alpha_0 := 16e^3$, and choose $r = \frac{1}{4}\alpha$. We find that the coincidence set for V equals $\mathbb{A}(\frac{1}{2}\sqrt{\alpha}, \sqrt{\alpha})$ and that

$$V(2) = -\frac{1}{8} \log \frac{\alpha}{16e^{\frac{4}{\alpha}}} \leq -\frac{1}{8} \log e^2 = -\frac{1}{4}.$$

This completes the proof □

Proposition 4.5. *Assume that \mathcal{G} is contained in the unit disk. Then the equilibrium measure takes the form*

$$d\mu_{\alpha, G} = d\mu_{\alpha, G}^s + \frac{1}{\pi\alpha} \chi_{\mathbb{D}(0, \sqrt{\alpha}) \setminus \Omega} dA,$$

where $\Omega = \Omega_\alpha$ is an open set containing \mathcal{G} and where the singular part of the measure is supported on the boundary of \mathcal{G} . Moreover, there exists a universal constant α_0 , such that the closed annulus $\overline{\mathbb{A}(\frac{1}{2}\sqrt{\alpha}, \sqrt{\alpha})}$ belongs to the support of $\mu_{\alpha, G}$ whenever $\alpha \geq \alpha_0$.

Remark 4.6. (a) The domain Ω is the forbidden region defined in the introduction. (b) We may take $\alpha_0 = 16e^3$. This is likely far from sharp, but easy to obtain.

Proof. The decomposition of the measure $\mu_{\alpha, \mathcal{G}}$ is evident in view Proposition 3.1, and the fact that $U^{\mu_{\alpha, \mathcal{G}}}$ may be regarded as the solution to the mixed obstacle problem on a bounded domain $D = \mathbb{D}(0, \sqrt{\alpha})$ with boundary data $f = U^{\mu_{\alpha, \mathcal{G}}}|_{\mathbb{T}(0, \sqrt{\alpha})}$. To see why the latter statement holds, we define the function

$$v_0(z) = \sup \{v(z) : v \in \text{SH}(D) \cap \mathcal{K}_{\psi, \mathcal{G}}^f\}$$

and its upper semicontinuous regularization v_0^* , which by the regularity of ψ has $v_0^* = v_0$ except possibly on a polar subset of $\partial \mathcal{G}$. Abusing notation slightly, we denote the regularized function by v_0 , and note that this function solves the obstacle problem for $\mathcal{K}_{\psi, \mathcal{G}}^f$. Moreover, $v_0 \geq U^{\mu_{\alpha, \mathcal{G}}}$ with equality on ∂D . If we glue v_0 and $U^{\mu_{\alpha, \mathcal{G}}}$ together along ∂D we obtain a majorant w of $U^{\mu_{\alpha, \mathcal{G}}}$. If we can show that w is subharmonic, it follows that the Riesz measure of which meets all requirements for the defining minimization problem for $\mu_{\alpha, \mathcal{G}}$. But this follows from the sub-mean value property, which trivially holds on D and on the interior of D^c , while on ∂D we have

$$w(z) = U^{\mu_{\alpha, \mathcal{G}}}(z) \leq \frac{1}{|B|} \int_B U^{\mu_{\alpha, \mathcal{G}}} \, dA \leq \frac{1}{|B|} \int_B w \, dA.$$

It remains to prove the statement concerning the coincidence set. We first observe that the singular mass $\mu_{\alpha, \mathcal{G}}^s(\mathbb{C})$ meets the bound

$$\mu_{\alpha, \mathcal{G}}^s(\mathbb{C}) \leq \frac{e}{\alpha},$$

by Lemma 4.3. Secondly, we observe that away from \mathcal{G} , say for definiteness outside $\mathbb{D}(0, 2)$, we may reconstruct the function $U = U^{\mu_{\alpha, \mathcal{G}}} - B_\alpha(\mu_{\alpha, \mathcal{G}})$ from its boundary values $f = U|_{\mathbb{T}(0, 2)}$ by solving an obstacle problem in the same way as above, so that

$$U(z) = \sup \{v(z) : v \in \text{SH}_1(\mathbb{C} \setminus \mathbb{D}(0, 2)), u \leq \frac{|z|^2}{2\alpha}, u|_{\mathbb{T}(0, 2)} = f\}.$$

We will complete the proof by estimating the boundary values of U from below by a constant $C(\alpha)$, and show that the solution U dominates the solution V of Proposition 4.4, whose coincidence set contains the annulus $\mathbb{A}(\frac{1}{2}\sqrt{\alpha}, \sqrt{\alpha})$. We proceed to estimate $B_\alpha(\mu_{\alpha, \mathcal{G}})$ from above. We know that $B_\alpha(\mu_{\alpha, \mathcal{G}})$ is attained on both \mathcal{G} and \mathcal{G}^c . On \mathcal{G} , we have that

$$U^{\mu_{\alpha, \mathcal{G}}^s}(z) \leq \mu_{\alpha, \mathcal{G}}^s(\mathbb{C}) \sup_{w \in \partial \mathcal{G}} \log |z - w| \leq \frac{e \log 2}{\alpha}.$$

This allows us to estimate

$$U^{\mu_{\alpha, \mathcal{G}}}(z) - \frac{|z|^2}{2\alpha} \leq (U^{\mu_{\alpha, \mathcal{G}}}(z) - \frac{|z|^2}{2\alpha}) - U^{\frac{1}{\alpha}\chi_\Omega}(z) + \frac{e \log 2}{\alpha} = c_\alpha - U^{\frac{1}{\alpha}\chi_\Omega}(z) + \frac{e \log 2}{\alpha},$$

where c_α is given by (4.3.2) and where $\mu_\alpha = \frac{1}{\alpha}\chi_{\mathbb{D}(0, \sqrt{\alpha})}$. From the estimate of the singular mass in Lemma 4.3 and the fact that $\mu_{\alpha, \mathcal{G}}$ is a probability measure, we infer that $|\Omega| \leq e$.

We next observe that there are two *extremal distributions of mass*, which allow us to bound $U^{\chi_\Omega}(z)$. Indeed, by monotonicity of the logarithmic kernel, it holds that $U^{\chi_\Omega}(z) \leq U^{\mu_\alpha^{z, 1}}(z)$, where $\mu_\alpha^{z, 1} = |\Omega|\delta_w$ where $w = -\sqrt{\alpha}\frac{z}{|z|}$, which lies at maximal distance from z within the allotted domain. Similarly, now using the fact that $\mu_{\alpha, \mathcal{G}}^c$ has uniform density with respect to area measure on some plane region, we find that $U^{\chi_\Omega}(z) \geq U^{\chi_{\mathbb{D}(z, \sqrt{|\Omega|})}}(z)$. Performing the necessary computations, we see that

$$\frac{1 - e}{2\alpha} \leq U^{\frac{1}{\alpha}\chi_\Omega}(z) \leq \frac{e \log(2\sqrt{\alpha})}{\alpha}, \quad z \in \mathcal{G}.$$

In conclusion, we have that

$$B_\alpha(\mu_{\alpha,\mathcal{G}}) \leq c_\alpha + \frac{e(2 \log 2 + 1) - 1}{2\alpha}.$$

We turn to estimating $U|_{\mathbb{T}(0,2)}$ from below, and first notice that the potential of the singular measure $\mu_{\alpha,\mathcal{G}}^s$ is positive on the circle $\mathbb{T}(0,2)$. As was also observed above, we have $U^{\mu_{\alpha,\mathcal{G}}^c}(z) = U^{\alpha^{-1}\chi_{\mathbb{D}(0,\sqrt{\alpha})}}(z) - U^{\frac{1}{\alpha}\chi_\Omega}$, so by our previous estimate on $B_\alpha(\mu_{\alpha,\mathcal{G}})$ we have that on $\mathbb{T}(0,2)$

$$(4.3.3) \quad U^{\mu_{\alpha,\mathcal{G}}}(z) - B_\alpha(\mu_{\alpha,\mathcal{G}}) \geq -\frac{2e \log(2\sqrt{\alpha}) + e(2 \log 2 + 1) - 5}{2\alpha}.$$

The lower bound (4.3.3) for $U|_{\mathbb{T}(0,2)}$ is monotonically decreasing in α for $\alpha > e$, so we may replace α by $\alpha_0 = 16e^3$. For this value the bound (4.3.3) gives $U|_{\mathbb{T}(0,2)} = \frac{5 - e(4 + \log(256))}{32e^3} > -\frac{1}{30} > -\frac{1}{4}$, so that the function $V = V_{r,\alpha}$ of Proposition 4.4 is a competitor for the obstacle problem. But then we have $U \geq V$ on \mathbb{C} , and it follows that the support of $\mu_{\alpha_0,\mathcal{G}}$ covers the annulus $\mathbb{A}(\frac{1}{2}\sqrt{\alpha_0}, \sqrt{\alpha_0})$. \square

We next show that $\mu_{\alpha,\mathcal{G}}$ stabilizes as the parameter α grows.

Proposition 4.7. *Assume that $\alpha \geq \alpha_0$. Then we have that*

$$\mu_{\alpha,\mathcal{G}} = \frac{\alpha_0}{\alpha} \mu_{\alpha_0,\mathcal{G}} + \frac{1}{\pi\alpha} \chi_{\mathbb{A}(\frac{1}{2}\sqrt{\alpha_0}, \sqrt{\alpha_0})} dA.$$

Proof. Provided that $\alpha \geq \alpha_0$, we have in view of Proposition 4.5 that $\mu_{\alpha,\mathcal{G}}$ takes the form

$$\mu_{\alpha,\mathcal{G}} = \frac{1}{4} \mu^{(0)} + \frac{1}{\pi\alpha} \chi_{A_\alpha} dA$$

where $\mu^{(0)}$ is a probability measure supported on $\mathbb{D}(0, \frac{1}{2}\sqrt{\alpha})$ whose potential is constant on $\mathbb{T}(0, \frac{1}{2}\sqrt{\alpha})$, and where A_α is shorthand for the annulus $\mathbb{A}(\frac{1}{2}\sqrt{\alpha_0}, \sqrt{\alpha_0})$. Observe that $\mu_{\alpha/4,\mathcal{G}}$ also is a probability measure on $\mathbb{D}(0, \frac{1}{2}\sqrt{\alpha})$ with constant potential on $\mathbb{T}(0, \frac{1}{2}\sqrt{\alpha})$, in view of Proposition 4.5.

The potential of the measure $\frac{1}{\pi\alpha} \chi_{A_\alpha} dA$ is constant on $\mathbb{D}(0, \frac{1}{2}\sqrt{\alpha})$ so in particular on \mathcal{G} , and it equals $c_\alpha - \frac{1}{4}c_{\alpha/4}$ there. Since we know that the value $B_\alpha(\mu_{\alpha,\mathcal{G}})$ is attained on \mathcal{G} and equals c_α , we may conclude that

$$c_\alpha = B_\alpha(\mu_{\alpha,\mathcal{G}}) = B_\alpha(\frac{1}{4}\mu^{(0)}) + c_\alpha - \frac{1}{4}c_{\alpha/4}.$$

Moreover, we have, by scaling,

$$B_\alpha(\frac{1}{4}\mu^{(0)}) = \frac{1}{4}B_{\alpha/4}(\mu^{(0)}),$$

so that $B_{\alpha/4}(\mu^{(0)}) = c_{\alpha/4} = B_{\alpha/4}(\mu_{\alpha/4,\mathcal{G}})$.

Finally, since the potential $U^{\mu^{(0)}}$ equals $\log|z|$ on the support of χ_{A_α} , it follows that the energy $-\Sigma(\mu_\alpha)$ may be rewritten as

$$-\Sigma(\mu_\alpha) = -\frac{1}{16}\Sigma(\mu^{(0)}) + C(\alpha)$$

for a constant $C(\alpha)$ which does not depend on the choice of $\mu^{(0)}$, within the given restrictions mentioned above. It follows that $\mu^{(0)}$ minimizes the energy over all measures $\mu \in \mathcal{M}_\mathcal{G}$ with support on $\mathbb{D}(0, \frac{1}{2}\sqrt{\alpha})$ whose potentials are constant on $\mathbb{T}(0, \frac{1}{2}\sqrt{\alpha})$, under the additional constraint $B_{\alpha/4}(\mu^{(0)}) = B_{\alpha/4}(\mu_{\alpha/4,\mathcal{G}})$. But $\mu_{\alpha/4,\mathcal{G}}$ belongs to this class, and minimizes $-\Sigma(\mu)$ over the larger class of measures $\mu \in \mathcal{M}_\mathcal{G}$ with $B_{\alpha/4}(\mu) = c_{\alpha/4}$. Hence, it follows that $\mu^{(0)} = \mu_{\alpha/4,\mathcal{G}}$.

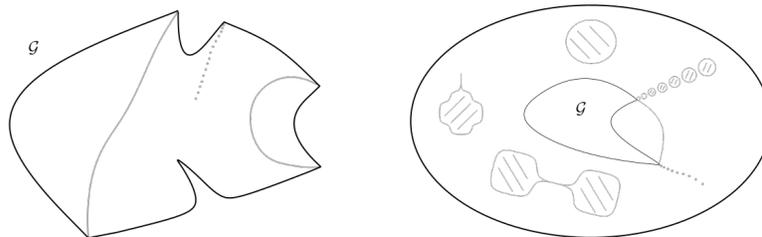


FIGURE 5.1. The coincidence sets on \mathcal{G} (left) and $\mathbb{C} \setminus \bar{\mathcal{G}}$ (right), with contact between the free boundary \mathcal{S} and the fixed boundary $\partial\mathcal{G}$ only near corners. Taken together, the figures illustrate all different types of free boundary points for an analytic obstacle.

Whenever $\frac{1}{4^{k+1}}\alpha \geq \alpha_0$, this process can be repeated to express $\mu_{4^{-k}\alpha, \mathcal{G}}$ in terms of $\mu_{4^{-k-1}\alpha, \mathcal{G}}$ by adding an annular shell of uniform mass. For the unique integer k for which $\alpha_0 \in (4^{-k-1}\alpha, 4^{-k}\alpha)$, the above procedure instead leads to

$$\mu_{4^{-k}\alpha, \mathcal{G}} = \frac{\alpha_0}{4^{-k}\alpha} \mu_{\alpha_0, \mathcal{G}} + \frac{1}{4^{-k}\alpha\pi} \chi_{\mathbb{A}(\frac{1}{2}\sqrt{\alpha_0}, 4^{-k}\sqrt{\alpha})} d\mathbb{A}.$$

This completes the proof. \square

5. SEPARATION OF THE FREE BOUNDARY FROM THE THIN OBSTACLE

5.1. Local control of the Riesz mass on the coincidence set. Denote by \mathcal{G} a bounded simply connected domain whose boundary $\Gamma = \partial\mathcal{G}$ is piecewise C^2 -smooth without cusps. Denote the collection of all corners of Γ by \mathcal{E} , and by \mathcal{E}_0 the collection of all corners with interior angle $\pi\sigma$ for $\sigma \in (0, 1)$. We let $\mathcal{E}_1 = \mathcal{E} \setminus \mathcal{E}_0$.

We continue the study the extremal measure $\mu_{\alpha, \mathcal{G}}$ for the minimization of I_α over the set of probability measures which give no mass to \mathcal{G} . To ease notation, we let $\mu_0 = \mu_{\alpha, \mathcal{G}}$, $u_0 = U^{\mu_{\alpha, \mathcal{G}}}$ and $\psi = \psi_\alpha$. We will assume throughout this section that $\alpha \geq \alpha_0(\mathcal{G})$ (see Proposition 4.5). Recall that u_0 is said to be non-degenerate if $u_0 < \psi$ along Γ , c.f. (1.5.3), and denote by $d(z, E)$ the distance $\inf_{w \in E} |z - w|$, where $z \in \mathbb{C}$ and $E \subset \mathbb{C}$.

Theorem 5.1. *Assume that Γ is piecewise C^2 -smooth without cusps. Then it holds that*

$$(5.1.1) \quad d(z, \partial\mathcal{G}) \lesssim d(z, \mathcal{E}_0)^{\max\{1, \gamma\}}, \quad z \in \mathcal{I},$$

where $\gamma = \frac{1-\sigma}{\sigma}$ and σ is the smallest angle for corner points on $\partial\mathcal{G}$. If Γ is C^2 -smooth, then u_0 is non-degenerate.

For $j = 0, 1$ we denote by \mathcal{E}_j^η the fattened singular sets

$$\mathcal{E}_j^\eta = \bigcup_{z \in \mathcal{E}_j} \mathbb{D}(z, \eta)$$

and put $\mathcal{E}^\eta = \mathcal{E}_0^\eta \cup \mathcal{E}_1^\eta$. Then for any fixed $\eta > 0$ it holds that $d(\mathcal{I}, \Gamma \setminus \mathcal{E}^\eta) > 0$. The consequence of the theorem is illustrated in Figure 5.1.

The idea is to use the first order variational formula for perturbations of (3.1.1), and given a sequence of points which approach the boundary at a rate in breach of

Theorem 5.1, try to find suitable perturbations which decrease the Dirichlet energy while yielding admissible measures ($\mu_\epsilon(\mathcal{G}) = 0$).

The key to obtaining such a result is a local estimate of the distributional Laplacian Δu_0 near a point $z_0 \in \mathcal{S}$, valid in particular for $z_0 \in \Gamma \cap \mathcal{S}$. Here, we recall that \mathcal{S} is the coincidence set defined in (2.8.2). We denote by \mathcal{T} the coincidence set with the full obstacle on Γ

$$\mathcal{T} = \{z \in \Gamma : u(z) = \psi(z)\},$$

and by \mathcal{T}_r the r -neighbourhood of \mathcal{T} on Γ .

Lemma 5.2. *For the full obstacle $\psi_\alpha(z) = \frac{1}{2\alpha}|z|^2$, it holds for any point $z_0 \in \mathcal{S}$ that*

$$\mu_0(\mathbb{D}(z_0, r)) \leq \frac{1}{2\alpha}e^2r^2, \quad 0 < r < 1.$$

Moreover, it holds that

$$\frac{\mu(\mathcal{T}_r)}{|\mathcal{T}_r|} \leq \frac{7}{\alpha}e^2r, \quad r \rightarrow 0.$$

Proof. We set $v_0 = \alpha u_0$, $\nu_0 = \alpha \mu_0$ and $\psi = \frac{1}{2}|z|^2$. We have

$$v_0(z) - v_0(0) = \int_{\mathbb{C}} (\log |z - w| - \log |w|) d\nu_0(w).$$

Averaging over the small circle $|z| = rm$ we find that

$$\frac{1}{r} \int_{|z|=r} (v_0(z) - v_0(z_0)) ds(z) = \int_{\mathbb{D}(0,r)} H(r, w) d\nu_0(w),$$

where

$$H(r, w) = \frac{1}{r} \int_{|z|=r} (\log |z - w| - \log |w|) ds(z) = \log \frac{r}{|w|} \chi_{\mathbb{D}(0,r)}.$$

The kernel $H(r, w) = \log \frac{r}{|w|}$ is bounded from below by 1 on $[0, re^{-1}]$ and is positive everywhere on $[0, r]$, so

$$\int_{\mathbb{D}(z_0,r)} H(r, w) d\nu_0 \geq \nu_0(\mathbb{D}(z_0, r/e)).$$

We next notice that

$$\frac{1}{r} \int_{\mathbb{T}(0,r)} (\psi(z) - \psi(z_0)) ds = \frac{1}{2\pi} \int_{\mathbb{D}(0,r) \setminus \mathcal{S}} H(r, w) \Delta \psi(z) dA(w).$$

Since $\Delta \psi = 2$, we find that the right-hand side is bounded above by

$$\frac{1}{\pi} \int_{\mathbb{D}(0,r)} H(r, w) dA(w) = \frac{r^2}{2}.$$

Putting the desired estimates together, we find that whenever $v_0(z_0) = \varphi(z_0)$, we have

$$\begin{aligned} 0 &\geq \frac{1}{r} \int_{\mathbb{T}(0,r)} (v_0(z) - \psi(z)) ds(z) \\ &= \frac{1}{r} \int_{\mathbb{T}(0,r)} (v_0(z) - v_0(z_0) - (\psi(z) - \psi(z_0))) ds(z) \geq \nu_0^s(\mathbb{D}(0, r/e)) - \frac{r^2}{2}, \end{aligned}$$

which after replacing r by er yields the desired bound

$$(5.1.2) \quad \nu_0^s(\Gamma_r) \leq \frac{1}{2}e^2r^2.$$

In order to reach the final conclusion, we may use the Vitali covering lemma to find a collection of $N(r) < \infty$ disjoint balls B_j of radius $\frac{1}{5}r$, centered at points $z_j \in \mathcal{T}$, such that the balls B'_j rescaled by 5 cover $\bigcup_{z \in \mathcal{T}} \mathbb{D}(z, r/5)$. Notice in particular that the latter set covers $\mathcal{T}_{r/5}$. We use the notation $\Gamma_r^{(z_j)} = \Gamma \cap \mathbb{D}(z_j, r)$. The set $\Gamma_r^{(z_j)}$ are disjoint and satisfy $|\Gamma_r^{(z_j)}| \geq \frac{2}{5}r$, while the collection $\Gamma_r^{(z_j)}$ cover $\mathcal{T}_{r/5}$. If we invoke the estimate (5.1.2), we find that

$$\rho_{r/5} = \frac{\nu_0(\mathcal{T}_{r/5})}{|\mathcal{T}_{r/5}|} \leq \frac{\sum_j \nu_0(\Gamma_r^{(z_j)})}{\sum_j |\Gamma_r^{(z_j)}|} \leq \frac{\frac{1}{2}e^2 r^2 N(r)}{\frac{2}{5}r N(r)} \leq \frac{5e^2}{4}r, \quad r \rightarrow 0,$$

which after rescaling yields that

$$\rho_r \leq \frac{25e^2}{4}r.$$

This completes the proof. \square

5.2. Quantitative non-degeneracy. Recall that $\partial\mathcal{G}$ is a piecewise smooth curve with finitely many corners $\{z_j\} = \mathcal{E}$, and let $w \in \partial\mathcal{G}$. We need estimates of $\omega(z, \mathbb{D}(w, \epsilon) \cap \partial\mathcal{G}, \mathcal{G})$ in terms of $d(z, \mathcal{E})$ and $d(z, \partial\mathcal{G})$. To this end, use Proposition 2.3 to control the quantity

$$\frac{d\omega(z, \cdot, \mathcal{G})}{ds} = P_{\mathcal{G}}(z, \cdot) = |\phi'_z(\cdot)|,$$

where ϕ_z is a conformal mapping of \mathcal{G} onto the unit disk so that $\phi_z(z) = 0$. Denote by $m_\zeta(z)$ the Möbius transformation $m_\zeta = (z - \zeta)/(1 - \bar{\zeta}z)$. If $\phi = \phi_{z_0}$ for some fixed interior point z_0 of \mathcal{G} , we have

$$\phi'_z(w) = \partial_w m_{\phi(z)}(\phi(w)) = \partial_w \frac{\phi(w) - \phi(z)}{1 - \overline{\phi(z)}\phi(w)} = \phi'(w) \frac{1 - |\phi(z)|^2}{(1 - \overline{\phi(z)}\phi(w))^2}$$

Thus, provided that z is confined to the set $\mathcal{G} \setminus \mathcal{E}_0^\eta$ for a fixed $\eta > 0$, we find that when w approaches a point $\zeta \in \mathcal{E}$, the density of harmonic measure satisfies

$$|\phi'_z(w)| \gtrsim (1 - |\phi(z)|^2)|\phi'(w)| \gtrsim d(z, \partial\mathcal{G})|\phi'(w)|,$$

where we have used Remark 2.4 in the last step. Hence, it follows from Proposition 2.3 that

$$(5.2.1) \quad \frac{d\omega(z, w, \mathcal{G})}{ds(w)} \gtrsim d(z, \partial\mathcal{G})|w - \zeta|^\gamma, \quad z \in \mathcal{G} \setminus \mathcal{E}_0^\eta$$

as $w \rightarrow \zeta \in \mathcal{E}_0$, where $\gamma = (1 - \sigma)/\sigma \in (0, \infty)$.

