OFF-SPECTRAL ANALYSIS OF BERGMAN KERNELS

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Abstract. The asymptotic analysis of Bergman kernels with respect to exponentially varying measures near emergent interfaces has attracted recent attention. Such interfaces typically occur when the associated limiting Bergman density function vanishes on a portion of the plane, *the off-spectral region*. This type of behaviour is observed when the metric is negatively curved somewhere, or when we study partial Bergman kernels in the context of positively curved metrics. In this work, we cover these two situations in a unified way, for exponentially varying planar measures on the complex plane. We obtain uniform asymptotic expansions of *root functions*, which are essentially normalized partial Bergman kernels at an off-spectral point, valid in the entire off-spectral component and protruding into the spectrum as well, which allows us to show error function transition behaviour of the original kernel along the interface. In contrast, previous work on asymptotic expansions of Bergman kernels is typically local, and valid only in the bulk region of the spectrum.

1. Introduction

1.1. **Bergman kernels and emergent interfaces.** This article is a companion to our recent work [12] on the structure of planar orthogonal polynomials. We will make frequent use of methods developed there, and recommend that the reader keep that article available for ease of reference.

Recently, the asymptotic behaviour of Bergman kernels near emergent interfaces has attracted considerable attention. Such interfaces may occur, e.g., when the metric develops a singularity, or if it is negatively curved somewhere. Similar interfaces also appear in the context of partial Bergman kernels, that is, the kernel for the orthogonal projections of a weighted L^2-space onto a proper subspace of the full Bergman space. In this work, we intend to cover both situations in a unified manner, in the setting of exponentially varying weights on the complex plane. The partial Bergman kernels we consider here fall into one of two categories: polynomial kernels, and kernels for spaces defined by a prescribed degree of vanishing at a given point. In both cases, the transition of the Bergman kernel is described in terms of the error function, which is suggestive of an interpretation of this transition as the result of diffusion. We also obtain an extension of the orthogonal foliation flow developed in [12] in the context of the orthogonal polynomials, which immediately yields a perturbation result for orthogonal polynomials allowing for a twist of the base metric.

The key to our obtaining these results is the expansion of the Bergman kernel in the orthogonal basis provided by the normalized reproducing kernels $\mathrm{k}_n(z, w_0)$ with prescribed degree n of vanishing at an *off-spectral* point w_0, i.e., a point in the forbidden (asymptotically massless) region on one side of the interface, which is

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globally valid in a neighbourhood of the entire forbidden component. In particular, the analysis is not local.

We now briefly recall the objects of study. The Bergman space A^2_{mQ} is defined as the collection of all entire functions f in with finite weighted L^2-norm

$$\|f\|^2_{mQ} := \int_{\mathbb{C}} |f(z)|^2 \mathrm{e}^{-2mQ(z)} \mathrm{dA}(z) < +\infty,$$

where dA denotes the planar area element normalized so that the unit disk $\mathbb{D}$ has unit area, and where Q is a potential with certain growth and regularity properties (see Definition 1.4.1). We denote the reproducing kernel for A^2_{mQ} by K_m, and for a given point $w_0 \in \mathbb{C}$, we consider the normalized kernel

$$\mathrm{k}_{m,w_0}(z) := K_m(w_0, w_0)^{-1/2} K_m(z, w_0), \tag{1.1.1}$$

which gets norm 1 in A^2_{mQ}. There is a notion of the *spectrum* $\mathcal{S}$, also called the *spectral droplet*. This set is closed, and defined in terms of an obstacle problem. Let $\mathrm{SH}(\mathbb{C})$ denote the cone of all subharmonic functions on the plane $\mathbb{C}$, and consider the function

$$\hat{Q}(z) := \sup\big\{q(z) \,:\, q \in \mathrm{SH}(\mathbb{C}), \text{ and } q \le Q \text{ on } \mathbb{C}\big\}.$$

Given that Q is $C^{1,1}$-smooth and has some modest growth at infinity, it is known that $\hat{Q} \in C^{1,1}$ as well, and it is a matter of definition that $\hat{Q} \le Q$ pointwise (see, e.g., [9]). Here, $C^{1,1}$ denotes the standard smoothness class of differentiable functions with Lipschitz continuous first order partial derivatives. We define the *spectrum* (or the *spectral droplet*) as the contact set

$$\mathcal{S} := \big\{z \in \mathbb{C} \,:\, \hat{Q}(z) = Q(z)\big\}. \tag{1.1.2}$$

We will need these notions in the context of partial Bergman kernels as well. For a non-negative integer n and a point $w_0 \in \mathbb{C}$, we consider the subspace A^2_{mQ,n,w_0} of A^2_{mQ}, consisting of those functions that vanish to order at least n at w_0. It may happen for some n that this space is trivial, for instance when the potential Q has logarithmic growth only, because then the space A^2_{mQ} consists of polynomials of a bounded degree. We denote its reproducing kernel by K_{m,n,w_0}, and observe that $K_{m,0,w_0} = K_m$. We shall need also the *root function of order n at w_0*, denoted k_{m,n,w_0}, which is the unique solution to the optimization problem

$$\max\big\{ \operatorname{Re} f