We are now ready for a proof of the main result of this section.

Proof of Theorem 5.1. We begin with a sequence of points $w_k \in \mathcal{I}$ approaching a boundary point z_0 , and aim to control $d(w_k, \partial\mathcal{G})$ from below. We begin to establish the bound (5.1.1) when $z_0 \in \mathcal{E}_0$, and then show that $z_0 \in \Gamma \setminus \mathcal{E}_0^\eta$ is impossible.

Thus let $z_0 \in \mathcal{E}_0$, and suppose that the claim of the theorem is false, so that there exists a sequence (w_k) of points in \mathcal{I} converging to $z_0 \in \partial\mathcal{G}$, for which

$$d(w_k, \partial\mathcal{G}) \leq |w_k - z_0|^{\max\{\gamma, 1\}} a_k,$$

where a_k are positive numbers tending to zero. We let $r_k = 2d(w_k, \partial\mathcal{G})$ and define a subarc of $\partial\mathcal{G}$ by

$$T_k = \mathbb{D}(w_k, r_k) \cap \partial\mathcal{G}$$

which since $\partial\mathcal{G}$ is piecewise smooth without cusps has length comparable to r_k .

CLAIM 1. We show that there exists an arc $U \subset \Gamma \setminus \mathcal{E}^\eta$ for some $\eta > 0$ with positive measure $\mu_0^s(U) = \delta > 0$. Such a set exists, since if it did not, then $\mathcal{I} \cap \partial\mathcal{G}$ would be everywhere dense in $\text{supp}(\mu_0)$. But then $u_0(z) = \psi(z)$ on $\text{supp}(\mu_0)$ by semi-continuity of u_0 , and in view of Lemma 5.2 we have for the set

$$\mathcal{T}_r = \{z \in \Gamma : d(z, \mathcal{I}) \leq r\}$$

that

$$\mu_0(\Gamma) = \mu_0(\mathcal{T}) \leq \mu_0(\mathcal{T}_r) \leq 7e^2 |\mathcal{T}_r| r \quad \text{for all } r > 0$$

so $\mu_0(\Gamma) = 0$. But then the obstacle is inactive, so it follows that $u_0 = \psi$, which is impossible in view of $\Delta u_0|_{\mathcal{G}} = 0$.

We now define a perturbation h_k , by

$$h_k(z) = \omega(z, U, \mathcal{G}) - r_k^{-2} a_k^{\frac{1}{2}} \omega(z, T_k, \mathcal{G}),$$

and claim that for large enough k , the function h_k is an admissible perturbation which decreases the Dirichlet energy. We will show (Claim 2) that $h_k \leq 0$ on \mathcal{I} , and (Claim 3) that

$$\int_{\partial\mathcal{G}} h_k(z) d\mu_0^s(z) > 0.$$

CLAIM 2. We show the inequality $h_k \leq 0$ on \mathcal{I} by showing that the function h is non-positive except for a small neighbourhood of U in \mathcal{G} . This follows by an application of the maximum principle in the domain

$$\tilde{\mathcal{G}} = \{z \in \mathcal{G} : d(z, U) \geq C_0\},$$

where C_0 is small enough that \mathcal{I} belongs to the closure of $\tilde{\mathcal{G}}$ and $\mathcal{G} \setminus \tilde{\mathcal{G}}$ is simply connected. For this argument to work, we need to show that $h_k \leq 0$ on the boundary of \mathcal{G} away from the set U , as well as on the cross-cut $\partial\tilde{\mathcal{G}} \setminus \partial\mathcal{G}$. We may immediately observe that $h_k \equiv 0$ on $\partial\mathcal{G} \setminus (U \cup T_k)$ and negative on T_k by definition.

Turning to analyzing h_k on the cross cut, we simply use the estimate (5.2.1), to find that

$$(5.2.2) \quad \omega(z, T_k, \mathcal{G}) \gtrsim d(z, \partial\mathcal{G}) \int_{T_k} |w - z_0|^\gamma ds(w) \cong d(z, \partial\mathcal{G}) |w_k - z_0|^\gamma, \quad z \in \partial\tilde{\mathcal{G}} \setminus \partial\mathcal{G}.$$

But this yields that

$$h_k(z) \lesssim d(z, \partial\mathcal{G}) \left(1 - a_k^{\frac{1}{2}} r_k^{-2} |w_k - z_0|^\gamma\right), \quad z \in \partial\tilde{\mathcal{G}} \setminus \partial\mathcal{G},$$

where we use the trivial bound $\omega(z, U, \partial\mathcal{G}) \lesssim d(z, \partial\mathcal{G})$, which holds since $U \subset \partial\mathcal{G} \setminus \mathcal{E}^\eta$. Moreover, since by assumption

$$r_k \leq |w_k - z_0|^{\max\{\gamma, 1\}} a_k \leq |w_k - z_0|^\gamma a_k,$$

where $a_k \rightarrow 0$, it holds that

$$a_k r_k^{-1} |w_k - z_0|^\gamma \geq 1,$$

so that

$$a_k^{\frac{1}{2}} r_k^{-2} |w_k - z_0|^\gamma \geq r_k^{-1} a_k^{-\frac{1}{2}} \rightarrow \infty$$

and so $h_k < 0$ also here. It follows that for large enough k , say $k \geq k_0$, the function h_k is an admissible perturbation.

CLAIM 3. We need to study the quantity

$$\int_{\partial\mathcal{G}} h_k(z) d\mu_0^s(z) = \mu_0^s(U) - a_k^{\frac{1}{2}} r_k^{-2} \mu_0^s(T_k).$$

In view of the regularity estimate of Lemma 5.2 we have the control

$$\mu_0(\mathbb{D}(w, r)) \lesssim r^2, \quad w \in \mathcal{S}.$$

In addition, we have the inclusion $T_k \subset \mathbb{D}(w_k, r_k)$ where we recall that $w_k \in \mathcal{I} \subset \mathcal{S}$. As a consequence, it follows that

$$\mu_0(T_k) \lesssim r_k^2.$$

But then it follows, from the fact that $\mu_0^s(U) = \delta > 0$, that

$$\int h_k d\mu_0^s > 0$$

for k large enough, say $k \geq k_1$.

For $k \geq \max\{k_0, k_1\}$ the perturbation is both admissible and decreases the Dirichlet energy, which is impossible. Hence the result follows for corner points of the first kind, $z_0 \in \mathcal{E}_0$.

We thus assume that $z_0 \in \partial\mathcal{G} \setminus \mathcal{E}_0$. Also in this case we argue by contradiction, so assume that there exists $a_k = o(1)$ and $w_k \in \mathcal{I}$ approaching z_0 with

$$|w_k - z_0| = a_k.$$

We define the perturbation

$$h_k = \omega(z, U, \mathcal{G}) - a_k^{-\frac{3}{2}} \omega(z, T_k, \mathcal{G}),$$

and perform the same estimates as above, the only difference being that $|\phi'(w)|$ is bounded away from zero near z_0 , so that instead of (5.2.2) we now have the lower bound

$$\omega(z, T_k, \mathcal{G}) \gtrsim d(z, \partial\mathcal{G}) \int_{T_k} ds(w) \cong d(z, \partial\mathcal{G}), \quad z \in \partial\tilde{\mathcal{G}} \setminus \partial\mathcal{G}.$$

The rest of the argument goes through unchanged, so that as a consequence, we find that for large enough k , the function h_k is an admissible perturbation which decreases the Dirichlet energy. Hence we conclude that $a_k = o(1)$ is an impossibility, and the claim follows. \square

6. THE EMERGENCE OF QUADRATURE DOMAINS

6.1. The perturbation scheme. In view of Theorem 4.1, Proposition 4.5, and Remark 2.16, the potential $u_0 = U^{\mu_{\alpha, \mathcal{G}}}$ of the extremal measure $\mu_0 = \mu_{\alpha, \mathcal{G}}$ solves the obstacle problem (see Definition 2.10) for the class $\mathcal{K}_{\psi_{\alpha, g}}(D)$, $g \in H^{\frac{1}{2}}(\Gamma)$ and where we use the notation $D = \mathbb{D}(0, \sqrt{\alpha})$.

In order to extract more precise information about μ_0 , we will perturb the implicit thin obstacle g by a function ϵh , and solve the mixed obstacle problem with thin obstacle $g_{\epsilon} = g + \epsilon h$ and full obstacle ψ_{α} on D . This procedure produces a family of functions u_{ϵ} , the Riesz masses of which provide a family μ_{ϵ} of competitors for the minimization problem, under suitable conditions on the perturbation h . We then hope to reveal information about the minimizer by comparing $I_{\alpha}(\mu_{\epsilon})$ with $I_{\alpha}(\mu_0)$ using Corollary 3.7.

We recall that when the measure μ_0 is non-degenerate in the sense of (1.5.3), Sakai's regularity theorem (see Theorem 2.17) implies that the interior coincidence

set is finite. Moreover, in view of Lemma 4.3 this set is non-empty. Whenever h is harmonic on \mathcal{G} with $h(\lambda) < 0$ for $\lambda \in \mathcal{I}$, it turns out that $g_\epsilon = g + \epsilon h$ is an admissible perturbation for small $\epsilon > 0$.

Proposition 6.1. *Assume that $\mu_{\alpha, \mathcal{G}}$ is non-degenerate, that \mathcal{G} is contained in the unit disk, that $\alpha \geq \alpha_0$, and let*

$$g = U^{\mu_{\alpha, \mathcal{G}}}|_{\partial \mathcal{G}}.$$

Denote by h the boundary values on $\partial \mathcal{G}$ of a function harmonic in a neighbourhood of \mathcal{G} and negative on \mathcal{I} . Then for small enough $\epsilon > 0$, the Riesz masses μ_ϵ of the solutions u_ϵ to the obstacle problem with thin obstacle g_ϵ and global obstacle ψ_α are admissible for the hole event, and we have that

$$\int_{\partial \mathcal{G}} h \, d\mu_{\alpha, \mathcal{G}}^s \leq 0.$$

Proof. Since h is negative on \mathcal{I} , it follows (by an argument familiar from the proof of Theorem 4.1) that the Riesz mass μ_ϵ of u_ϵ is admissible for the hole event. Moreover, it is clear that $B_\alpha(\mu_\epsilon) = B_\alpha(\mu_0) = c_\alpha$, by virtue of the coincidence set containing the non-trivial annulus $\mathbb{A}(\frac{1}{2}\sqrt{\alpha_0}, \sqrt{\alpha})$. Indeed, the proof of Proposition 4.5 applies, since the boundary values have been perturbed by at most by $\epsilon \|h\|_\infty$. In particular (4.3.3) holds with $o(1)$ added on the right hand side, which does not affect the remainder of the argument.

The functions u_ϵ are minimizers of the Dirichlet energy over the sets $\mathcal{K}_{\psi, g_\epsilon}$, so that the variational formula of Corollary 3.7 applies, and yields that

$$I_\alpha(\mu_\epsilon) = I_\alpha(\mu_{\alpha, \mathcal{G}}) - 4\pi\epsilon \int_{\partial \mathcal{G}} h \, d\mu_0^s + o(\epsilon).$$

That the boundary integral is negative is immediate from minimality of μ_0 for the functional I_α over the class of admissible measures for the hole event. This proves the lemma. \square

6.2. Proof of the main theorem. We first treat the case when μ_0 is non-degenerate.

Lemma 6.2. *Assume that $\alpha \geq \alpha_0$ and that $\mu_0 = \mu_{\alpha, \mathcal{G}}$ is non-degenerate. Then μ_0^s there exist a finite set $\Lambda \subset \mathcal{G}$ and positive weights $(\rho_\lambda)_{\lambda \in \Lambda}$ such that*

$$\mu_0^s = \sum_{\lambda \in \Lambda} \rho_\lambda \omega_{\mathcal{G}, \lambda}.$$

Proof. Since $U^{\mu_0} < B_\alpha(\mu_0)$ on $\partial \mathcal{G}$, Sakai's theorem implies that \mathcal{I} is a finite subset of \mathcal{G} . If h is any harmonic function in \mathcal{G} with $h|_{\mathcal{I}} \equiv 0$, then for $\delta > 0$ we apply Proposition 6.1 to $h - \delta$ and $-h - \delta$ to find that

$$-\mu_0^s(\partial \mathcal{G})\delta < \int h \, d\mu_0^s < \mu_0^s(\partial \mathcal{G})\delta.$$

Since $\delta > 0$ was arbitrary, it follows that

$$\int h \, d\mu_0^s = 0.$$

Let now h be any smooth function on $\partial\mathcal{G}$, and extend h to a harmonic function on \mathcal{G} . The functions $(f_\lambda)_{\lambda \in \Lambda}$ defined by

$$f_\lambda = \operatorname{Re} \prod_{\lambda' \neq \lambda} (z - \lambda') / (\lambda - \lambda'), \quad \lambda \in \mathcal{I}$$

supply a basis for harmonic interpolation on Λ . We use this basis to show that the integral of h against μ_0^s is given by point evaluations on \mathcal{I} :

$$(6.2.1) \quad \int h \, d\mu_0^s = \int \left(h - \sum_{\lambda \in \mathcal{I}} h(\lambda) f_\lambda(z) \right) d\mu_0^s + \sum_{\lambda \in \mathcal{I}} \rho_\lambda h(\lambda) = \sum_{\lambda \in \mathcal{I}} \rho_\lambda h(\lambda),$$

where $\rho_\lambda = \int f_\lambda \, d\mu_0^s$. To see that the weights are non-negative, we observe that if some weight ρ_{j_0} is negative, then we set

$$h = -f_{\lambda_0} - \delta' \sum_{\lambda \neq \lambda_0} f_\lambda$$

for some $\delta' > 0$. We know by Proposition 6.1 that $\int h \, d\mu_{\alpha, \mathcal{G}}^s < 0$, while at the same time we see that

$$\int (-f_{\lambda_0} - \delta' \sum_{\lambda \neq \lambda_0} f_\lambda) d\mu_0^s = |\rho_{\lambda_0}| + O(\delta')$$

which is positive for small enough δ' , which is a contradiction. Let $\Lambda = \{\lambda \in \mathcal{I} : \rho_\lambda > 0\}$.

By (6.2.1), integrating h with respect to the singular part of the equilibrium measure is the same as integrating with respect to the measure $\sum_{\lambda \in \Lambda} \rho_\lambda \omega_{\lambda, \mathcal{G}}$. Since h can be taken as an arbitrary continuous function, this implies that the two measures agree, which completes the proof that μ_0^s is a finite sum of harmonic measures. \square

The above lemma allows us to proceed to the proof of Theorem 1.8.

Proof of Theorem 1.8. Suppose first that $\partial\mathcal{G}$ is C^2 -smooth. In this case, we may apply Theorem 5.1 to conclude that $\mu_0 = \mu_{\alpha, \mathcal{G}}$ is non-degenerate in the sense of (1.5.3), and by Lemma 6.2 the singular component μ_0^s of the measure μ_0 takes the form claimed in the statement of the theorem. We also know that the continuous part takes the form

$$\mu_0^c = \frac{1}{\alpha} \chi_{\mathbb{D}(0, \sqrt{\alpha}) \setminus \Omega},$$

where $\mathcal{G} \subset \Omega \subset \mathbb{D}(0, \sqrt{\alpha})$ and where

$$\Omega = \mathbb{D}(0, \sqrt{\alpha}) \setminus (\mathcal{G} \cup \operatorname{supp}(\mu_0^c)) = (\mathbb{D}(0, \sqrt{\alpha}) \setminus \mathcal{S}) \cap \mathcal{G}^c.$$

We set $U = \alpha U^{\mu_0^c}$ and notice that U is the envelope

$$(6.2.2) \quad U(z) = \sup \left\{ u(z) : \frac{1}{\alpha} u \in \operatorname{SH}_{\tau(\alpha)}, u(w) \leq \frac{1}{2}|w|^2 - U^\nu(w) \right\}$$

where $\nu = \sum_{\lambda} \rho_\lambda \delta_\lambda$ and $\tau(\alpha) = 1 - \nu(\mathcal{G})/\alpha$. If we replace the family $\operatorname{SH}_{\tau(\alpha)}$ of subharmonic functions in (6.2.2) by the larger family SH , we obtain a function V with $V \geq U$. However, we have $V = U$ in $\mathbb{D}(0, \sqrt{\alpha})$. Indeed, by Proposition 4.5, the coincidence set $\{U(w) = \frac{1}{2}|w|^2 - U^\nu(w)\}$ contains some non-trivial annulus $\mathbb{A}(\frac{1}{2}\sqrt{\alpha_0}, \sqrt{\alpha})$. Hence, if we set

$$U_0(z) = \sup \left\{ u(z) : u \in \operatorname{SH}(\mathbb{D}(0, \sqrt{\alpha})), u(w) \leq \frac{1}{2}|w|^2 - U^\nu(w) \right\},$$

then by a standard pasting argument U_0 may be glued together with $U|_{\mathbb{C} \setminus \mathbb{D}(0, \sqrt{\alpha})}$ to yield an admissible function for the original problem over $\text{SH}_{\tau(\alpha)}$, so we have $U|_{\mathbb{D}(0, \sqrt{\alpha})} = U_0$. As a consequence, $U = U_0 \geq V$ on $\mathbb{D}(0, \sqrt{\alpha})$, so

$$U(z) = \sup \left\{ u(z) : u \in \text{SH}, u(w) \leq \frac{1}{2}|w|^2 - U^\nu(w) \right\}.$$

It follows from Theorem 2.8 that Ω is a quadrature domain with respect to ν . Hence any non-degenerate equilibrium measure takes the form claimed by Theorem 1.8, and the proof is complete in the smooth case.

Assume now that \mathcal{G} is a bounded simply connected domain with piecewise smooth boundary without cusps. We then approximate \mathcal{G} from within by C^2 -smooth domains \mathcal{G}_t (e.g. as in [72, Theorem 1.12]) with associated extremal measures $\mu_{0,t}$ of the form

$$\mu_{0,t} = \frac{1}{\alpha} \text{Bal}(\nu_t, \mathcal{G}_t^c) + \frac{1}{\pi\alpha} \chi_{\mathbb{D}(0, \sqrt{\alpha}) \setminus \Omega_{\nu_t}} dA,$$

for ν_t finitely supported atomic measures. Using convergence properties of subharmonic functions (see [39, Theorem 3.2.13]) and standard Hilbert space techniques to ensure I_α -minimality for weak limit points of $(\mu_{0,t})$ (see [10, Proposition 3.5]), one concludes that $\mu_{0,t} \rightarrow \mu_0$ weakly, and that the sequence ν_t has a limit point ν supported on \mathcal{I} . From this one can conclude that μ_0 takes the required form

$$d\mu_0 = \frac{1}{\alpha} \text{Bal}(\nu, \mathcal{G}^c) + \frac{1}{\pi\alpha} \chi_{\mathbb{D}(0, \sqrt{\alpha}) \setminus \Omega_\nu} dA.$$

The properties of the support of ν follow from Sakai's theorem. This completes the proof. \square

We mention here that there are indeed holes for which the associated forbidden region is a non-trivial quadrature domain. Namely, for a hole \mathcal{G} obtained by two disjoint disks at an appropriate distance, or smashing together two overlapping disks, one obtains as forbidden region the so-called Neumann ovals. This is discussed below, in § 10.

7. GLOBAL HÖLDER REGULARITY OF THE POTENTIAL

7.1. Control of the density near corner points. We continue the study of the minimizer $\mu_0 = \mu_{\alpha, \mathcal{G}}$ of I_α over $\mathcal{M}_{\mathcal{G}}$, and denote by $u_0 = U^{\mu_0}$ its potential. We assume that $\partial\mathcal{G}$ is piecewise smooth without cusps, and that $\mathcal{G} \subset \mathbb{D}$. We fix $\alpha \geq \alpha_0$ (see Proposition 4.5).

When the boundary $\partial\mathcal{G}$ is a C^2 -smooth Jordan curve, Theorem 1.8 combined with Theorem 5.1 shows that the solution u_0 is Lipschitz continuous, and even $C^{1,1}(D^\pm)$ for each of the two components D^\pm of $\mathbb{C} \setminus \partial\mathcal{G}$. Here, we will explore the regularity properties of u_0 when $\partial\mathcal{G}$ is merely known to be piecewise smooth without cusps.

We recall the notation \mathcal{E} for the set of corner points, which splits into the set \mathcal{E}_0 of corners with interior angle $\pi\sigma \in (0, \pi)$, and the set \mathcal{E}_1 of corners with angle $\pi\sigma \in [\pi, 2\pi)$. We recall the notation \mathcal{E}_0^η , \mathcal{E}_1^η , and \mathcal{E}^η for the fattened singular sets, and we fix a number $\eta = \eta(\mathcal{G}) > 0$ small enough that the disks $(\mathbb{D}(z, 2\eta))_{z \in \mathcal{E}}$ are disjoint.

We also recall the notation μ_0^s for the singular part of μ_0 , which coincides with the restriction $\mu_0|_{\partial\mathcal{G}}$. In Theorem 5.1 we established that the interior coincidence set \mathcal{I} may only approach the boundary through the set \mathcal{E}_0 . Thus, on $\mathbb{D}(z_0, \eta)$ we have

$$d(z, \partial\mathcal{G}) \lesssim |z - z_0|^{\max\{\gamma, 1\}}, \quad z \in \mathcal{I} \cap \mathbb{D}(z_0, \eta),$$

where $\frac{1-\sigma}{\sigma}$ and $\sigma = \sigma(z_0) \in (0, 1)$ is such that the corner at z_0 has opening $\sigma\pi$. By using the structural formula

$$\mu_0^s = \frac{1}{\alpha} \text{Bal}(\nu, \mathcal{G}^c)$$

where ν is a measure supported on \mathcal{I} , we will be able to deduce that u_0 is Hölder continuous outside \mathcal{E}_0^η , and with a more precise argument we will deduce Hölder regularity also on \mathcal{E}^η .

In order to achieve this, we will control the density of μ_0^s with respect to arc length as we move towards a singularity $z_0 \in \mathcal{E}_0$. For this we need the following preliminary result.

Lemma 7.1. *For $\partial\mathcal{G}$ a simple closed piecewise smooth curve without cusps, it holds that*

$$\nu(\mathbb{D}(z, r)) \lesssim \alpha r^2, \quad z \in \mathcal{I}, \quad r \geq 2d(z, \partial\mathcal{G}).$$

Proof. Denote by $z_j \in \mathcal{I}$ a sequence of points converging to $\partial\mathcal{G}$, and denote by r_j the associated radii with $r_j \geq 2d(z_j, \partial\mathcal{G})$. Taking subsequences, we may assume that $z_j \rightarrow z_0$ for some $z_0 \in \partial\mathcal{G}$. In view of Theorem 5.1 we have $z_0 \in \mathcal{E}_0$. Now, for any $z \in \mathbb{D}(z_j, r_j) \cap \mathcal{G}$ we must have $d(z, \partial\mathcal{G}) \leq \frac{3}{2}r_j$. We know that a fixed portion of $\mathbb{T}(z, 2r_j)$ must lie outside \mathcal{G} for j large enough. To see why, notice first that the closest point w of $\partial\mathcal{G}$ to z satisfies $|z - w| \leq \frac{3}{2}r_j$. Since $z_0 \in \mathcal{E}_0$, it is geometrically obvious that w is a regular boundary point (*not* a corner) and so the line passing through z and w intersects $\partial\mathcal{G}$ orthogonally. But after a rotation and rescaling by $\frac{1}{2}r_j^{-1}$, this means that $\mathbb{T}(z, 2r_j) \cap \mathcal{G}^c$ looks asymptotically like an arc

$$\{z \in \mathbb{T} : \text{Im } z \geq \frac{|z-w|}{2r_j}\} \supset \{z \in \mathbb{T} : \text{Im } z \geq \frac{3}{4}\}.$$

The significance of this is that the point of exit from $\mathbb{D}(z, 2r_j)$ of a Brownian motion started at z is uniformly distributed on $\mathbb{T}(z, 2r_j)$, so at least a fixed portion, say $c_0 > 0$, of Brownian motions in \mathcal{G} started at $\mathbb{D}(z_j, r_j)$ will exit $\partial\mathcal{G}$ already within $\mathbb{D}(z_j, 2r_j)$. But taking this together with $\mu_0^s = \frac{1}{\alpha} \text{Bal}(\nu, \mathcal{G}^c)$ tells us that

$$\mu_0^s(\mathbb{D}(z_j, 2r_j)) \geq \frac{c_0}{\alpha} \nu(\mathbb{D}(z_j, r_j)).$$

Finally, in view of Lemma 5.2 we have the inequality

$$\mu_0(\mathbb{D}(z_j, 2r_j)) \lesssim r_j^2,$$

from which the claim of the lemma follows. \square

We are now ready for the first significant result of this section.

Theorem 7.2. *Let $\alpha \geq \alpha_0$. Then, as $z \rightarrow z_0 \in \mathcal{E}_0$ with interior angle $\pi\sigma$, the density $\rho_0 = \alpha \frac{d\mu_0^s}{ds}$ with respect to arc-length on $\partial\mathcal{G}$ satisfies*

$$\rho_0(z) \lesssim \begin{cases} |z - z_0|, & 0 < \sigma < \frac{1}{2} \\ |z - z_0| |\log |z - z_0||, & \sigma = \frac{1}{2} \\ |z - z_0|^{\frac{1-\sigma}{\sigma}}, & \frac{1}{2} < \sigma < 1, \end{cases}$$

for $z \in \mathbb{D}(z_0, \eta)$. In particular, the density is bounded on $\mathbb{D}(z, \eta)$.

Proof. The density at $z \in \partial\mathcal{G}$ is given by (cf. (2.6.2))

$$\rho_0(\cdot) = \int_{\mathcal{I}} \frac{d\omega(\xi, \cdot, \Omega)}{ds} d\nu(\xi) = \int_{\mathcal{I}} P_{\mathcal{G}}(\xi, \cdot) d\nu(\xi)$$

where $P_{\mathcal{G}}(\xi, z)$ denotes the Poisson kernel for \mathcal{G} , $\xi \in \mathcal{G}$ and $z \in \partial\mathcal{G}$, so that $\xi \mapsto P_{\mathcal{G}}(\xi, z)$ is harmonic. Around z , we define annular shells $\mathbb{A}_j := \mathbb{A}(2^{-j}, 2^{-(j-1)}) + z$. By Lemma 7.1 we have $\nu(\mathbb{A}_j) \leq \nu(\mathbb{D}(z_j, 2^{-(j-1)})) \lesssim 4^{-j}$. We estimate

$$\rho_0(z) \lesssim \sum_{j \geq -K_0} 4^{-j} \sup_{\xi \in \mathbb{A}_j} P_{\mathcal{G}}(\xi, z),$$

where K_0 is the smallest number such that $\mathcal{G} \subset \mathbb{D}(z, 2^{K_0})$. This is bounded by an absolute constant K_1 independent of z , by compactness of \mathcal{G} .

We split the annular shells into two categories, which we treat in different ways.

TYPE I. The first kind consists of annuli \mathbb{A}_j for j such that $2^{-j} < 2|z - z_0|$. Then the corner point is far enough from z at the scale determined by $d(z, \partial\mathcal{G})$ to have little impact, and we will estimate harmonic measures from above by harmonic measures in a half-plane-like regular domain containing \mathcal{G} . The construction is illustrated in Figure 7.1.

Specifically, denote by Γ_0 be the subarc of $\partial\mathcal{G}$ containing z , which embeds into the open C^2 -smooth arc $\tilde{\Gamma}_0$. We let L be a C^2 -smooth extension of the arc $\tilde{\Gamma}_0 \cap \mathbb{D}(z_0, \eta)$ which splits the plane into two unbounded components and keeps \mathcal{G} on one side. Denote by Ω the component of $\mathbb{C} \setminus L$ containing \mathcal{G} . We fix $z_1 \in \Omega \setminus \mathbb{D}(z_0, \eta)$ and consider a family f_z of conformal maps of Ω onto the upper half-plane \mathbb{H} conformally with $f_z(z) = 0$ and $f_z(z_1) = i$. In view of the smoothness of L , the fact that the family (z) remains in a smooth subarc of $\partial\Omega$ and Kellog's theorem (Theorem 2.1), (f_z) is locally uniformly $C^{1,1-\epsilon}$ -smooth for any $\epsilon > 0$. We have that

$$P_{\mathcal{G}}(\xi, z) \leq P_{\Omega}(\xi, z) = P_{\mathbb{H}}(f_z(\xi), 0) |f'_z(z)|$$

where we use (2.4.1) in the first step. Moreover, by using the explicit form of the Poisson kernel $P_{\mathbb{H}}$ (see e.g. [22, p. 4]) we find that

$$P_{\mathbb{H}}(f_z(\xi), 0) |f'_z(z)| \lesssim \frac{\text{Im}(f_z(\xi))}{|f_z(\xi)|^2} \leq \frac{1}{|f_z(\xi)|},$$

where we use the fact that for any $\epsilon > 0$, $(f_z)_z$ is a uniformly $C^{1,1-\epsilon}$ -smooth family of conformal mappings of Ω onto the closed half-plane, so that in particular we have that $|f'_z(z)| \asymp |f'_{z_0}(z)|$ is bounded and bounded away from zero (the lower bound is needed below). In summary we obtain the bound

$$P_{\mathcal{G}}(\xi, z) \lesssim \frac{1}{|f_z(\xi)|}.$$

For the range of j considered here, we have for any $\delta_0 > 0$ that

$$|f_z(\xi)| = |f'_z(z)(\xi - z)| + O(|\xi - z|^{2-\delta_0}) \gtrsim |\xi - z|$$

e.g. by Taylor's formula and Kellog's theorem (Theorem 2.1 above). We stress that because for any $\epsilon > 0$ the family (f_z) is uniformly $C^{1,1-\epsilon}$ -smooth, the implicit constants involved are bounded and independent of z . As a consequence, the Poisson kernel meets the uniform bound

$$P_{\mathcal{G}}(\xi, z) \lesssim |z - \xi|^{-1} \lesssim 2^j, \quad \xi \in \mathbb{A}_j.$$

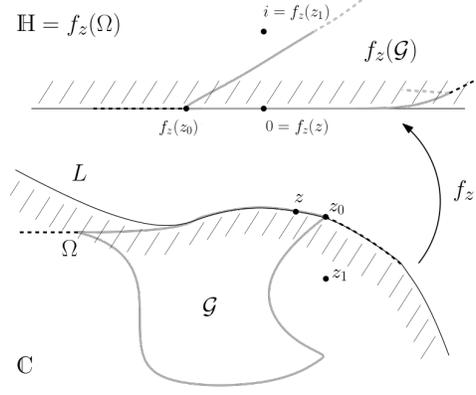


FIGURE 7.1. The construction of the domain Ω enclosing \mathcal{G} , and associated conformal maps $f_z : \Omega \rightarrow \mathbb{H}$ with $f_z(z) = 0$.

The total contribution to the density of these annuli is small:

$$\sum_{j \geq \lceil \log_2 2|z-z_0| \rceil} 4^{-j} \sup_{\xi \in \mathbb{A}_j} P_{\mathcal{G}}(\xi, z) \lesssim \sum_{j \geq \lceil \log_2 2|z-z_0| \rceil} 2^{-j} \lesssim |z-z_0|.$$

TYPE II. For the annuli \mathbb{A}_j for which $2^{-j} \geq 2|z-z_0|$, we need to use the fact that harmonic measure decays near corners. We recall (see e.g. [22]) that $P_{\mathcal{G}}(\xi, z)$ may be expressed in terms of a conformal mapping $\phi : \mathcal{G} \rightarrow \mathbb{D}$ via

$$P_{\mathcal{G}}(\xi, z) = |\phi'(z)| \frac{1 - |\phi(\xi)|^2}{|\phi(z) - \phi(\xi)|^2}.$$

and that there exists a constant $a > 0$ such that

$$|\phi(w) - \phi(z_0)| = a|w - z_0|^{\frac{1}{\sigma}}(1 + o(1)).$$

In particular, if $|\xi - z_0| \geq 2|z - z_0|$, then by the reverse triangle inequality we also have

$$|\phi(z) - \phi(\xi)| \geq (|\phi(\xi) - \phi(z_0)| - |\phi(z) - \phi(z_0)|) \gtrsim |\xi - z_0|^{\frac{1}{\sigma}}.$$

Moreover, we have

$$1 - |\phi(\xi)|^2 \lesssim |\xi - z_0|^{\frac{1}{\sigma}}.$$

Since for $j \leq \lceil \log_2 2|z-z_0| \rceil$ we have $|\xi - z_0| \geq 2|z-z_0|$ whenever $\xi \in \mathbb{A}_j(z)$, it holds that

$$P_{\mathcal{G}}(\xi, z) \lesssim \frac{|z-z_0|^{\frac{1}{\sigma}}}{|\xi-z_0|^{\frac{1}{\sigma}} |z-z_0|} = \frac{|z-z_0|^{\frac{1}{\sigma}}}{|\xi-z_0|^{\frac{1}{\sigma}} |z-z_0|},$$

so the contribution to the density of this part of the sum may be bounded as follows:

$$I(z, z_0) := \sum_{j=-K_0}^{\lceil \log_2 2|z-z_0| \rceil} 4^{-j} \sup_{\xi \in \mathbb{A}_j} \frac{|z-z_0|^{\frac{1}{\sigma}}}{|\xi-z_0|^{\frac{1}{\sigma}} |z-z_0|} \lesssim |z-z_0|^{\frac{1-\sigma}{\sigma}} \sum_{0 \leq j \leq \lceil \log_2 2|z-z_0| \rceil} 2^{(\frac{1-\sigma}{\sigma}-1)j}.$$

Computing the sum, using the bound (with ρ fixed)

$$\sum_{0 \leq j \leq R} \rho^j \lesssim \begin{cases} \rho^R, & \rho > 1, \\ R, & \rho = 1, \\ 1, & \rho < 1, \end{cases}$$

we find that

$$I(z, z_0) \lesssim |z - z_0|^{\frac{1-\sigma}{\sigma}} \times \begin{cases} 1, & \sigma > \frac{1}{2} \\ |\log |z - z_0||, & \sigma = \frac{1}{2} \\ |z - z_0|^{1-\frac{1-\sigma}{\sigma}}, & \frac{1}{2} < \sigma < 1. \end{cases}.$$

This completes the proof. \square

7.2. Global Hölder regularity of the solution. The main result of this section is now straightforward.

Theorem 7.3. *Assume that \mathcal{G} is piecewise smooth without cusps. Then the potential $U^{\mu_\alpha, \mathcal{G}}$ is Lipschitz continuous on $\mathbb{C} \setminus \mathcal{E}^\eta$ and Hölder continuous on \mathcal{E}^η . In particular u_0 is globally Hölder continuous.*

Proof. For a given boundary point $z_0 \in \Gamma \setminus \mathcal{E}_0^\eta$, the family $(\phi_z)_{z \in \mathcal{I}}$ of Riemann mappings $\phi_z : \mathcal{G} \rightarrow \mathbb{D}$ with $\phi_z(z) = 0$, are all of the same regularity near z_0 . On the smooth part $\Gamma \setminus \mathcal{E}^\eta$, they are all uniformly Lipschitz regular in a neighbourhood of z_0 , so the potential of harmonic measure seen from any point of \mathcal{I} is Lipschitz regular near z_0 . But then the potential of the whole balayage measure $U^{\mu_0^\#}$ is also Lipschitz. Similarly, for $z \in \mathcal{E}_1^\eta$ near a corner with opening angle $\pi\sigma \in [\pi, 2\pi)$, the mappings ϕ_z are uniformly Hölder continuous by Remark 2.4.

We turn to analyzing the regularity in \mathcal{E}_0^η . It turns out that we merely need to use that the density ρ_0 of μ_0 with respect to arc-length is bounded. Indeed, in view of Proposition 3.1 we have for $v_0 = \alpha u_0$

$$|v_0(z) - v_0(w)| \lesssim \int_{\partial\mathcal{G}} \left| \log \left| \frac{z-\xi}{w-\xi} \right| \right| \rho_0(\xi) ds(\xi) + |z - w|,$$

since the continuous part of the measure μ_0 has a C^1 -smooth (in particular Lipschitz continuous) potential. By symmetry, it is enough to show that

$$\int_{\partial\mathcal{G} \cap \{|z-\xi| > |w-\xi|\}} \log \left| \frac{z-\xi}{w-\xi} \right| \rho_0(\xi) ds(\xi) \lesssim |z - w| \log |z - w|.$$

Since by Theorem 7.2 the density ρ_0 is bounded on $\mathbb{D}(z_0, \eta)$, we have the uniform bound

$$|\partial\mathcal{G} \cap \mathbb{D}(z, \epsilon)| \lesssim \epsilon, \quad z \in \mathbb{D}(z_0, \eta),$$

and in addition it holds that

$$\begin{aligned} \int_{\partial\mathcal{G} \cap \{|z-\xi| > |w-\xi|\}} \log \left| \frac{z-\xi}{w-\xi} \right| \rho_0(\xi) ds(\xi) \\ \lesssim \|\rho_0\|_{L^\infty(\mathbb{D}(z_0, \eta))} \int_0^\infty \left| \partial\mathcal{G} \cap \left\{ \xi : \left| \frac{z-\xi}{w-\xi} \right| > e^\lambda \right\} \right| d\lambda. \end{aligned}$$

We split the integral into two: one over the set $0 \leq \lambda \leq |z - w|$ and one over $|z - w| < \lambda < \infty$. For the former, we simply note that the integrand is bounded, so

$$\int_0^{|z-w|} \left| \partial\mathcal{G} \cap \left\{ \xi : \left| \frac{z-\xi}{w-\xi} \right| > e^\lambda \right\} \right| d\lambda \lesssim |z - w|.$$

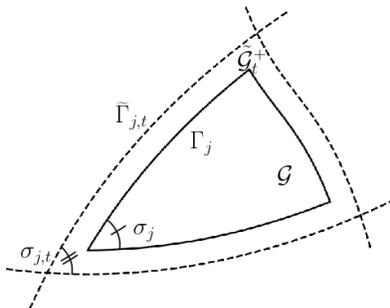


FIGURE 8.1. The construction for approximating \mathcal{G} from the outside by \mathcal{G}_t^+ .

For the remaining integral, notice that the set $\{\xi : \frac{|z-\xi|}{|w-\xi|} > e^\lambda\}$ is contained in the disk $\mathbb{D}(w, (e^\lambda - 1)^{-1}|z - w|)$, so we find that

$$|v_0(z) - v_0(w)| \lesssim |z - w| + \int_{|z-w|}^{\infty} |z - w| \frac{d\lambda}{e^\lambda - 1} \lesssim |z - w| |\log |z - w||,$$

which shows that for any $\epsilon > 0$, the function u_0 is Hölder continuous with exponent $1 - \epsilon$ on \mathcal{E}_0^η . This completes the proof. \square

8. QUANTITATIVE STABILITY UNDER DOMAIN PERTURBATIONS

8.1. A word on Hausdorff distance and approximation of domains. For a domain \mathcal{G} whose boundary is piecewise smooth and without cusps, we want to consider domains which approach \mathcal{G} from within and from the outside, respectively. We define the Hausdorff distance $d_H(\mathcal{G}, \mathcal{G}')$ between domains \mathcal{G} and \mathcal{G}' as

$$d_H(\mathcal{G}, \mathcal{G}') = \max \left\{ \sup_{z \in \mathcal{G}} \inf_{w \in \mathcal{G}'} |z - w|, \sup_{z \in \mathcal{G}'} \inf_{w \in \mathcal{G}} |z - w| \right\}.$$

The approximation of \mathcal{G} from within may be accomplished by considering level curves Γ_t^- of the moduli of the conformal mapping $\varphi : \mathcal{G} \rightarrow \mathbb{D}$:

$$\Gamma_t^- = \{z \in \mathbb{C} : |\varphi(z)| = e^{-t}\}, \quad t > 0.$$

Denote by \mathcal{G}_t^- the bounded component of $\mathbb{C} \setminus \Gamma_t^-$. The boundary of the domain \mathcal{G}_t^- is a smooth Jordan curve, and the domains approach \mathcal{G} well as $t \rightarrow 0$, in the sense that

$$d_H(\mathcal{G}_t^-, \mathcal{G}) \leq t^\beta$$

for some $\beta > 0$ which depends only on the angles of points $z \in \mathcal{E}$.

The approximation of \mathcal{G} from outside pertains to obtaining stability bounds for outwards perturbations of \mathcal{G} . These are not needed for our applications, but we include this result for general interest. The approximation may be done with a regularized outer parallel curve construction, which goes as follows (see Figure 8.1). For each C^2 -smooth subarc Γ_j of $\partial\mathcal{G}$, let $\tilde{\Gamma}_j$ denote an open C^2 -smooth arc containing Γ_j , and let $\tilde{\Gamma}_{j,t}$ denote the curve given by the parameterization ($\epsilon \leq s \leq 1 - \epsilon$)

$$(8.1.1) \quad (x(s, t), y(s, t)) = \left(x(s) + \frac{ty'_\epsilon(s)}{\sqrt{x'_\epsilon(s)^2 + y'_\epsilon(s)^2}}, y(s) - \frac{t}{\sqrt{x'_\epsilon(s)^2 + y'_\epsilon(s)^2}} \right),$$

where $(x(s), y(s))$ for $0 \leq s \leq 1$ parametrizes the arc $\tilde{\Gamma}_j$, and where for a function F , $F'_\epsilon(z)$ denotes its averaged derivative over the ϵ -neighbourhood of z . If t is small enough (depending only on \mathcal{G}) for

$$\partial_s x(s, t) = x'(s) + \frac{ty''_\epsilon(s)}{((x'_\epsilon)^2 + (y'_\epsilon(s)^2)^{\frac{3}{2}}} > 0, \quad \epsilon \leq s \leq 1 - \epsilon$$

to hold, then $\tilde{\Gamma}_{j,t}$ is a C^1 -smooth curve. In fact, and this is the reason for regularization, $\tilde{\Gamma}_{j,t}$ is C^2 -smooth under this condition. If ϵ is chosen small enough, then $(x'_\epsilon, -y'_\epsilon)$ points approximately in the outwards normal direction to Γ , in which case the curve $\tilde{\Gamma}_{j,t}$ lies at a distance of at least, $\frac{1}{2}t$ from \mathcal{G} .

The arcs $\tilde{\Gamma}_{j,t}$ and $\tilde{\Gamma}_{j+1,t}$ meet with angles $\sigma_t(j)$ at corners, which are in a one-to-one correspondence with those of $\partial\mathcal{G}$. That is, $\sigma_t(j) = \sigma(\Gamma_j, \Gamma_{j+1}) + o(1)$ as $t \rightarrow 0^+$, where $\sigma(\Gamma_j, \Gamma_{j+1})$ denotes the angle formed by the subarcs Γ_j and Γ_{j+1} of $\partial\mathcal{G}$. The resulting piecewise smooth Jordan curve Γ_t encloses a piecewise smooth Jordan domain \mathcal{G}_t^+ without cusps (for small enough t). In addition, if $w_t = w_{t,j}$ denotes the parameterization (8.1.1), then it is clear that $w_t'' \rightarrow w_0''$ e.g. in $L^1(0, 1)$ (cf. Remark 2.5).

8.2. Stability of the Zeitouni-Zelditch functional. Assume that \mathcal{G}_0 is a bounded simply connected domain whose boundary $\partial\mathcal{G}$ is piecewise smooth without cusps. Denote by $(\mathcal{G}_t)_{t>0}$ a family of simply connected Lipschitz domains, such that $\mathcal{G}_t \rightarrow \mathcal{G}$ in the sense of convergence in Hausdorff distance as $t \rightarrow 0$, and normalized such that

$$d_H(\mathcal{G}_t, \mathcal{G}_0) \leq t, \quad t > 0.$$

In this section, we quantify how well the equilibrium measures $\mu_t = \mu_{\alpha, \mathcal{G}_t}$ approximates μ_0 in the sense of the functional I_α .

Proposition 8.1. *Under the above conditions, it holds that*

$$I_\alpha(\mu_t) = I_\alpha(\mu_0) + O(t^\beta),$$

for some constant $\beta > 0$ depending only on \mathcal{G}_0 , as $t \rightarrow 0$.

The number β can be explicitly related to angles at the corner points $z \in \mathcal{E}$, but we refrain from doing so for reasons of length.

Proof. The proof is rather long, and is split into steps for the reader's convenience. We treat separately the cases when $\mathcal{G}_t \subset \mathcal{G}_0$ and when $\mathcal{G}_0 \subset \mathcal{G}_t$, and finally argue that this implies the general result.

STEP 1. We first look at the case when $\mathcal{G}_0 \subset \mathcal{G}_t$ for $t > 0$, so that in particular μ_t is a competitor for the minimization problem defining μ_0 . We may, by monotonicity of $I_\alpha(\mu_{\alpha, \mathcal{G}})$ with respect to domain inclusion, assume that \mathcal{G}_t is enclosed by curves Γ_t^+ constructed as in § 8.1. By sweeping the measure $\chi_{\mathcal{G}_t} \mu_0$ to \mathcal{G}_t^c , we moreover obtain a good upper bound for $I_\alpha(\mu_t)$ while $I_\alpha(\mu_0)$ supplies a lower bound

$$I_\alpha(\mu_0) \leq I_\alpha(\mu_t) \leq I_\alpha(\mu_t^*)$$

where $\mu_t^* = \text{Bal}(\mu_0, \mathcal{G}_t^c)$, and it remains to explain how the right-hand side may be estimated. We recall that $\text{Bal}(\mu_0, \mathcal{G}_t^c)$ is interpreted as

$$\text{Bal}(\mu_0, \mathcal{G}_t^c) = \text{Bal}(\chi_{\mathcal{G}_t} \mu_0, \mathcal{G}_t^c) + \chi_{\mathcal{G}_t^c} \mu_0.$$

The balayage operation decreases the energy $-\Sigma(\mu)$ of a measure μ , so it suffices to estimate $B_\alpha(\mu_t^*)$. We claim that there exists a positive number $\beta > 0$ and a constant $C < \infty$ depending only on \mathcal{G} such that

$$0 \leq \sup_{z \in \mathcal{G}_t} U^{\mu_t^* - \mu_0}(z) \leq Ct^\beta,$$

as $t \rightarrow 0$. Notice first that the potential of the balayage measure $\eta_{t,\zeta} = \text{Bal}(\delta_\zeta, \mathcal{G}_t^c)$ is given by (cf. (2.6.1))

$$U^{\eta_{t,\zeta}}(z) = \log |z - \zeta| - g_{\mathcal{G}_t}(z, \zeta),$$

where $g_{\mathcal{G}_t}$ is the Green function for \mathcal{G}_t , and hence we may write

$$U^{\mu_t^* - \mu_0}(z) = \int_{\mathcal{G}_t \setminus \mathcal{G}_0} (U^{\eta_{t,\zeta}}(z) - \log |z - \zeta|) d\mu_0(\zeta) = - \int_{\mathcal{G}_t \setminus \mathcal{G}_0} g_{\mathcal{G}_t}(z, \zeta) d\mu_0(\zeta).$$

That this identity holds may moreover be seen by noting that $U = U^{\mu_t^* - \mu_0}$ is the unique solution to the boundary value problem

$$\begin{cases} \Delta U = -\mu_0 & \text{in } \mathcal{G}_t \\ U = 0 & \text{on } \partial \mathcal{G}_t. \end{cases}$$

We let $E(z, \lambda, t)$ denote the sub-level set

$$E(z, \lambda, t) = \{\zeta \in \mathcal{G}_t \setminus \mathcal{G}_0 : g_{\mathcal{G}_t}(z, \zeta) \leq -\lambda\}$$

and find that

$$\sup_{z \in \mathcal{G}_t} U^{\mu_t^* - \mu_0}(z) = \sup_{z \in \mathcal{G}_t \setminus \mathcal{G}_0} \left(\int_0^\infty \mu_0(E(z, \lambda, t)) d\lambda \right),$$

where we have used the maximum principle to restrict the supremum to $\mathcal{G}_t \setminus \mathcal{G}_0$ and the layer cake formula to obtain the integral on the right-hand side.

We proceed to estimate the size of $E(z, \lambda, t)$, and to this end we use the fact that \mathcal{G}_t is simply connected. Hence, there exists a conformal mapping $\varphi_t : \mathcal{G}_t \rightarrow \mathbb{D}$, which takes a fixed interior point z_0 of \mathcal{G} to the origin. The Green function is then given by

$$-g_{\mathcal{G}_t}(z, \zeta) = \log \left| \frac{1 - \overline{\varphi_t(z)}\varphi_t(\zeta)}{\varphi_t(\zeta) - \varphi_t(z)} \right|, \quad (z, \zeta) \in \mathcal{G}_t \times \mathcal{G}_t.$$

By construction, the domains \mathcal{G}_t approximate \mathcal{G} well in the sense of Remark 2.5, so that by Remark 2.4, there exist positive constants c_1, c_2, β_0 and β_1 with $\beta_0 > \beta_1$ depending on \mathcal{G} but not on t , such that

$$c_1 |z - \zeta|^{\beta_0} \leq |\varphi_t(\zeta) - \varphi_t(z)| \leq c_2 |z - \zeta|^{\beta_1}, \quad (z, \zeta) \in \mathcal{G}_t \times \mathcal{G}_t$$

as $t \rightarrow 0^+$. A point $\zeta \in \mathcal{G}_0$ belongs to $E(z, \lambda, t)$ if and only if

$$(8.2.1) \quad \left| \frac{1 - \overline{\varphi_t(z)}\varphi_t(\zeta)}{\varphi_t(\zeta) - \varphi_t(z)} \right| > e^\lambda.$$

But for $z \in \mathcal{G}_t \setminus \mathcal{G}_0$ we have the bound $1 - |\varphi_t(z)|^2 = O(t^{\beta_1})$, so by the triangle inequality it holds that

$$\left| \frac{1 - \overline{\varphi_t(z)}\varphi_t(\zeta)}{\varphi_t(\zeta) - \varphi_t(z)} \right| = \left| \frac{1 - |\varphi_t(z)|^2 + \overline{\varphi_t(z)}(\varphi_t(z) - \varphi_t(\zeta))}{\varphi_t(\zeta) - \varphi_t(z)} \right| \leq 1 + \frac{Ct^{\beta_1}}{|z - \zeta|^{\beta_0}}.$$

In view of this estimate, the inequality (8.2.1) implies that

$$|z - \zeta| \lesssim t^{\beta_1/\beta_0} (e^\lambda - 1)^{-\frac{1}{\beta_0}},$$

for some $\beta_2 > 0$. It follows that $E(z, \lambda, t) \subset \mathbb{D}(z, D_0 t^{\beta_2} (e^\lambda - 1)^{-\frac{1}{\beta_0}})$ for some constant D_0 . We split the integral in λ over two regions: one where $\lambda < t^\tau$ and one where $\lambda > t^\tau$, for a constant $\tau > 0$ to be determined. The first is we estimate as

$$\int_0^{t^\tau} \mu_0 \left((\mathcal{G}_t \setminus \mathcal{G}_0) \cap \mathbb{D}(z, D_0 t^{\beta_2} (e^{-\lambda} - 1)^{-\frac{1}{\beta_0}}) \right) d\lambda \leq \int_0^{t^\tau} \mu_0(\mathcal{G}_t) d\lambda \lesssim t^\tau,$$

where we use the fact that μ_0 is a probability measure. Using Theorem 7.2 to control the singular part of the measure, and Proposition 3.1 to control the continuous part, we find that it holds that

$$\mu(\mathbb{D}(z, \rho)) \lesssim \rho^{\beta_4}, \quad \text{for all } z \in \partial\mathcal{G}, \rho > 0$$

for some $\beta_4 > 0$. Hence, for the second integral (over the region $\lambda > t^\tau$), we estimate

$$\int_{t^\tau}^\infty \mu_0 \left(\mathbb{D}(z, D_0 t^{\beta_2} (e^\lambda - 1)^{-\frac{1}{\beta_0}}) \right) d\lambda \lesssim \int_{t^\tau}^\infty t^{\beta_2 \beta_3} (e^\lambda - 1)^{-\frac{\beta_3}{\beta_0}} d\lambda$$

for some β_3 . A simple change of variables shows that we have

$$\int_{t^\tau}^\infty (e^\lambda - 1)^{-\frac{\beta_3}{\beta_0}} d\lambda \lesssim t^{-\frac{\tau \beta_3}{\beta_0}},$$

so we may estimate the integral by

$$\int_{t^\tau}^\infty \mu \left(\mathbb{D}(z, D_0 t^{\beta_2} (e^\lambda - 1)^{-\frac{1}{\beta_0}}) \right) d\lambda \lesssim t^{\beta_3 \beta_2 - \frac{\tau \beta_3}{\beta_0}} = t^{\beta_4}$$

where $\beta_4 > 0$ provided that τ is chosen small enough. We find that

$$\sup_{z \in \mathcal{G}_t \setminus \mathcal{G}_0} |U^{\mu_t^* - \mu_0}(z)| \leq t^\beta$$

for some $\beta > 0$. It follows that

$$B_\alpha(\mu_t) = B_\alpha(\mu_0) + O(t^\beta),$$

whenever \mathcal{G}_t approximates \mathcal{G}_0 from the outside, which completes the proof.

STEP 2. We turn to the case when $\mathcal{G}_t \subset \mathcal{G}_0$ for all $t > 0$. By monotonicity with respect to domain inclusions, we may without loss of generality fix φ to be a conformal mapping $\varphi : \mathcal{G}_0 \rightarrow \mathbb{D}$, and make the specific choice $\mathcal{G}_t = \varphi^{-1}(e^{-t^\tau} \mathbb{D})$ for some appropriate choice of a constant $\tau > 0$.

We set $\mu_t^* = \text{Bal}(\mu_t, \mathcal{G}_0^c)$, and notice that $I_\alpha(\mu_{\alpha, \mathcal{G}_0})$ is sandwiched in between $I_\alpha(\mu_t)$ and $I_\alpha(\mu_t^*)$ by monotonicity in the domain \mathcal{G}_0 and optimality of $\mu_{\alpha, \mathcal{G}}$ for the minimization of I_α . Hence, we need to bound the difference

$$I_\alpha(\mu_t^*) - I_\alpha(\mu_t),$$

by a positive power of t . As before, the balayage operation only decreases the energy $-\Sigma(\mu)$, so it suffices to estimate $U^{\mu_t^* - \mu_t}(z)$ for $z \in \mathcal{G}_0$. We may assume that $\mu_t^c(\mathcal{G}_0) = 0$, since otherwise we split the measure μ_t into its singular and continuous parts, and treat the latter separately as follows: By the maximum principle, it is

enough to consider the Balayage potential for $z \in \mathcal{G}_0 \setminus \mathcal{G}_t$, and for such z we clearly have

$$0 \leq U^{\text{Bal}(\mu_t^c, \mathcal{G}_0^c) - \mu_t^c}(z) \leq \frac{1}{\pi\alpha} \int_{\mathcal{G}_0 \setminus \mathcal{G}_t} |g_{\mathcal{G}_0}(z, w)| dA(w) \lesssim t^\beta$$

by mirroring the computations from the first part of the proof.

Hence, U^{μ_t} may be assumed to be harmonic in $\mathcal{G}_0 \setminus \partial\mathcal{G}_t$, so by the maximum principle, it is sufficient to estimate $U^{\mu_t^* - \mu_t}$ for $z \in \mathcal{G}_0 \setminus \mathcal{G}_t$. In fact, by an additional application of the maximum principle we find that it is enough to estimate this for $z \in \partial\mathcal{G}_t$. We use the fact that $\mu_t^s = \text{Bal}(\nu_t, \mathcal{G}_t^c)$ for some atomic measure ν_t , finitely supported on \mathcal{G}_t . Moreover, $\mu_t^* = \text{Bal}(\mu_t, \mathcal{G}_0^c)$. Therefore, the potentials of μ_t^s and ν_t agree outside \mathcal{G}_0 , so in particular their balayage measures to \mathcal{G}_0^c are equal:

$$\mu_t^* = \mu_t^c + \text{Bal}(\mu_t^s, \mathcal{G}_0^c) = \mu_t^c + \text{Bal}(\nu_t, \mathcal{G}_0^c).$$

As a consequence, we find that

$$U^{\mu_t^* - \mu_t}(z) = U^{\text{Bal}(\nu_t, \mathcal{G}_0^c) - \text{Bal}(\nu_t, \mathcal{G}_t^c)}(z) = \int (g_{\mathcal{G}_0}(z, w) - g_{\mathcal{G}_t}(z, w)) d\nu_t(w).$$

Due to the special choice of approximating domains \mathcal{G}_t , the conformal mapping φ_t of \mathcal{G}_t onto \mathbb{D} is given by $\varphi_t(z) = e^{t^\tau} \varphi(z)$ for an appropriate positive parameter τ , so we find that for $z \in \partial\mathcal{G}_t$ and $w \in \mathcal{G}_t$ we have

$$\begin{aligned} |g_{\mathcal{G}_0}(z, w) - g_{\mathcal{G}_t}(z, w)| &= \log \left| \frac{1 - e^{-2t^\tau} \overline{\varphi(w)} \varphi(z)}{1 - \overline{\varphi(w)} \varphi(z)} \right| - t^\tau \\ &\lesssim \log \left(1 + \frac{t^\tau}{|1 - \overline{\varphi(w)} \varphi(z)|} \right) + O(t^\tau). \end{aligned}$$

If m_ξ denotes the Möbius transformation, the conformal mapping $\varphi_w(z) = m_{\varphi(w)}(\varphi(z))$ maps \mathcal{G}_0 into \mathbb{D} , so the ratio $|\varphi(z) - \varphi(w)|/|1 - \overline{\varphi(w)} \varphi(z)|$ is bounded by one. From this we may conclude that

$$|1 - \overline{\varphi(w)} \varphi(z)| \geq |\varphi(z) - \varphi(w)| \gtrsim |z - w|^{\beta_1},$$

where the last inequality follows from the assumed regularity of the boundary $\partial\mathcal{G}_0$ and Remark 2.4. By possibly changing the value of β , we find that

$$1 + \frac{t^\tau}{|1 - \overline{\varphi(z)} \overline{\varphi(w)}|} \lesssim 1 + \frac{t^\beta}{|z - w|^\beta}, \quad z \in \partial\mathcal{G}_t.$$

Next, we recall Lemma 7.1, which says that there exists some constant C_1 (universal) so that

$$\nu_t(\mathbb{D}(z, r)) \leq C_1 r^2, \quad r \geq 2d(z, \partial\mathcal{G}_t), \quad z \in \mathcal{I}_t.$$

We next split the integral against ν_t as follows: for each $j \geq 0$, we let $r_j = 2^{-j} \text{diam}(\mathcal{G}_t)$ and set $A_j(z) = z + \mathbb{A}(r_j, r_{j+1})$. Then, whenever $A_j(z) \cap \mathcal{I}_t$ contains some point z_j , we have

$$\nu_t(A_j(z)) \leq \nu_t(\mathbb{D}(z_j, r_j)) \leq C_1 r_j^2,$$

and if no such z_j may be found, it holds that $\nu_t(A_j(z)) = 0$. We may then estimate the sought-after quantity:

$$\begin{aligned} U^{\mu_t^* - \mu_t}(z) &\leq \sum_{j \geq 0} \nu_t(A_j(z)) \sup_{w \in A_j(z)} (g_{\mathcal{G}_0}(z, w) - g_{\mathcal{G}_t}(z, w)) \\ &\leq C_2 \sum_{j \geq 0} \log \left(1 + \frac{t^\beta}{2^{-\beta j}} \right) 2^{-2j} + O(t^\tau), \end{aligned}$$

where C_2 is some constant; this is easily seen to be of order $O(t^\beta)$ as $t \rightarrow 0$ for β chosen small enough. This completes the proof also in this case.

STEP 3. Finally, we let \mathcal{G}_t be any family of smooth domains at Hausdorff distance at most t from \mathcal{G}_0 . We can then find domains \mathcal{G}_t^\pm by the same process as above, with

$$\mathcal{G}_t^- \subset \mathcal{G}_0, \quad \mathcal{G}_t \subset \mathcal{G}_t^+$$

with corresponding extremal measures μ_t^\pm for I_α , such that $I_\alpha(\mu_0)$ and $I_\alpha(\mu_t)$ are both bounded between $I_\alpha(\mu_t^-)$ and $I_\alpha(\mu_t^+)$:

$$I_\alpha(\mu_t^-) \leq I_\alpha(\mu_0), \quad I_\alpha(\mu_t) \leq I_\alpha(\mu_t^+),$$

with $d_H(\mathcal{G}_t^\pm, \mathcal{G}_0) \leq t$, and we may apply the above approach to the measures μ_t^\pm to conclude

$$I_\alpha(\mu_t) = I_\alpha(\mu_0) + O(t^\beta), \quad t \rightarrow 0,$$

which completes the proof. \square

9. THE LIMITING CONDITIONAL ZERO DISTRIBUTION

9.1. **The general main theorem for random zeros.** For the formulation of the main theorem, we need several notions. We denote by

$$\mu_L^{\mathbb{C}} = \sum_{w : F_L(w)=0} \delta_w$$

the empirical measure of the zeros of F_L , and by $\mu_{L, \mathcal{G}}^{\mathbb{C}}$ the same measure conditioned on the hole event $\mathcal{H}_L(\mathcal{G}) = \{F_L \neq 0 \text{ in } \mathcal{G}\}$.

We define the *Schwarz potential* $u_{\mathcal{G}}$ with respect to the quadrature domain Ω_ν as the solution to the boundary value problem

$$\begin{cases} \Delta u_{\mathcal{G}} = 1 - \nu & \text{on } \Omega_\nu; \\ u_{\mathcal{G}} = 0 & \text{on } \partial\Omega_\nu. \end{cases}$$

Theorem 9.1. *Let \mathcal{G} be a Jordan domain with piecewise C^2 -smooth boundary without cusps and let $\mu_{\mathcal{G}}^{\mathbb{C}}$ be given by*

$$d\mu_{\mathcal{G}}^{\mathbb{C}} = d\text{Bal}(\nu, \mathcal{G}^c) + \frac{1}{\pi} \chi_{\mathbb{C} \setminus \Omega_\nu} dA,$$

where ν and Ω_ν are as in Theorem 1.8. Then the empirical measures $L^{-2} \mu_{L, \mathcal{G}}^{\mathbb{C}}$ converge to $\mu_{\mathcal{G}}^{\mathbb{C}}$ vaguely in distribution as $L \rightarrow \infty$. In addition, the hole probability satisfies

$$\frac{1}{L^4} \log \mathcal{P}(\mathcal{H}_L(\mathcal{G})) = -\frac{1}{\pi \alpha^2} \int_{\Omega_\nu} u_{\mathcal{G}} dA + O(L^{-2} \log L^2)$$

as $L \rightarrow \infty$.

Let φ be a continuous test function with compact support. We write

$$n_L(\varphi) = \sum_{w:F_L(w)=0} \varphi(w),$$

for the *linear statistic* of the zeros of F_L with respect to φ . We also denote by $\mathcal{P}_L^{\mathcal{G}}$ the probability measure conditioned on the event $\mathcal{H}_L(\mathcal{G})$, and for a compactly supported smooth test function we write

$$\mathcal{D}(\varphi) = \int_{\mathbb{C}} |\nabla \varphi|^2 dA.$$

Vague convergence in distribution of the conditional empirical measures $\mu_{L,\mathcal{G}}^{\mathbb{C}}$ is equivalent to the convergence in distribution of the random variables $L^{-2}n_L(\varphi)$ to the limit $\int \varphi d\mu_{\mathcal{G}}^{\mathbb{C}}$. Using the same general approach as in [24], we prove Theorem 9.1 by bounding from above the probability that, conditional on the hole event $\mathcal{H}_L(\mathcal{G})$, the linear statistic $n_L(\varphi)$ is far from $L^2 \int \varphi d\mu_{\mathcal{G}}^{\mathbb{C}}$.

Let $\epsilon > 0$. For a compactly supported smooth test function φ , we prove in this section the (conditional) large deviation upper bound

$$\mathcal{P}_L^{\mathcal{G}}\left(\left\{|n_L(\varphi) - L^2 \int \varphi d\mu_{\mathcal{G}}^{\mathbb{C}}| > \epsilon L^2\right\}\right) \leq \exp\left(-\frac{c\epsilon^2 L^4}{\mathcal{D}(\varphi)} + O(L^2 \log^2 L)\right)$$

as $L \rightarrow \infty$.

9.1.1. Negligible events. In the proof of Theorem 9.1 we may assume without loss of generality that \mathcal{G} is contained in the unit disk. We will say that an event is *negligible* (with respect to the hole probability, depending L) if its probability is at most $\exp(-2L^4)$. The precise constant 2 is not important, but we do use the fact that the hole probability for \mathcal{G} is bounded from below by the hole probability for \mathbb{D} , which decays asymptotically like $\exp(-\frac{\epsilon^2}{4}L^4)$.

Remark 9.2. Notice that the union of polynomially many (in L) negligible events consists of a negligible event, for L large.

9.2. A guide to the proof of Theorem 9.1. Roughly, the proof may be split into four steps:

- STEP 1. A truncation argument, which replaces the GEF F_L by a polynomial.
- STEP 2. Obtaining a lower bound for the hole probability.
- STEP 3. Deriving an effective large deviation upper bound for linear statistics of the zeros. This also provides an upper bound for the hole probability.
- STEP 4. Deducing the convergence of conditional empirical measures

STEPS 1, 3 and 4 are very similar to the corresponding arguments in [24]. The use of truncation in STEP 1 leads to small (random) perturbation in the location of the zeros of the polynomial compared with those of F_L . The technical difficulties induced by this perturbation are rather mild in [24], since there the domain \mathcal{G} is a disk (which, by its convexity, is stable under small perturbations). Here we have to rely instead on our quantitative stability results from § 8 (these are also used in STEP 2).

In [24] the lower bound for the probability of the hole event is obtained explicitly, by constructing an appropriate event in terms of the random variables $\{\xi_n\}$. This construction depends crucially on the fact that the domain \mathcal{G} is a disk (by using the circular symmetry of Taylor series). Our STEP 2 requires a completely different

approach, which is based on discretization of the continuous minimizer $\mu_{\alpha, \mathcal{G}}$ of the functional I_α .

9.3. Truncation of the power series. Since it is difficult to handle directly the zeros of the GEF F_L , we first approximate (some of) them by zeros of the *Weyl polynomial*

$$P_{N,L}(z) = \sum_{n=0}^N \xi_n \frac{(Lz)^n}{\sqrt{n!}}, \quad z \in \mathbb{C}.$$

For this to work we need to control the size of

$$T_N(z) = T_{N,L}(z) = \sum_{n=N+1}^{\infty} \xi_n \frac{(Lz)^n}{\sqrt{n!}}, \quad z \in \mathbb{C},$$

that is, the tail of the Gaussian Taylor series F_L . Note that $P_{N,L}$ and T_N are independent Gaussian analytic functions. We use the following crude bound for the tail (e.g. [24, Lemma 3.3]).

Proposition 9.3. *Let $A, B \geq 1$ be fixed parameters. Let $L > 0$ be sufficiently large and put $N = \lceil L^2 \log L \rceil$. With probability at least $1 - \exp(-CL^6)$ we have*

$$\sup_{z \in \mathbb{D}(0, B)} |T_N(z)| \leq \exp(-AL^2 \log L),$$

where $C > 0$ is an absolute constant.

Remark 9.4. A small simplification compared with [24] is that we choose the parameter N to be non-random.

In order to control the perturbation of the zeros of F_L we will apply Rouché's theorem. Thus, we need a lower bound for $|F_L|$ away from its zeros. Let H be an entire function, and $B > 0, \eta \in (0, \frac{1}{4}]$ real parameters. Denote by w_1, \dots, w_m the zeros of G in $\mathbb{D}(0, B)$, including multiplicities. We define

$$m_H(B; \eta) = \min \{ |H(z)| : z \in \mathbb{D}(0, B), d(z, w_j) \geq \eta \text{ for all } j \in \{1, \dots, m\} \}.$$

The following result is obtained by combining Lemmas 3.5 and 3.6, Corollary 3.4, and Theorem 3.7 from [24].

Proposition 9.5. *Let $B, \tau \geq 1$ be fixed parameters, and $L > 0$ be sufficiently large. Then, with probability at least $1 - \exp(-CB^4L^4)$, we have that*

$$m_{F_L}(B, L^{-\tau}) \geq \exp(-C\tau B^2 L^2 \log L),$$

where $C > 0$ is an absolute constant.

9.4. Joint density of the zeros of $P_{N,L}$. Thinking for the moment of $N \in \mathbb{N}, L > 0$ as free parameters, we write

$$(9.4.1) \quad P_{N,L}(w) = \sum_{n=0}^N \xi_n \frac{(Lw)^n}{\sqrt{n!}} = \xi_N \frac{L^N}{\sqrt{N!}} \prod_{j=1}^N (w - z_j) =: \xi_N \frac{L^N}{\sqrt{N!}} Q_{\mathbf{z}}(w).$$

By a change of variables (see e.g. [24, Appendix A, Lemma A.1]) the joint density of the zeros with respect to the product measure $d\mathbf{A}(\mathbf{z}) := d\mathbf{A}^{\otimes N}(z_1, \dots, z_N)$ takes the form

$$(9.4.2) \quad f_{N,L}(\mathbf{z}) = \frac{1}{Z_{N,L}} |\Delta(\mathbf{z})|^2 \left(\frac{L^2}{\pi} \int_{\mathbb{C}} |Q_{\mathbf{z}}(w)|^2 e^{-L^2|w|^2} d\mathbf{A}(w) \right)^{-(N+1)},$$

where $|\Delta(\mathbf{z})| = \prod_{i < j} |z_i - z_j|$ is the Vandermonde determinant, and $\mathcal{Z}_{N,L} = \pi^N L^{N(N+1)} N!^{-1} \prod_{k=1}^N (k!)^{-1}$ is a normalizing constant. In alignment with [24] we use the notation

$$S(\mathbf{z}) = \frac{L^2}{\pi} \int_{\mathbb{C}} |Q_{\mathbf{z}}(w)|^2 e^{-L^2|w|^2} dA(w).$$

Remark 9.6. Henceforth, the empirical probability measure of the points $\mathbf{z} = (z_1, \dots, z_N)$ will be denoted by

$$\mu_{\mathbf{z}} = \frac{1}{N} \sum_{j=1}^N \delta_{z_j},$$

where δ_{z_j} is a point mass at z_j .

9.5. The conditional limiting measure. Put $\alpha = NL^{-2}$. Recall that,

$$I_{\alpha}(\mu) = -\Sigma(\mu) + 2B_{\alpha}(\mu) = -\Sigma(\mu) + 2 \sup_{z \in \mathbb{C}} (U^{\mu}(z) - \frac{|z|^2}{2\alpha}),$$

and denote by $\mu_{\alpha} = \chi_{\mathbb{D}(0, \sqrt{\alpha})} dA$ the global minimizer of I_{α} . Also recall that the minimum value of I_{α} over the class of probability measures $\mathcal{M}_{\mathcal{G}}$ which charge zero mass to \mathcal{G} is attained uniquely at $\mu_{\alpha, \mathcal{G}}$.

Roughly speaking, the idea of Zeitouni and Zelditch in [73] is to approximate the Vandermonde term $|\Delta(\mathbf{z})|^2$ in (9.4.2) by $\exp(N^2 \Sigma(\mu_{\mathbf{z}}))$ (appropriately regularized), and to replace $S(\mathbf{z})$ by $\exp(2NB_{\alpha}(\mu_{\mathbf{z}}))$ (for a more precise statement, see e.g. Proposition 9.9).

Looking at (9.4.2) at the logarithmic scale, and expressing the normalizing constant $\mathcal{Z}_{N,L}$ in terms of $I_{\alpha}(\mu_{\alpha})$, the probability of the hole event in \mathcal{G} for $P_{N,L}$ is equal, up to smaller error terms, to the maximum of $-N^2(I_{\alpha}(\mu) - I_{\alpha}(\mu_{\alpha}))$ over $\mu \in \mathcal{M}_{\mathcal{G}}$, that is (by Lemma 9.7 below) to $-L^4 \int_{\Omega_{\nu}} u_{\mathcal{G}} \frac{dA}{\pi}$. Moreover, the probability of zero configurations which are not ‘close’ to $\mu_{\alpha, \mathcal{G}}$ is negligible with respect to the hole probability, so that (following [24]), we show that the limiting measure of the zeros of F_L on the hole event in \mathcal{G} is given by the Radon measure

$$(9.5.1) \quad \mu_{\mathcal{G}}^{\mathbb{C}} := \lim_{\alpha \rightarrow \infty} \alpha \mu_{\alpha, \mathcal{G}}.$$

That the above limit exists can be seen by appealing to Proposition 4.7. Moreover, by that proposition we see that

$$\mu_{\mathcal{G}}^{\mathbb{C}} = \alpha_0 \mu_{\alpha_0, \mathcal{G}} + \frac{1}{\pi} \chi_{\{z: |z| \geq \sqrt{\alpha_0}\}} dA.$$

Recall that there is a finite measure $\nu = \nu_{\mathcal{G}}$ supported in \mathcal{G} , such that

$$\mu_{\alpha, \mathcal{G}} = \frac{1}{\alpha} \text{Bal}(\nu, \mathcal{G}^c) + \frac{1}{\pi \alpha} \chi_{\mathbb{D}(0, \sqrt{\alpha}) \setminus \Omega_{\nu}} dA,$$

where Ω_{ν} denotes the subharmonic quadrature domain with respect to ν (which contains \mathcal{G}). Define the *Schwarz potential* $u = u_{\mathcal{G}}$ associated to the data (ν, Ω_{ν}) as the unique solution to the PDE,

$$\begin{cases} \frac{1}{2\pi} \Delta u = 1 - \nu & \text{on } \Omega_{\nu}, \\ u = 0 & \text{on } \partial \Omega_{\nu}. \end{cases}$$

Lemma 9.7. *With the above definitions, we have*

$$\alpha^2 (I_{\alpha}(\mu_{\alpha, \mathcal{G}}) - I_{\alpha}(\mu_{\alpha})) = \frac{1}{\pi} \int_{\Omega_{\nu}} u_{\mathcal{G}} dA.$$

Proof. We use the notation

$$\Sigma(\mu, \nu) = \int U^\mu d\nu = \int U^\nu d\mu,$$

provided that $U^\mu \in L^1(\nu)$ and vice versa. It then holds that

$$\begin{aligned} I_\alpha(\mu_{\alpha, \mathcal{G}}) - I_\alpha(\mu_\alpha) &= \Sigma(\mu_\alpha) - \Sigma(\mu_{\alpha, \mathcal{G}}) = \Sigma(\mu_\alpha) - \Sigma(\mu_{\alpha, \mathcal{G}}, \mu_{\alpha, \mathcal{G}}^s + \frac{1}{\alpha} \chi_{\mathbb{D}_{\sqrt{\alpha}} \setminus \Omega_\nu}) \\ &= \Sigma(\mu_\alpha) - \Sigma(\mu_{\alpha, \mathcal{G}}, \frac{1}{\alpha} \nu + \frac{1}{\alpha} \chi_{\mathbb{D}_{\sqrt{\alpha}} \setminus \Omega_\nu}), \end{aligned}$$

where the last equality holds because $\mu_{\alpha, \mathcal{G}}^s = \frac{1}{\alpha} \text{Bal}(\nu, \mathcal{G}^c)$ and since the potential is harmonic across \mathcal{G} . Next, observe that $U^{\mu_{\alpha, \mathcal{G}}} = U^{\mu_\alpha}$ on the support of the measure $\nu + \chi_{\mathbb{D}_{\sqrt{\alpha}} \setminus \Omega_\nu}$. In view of the identity $U^{\chi_{\Omega_\nu} - \nu} = u_{\mathcal{G}}$ we may write

$$I_\alpha(\mu_{\alpha, \mathcal{G}}) - I_\alpha(\mu_\alpha) = \frac{1}{\alpha} \Sigma(\mu_\alpha, \chi_{\Omega_\nu} - \nu) = \frac{1}{\pi \alpha^2} \int_{\Omega_\nu} u_{\mathcal{G}} dA.$$

This completes the proof. \square

9.6. Fekete points and discretization of the limiting measure. Before embarking on the proof of a lower bound for the hole probability, we explain how to construct a discrete approximation for the conditional limiting measure.

Consider the measure $\mu_{\alpha, \mathcal{G}}$ for $\alpha \geq \alpha_0$ and its logarithmic potential $U^{\mu_{\alpha, \mathcal{G}}}$. By Theorem 7.3 and Proposition 4.7, there is a constant $\gamma = \gamma_{\mathcal{G}} \in (0, 1]$ such that the potential $U^{\alpha \mu_{\alpha, \mathcal{G}}} |_{\{\mathbb{D}(0, \sqrt{\alpha_0})\}}$ is Hölder continuous with exponent γ and norm equal to C_U^γ (both γ and C_U^γ do not depend on α). Since $U^{\mu_{\alpha, \mathcal{G}}} = U^{\mu_\alpha}$ outside $\mathbb{D}(0, \sqrt{\alpha_0})$, we conclude that

$$\|U^{\mu_{\alpha, \mathcal{G}}}\|_{C^{0, \gamma}} = O(\alpha^{-\gamma/2}).$$

Lemma 9.8. *Let \mathcal{G} be a domain contained in the unit disk and satisfying the conditions of Theorem 9.1 and put $N = \lceil L^2 \log L \rceil = \alpha L^2$. For all L sufficiently large (depending on \mathcal{G}) there is a set of points $\mathcal{F}_N^{\mathcal{G}} = (z_1, \dots, z_N) = \mathbf{z}$ with the following properties:*

- (1) *The points lie outside of \mathcal{G} : $\mathcal{F}_N^{\mathcal{G}} \subset \mathbb{D}(0, \sqrt{\alpha}) \setminus \mathcal{G}$.*
- (2) *The points are separated:*

$$\inf_{j \neq k} |z_j - z_k| \geq L^{-3/\gamma}.$$

- (3) *The logarithmic energy of $\mu_{\mathbf{z}}$ satisfies the bound*

$$-\Sigma^*(\mu_{\mathbf{z}}) \leq -\Sigma(\mu_{\alpha, \mathcal{G}}) + \frac{2}{\gamma L^2}.$$

- (4) *The logarithmic potential of $\mu_{\mathbf{z}}$ meets the bound*

$$U^{\mu_{\mathbf{z}}}(\zeta) \leq U^{\mu_{\alpha, \mathcal{G}}}(\zeta) + \frac{12}{\gamma L^2} \quad \forall \zeta \in \mathbb{C}.$$

Proof. We apply Proposition A.2 and Theorem A.3 with $\Lambda = \mathbb{D}(0, \sqrt{\alpha}) \setminus \mathcal{G}$, $Q = U^{\mu_{\alpha, \mathcal{G}}}$ (so that $\mu_{Q, \Lambda} = \mu_{\alpha, \mathcal{G}}$) and we denote by $\mathcal{F}_N^{\mathcal{G}} = (z_1, \dots, z_N) = \mathbf{z}$ the corresponding (weighted) Fekete points, restricted to Λ . Property (1) holds by definition. We have,

$$\inf_{z' \neq z'' \in \mathcal{F}_N^{\mathcal{G}}} |z' - z''| \geq \frac{1}{2} \exp(-C\alpha^{-\gamma/2}) N^{-1/\gamma},$$

which gives Property (2), when L is sufficiently large. Moreover,

$$-\Sigma^*(\mu_{\mathbf{z}}) \leq -\Sigma(\mu_{\alpha, \mathcal{G}}) + E_1(N, \alpha),$$

where $E_1(N, \alpha) = N^{-1}[C\alpha^{-\gamma/2} + 2\log 2\sqrt{\alpha} + \gamma^{-1}\log N]$. In addition, by the proof of Proposition 4.5 we have that $D(Q, \Lambda) = \sup_{z \in \Lambda} |U^{\mu_{\alpha, \mathcal{G}}}| = O(\log \alpha)$. Therefore,

$$U^{\mu_{\mathbf{z}}}(\zeta) \leq U^{\mu_{Q, \Lambda}}(\zeta) + E_2(N, \alpha) \quad \forall \zeta \in \mathbb{C},$$

where $E_2(N, \alpha) \leq 2N^{-1}[C\alpha^{-\gamma/2} + \log 2\sqrt{\alpha} + (2 + 3\gamma^{-1})\log N + C\log \alpha]$. Properties (3) and (4) are established, for L sufficiently large, by examining the relations between L, N and α . \square

The following properties of Fekete points (and small perturbations of them) are crucial for the proof of the lower bound. Recall that $|\Delta(\mathbf{z})|^2 = \prod_{i \neq j} |z_i - z_j| = \exp(N^2 \Sigma^*(\mu_{\mathbf{z}}))$ and $S(\mathbf{z}) = \pi^{-1} L^2 \int_{\mathbb{C}} |Q_{\mathbf{z}}(w)|^2 e^{-L^2 |w|^2} dA(w)$.

Proposition 9.9. *Let $\mathcal{F}_N^{\mathcal{G}} = \mathbf{z} = (z_1, \dots, z_N)$ be a set of Fekete points, and $\tau > 3/\gamma + 2$ as in the previous lemma. Moreover, let the points $\mathbf{w} = (w_1, \dots, w_N) \in \mathbb{D}(0, \sqrt{\alpha})^N$ satisfy $|w_i - z_i| \leq L^{-\tau}$ for all $i \in \{1, \dots, N\}$. For L sufficiently large, we have*

- (1) $|\Delta(\mathbf{w})|^2 \geq \exp(N^2 \Sigma(\mu_{\alpha, \mathcal{G}}) - (C/\gamma)(N/L)^2)$.
- (2) $S(\mathbf{w}) \leq \exp(2NB_{\alpha}(\mu_{\alpha, \mathcal{G}}) + \frac{C\alpha}{\gamma})$.

Proof. The reverse triangle inequality

$$|w_j - w_k| \geq |z_j - z_k| - 2 \max_i |w_i - z_i| \geq |z_j - z_k| - 2L^{-\tau},$$

shows that Properties (2) and (3) in Lemma 9.8 hold also for \mathbf{w} , perhaps with different (absolute) constants on the respective right hand sides. In particular, this proves Property 1.

We now show that Property (4) of Lemma 9.8 also holds for \mathbf{w} (with a modified constant). Let $\gamma \in (0, 1]$ be the Hölder exponent of $U^{\mu_{\alpha, \mathcal{G}}}$ and let σ_t be the normalized Lebesgue measure on the circle $\mathbb{T}(0, t)$. If we take $t = L^{-2/\gamma}$, then

$$U^{\mu_{\mathbf{z}} * \sigma_t}(\zeta) = \int_{\mathbb{T}} U^{\mu_{\mathbf{z}}}(\zeta + te^{i\theta}) \frac{d\theta}{2\pi} \leq U^{\mu_{\alpha, \mathcal{G}}}(\zeta) + \frac{24}{\gamma L^2}, \quad \zeta \in \mathbb{C},$$

for L sufficiently large, where we used Property (4) of Lemma 9.8, and the choice of t . Put $a \vee b := \max\{a, b\}$. Observe that for L sufficiently large,

$$|\log(|\zeta - w_j| \vee t) - \log(|\zeta - z_j| \vee t)| \leq \frac{2L^{-\tau}}{t} \leq \frac{1}{\gamma L^2}, \quad \zeta \in \mathbb{C}.$$

Since

$$U^{\mu_{\mathbf{w}}}(\zeta) \leq U^{\mu_{\mathbf{z}} * \sigma_t}(\zeta) = \frac{1}{N} \sum_{k=1}^N \log(|\zeta - w_j| \vee t) \leq U^{\mu_{\mathbf{z}} * \sigma_t}(\zeta) + \frac{1}{\gamma L^2},$$

we obtain the required bound for $U^{\mu_{\mathbf{w}}}$.

We return to the proof of Property (2). Write

$$S(\mathbf{w}) = \int_{\mathbb{C}} \exp(2N U^{\mu_{\mathbf{w}}}(\zeta) - L^2 |\zeta|^2) \frac{L^2}{\pi} dA(\zeta) =: I_1 + I_2$$

where I_1 in the integral over $D(0, \sqrt{\alpha})$ and I_2 over $\mathbb{C} \setminus D(0, \sqrt{\alpha})$. For I_1 we have the bound

$$\int_{\mathbb{D}(0, \sqrt{\alpha})} \exp\left(2N(U^{\mu_{\alpha, \mathcal{G}}}(\zeta) - \frac{|\zeta|^2}{2\alpha}) + \frac{c_N}{\gamma L^2}\right) \frac{L^2}{\pi} dA(\zeta) \leq \alpha L^2 \cdot \exp\left(2NB_{\alpha}(\mu_{\alpha, \mathcal{G}}) + \frac{c_N}{\gamma}\right).$$

We bound I_2 by

$$\int_{\{|\zeta| \geq \sqrt{\alpha}\}} \exp\left(2N \log |\zeta| - L^2 |\zeta|^2\right) \frac{L^2}{\pi} dA(\zeta) = \mathcal{E}[|W|^{2N} \mathbf{1}_{\{|W| \geq \sqrt{\alpha}\}}],$$

where $W \sim N_{\mathbb{C}}(0, L^{-1})$. Now an application of the Cauchy-Schwarz inequality shows that $I_2 = o(I_1)$ if L is large. \square

9.7. Lower bound for the hole probability. Given $\delta > 0$ we write

$$\mathcal{G}_{\delta}^+ = \{w \in \mathbb{C} : d(w, \mathcal{G}) \leq \delta\},$$

for the δ -neighbourhood of \mathcal{G} . Combining Propositions 9.3 and 9.5 we see that it is enough to construct an event $\mathcal{H}_{\mathcal{G}, L}^N$ on which the polynomial $P_{N, L}$ has no zeros inside \mathcal{G}_{δ}^+ , with $\delta = L^{-\tau}$, and $\tau = \frac{6}{\gamma}$. Notice we may choose B sufficiently large so that in Proposition 9.5 the exceptional event is negligible. Then we choose A in Proposition 9.3 large so that $P_{N, L}$ dominates the tail T_N in $\mathbb{D}(0, B)$ which contains \mathcal{G}_{δ}^+ .

The construction of the event $\mathcal{H}_{\mathcal{G}, L}^N$ is based on a collection of small perturbations of weighted Fekete points $\mathcal{F}_N^{\mathcal{G}} = (z_1, \dots, z_N)$ from the previous section. More precisely, by the uniform cone condition, there is a constant $c_{\mathcal{G}} > 0$ such that

$$|\mathcal{N}(z_j, \delta)| := |\mathbb{D}(z_j, 2\delta) \cap (\mathcal{G}_{\delta}^+)^c \cap \mathbb{D}(0, \sqrt{\alpha})| \geq c_{\mathcal{G}} \delta^2, \quad \forall j \in \{1, \dots, N\}.$$

Note that such a lower bound is immediate if z_j is sufficiently far from $\partial\mathcal{G}$. We define the event $\mathcal{H}_{\mathcal{G}, L}^N$ by

$$\mathcal{H}_{\mathcal{G}, L}^N = \{\mathbf{w} = (w_1, \dots, w_N) \in \mathbb{C}^N : w_j \in \mathcal{N}(z_j, \delta) \quad \forall j \in \{1, \dots, N\}\},$$

and observe that $|\mathcal{H}_{\mathcal{G}, L}^N| \geq (c_{\mathcal{G}} \delta^2)^N \geq \exp(-13\gamma^{-1} L^2 \log^2 L)$, for L sufficiently large.

The expression (9.4.2) for the joint density of the zeros, and (9.4.1) give us

$$\mathcal{P}(\mathcal{H}_{\mathcal{G}, L}^N) = \int_{\mathcal{H}_{\mathcal{G}, L}^N} f_{N, L}(\mathbf{w}) dA(\mathbf{w}) = \frac{1}{\mathcal{Z}_{N, L}} \int_{\mathcal{H}_{\mathcal{G}, L}^N} |\Delta(\mathbf{w})|^2 S(\mathbf{w})^{-(N+1)} dA(\mathbf{w}).$$

The normalizing constant $\mathcal{Z}_{N, L}$ is given by (see e.g. [24, Appendix A, Lemma A.1])

$$\begin{aligned} \mathcal{Z}_{N, L} &= \frac{\pi^N L^{N(N+1)}}{N!} \prod_{k=1}^N \frac{1}{k!} = \exp\left(-\frac{1}{2} N^2 \log \alpha + \frac{3}{4} N^2 + O(L^2 \log^2 L)\right) \\ &= \exp\left(-N^2 I_{\alpha}(\mu_{\alpha}) + O(L^2 \log^2 L)\right), \end{aligned}$$

where we recall $\mu_{\alpha} = \chi_{\mathbb{D}(0, \sqrt{\alpha})} dA$ is the global minimizer of I_{α} .

By appealing to Proposition 9.9, we get the bound

$$\mathcal{P}(\mathcal{H}_{\mathcal{G}, L}^N) \geq |\mathcal{H}_{\mathcal{G}, L}^N| \exp\left(N^2 (I_{\alpha}(\mu_{\alpha}) + \Sigma(\mu_{\alpha, \mathcal{G}}) - 2B_{\alpha}(\mu_{\alpha, \mathcal{G}})) - E(N, L)\right),$$

where

$$\begin{aligned} E(N, L) &= (C/\gamma)(N/L)^2 + 2NB_\alpha(\mu_\alpha, \mathcal{G}) + (C/\gamma)(N+1)\alpha + O(L^2 \log^2 L) \\ &= (C/\gamma)L^2\alpha^2 + N(\log \alpha + 1) + (C/\gamma)(N+1)\alpha + O(L^2 \log^2 L) \\ &= (1/\gamma)O(L^2 \log^2 L). \end{aligned}$$

We have shown that

$$\mathcal{P}(\mathcal{H}_{\mathcal{G}, L}^N) \geq \exp\left(-L^4\alpha^2(I_\alpha(\mu_\alpha, \mathcal{G}) - I_\alpha(\mu_\alpha)) + (1/\gamma) \cdot O(L^2 \log^2 L)\right),$$

and therefore the lower bound for the hole probability in Theorem 9.1 follows from Lemma 9.7.

9.8. Large deviation upper bound for linear statistics. Here we give a brief account of the proof of the large deviation upper bound. This proof is largely based on [24, § 7.3], with some important differences, concerning the approximation of \mathcal{G} , and some rather minor technical simplifications.

9.8.1. Preliminary large deviation bound. As before, we let $N = \alpha L^2 = \lceil L^2 \log L \rceil$. We use $\mathbf{z} = (z_1, \dots, z_N)$ to denote a possible zero configuration (that is, the zeros of $P_{N, L}$). We also recall the expression for the joint density of the zeros

$$f_{N, L}(\mathbf{z}) = \frac{1}{\mathcal{Z}_{N, L}} |\Delta(\mathbf{z})|^2 S(\mathbf{z})^{-(N+1)},$$

where $\mathcal{Z}_{N, L} = \exp(-N^2 I_\alpha(\mu_\alpha) + O(L^2 \log^2 L))$. Outside a probabilistically negligible event E_1^N (cf. [24, Claim 4.6]) we have

$$S(\mathbf{z}) \leq \exp(CL^6).$$

We define

$$S^\dagger(\mathbf{z}) = \sup_{w \in \mathbb{C}} \{|Q_{\mathbf{z}}(w)|^2 e^{-L^2|w|^2}\} = \exp(2NB_\alpha(\mu_{\mathbf{z}}));$$

note that in [24] the above expression is denoted by $A(\mathbf{z})$. The Bernstein-Markov property $S^\dagger(\mathbf{z}) \leq S(\mathbf{z})$ holds (e.g. see [24, Appendix A, Lemma A.4]), and by [24, Claim 4.5], for $b > 1$

$$\int_{\mathbb{C}^N} S^\dagger(\mathbf{z})^{-b} dA(\mathbf{z}) \leq \exp\left(bL^2 + N \log\left(\frac{Cb}{b-1}\right)\right).$$

Recall the definition of the discrete energy functional

$$I_\alpha^*(\mu_{\mathbf{z}}) = -\Sigma^*(\mu_{\mathbf{z}}) + 2B_\alpha(\mu_{\mathbf{z}}),$$

and observe that

$$|\Delta(\mathbf{z})|^2 S(\mathbf{z})^{-N} \leq \exp\left(N^2(\Sigma^*(\mu_{\mathbf{z}}) - 2B_\alpha(\mu_{\mathbf{z}}))\right) = \exp(-N^2 I_\alpha^*(\mu_{\mathbf{z}})).$$

Let $\mathcal{A}^N \subset \mathbb{C}^N$ be a collection of possible positions of the zeros of $P_{N, L}$, we now obtain a large deviation upper bound for $\mathcal{P}(\mathcal{A}^N)$ as follows

$$\begin{aligned} \mathcal{P}(\mathcal{A}^N) &= \int_{\mathcal{A}^N} f_{N, L}(\mathbf{z}) dA(\mathbf{z}) \\ &\leq J_1 \exp\left(-N^2 \inf_{\mathbf{z} \in \mathcal{A}^N} (I_\alpha^*(\mu_{\mathbf{z}}) - I_\alpha(\mu_\alpha)) + O(L^2 \log^2 L)\right) + \mathcal{P}(E_1^N), \end{aligned}$$

where

$$\begin{aligned} J_1 &= \int_{\mathbb{C}^N \setminus E_1^N} S(\mathbf{z})^{-1} dA(\mathbf{z}) \leq \exp(CL^6/N^2) \int_{\mathbb{C}^N} S^\dagger(\mathbf{z})^{-1-N^{-2}} dA(\mathbf{z}) \\ &\leq \exp(CL^6/N^2 + (1 + N^{-2})L^2 + CN \log N) = \exp(O(L^2 \log^2 L)). \end{aligned}$$

We now replace I_α^* by I_α . Put $\mu_{\mathbf{z}}^t = \mu_{\mathbf{z}} * \sigma_t$, where σ_t is the normalized Lebesgue measure on the circle $\mathbb{T}(0, t)$. Using [24, Claims 4.7, 4.8] we get for t small

$$I_\alpha^*(\mu_{\mathbf{z}}) \geq I_\alpha(\mu_{\mathbf{z}}^t) - C\left(\frac{1}{N} \log \frac{1}{t} + \frac{t}{\sqrt{\alpha}} + \frac{1}{L^2}\right).$$

We put $t = L^{-\tau'}$, with $\tau' \geq 2$ fixed, so that the error term above is $O(L^{-2})$.

Putting all of this together we have shown

$$(9.8.1) \quad \mathcal{P}(\mathcal{A}^N) \leq \exp\left(-N^2 \inf_{\mathbf{z} \in \mathcal{A}^N} (I_\alpha(\mu_{\mathbf{z}}^t) - I_\alpha(\mu_\alpha)) + O(L^2 \log^2 L)\right) + \mathcal{P}(E_1^N),$$

where $\mathcal{P}(E_1^N)$ is negligible with respect to the hole probability.

9.8.2. *Upper bound for linear statistics.* Fix a parameter $\tau \geq 1$, let $\delta = L^{-\tau}$ and define

$$\mathcal{G}_\delta^- = \{w \in \mathcal{G} : d(w, \mathcal{G}^c) \geq \delta\},$$

to be the δ -interior of \mathcal{G} . Using Propositions 9.3 and 9.5 (with B large), together with Rouché's theorem, we find that outside a probabilistically negligible exceptional event $n_{P_N}(\mathcal{G}_\delta^-) \leq n_L(\mathcal{G})$. Moreover, let φ be a continuous function supported in $\mathbb{D}(0, B-1)$, with modulus of continuity given by $\omega(\varphi; \cdot)$, then by [24, Corollary 3.4]

$$|n_L(\varphi) - n_{P_N}(\varphi)| \leq L^4 \omega(\varphi; \delta),$$

outside an exceptional event.

Our goal is to bound from above the probability of the event

$$H_{L, \mathcal{G}, \epsilon} := \left\{ F_L \neq 0 \text{ in } \mathcal{G}, \quad \left| n_L(\varphi) - L^2 \int \varphi d\mu_{\mathcal{G}}^{\mathbb{C}} \right| > \epsilon L^2 \right\}.$$

Remark 9.10. To simplify matters we will take $\epsilon > 0$ fixed. Checking the details of the proof below, one can see that $L^{-1} \log L = o(\epsilon)$ is the actual requirement. Similarly, as in [24, § 7] it is possible to take φ depending on L , but we will not pursue this here (the details are similar).

Since φ is compactly supported, for α sufficiently large we have by (9.5.1) that

$$L^2 \int \varphi d\mu_{\mathcal{G}}^{\mathbb{C}} = N \int \varphi d\mu_{\alpha, \mathcal{G}}.$$

In addition, $n_{P_N}(\varphi) = N \int \varphi d\mu_{\mathbf{z}}$, thus,

$$|n_{P_N}(\varphi) - N \int \varphi d\mu_{\mathbf{z}}^t| \leq N\omega(\varphi, t).$$

Put $\mathcal{G}' = \mathcal{G}_{\delta+t}^-$, and recall $\delta = L^{-\tau}$, $t = L^{-\tau'}$. Using [24, Lemma 3.14, Claim 5.9], we get

$$\begin{aligned} \left| \int \varphi d\mu_{\alpha, \mathcal{G}} - \int \varphi d\mu_{\alpha, \mathcal{G}'} \right| &\leq \frac{1}{\sqrt{2\pi}} \sqrt{\mathcal{D}(\varphi)} \sqrt{-\Sigma(\mu_{\alpha, \mathcal{G}} - \mu_{\alpha, \mathcal{G}'})} \\ &\leq \frac{1}{\sqrt{2\pi}} \sqrt{\mathcal{D}(\varphi)} \sqrt{I(\mu_{\alpha, \mathcal{G}}) - I(\mu_{\alpha, \mathcal{G}'})}. \end{aligned}$$

If τ, τ' are chosen sufficiently large (depending only on \mathcal{G}) then by Proposition 8.1 we have $I(\mu_{\alpha, \mathcal{G}}) - I(\mu_{\alpha, \mathcal{G}'}) = O(L^{-4})$. Collecting all these estimates, we find that on the event $H_{L, \mathcal{G}, \epsilon}$ (and discarding an exceptional event of negligible probability), we have, for L sufficiently large,

$$\mu_{\mathbf{z}}^t(\mathcal{G}') = 0 \text{ and } \left| \int \varphi d\mu_{\mathbf{z}}^t - \int \varphi d\mu_{\alpha, \mathcal{G}'} \right| > \frac{\epsilon}{\alpha} - \omega(\varphi, t) - \frac{L^4}{N} \omega(\varphi, \delta) - CL^{-2} > \frac{\epsilon}{2\alpha}.$$

Using [24, Lemma 3.14, Claim 5.9] and Proposition 8.1 once again, we get

$$I(\mu_{\mathbf{z}}^t) \geq I(\mu_{\alpha, \mathcal{G}'}) + \frac{c\epsilon^2}{\mathcal{D}(\varphi)\alpha^2} \geq I(\mu_{\alpha, \mathcal{G}}) + \frac{c'\epsilon^2}{\mathcal{D}(\varphi)\alpha^2}.$$

Finally, by (9.8.1) we conclude that

$$\mathcal{P}(H_{L, \mathcal{G}, \epsilon}) \leq \exp\left(-N^2(I(\mu_{\alpha, \mathcal{G}}) - I(\mu_{\alpha})) - \frac{c\epsilon^2}{\mathcal{D}(\varphi)}L^4 + O(L^2 \log^2 L)\right),$$

which combined with Lemma 9.7 proves the large deviation upper bound for linear statistics.

9.9. Convergence of the conditional empirical measures. The proofs here are essentially the same as the ones in [24, § 7.4]. We provide few details below, and refer the interested reader to that paper for the rest.

From the above results it is possible to deduce that

$$\left| \mathcal{E}[n_{F_L}(\varphi)|H_{L, \mathcal{G}}] - L^2 \int \varphi d\mu_{\mathcal{G}}^{\mathbb{C}} \right| \leq C_{\varphi} L \log^2 L,$$

where C_{φ} depends on the test function φ . In order to prove the vague convergence in distribution of the conditional empirical measures we have to show that the random variables $L^{-2}n_{F_L}(\psi)|_{H_{L, \mathcal{G}}}$ converge in distribution to $\int \psi d\mu_{\mathcal{G}}^{\mathbb{C}}$ for every continuous test function with compact support ψ (see e.g. [41, Chapter 4]). This is achieved using an approximation of ψ in sup-norm by a smooth test function φ which is compactly supported in a slightly larger set, and using the large deviation upper bound.

This completes our outline of the proof of Theorem 9.1. \square

10. THE INVERSE PROBLEM, DISK-LIKE DOMAINS AND EXAMPLES

10.1. A sufficient condition for extremality. In this section, we discuss some cases in which one can say more about the measure $\mu_{\alpha, \mathcal{G}}$ than provided by the general classification encountered in Theorem 1.8. In order to show that a measure is indeed an extremal measure for the minimization of I_{α} over the class $\mathcal{M}_{\mathcal{G}}$, we will verify that the variational inequalities for I_{α} found in [24] are met. We summarize this variational principle in a proposition.

Proposition 10.1. *Let \mathcal{M} be a non-empty closed convex set, whose elements are compactly supported probability measures on \mathbb{C} with finite logarithmic energy. The minimization problem*

$$\text{minimize } I_{\alpha}(\mu) \quad \text{subject to } \mu \in \mathcal{M}$$

admits a unique solution $\mu_0 = \mu_0(\mathcal{M})$. Moreover, μ_0 is characterized by the property that for all $\nu \in \mathcal{M}$, we have

$$B_{\alpha}(\nu) - \int U^{\nu} d\mu_0 \geq B_{\alpha}(\mu_0) - \int U^{\mu_0} d\mu_0.$$

with equality if and only if $\nu = \mu_0$.

The proof of existence of minimizers and of the \Rightarrow direction is supplied in [24].

Proof of \Leftarrow . Assume that μ_0 meets the variational inequality for all $\nu \in \mathcal{M}$. We may assume that \mathcal{M} contains other measures than μ_0 , or the result follows directly. Then we find that for any admissible $\nu \neq \mu_0$, we have

$$\begin{aligned} I_\alpha(\nu) &= 2B_\alpha(\nu) - \Sigma(\nu) = 2B_\alpha(\nu) - 2 \int U^{\mu_0} d\nu + 2 \int U^{\mu_0} d\nu - \Sigma(\nu) \\ &\geq 2B_\alpha(\mu_0) - 2\Sigma(\mu_0) + 2 \int U^{\mu_0} d\nu - \Sigma(\nu) = I_\alpha(\mu_0) - \Sigma(\mu_0 - \nu) > I_\alpha(\mu_0) \end{aligned}$$

where we use the positivity of $-\Sigma(\eta) = \mathcal{D}(U^\eta)$ for a signed measure η with total mass zero, so μ_0 is the minimum. \square

Using these variational inequalities for the extremal problem, we can find sufficient conditions for extremality on the hole event, in a form which is convenient to approach the inverse problem (Problem 1.2).

Lemma 10.2. *Let ν be a measure supported in \mathcal{G} , and suppose that Ω_ν is a subharmonic quadrature domain with respect to ν , containing \mathcal{G} . Let μ be the measure*

$$d\mu = \frac{1}{\alpha} d\text{Bal}(\nu, \mathcal{G}^c) + \frac{1}{\pi\alpha} \chi_{\mathbb{D}(0, \sqrt{\alpha}) \setminus \Omega_\nu} dA.$$

Assume moreover that $B_\alpha(\mu) = c_\alpha$ and that

$$\text{supp}(\nu) \subset \mathcal{I} = \{z \in \mathcal{G} : U^\mu(z) = \frac{1}{\alpha}|z|^2 + c_\alpha\}.$$

Then μ is the minimizer of I_α over $\mathcal{M}_{\mathcal{G}}$, i.e. $\mu = \mu_{\alpha, \mathcal{G}}$.

Proof. Let η be any competing measure in $\mathcal{M}_1(G)$. We split $\mu = \mu^s + \mu^c$ according to the Lebesgue decomposition, with $\text{supp}(\mu^s) \subset \partial\mathcal{G}$, and note that

$$\begin{aligned} \int U^\eta d\mu &= \int_{\mathbb{D}(0, \sqrt{\alpha}) \setminus \Omega_\nu} U^\eta(z) \frac{dA(z)}{\pi\alpha} + \frac{1}{\alpha} \int_{\partial\mathcal{G}} U^\eta d\text{Bal}(\nu, \mathcal{G}^c) \\ &\leq \left(1 - \frac{|\Omega_\nu|}{\alpha}\right) B_\alpha(\eta) + \int_{\mathbb{D}(0, \sqrt{\alpha}) \setminus \Omega_\nu} \frac{|z|^2}{2\alpha} \frac{dA(z)}{\pi\alpha} + \frac{1}{\alpha} \int_{\mathcal{I}} U^\eta d\nu, \end{aligned}$$

where we use the harmonicity of U^η in \mathcal{G} to pass from integration against the balayage measure to integration against ν . By adding and subtracting a quantity independent of the competing measure η , we may further rewrite

$$\frac{1}{\alpha} \int_{\mathcal{I}} U^\eta d\nu \leq \frac{|\nu|}{\alpha} B_\alpha(\eta) + \frac{1}{\alpha} \int_{\mathcal{I}} \frac{|z|^2}{2\alpha} d\nu = \frac{|\Omega_\nu|}{\alpha} B_\alpha(\eta) + \frac{1}{\alpha} \int_{\mathcal{I}} \frac{|z|^2}{2\alpha} d\nu,$$

so it follows that

$$\int U^\eta d\mu \leq \left(1 - \frac{|\Omega_\nu|}{\alpha}\right) B_\alpha(\eta) + \frac{|\Omega_\nu|}{\alpha} B_\alpha(\eta) + E = B_\alpha(\eta) + E,$$

where E is the η -independent quantity

$$E = \frac{1}{\alpha} \int_{\mathcal{I}} \frac{|z|^2}{2\alpha} d\nu + \int_{\mathbb{D}(0, \sqrt{\alpha}) \setminus \Omega_\nu} \frac{|z|^2}{2\alpha} \frac{dA(z)}{\pi\alpha}.$$

Similarly when we replace η by μ , we have equality in each of the above steps, and find that

$$\int U^\mu d\mu = B_\alpha(\mu) + E.$$

Summarizing what we have established, we have

$$B_\alpha(\eta) - \int U^\eta d\mu \geq -E$$

while

$$B_\alpha(\mu) - \int U^\mu d\mu = -E$$

so it follows that for any probability measure η with $\eta(\mathcal{G}) = 0$, it holds that

$$B_\alpha(\eta) - \int U^\eta d\mu \geq B_\alpha(\mu) - \int U^\mu d\mu,$$

which completes the proof (by Proposition 10.1). \square

10.2. Disk-like domains. Recall that a bounded simply connected domain \mathcal{G} is said to be disk-like with radius r and center $z_0 \in \mathcal{G}$ if the conformal mapping $\varphi = \varphi_{z_0} : \mathcal{G} \rightarrow \mathbb{D}$, normalized by $\varphi(z_0) = 0$ and $\varphi'(z_0) > 0$, meets the bound

$$(10.2.1) \quad |\varphi(z)| \geq \frac{|z - z_0|}{r} e^{-\frac{|z - z_0|^2}{2er^2}}, \quad z \in \overline{\mathcal{G}},$$

where r is given by $r = |\varphi'(z_0)|^{-1}$. In fact, for a disk-like domain \mathcal{G} , the point z_0 is a local minimizer of $z \mapsto \varphi'_z(z)$, which is to say that z_0 is a *conformal center* of \mathcal{G} . The number r is the *inner conformal radius* with respect to (\mathcal{G}, z_0) . For more details on these notions, we refer to [58, § 6.3].

The potential of the Balayage measure $\text{Bal}(\delta_w, \mathcal{G}^c)$ is given by

$$U^{\text{Bal}(\delta_w, \mathcal{G}^c)}(z) = \log |z - w| - g_{\mathcal{G}}(z, w) = \log \frac{|z - w|}{|\phi_w(z)|},$$

where $g_{\mathcal{G}}$ denotes the Green function for \mathcal{G} with pole at w , and ϕ_w is a conformal mapping of \mathcal{G} onto \mathbb{D} with $\phi_w(w) = 0$.

Proposition 10.3. *Let \mathcal{G} be an disk-like domain of radius r and center z_0 , and let α be large enough for the disk $\mathbb{D}(z_0, \sqrt{er})$ to be contained in the disk $\mathbb{D}(0, \sqrt{\alpha})$. Then the minimizer $\mu_{\alpha, \mathcal{G}}$ of I_α over $\mathcal{M}_{\mathcal{G}}$ is given by*

$$d\mu_{\alpha, \mathcal{G}} = \frac{er^2}{\alpha} d\omega_{z_0, \mathcal{G}} + \frac{1}{\pi\alpha} \chi_{\mathbb{D}(0, \sqrt{\alpha}) \setminus \mathbb{D}(z_0, \sqrt{er})} dA,$$

where $\omega_{z_0, \mathcal{G}}$ denotes harmonic measure from the point z_0 in \mathcal{G} . Conversely, let $\mathcal{G} \subset \mathbb{D}$ denote a Jordan domain with piecewise smooth boundary without cusps whose forbidden region is a disk. Then \mathcal{G} is disk-like.

Remark 10.4. Curiously, it is not immediately clear that \mathcal{G} could not be disk-like with respect to several different pairs (r_j, z_j) . However, it follows from Proposition 10.3 that it is so, since the minimizer of $I_\alpha(\mu)$ over $\mathcal{M}_{\mathcal{G}}$ is unique.

Remark 10.5. In addition, the minimal value of the functional satisfies

$$I_\alpha(\mu_{\alpha, \mathcal{G}}) - I_\alpha(\mu_\alpha) = \frac{1}{\pi\alpha} \int_{\mathbb{D}(z_0, \sqrt{er})} \frac{|z - z_0|^2}{2\alpha} dA(z) = \frac{e^2 r^4}{4\alpha^2},$$

where μ_α denotes the unconstrained equilibrium measure $\mu_\alpha = (\pi\alpha)^{-1}\chi_{\mathbb{D}(0,\sqrt{\alpha})}dA$. In view of Theorem 9.1 this justifies the remark following Theorem 1.4 concerning hole probabilities.

Proof. The first claim of the theorem is that whenever α is large enough so that $\mathbb{D}(z_0, r\sqrt{\epsilon}) \subset \mathbb{D}(0, \sqrt{\alpha})$, the extremal measure for I_α over \mathcal{M}_G is the measure

$$d\mu_0 = \frac{er^2}{\alpha}\omega_{z_0, \mathcal{G}} + \frac{1}{\pi\alpha}\chi_{\mathbb{D}(0,\sqrt{\alpha}) \setminus \mathbb{D}(z_0, r\sqrt{\epsilon})} dA.$$

Here, it is important to notice that $\mathcal{G} \subset \mathbb{D}(z_0, \sqrt{\epsilon}r)$. Indeed, if this is not the case then there exists a point $z \in \mathcal{G} \cap \mathbb{T}(z_0, \sqrt{\epsilon}r)$. As a consequence, $|z - z_0|^2 = er^2$, so that

$$|\varphi(z)| \geq \frac{|z - z_0|}{r} e^{-\frac{|z - z_0|^2}{2er^2}} = 1,$$

which says that $z \in \partial\mathcal{G}$, which is a contradiction.

We will apply Lemma 10.2, and hence we begin to examine the relative potential R_{μ_0} , where for a measure μ the relative potential R_μ is given by

$$R_\mu(z) = U^\mu(z) - \frac{1}{2\alpha}|z|^2.$$

We have, with $c_\alpha = \frac{1}{2}(\log \alpha - 1)$, that

$$\begin{aligned} U^{\mu_0}(z) - \frac{|z|^2}{2\alpha} &= c_\alpha + U^{\mu_0^s}(z) - U^{\frac{1}{\pi\alpha}\chi_{\mathbb{D}(z_0, \sqrt{\epsilon}r)}}(z) \\ &= c_\alpha + \frac{er^2}{\alpha} \log \frac{|z - z_0|}{|\varphi(z)|} - \frac{er^2}{\alpha} \log r - \frac{|z - z_0|^2}{2\alpha}. \end{aligned}$$

At the point z_0 , the value of the relative potential equals c_α , as is seen by using the fact that $\varphi'(z_0) = \frac{1}{r}$.

We claim that the distortion bound (10.2.1) says that the relative potential reaches its maximum on the entire set $\{z_0\} \cup \text{supp}(\mu_0^c)$. On the complement \mathcal{G}^c we may write the relative potential as

$$U^\mu(z) - \frac{|z|^2}{2\alpha} = c_\alpha + \frac{1}{\alpha} \left(U^{er^2\delta_{z_0}}(z) - U^{\frac{1}{\pi}\chi_{\mathbb{D}(z_0, \sqrt{\epsilon}r)}}(z) \right)$$

which is bounded above by c_α . Moreover, since the point mass $er^2\delta_{z_0}$ and the area measure $\pi^{-1}\chi_{\mathbb{D}(z_0, \sqrt{\epsilon}r)}dA$ have the same potential outside $\mathbb{D}(z_0, \sqrt{\epsilon}r)$, we have equality on the set $\mathbb{C} \setminus \mathbb{D}(z_0, \sqrt{\epsilon}r)$. As a consequence of the above, it is enough to study $R_{\mu_0}(z)$ on \mathcal{G} . Subtracting c_α , the condition that $R_{\mu_0}(z)$ attains its maximum on z_0 reads

$$\frac{er^2}{\alpha} \log \frac{|z - z_0|}{|\varphi(z)|} - \frac{er^2}{\alpha} \log r - \frac{|z|^2}{2\alpha} \leq 0, \quad z \in \mathcal{G}$$

or, equivalently

$$|\varphi(z)| \geq \frac{|z - z_0|}{r} e^{-\frac{|z - z_0|^2}{2er^2}}, \quad z \in \mathcal{G}$$

which is precisely the assumed disk-likeness condition.

We next turn to the converse statement. From the obstacle problem characterization of the equilibrium measure, it is clear that $U^{\mu_{\alpha, \mathcal{G}}}$ equals $\frac{|z|^2}{2\alpha} + c_\alpha$ on the support of the continuous part of the measure. The piecewise smoothness assumption implies, via Theorem 1.8, that the singular measure is the balayage to $\partial\mathcal{G}$ of a

measure ν on \mathcal{G} . The assumption that the forbidden region is a disk, say $\mathbb{D}(0, R)$, now shows that

$$U^{\mu_{\alpha, \mathcal{G}}^s}(z) = U^{\frac{1}{\pi\alpha}\chi_{\mathbb{D}(0, R)}}(z), \quad z \in \mathbb{D}(0, \sqrt{\alpha}) \setminus \mathbb{D}(0, R).$$

From this it follows that the atomic measure is a single point mass. Indeed, on the annulus $\mathbb{D}(0, \sqrt{\alpha}) \setminus \mathbb{D}(0, R)$ we have

$$R_{\mu_{\alpha, \mathcal{G}}}(z) = \frac{1}{\alpha} \int \log |z - w| d\nu - \frac{R^2}{\alpha} \log |z| = 0.$$

But since ν has regular support (a loop-free finite union of analytic curves and countably many points), the set $D(R, \nu) := \mathbb{D}(0, R) \setminus \text{supp}(\nu) \cup \{0\}$ is connected and open. But then

$$(10.2.2) \quad \int \log |z - w| d\nu = R^2 \log |z|, \quad z \in D(R, \eta)$$

by harmonic continuation. But then the two functions in (10.2.2) agree as distributions, so taking the Laplacian of both sides we see that this means that $\nu = R^2\delta_0$. As a consequence of this, the measure $\mu_{\alpha, \mathcal{G}}$ takes the form

$$\mu_{\alpha, \mathcal{G}} = \frac{R^2}{\alpha} d\omega_{0, \mathcal{G}} + \frac{1}{\pi\alpha} \chi_{\mathbb{D}(0, \sqrt{\alpha}) \setminus \mathbb{D}(0, R)} dA.$$

We can then compute the logarithmic potential of $\mu_{\alpha, \mathcal{G}}$ in terms of the Riemann mapping φ of \mathcal{G} , which takes the origin to the origin with positive derivative. Expressing the fact that $U^{\mu_{\alpha, \mathcal{G}}}(z) - \frac{|z|^2}{2\alpha} \leq c_\alpha$ in terms the conformal mapping yields that \mathcal{G} is disk-like. \square

Proof of Theorem 1.4. This is now immediate in view of Theorem 1.6 and Proposition 10.3. \square

Examples of domains \mathcal{G} which are disk-like include the square, ellipses up to a critical eccentricity (see [55]), and a certain eye shaped domain. The latter is worth a special mention, as it is an extremal domain among disk-like holes. With $r = 1$ and $z_0 = 0$ we set

$$\varphi(z) = ze^{-\frac{z^2}{2e}}.$$

The conformal mapping $f: \mathbb{D} \rightarrow \mathcal{G}$ is the inverse $f = \varphi^{-1}$, and is given by

$$f(w) = -i\sqrt{e}W\left(-\frac{z^2}{e}\right), \quad w \in \mathbb{D}$$

where W is the principal branch of the Lambert W -function. The function f is a conformal mapping, which maps \mathbb{D} onto an elongated eye shaped domain which touches $\partial\mathbb{D}(0, \sqrt{e})$ at its *corners* at $\{-\sqrt{e}, \sqrt{e}\}$. The relative potential takes the extremal value c_α on the entire line segment $[-\sqrt{e}, \sqrt{e}]$. Hence this domain is degenerate in the sense of (1.5.3).

10.3. A family of Neumann ovals. We next discuss a family of holes whose forbidden regions are *Neumann ovals*. For $t > 1$, we let \mathcal{G}_t denote the disjoint union of two unit disks centered at $-t$ and t . For $0 < t < 1$ we continue this family of domains by letting \mathcal{G}_t be the unique subharmonic quadrature domain with nodes $\{-t, t\}$ and a unit mass at each one of them. More generally, the symmetric Neumann oval $\Omega_{r, t}$ with respect to $r(\delta_t + \delta_{-t})$ is the Jordan domain whose boundary is given by the equation

$$(x^2 + y^2)^2 - 2r^2(x^2 + y^2) - 2t^2(x^2 - y^2) = 0, \quad (x, y) \in \mathbb{R}^2,$$

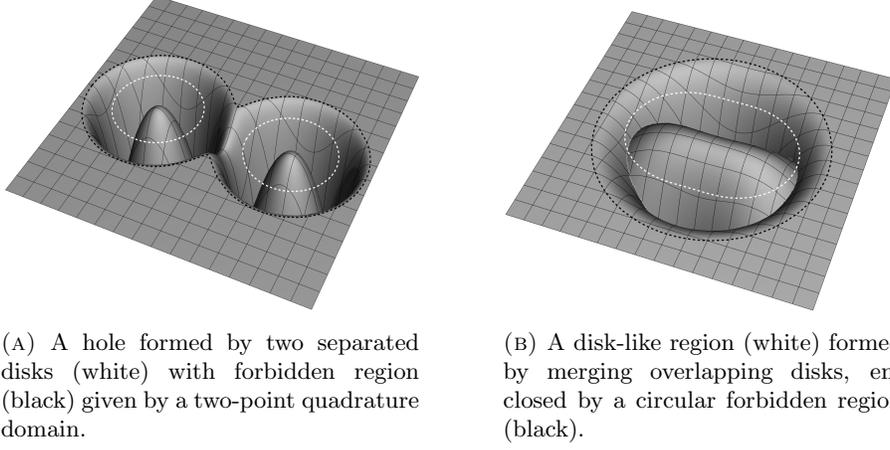


FIGURE 10.1

see for instance [65]. Denote by μ_t the minimizer of I_α over $\mathcal{M}_{\mathcal{G}_t}$. For $t > \sqrt{e}$ and α large enough (with respect to t), it is evident that each component of the hole expels a circular forbidden region of radius \sqrt{e} , concentric with the hole component. At $t = \sqrt{e}$, the forbidden regions touch, and their union equals the degenerate Neumann oval with mass $2\sqrt{e}$ and nodes at $\{-\sqrt{e}, \sqrt{e}\}$. As t decreases from this critical point, we expect a nontrivial family of solutions.

Example 10.6. There exist functions $r = r(t)$, $\lambda = \lambda(t)$ such that $\mu_t = \mu_{t,r(t),\lambda(t)}$, where

$$\mu_{t,r,\lambda} = \frac{r}{\alpha} \text{Bal}(\delta_{-\lambda} + \delta_\lambda, G_t^c) + \frac{1}{\pi\alpha} \chi_{\mathbb{D}(0,\sqrt{\alpha}) \setminus \Omega_{r,\lambda}} \, dA.$$

Moreover, the function $\lambda(t)$ is decreasing in t and there exists a number $t_0 > 0$ such that \mathcal{G}_t is disk-like for $t \leq t_0$, and $\lambda(t) = 0$ for such t .

Since the article is already too long, we will not provide a proof of this statement. A numerical verification may be done with the help of computer algebra software and Lemma 10.2, see [55]. Indeed, in view of Lemma 10.2 we need only ensure that one may choose parameters $r = r(t)$ and $\lambda = \lambda(t)$ such that the relative potential

$$R_{\mu_{t,r,\lambda}} = U^{\mu_{t,r,\lambda}} - U^{\mu_\alpha}$$

attains its maximal value at λ , where we recall that $\mu_\alpha = \frac{1}{\pi\alpha} \chi_{\mathbb{D}(0,\sqrt{\alpha})} \, dA$. For illustrations of the Neumann oval family discussed above, see Figure 10.1. The hole is in both instances the disjoint union of two disks of the same radius. In the former the forbidden regions barely touch, while in the second figure the forbidden regions have merged to form a Neumann oval.

APPENDIX A. APPROXIMATION OF MEASURES WITH FEKETE POINTS

We extend the scope of Theorems 1 and 2 by Götze and Saff from [32] so that they will apply in our setting. These proofs were obtained together with S. Ghosh.

A.1. **Definitions.** and meets the conditions of Theorem 1.6, and put

$$\Lambda = \mathbb{D}(0, \sqrt{\alpha}) \setminus \mathcal{G},$$

where $\alpha > \alpha_0$ (as defined in Proposition 4.4).

We denote by Q a Hölder continuous weight function on \mathbb{C} with Hölder exponent $\gamma' > 0$, and let $\mu_{Q,\Lambda}$ denote the unique minimizer of the functional

$$J_Q(\mu) := -\Sigma(\mu) + 2 \int Q(z) d\mu(z)$$

among all probability measures μ supported on Λ . The measure $\mu_{Q,\Lambda}$ is the *equilibrium measure* for the weight Q on the set Λ . The potential $U^{\mu_{Q,\Lambda}}$ is known to be Hölder continuous with some exponent $\gamma'' > 0$. Without loss of generality we may assume that the Hölder exponent of both Q and $U^{\mu_{Q,\Lambda}}$ is at least γ for some constant $\gamma > 0$.

It is known (e.g. [63, Theorem I.3.1]) that $\mu_{Q,\Lambda}$ is uniquely characterized by the condition

$$\begin{cases} U^{\mu_{Q,\Lambda}}(z) = Q(z) + C(Q, \Lambda) & \text{on } \text{supp}(\mu_{Q,\Lambda}) \\ U^{\mu_{Q,\Lambda}}(z) \leq Q(z) + C(Q, \Lambda) & \text{on } \mathbb{C}, \end{cases}$$

where $C(Q, \Lambda)$ is a constant.

Given N points $\mathbf{z} = (z_1, \dots, z_N) \in \mathbb{C}^N$ let

$$\mu_{\mathbf{z}} = \frac{1}{N} \sum_{j=1}^N \delta_{z_j},$$

denote their empirical probability measure. Assuming these points are distinct, their *discrete logarithmic energy* is given by

$$-\Sigma^*(\mu_{\mathbf{z}}) = \int_{\mathbb{C}^2 \setminus \{w_1=w_2\}} \log \frac{1}{|w_1 - w_2|} d\mu_{\mathbf{z}}(w_1) d\mu_{\mathbf{z}}(w_2) = -\frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} \log |z_i - z_j|.$$

We define a *Fekete configuration* of points \mathcal{F}_N with respect to the weight (or external field) Q and confined to the set Λ as a minimizer $\mathcal{F}_N = (z_1, \dots, z_N) \subset \Lambda^N$ of the discrete weighted energy functional

$$J_Q^*(\mu_{\mathbf{z}}) := -\Sigma^*(\mu_{\mathbf{z}}) + 2 \int Q(w) d\mu_{\mathbf{z}}(w) = -\frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} \log |z_i - z_j| + \frac{2}{N} \sum_{j=1}^N Q(z_j).$$

By [63, Theorem III.1.2] it is known that

$$\mathcal{F}_N \subset \Lambda^* := \{w \in \Lambda : U^{\mu_{Q,\Lambda}}(w) = Q(w) + C(Q, \Lambda)\}.$$

Abusing notation, we will variously refer to the set \mathcal{F}_N by \mathbf{z} and (z_1, \dots, z_N) .

Remark A.1. Our use of weighted Fekete points is somewhat non-standard since the external field Q and the number of points N (and clearly Λ) both depend on the parameter α . This slightly complicates the proofs of the following results.

A.2. Separation of Fekete points and approximation of energy and potential. Using the same strategy of proof as [32] we prove the following results.

Proposition A.2 (Separation of Fekete points). *Let $\Lambda, Q, \gamma, \mu_{Q,\Lambda}, \mathcal{F}_N$ be defined as above. Then,*

$$\inf_{z' \neq z'' \in \mathcal{F}_N} |z' - z''| \geq \frac{1}{2} A_1 N^{-1/\gamma},$$

where

$$A_1 = \exp(-\|U^{\mu_{Q,\Lambda}}\|_{C^{0,\gamma}}).$$

Theorem A.3 (Approximation of energy and potential). *Let $\Lambda, Q, \gamma, \mu_{Q,\Lambda}, \mathcal{F}_N$ be defined as above, and put*

$$\begin{aligned} E_1(N, \gamma, \Lambda) &= \frac{1}{N} [\|U^{\mu_{Q,\Lambda}}\|_{C^{0,\gamma}} + 2 \log 2\sqrt{\alpha} + \gamma^{-1} \log N], \\ E_2(N, \gamma, \Lambda) &= \frac{2}{N} [4\|U^{\mu_{Q,\Lambda}}\|_{C^{0,\gamma}} + \log 2\sqrt{\alpha} + (2 + 3\gamma^{-1}) \log N + D(Q, \Lambda)], \end{aligned}$$

where

$$D(Q, \Lambda) = \sup_{z \in \Lambda} |U^{\mu_{Q,\Lambda}}(z)|.$$

Then,

$$-\Sigma^*(\mu_z) \leq -\Sigma(\mu_{Q,\Lambda}) + E_1(N, \gamma, \Lambda).$$

In addition, for $N \geq 11$, it holds that for all $w \in \mathbb{C}$

$$U^{\mu_{\mathcal{F}_N}}(w) \leq U^{\mu_{Q,\Lambda}}(w) + E_2(N, \gamma, \Lambda).$$

Moreover, if $\tau \geq 1 + 1/\gamma$ is fixed, then, whenever $d(w, \mathcal{F}_N) \geq N^{-\tau}$ we have

$$U^{\mu_{\mathcal{F}_N}}(w) \geq U^{\mu_{Q,\Lambda}}(w) - E_2(N, \gamma, \Lambda) - \frac{(\tau - 1/\gamma) \log N}{N}.$$

Proof of Proposition A.2. We begin by forming the interpolating polynomials

$$P_j(z) = \prod_{k \neq j} \frac{z - z_k}{z_j - z_k}, \quad z \in \mathbb{C}, \quad 1 \leq j \leq N.$$

That $\mathcal{F}_N = (z_1, \dots, z_N)$ is energy-minimal is equivalent to

$$(A.2.1) \quad -\sum_{j \neq k} \log |z_j - z_k| + 2N \sum_{j=1}^N Q(z_j) \leq -\sum_{j \neq k} \log |w_j - w_k| + 2N \sum_{j=1}^N Q(w_j),$$

whenever $(w_1, \dots, w_N) \in \Lambda^N$. In particular, for any $z \in \Lambda$ we may apply (A.2.1) with

$$\begin{cases} w_k = z_k, & k \neq j \\ w_j = w \end{cases}$$

for any fixed j to obtain

$$(A.2.2) \quad \sum_{k:k \neq j} \log |w - z_k| - \sum_{k:k \neq j} \log |z_j - z_k| \leq N(Q(w) - Q(z_j)), \quad 1 \leq j \leq N.$$

Let $w \in \text{supp}(\mu_{Q,\Lambda})$. Since $Q(w) = U^{\mu_{Q,\Lambda}}(w) - C(Q, \Lambda)$, we find by (A.2.2) that

$$\sum_{k:k \neq j} \log |w - z_k| - \sum_{k:k \neq j} \log |z_j - z_k| \leq N(U^{\mu_{Q,\Lambda}}(w) - U^{\mu_{Q,\Lambda}}(z_j)), \quad 1 \leq j \leq N.$$

If we denote by ν the sub-probability measure $\nu = \frac{1}{N} \sum_{k \neq j} \delta_{z_j}$ we may divide the previous equation by N and rewrite it as

$$U^\nu(w) - U^\nu(z_j) \leq U^{\mu_{Q,\Lambda}}(w) - U^{\mu_{Q,\Lambda}}(z_j), \quad w \in \text{supp}(\mu_{Q,\Lambda}).$$

The principle of domination [63, Theorem II.3.2] implies that the above inequality holds for all $w \in \mathbb{C}$, and consequently we obtain

$$|P_j(w)| \leq \exp\left(N(U^{\mu_{Q,\Lambda}}(w) - U^{\mu_{Q,\Lambda}}(z_j))\right).$$

This says that $|P_j(z)| \leq A_1^{-1} = \exp(\|U^{\mu_{Q,\Lambda}}\|_{C^{0,\gamma}})$ whenever $|z - z_j| \leq N^{-1/\gamma}$.

Now, if $|z - z_j| \leq \frac{1}{2}N^{-1/\gamma}$ we have by the standard Cauchy estimate that

$$|P_j'(z)| \leq \int_{\mathbb{T}(z, \frac{1}{2}N^{-1/\gamma})} \frac{|P_j(\zeta)|}{|\zeta - z|^2} d\sigma(\zeta) \leq 2A_1^{-1}N^{1/\gamma}.$$

If for two distinct points z_j and z_k in \mathcal{F}_N we were to have $|z_j - z_k| \leq \frac{1}{2}N^{-1/\gamma}$, then

$$1 = |P_j(z_j) - P_j(z_k)| \leq \int_{[z_j, z_k]} |P_j'(z)| dz \leq 2A_1^{-1}N^{1/\gamma}|z_j - z_k|.$$

But this shows that $|z_j - z_k| \geq \frac{1}{2}A_1N^{-1/\gamma}$, which completes the first step. \square

Proof of Theorem A.3. The proof is split into three steps. In the first step we obtain the upper bound for the discrete logarithmic energy. In the second step, we obtain a lower bound for the difference of the potentials. We conclude in the third step with a corresponding upper bound.

STEP 1. Dividing (A.2.2) by N^2 and summing over $1 \leq j \leq N$, we find that for $w \in \Lambda$

$$-\Sigma^*(\mu_{\mathbf{z}}) + \int Q d\mu_{\mathbf{z}} \leq -(1 - \frac{1}{N})U^{\mu_{\mathbf{z}}}(w) + Q(w) \leq -U^{\mu_{\mathbf{z}}}(w) + Q(w) + \frac{\log 2\sqrt{\alpha}}{N},$$

which gives after rearranging terms

$$(A.2.3) \quad -U^{\mu_{\mathbf{z}}}(w) + U^{\mu_{Q,\Lambda}}(w) \geq -\Sigma^*(\mu_{\mathbf{z}}) + \int U^{\mu_{Q,\Lambda}} d\mu_{\mathbf{z}} - \frac{\log 2\sqrt{\alpha}}{N}, \quad w \in \Lambda.$$

We would like to replace the discrete energy with a continuous one. We introduce the regularized measure $\mu_{\mathbf{z}}^r = \mu_{\mathbf{z}} * \sigma_r$, where σ is the normalized arc length measure on $\mathbb{T}(0, r)$, and $r > 0$ is a small parameter which remains to be chosen. A simple estimate shows that

$$(A.2.4) \quad \left| \int U^{\mu_{Q,\Lambda}} d(\mu_{\mathbf{z}}^r - \mu_{\mathbf{z}}) \right| \leq \frac{1}{N} \sum_{j=1}^N \int_{\mathbb{T}(z_j, r)} |U^{\mu_{Q,\Lambda}}(t) - U^{\mu_{Q,\Lambda}}(z_j)| d\sigma_r(t) \\ \leq \|U^{\mu_{Q,\Lambda}}\|_{C^{0,\gamma}} r^\gamma =: C_U^\gamma r^\gamma.$$

We next observe that by subharmonicity of the logarithm, it holds that

$$-\Sigma^*(\mu_{\mathbf{z}}) \geq -\Sigma(\mu_{\mathbf{z}}^r) - \frac{|\log r|}{N}.$$

For a signed measure ν of vanishing total mass, we have (e.g. [63, Lemma I.1.8])

$$-\Sigma(\nu) = \int |\nabla U^\nu|^2 dA(z),$$

where the right-hand side is evidently positive. Applying this to $\nu = \mu_{\mathbf{z}}^r - \mu_{Q,\Lambda}$ we find that

$$-\Sigma(\mu_{\mathbf{z}}^r) \geq -2 \int U^{\mu_{Q,\Lambda}} d\mu_{\mathbf{z}}^r + \Sigma(\mu_{Q,\Lambda}).$$

The first term on the right may be bounded, using (A.2.4):

$$-\int U^{\mu_{Q,\Lambda}} d\mu_{\mathbf{z}}^r \geq -\int U^{\mu_{Q,\Lambda}} d\mu_{\mathbf{z}} - C_U^\gamma r^\gamma.$$

Adding the above bounds yields

$$(A.2.5) \quad -\Sigma^*(\mu_{\mathbf{z}}) + \int U^{\mu_{Q,\Lambda}} d\mu_{\mathbf{z}}^r \geq -\int U^{\mu_{Q,\Lambda}} d\mu_{\mathbf{z}} + \Sigma(\mu_{Q,\Lambda}) - \frac{|\log r|}{N} - C_U^\gamma r^\gamma.$$

Returning to (A.2.3), invoking (A.2.5) and choosing $r = N^{-1/\gamma}$, it follows that

$$(A.2.6) \quad U^{\mu_{Q,\Lambda}}(w) - U^{\mu_{\mathbf{z}}}(w) \geq \epsilon_N - \gamma^{-1} \frac{\log N}{N} - \frac{C_U^\gamma}{N} - \frac{\log 2\sqrt{\alpha}}{N},$$

where

$$\epsilon_N = \Sigma(\mu_{Q,\Lambda}) - \int U^{\mu_{Q,\Lambda}} d\mu_{\mathbf{z}} = \int (U^{\mu_{Q,\Lambda}} - U^{\mu_{\mathbf{z}}}) d\mu_{Q,\Lambda}.$$

The inequality (A.2.6) initially holds only on Λ , but extends to an inequality on \mathbb{C} by the principle of domination, since $\Sigma(\mu_{Q,\Lambda})$ is finite.

Now let ν_1 be the (unweighted) equilibrium measure of the set Λ , and note that U^{ν_1} is equal to some constant value quasi-everywhere in Λ , and that it exceeds this value everywhere in \mathbb{C} . If we integrate $U^{\mu_{Q,\Lambda}}(w) - U^{\mu_{\mathbf{z}}}(w)$ against ν_1 , then by (A.2.6) we have

$$(A.2.7) \quad \epsilon_N \leq \gamma^{-1} \frac{\log N}{N} + \frac{C_U^\gamma}{N} + \frac{\log 2\sqrt{\alpha}}{N}.$$

Similarly, if we integrate (A.2.3) against ν_1 , then we get

$$-\Sigma^*(\mu_{\mathbf{z}}) \leq -\int U^{\mu_{Q,\Lambda}} d\mu_{\mathbf{z}} + \frac{\log 2\sqrt{\alpha}}{N},$$

which together with the definition of ϵ_N and (A.2.7) gives the required upper bound for the energy $-\Sigma^*(\mu_{\mathbf{z}})$.

STEP 2. In this step, we obtain a lower bound for ϵ_N . We start with a preliminary bound. Define the functions

$$h_j(w) = -\frac{1}{N} \sum_{k:k \neq j} \log |w - z_k|, \quad 1 \leq j \leq N.$$

Also recall that

$$D(Q, \Lambda) = \sup_{z \in \Lambda} |Q(z) + C(Q, \Lambda)| = \sup_{z \in \Lambda} |U^{\mu_{Q,\Lambda}}(z)|.$$

For the functions h_j , we show below that

$$(A.2.8) \quad |h_j(z_j) + U^{\mu_{Q,\Lambda}}(z_j) - \epsilon_N| \leq e(N, Q, \Lambda)$$

where

$$e(N, Q, \Lambda) = 2\gamma^{-1} \frac{\log N}{N} + \frac{\log 2\sqrt{\alpha}}{N} + \frac{2C_U^\gamma}{N} + \frac{D(Q, \Lambda)}{N}.$$

For $w \in \Lambda$ and $z_j \in \mathcal{F}_N$, arguing in the same way as in (A.2.2), we have by extremality of \mathcal{F}_N that

$$h_j(z_j) + U^{\mu_{Q,\Lambda}}(z_j) \leq h_j(w) + U^{\mu_{Q,\Lambda}}(w),$$

or equivalently that

$$h_j(z_j) \leq -U^{\mu_{Q,\Lambda}}(z_j) + h_j(w) + U^{\mu_{Q,\Lambda}}(w).$$

Integrating this against $\mu_{Q,\Lambda}$ we find that

$$\begin{aligned} h_j(z_j) &\leq -U^{\mu_{Q,\Lambda}}(z_j) - \frac{1}{N} \sum_{k:k \neq j} U^{\mu_{Q,\Lambda}}(z_k) + \Sigma(\mu_{Q,\Lambda}) \\ &\leq -U^{\mu_{Q,\Lambda}}(z_j) + \epsilon_N + \frac{U^{\mu_{Q,\Lambda}}(z_j)}{N} \leq -U^{\mu_{Q,\Lambda}}(z_j) + \epsilon_N + \frac{D(Q, \Lambda)}{N}. \end{aligned}$$

This completes the upper bound of (A.2.8).

To obtain the lower bound, let w satisfy $|w - z_j| = N^{-1/\gamma}$, and note that by (A.2.6) it holds that

$$h_j(w) = -U^{\mu_{\mathbf{z}}}(w) + \frac{\log |w - z_j|}{N} \geq -U^{\mu_{Q,\Lambda}}(z) + \epsilon_N - 2\gamma^{-1} \frac{\log N}{N} - \frac{C_U^\gamma}{N} - \frac{\log 2\sqrt{\alpha}}{N}.$$

This together with the Hölder continuity of $U^{\mu_{Q,\Lambda}}$ yields (recalling the definition of C_U^γ)

$$h_j(w) \geq -U^{\mu_{Q,\Lambda}}(z_j) - \frac{C_U^\gamma}{N} + \epsilon_N - 2\gamma^{-1} \frac{\log N}{N} - \frac{C_U^\gamma}{N} - \frac{\log 2\sqrt{\alpha}}{N}.$$

Since h is superharmonic, the same bound for h_j holds at the point $w = z_j$, which completes the proof of (A.2.8).

We return to bound the quantity ϵ_N from below, which was the main purpose of the current step. The claim is that

$$(A.2.9) \quad \epsilon_N \geq -\frac{f(N, Q, \Lambda)}{N},$$

where

$$f(N, Q, \Lambda) = 4C_U^\gamma + \log 2\sqrt{\alpha} + (2 + 3\gamma^{-1}) \log N + D(Q, \Lambda).$$

To see why this holds, put $\delta_1 = \frac{1}{4}A_1N^{-1/\gamma}$, where $A_1 = \exp(-C_U^\gamma) \leq 1$ is the constant from Proposition A.3, and using this proposition observe that

$$\sup_{|w-z_j| \leq \delta_1} |\nabla h_j(w)| \leq \sup_{|w-z_j| \leq \delta_1} \frac{1}{N} \sum_{k:k \neq j} \frac{1}{|w-z_k|} \leq (\delta_1)^{-1}.$$

Put $\eta = N^{-\beta}\delta_1$, with $\beta \geq 1$. By the Hölder continuity of $U^{\mu_{Q,\Lambda}}$, and the gradient bound for h_j , it follows that for w such that $|w - z_j| = \eta$ we have

$$|U^{\mu_{Q,\Lambda}}(w) - U^{\mu_{\mathbf{z}}}(w) - \epsilon_N| \leq |U^{\mu_{Q,\Lambda}}(z_j) + h_j(z_j) - \epsilon_N| + \frac{C_U^\gamma}{N} + \eta(\delta_1)^{-1} + \frac{|\log \eta|}{N}.$$

Put $\mathcal{D}(\eta) = \mathcal{D}(\eta; \beta) = \cup_j \mathbb{D}(z_j, \eta)$. Invoking the bound (A.2.8), we find that for $w \in \partial \mathcal{D}(\eta)$ it holds that

$$\begin{aligned} (A.2.10) \quad |U^{\mu_{Q,\Lambda}}(w) - U^{\mu_{\mathbf{z}}}(w) - \epsilon_N| &\leq \frac{C_U^\gamma}{N} + \eta(\delta_1)^{-1} + \frac{|\log \eta|}{N} + e(N, Q, \Lambda) \\ &\leq \frac{2\beta \log N}{N} + \frac{1}{N} [3\gamma^{-1} \log N + 4C_U^\gamma + D(Q, \Lambda) + \log 2\sqrt{\alpha}] =: \frac{2\beta \log N}{N} + e'(N, Q, \Lambda), \end{aligned}$$

assuming e.g. that $N \geq 11$.

Let ν_2 be the (unweighted) equilibrium measure of the set $\mathcal{D}(\eta)$. Note that U^{ν_2} has a constant value (everywhere) inside $\mathcal{D}(\eta)$, and that it exceeds this value outside it. Integrating $U^{\mu_{Q,\Lambda}}(w) - U^{\mu_{\mathbf{z}}}(w)$ against ν_2 , we see that ϵ_N is bounded from below by

$$\int [U^{\mu_{Q,\Lambda}}(w) - U^{\mu_{\mathbf{z}}}(w)] d\nu_2(w) - \frac{2\beta \log N}{N} - e'(N, Q, \Lambda) \geq -\frac{2\beta \log N}{N} - e'(N, Q, \Lambda).$$

Taking $\beta = 1$ we obtain (A.2.9), and this completes Step 2.

Now that the bound (A.2.9) has been established, the lower bound for the difference $U^{\mu_{Q,\Lambda}}(w) - U^{\mu_{\mathbf{z}}}(w)$ follows from (A.2.6), and reads

$$U^{\mu_{Q,\Lambda}}(w) - U^{\mu_{\mathbf{z}}}(w) \geq \epsilon_N - \gamma^{-1} \frac{\log N}{N} - \frac{C_U^\gamma}{N} - \frac{\log 2\sqrt{\alpha}}{N} \geq -\frac{2}{N} f(N, Q, \Lambda).$$

STEP 3. Now we take $\beta = \tau - 1/\gamma \geq 1$. We complete the proof of Theorem A.3 by obtaining an upper bound for $U^{\mu_{Q,\Lambda}}(w) - U^{\mu_{\mathbf{z}}}(w)$. This is readily derived from (A.2.10). This initially holds only on $\partial\mathcal{D}(\eta)$. However, on $\mathcal{D}(\eta)^c$, the function $U^{\mu_{Q,\Lambda}} - U^{\mu_{\mathbf{z}}}$ is subharmonic and vanishes at infinity. Applying the maximum principle, and using (A.2.7), we find that for all $w \in \mathbb{C}$

$$U^{\mu_{Q,\Lambda}}(w) - U^{\mu_{\mathbf{z}}}(w) \leq \epsilon_N + \frac{2\beta \log N}{N} + e'(N, Q, \Lambda) \leq \frac{2\beta \log N}{N} + \frac{2}{N} f(N, Q, \Lambda),$$

which completes the proof. \square

Acknowledgements. We would like to express our sincere gratitude to Subhro Ghosh for valuable discussions in the early stages of this project and for allowing us to reproduce the joint work on Fekete points in the appendix, to Fedor Nazarov for suggesting a version of the variational principle underlying the proof of Proposition 10.3, and to John Andersson for suggestions on how to approach matters of stability in thin obstacle problems. We have enjoyed stimulating discussions with Kari Astala, Yan Fyodorov, Håkan Hedenmalm, David Jerison, Avner Kiro, Nikolai Makarov, Sylvia Serfaty, Henrik Shahgholian, Mikhail Sodin, Pierpaolo Vivo, Oren Yakir and Ofer Zeitouni, for which we wish to thank them. We thank Mikhail Sodin and Oren Yakir for helpful suggestions that led to an improved presentation of the results.

The research of the first-named author was funded by ISF Grant 1903/18 and ERC Advanced Grant 692616. The second author was funded by KAW Foundation Grant 2017.0389, with travel support from ERC Advanced Grant 692616.

REFERENCES

- [1] Adams, R. A., Fournier, J. F. *Sobolev spaces*. Second edition. Pure and Applied Mathematics, **140**. Elsevier/Academic Press, Amsterdam, 2003.
- [2] Adhikari, K, Reddy, N. K., *Hole probabilities for finite and infinite Ginibre ensembles*, Int. Math. Res. Not. IMRN **21** (2017), 6694–6730.
- [3] Aharonov, D., Shapiro, H., *Domains in which analytic functions satisfy quadrature identities*, J. Analyse Math. **30** (1976), 39–73.
- [4] Akemann, G., Strahov, E. *Hole Probabilities and Overcrowding Estimates for Products of Complex Gaussian Matrices* J. Stat. Phys. **151** (2013) 987–1003.
- [5] Antezana, J., Buckley, J., Marzo, J., Olsen, J.-F. *Gap probabilities for the cardinal sine*. J. Math. Anal. Appl. **396** (2012), no. 2, 466–472.

- [6] Armstrong, S. N., Serfaty, S., Zeitouni, O., *Remarks on a constrained optimization problem for the Ginibre ensemble*, Potential Anal. **41** (2014), 945–958.
- [7] Aurzada, F., Simon, T. *Persistence probabilities and exponents*. Lévy matters. V, 183–224, Lecture Notes in Math., 2149, Lévy Matters, Springer, Cham, 2015.
- [8] Basu, R., Dembo, A., Feldheim, N., Zeitouni, O. *Exponential Concentration for Zeroes of Stationary Gaussian Processes*. Int. Math. Res. Not. IMRN, doi:10.1093/imrn/rny277 (2018).
- [9] Blank, I. LeCrone, J. *Perturbed Obstacle Problems in Lipschitz Domains: Linear Stability and Non-degeneracy in Measure*, Rocky Mountain J. Math. **49** no. 5, (2019) 1407–1418.
- [10] Brezis, H., *Functional Analysis, Sobolev spaces and Partial Differential Equations*, Universitext. Springer, New York, 2011.
- [11] Brezis, H., Ponce, A. C. *Kato’s inequality when Δu is a measure*, C. R. Acad. Sci. Paris, Ser. I **338** (2004) 599–604.
- [12] Buckley, J., Nishry, A., Peled, R., Sodin, M. *Hole probability for zeroes of Gaussian Taylor series with finite radii of convergence*. Probab. Theory Related Fields **171** (2018), no. 1-2, 377–430.
- [13] Butez, R., *Large deviations for the empirical measure of random polynomials: revisit of the Zeitouni-Zelditch theorem*, Electron. J. Probab. **21** no. 73 (2016), 37 pp.
- [14] Bogomolny, E., Bohigas, O., and Leboeuf, P. *Distribution of roots of random polynomials*. Phys. Rev. Lett. **68** (1992), no. 18, 2726–2729.
- [15] Caffarelli, L. A., Rivière, N. M., *Smoothness and analyticity of free boundaries in variational inequalities*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **3** (1976), no. 2, 289–310.
- [16] Caffarelli, L. A., Kinderlehrer, D., *Potential methods in variational inequalities*, J. Analyse Math. **37** (1980), 285–295.
- [17] Chafaï, D., Ferré, G., Stoltz, G. *Coulomb gases under constraint: some theoretical and numerical results*. arXiv:1907.05803 (2019).
- [18] Diaconis, P, and Fulton, W. *A growth model, a game, an algebra, Lagrange inversion, and characteristic classes*. Rend. Sem. Mat. Univ. Pol. Torino **49** (1991), 95–119.
- [19] Ding, Z., *A proof of the trace theorem of Sobolev spaces on Lipschitz domains*. Proc. Amer. Math. Soc. **124** (1996), no. 2, 591–600.
- [20] Feldheim, N., Feldheim, O., Jaye, B., Nazarov, F., Nitzan, S. *On the Probability That a Stationary Gaussian Process With Spectral Gap Remains Non-negative on a Long Interval*. Int. Math. Res. Not. IMRN, rny248, (2018).
- [21] Forrester, P. J., Honner, G. *Exact statistical properties of the zeros of complex random polynomials* J. Phys. A **32** (1999), no. 16, 2961–2981.
- [22] Garnett, J. B., Marshall, D. E., *Harmonic measure*, New Math Monographs. Cambridge University Press, Cambridge, 2005.
- [23] Ghosh, S., Nishry, A., *Point processes, hole events, and large deviations: random complex zeros and Coulomb gases*, Constr. Approx. **48** (2018), no. 1, 101–136.
- [24] Ghosh, S., Nishry, A., *Gaussian complex zeros on the hole event: the emergence of a forbidden region*, Comm. Pure. Appl. Math. **72** (2019), no. 1, 3–62.
- [25] Ghosh, S., Peres, Y. *Rigidity and tolerance in point processes: Gaussian zeros and Ginibre eigenvalues*. Duke Math. J. **166** (2017) 1789–1858.
- [26] Goluzin, G. M. *Geometric theory of functions of a complex variable*. Translations of Mathematical Monographs, Vol. **26**. American Mathematical Society, Providence, R.I. 1969.
- [27] Grishin, A., *Sets of regular growth of entire functions, I*, Teor. Funktsii, Funktsional. Anal. i Prilozhen (Kharkov) **40** (1983), 36–47.
- [28] Gustafsson, B., Putinar, M. *Selected topics on quadrature domains*. Physica D. **235** (2007), 90–100.
- [29] Gustafsson, B., Shahgholian, H., *Existence and geometric properties of a free boundary problem in potential theory*. J. Reine Angew. Math. **473** (1996) 137–179.
- [30] Gustafsson, B., Shapiro, H. S. *What is a quadrature domain?*. Quadrature domains and their applications, 1–25, Oper. Theory Adv. Appl., **156**, Birkhäuser, Basel, 2005.
- [31] Gustafsson, B., Teodorescu, R., Vasil’ev, A.. *Classical and Stochastic Laplacian Growth*. Advances in Mathematical Fluid Mechanics, Birkhäuser, 2014.
- [32] Götz, M., Saff, E.B. *Potential and Discrepancy Estimates for Weighted Extremal Points*, Constr. Approx. **16** (2000), 541–558.
- [33] Hannay, J. H. *The chaotic analytic function*. J. Phys. A: Math. Gen. **31**, no. 49 (1998) L755–L761.

- [34] Hedenmalm, H., Makarov, N. *Coulomb gas ensembles and Laplacian growth*. Proc. Lond. Math. Soc. **106** no. 3, (2013), 859–907.
- [35] Heinonen, J., Kilpeläinen, T., and Martio, O., *Nonlinear Potential Theory of Degenerate Elliptic Equations*. Unabridged republication of the 1993 original. Dover Publications, Inc., Mineola, NY, 2006.
- [36] Helffer, B. *Semi-classical analysis for the Schrödinger operator and applications*. Lecture Notes in Mathematics, vol. 1336. Springer, Berlin, 1988.
- [37] Hough, J.B., *Large deviations for the zero set of an analytic function with diffusing coefficients*, arXiv:0510237 (2005).
- [38] Hough, J. B., Krishnapur, M., Peres, Y., and Virág, B. *Zeros of gaussian analytic functions and determinantal point processes*, vol. 51, American Mathematical Soc., 2009.
- [39] Hörmander, L. *Notions of Convexity*, Progress in Mathematics, **127**. Birkhäuser Boston, Inc., Boston, MA, 1994.
- [40] Jancovici B., Lebowitz J. L., Manificat G. *Large charge fluctuations in classical Coulomb systems* J. Stat. Phys. vol. **72**, (1993) 773–787.
- [41] Kallenberg, O., *Random measures, theory and applications*. Probability Theory and Stochastic Modelling, **77**. Springer, Cham, 2017.
- [42] Kinderlehrer, D., Stampaccia, G., *An Introduction to Variational Inequalities and Their Applications*. Reprint of the 1980 original. Classics in Applied Mathematics, **31**. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000.
- [43] Kostlan, E. *On the distribution of roots of random polynomials*. From Topology to Computation: Proceedings of the Smalefest (Berkeley, CA, 1990), 419–431, Springer, New York, 1993.
- [44] Krapivsky, P. L., Meerson, B., Sasorov, P. V. *Void formation in diffusive lattice gases*, J. Stat. Mech. Theory Exp. (2012), no. 12, P12014, 30 pp.
- [45] Krishnapur, M. *Overcrowding estimates for zeroes of planar and hyperbolic Gaussian analytic functions*. J. Stat. Phys. **124** (2006), no. 6, 1399–1423.
- [46] Lee, S.-Y., Lyubich, M., Makarov, N. G., Mukherjee, S. *Dynamics of Schwarz reflections: the mating phenomena*. arXiv:1811.04979 (2018).
- [47] Lee, S.-Y., Lyubich, M., Makarov, N. G., Mukherjee, S. *Schwarz reflections and the Tricorn*. arXiv:1812.01573 (2018).
- [48] Levine, L., Peres, Y. *Scaling limits for internal aggregation models with multiple sources*. J. Anal. Math. **111**, (2010) 151–219.
- [49] Majumdar, S. N., Nadal, C., Scardicchio, A., Vivo, P. *How many eigenvalues of a Gaussian random matrix are positive?*, Phys. Rev. E **83** no. 4 (2011) 041105.
- [50] Mineev-Weinstein, M., Wiegmann, P. B., Zabrodin, A. *Integrable structure of interface dynamics*. Phys. Rev. Letters **84** no. 22 (2000), 5106–5109.
- [51] Nazarov, F., Sodin, M. *Fluctuations in random complex zeroes: asymptotic normality revisited*. Int. Math. Res. Not. IMRN 2011, no. 24 (2011) 5720–5759.
- [52] Nazarov, F., Sodin, M. *Random complex zeroes and random nodal lines*. Proceedings of the International Congress of Mathematicians. Vol. III, Hindustan Book Agency, New Delhi, 2010, pp. 1450–1484.
- [53] Nazarov, F., Sodin, M., Volberg, A. *Transportation to random zeros by the gradient flow*. GAFA Geom. Funct. Anal. **17** (2007) 887–935.
- [54] Nazarov, F., Sodin, M., Volberg, A. *The Jancovici-Lebowitz-Manificat law for large fluctuations of random complex zeroes*. Comm. Math. Phys. **284** (2008), no. 3, 833–865.
- [55] Nishry, A., Wennman, A., *The Neumann oval as forbidden region*, Mathematica notebook. Available at url: <https://people.kth.se/~aronw/Publications/NeumannHoleDiskLike.nb>.
- [56] Nishry, A., *Asymptotics of the hole probability for zeros of random entire functions*. Int. Math. Res. Not. IMRN 2010 no. 15 (2010) 2925–2946.
- [57] Peres, Y, Virág, B. *Zeros of the iid Gaussian power series: a conformally invariant determinantal process*. Acta Math. **194** no. 1 (2005) 1–35.
- [58] Pólya, G., Szegő, G. *Isoperimetric Inequalities in Mathematical Physics*. Annals of Mathematics Studies, no. **27**, Princeton University Press, Princeton, N. J., 1951.
- [59] Pommerenke, C., *Boundary behaviour of conformal maps*, Springer Verlag, Berlin, 1992.
- [60] Ross, J., Singer, M. *Asymptotics of partial density functions for divisors*. J Geom. Anal. **27** (2017) 1803–1854.

- [61] Sakai, M., *Regularity of free boundaries in two dimensions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **20** (1993), no. 3, 323–339.
- [62] Sakai, M., *Quadrature domains*, Lecture Notes in Mathematics, **934**. Springer Verlag, Berlin-New York, 1982.
- [63] Saff, E. B., Totik, V. *Logarithmic Potentials with External Fields*. Grundlehren der mathematischen Wissenschaften, Springer Verlag, Berlin, (1997).
- [64] Serfaty, S., Serra, J. *Quantitative stability of the free boundary in the obstacle problem*. Analysis & PDE **11** (2018) 1803–1839.
- [65] Shapiro, H. S., *The Schwarz function and its generalization to higher dimensions*. University of Arkansas Lecture Notes in the Mathematical Sciences, **9**. Wiley Interscience. John Wiley & Sons, Inc., New York, 1992.
- [66] Shiffman, B., Zelditch, S. *Random polynomials with prescribed Newton polytope*. J. Amer. Math. Soc. **17** no 1 (2003) 49–108.
- [67] Shirai, T. *Ginibre-type point processes and their asymptotic behavior*. J. Math. Soc. Japan Vol. **67**, No. 2 (2015) pp. 763–787.
- [68] Shub, M. and Smale, S. *Complexity of Bézout’s theorem. I. Geometric aspects*. J. Amer. Math. Soc. **6** (1993), no. 2, 459–501.
- [69] Sodin, M. *Hahn decomposition for the Riesz charge of δ -subharmonic functions*, Math. Scand. **83** (1998) 277–282.
- [70] Sodin, M., Tsirelson, B., *Random complex zeroes. I. Asymptotic normality*. Israel J. Math. **144** (2004), 125–149.
- [71] Sodin, M., Tsirelson, B., *Random complex zeroes, III. Decay of the hole probability*, Israel Journal of Mathematics **147.1** (2005): 371–379.
- [72] Verchota, G. *Layer potentials and regularity for the Dirichlet problem for Laplace’s equation in Lipschitz domains*. J. Funct. Anal. **59** (1984) 572–611.
- [73] Zeitouni, O. and Zelditch, S. *Large deviations of empirical measures of zeros of random polynomials*, International Mathematics Research Notices **20** (2010), 3935–3992.
- [74] Zelditch, S. *Large deviations of empirical measures of zeros on Riemann surfaces*. Int. Math. Res. Not. IMRN 2013, no. 3, (2013) 592–664.
- [75] Zhang, W., and Bao, J. *Regularity of very weak solutions for nonhomogeneous elliptic equation*. Commun. Contemp. Math. **15** (2013), no. 4, 1350012, 19 pp.
- [76] Warschawski, S. E. *On the Distortion in Conformal Mapping of Variable Domains*. Trans. Amer. Math. Soc. **82** (1956), No. 2, 300–322.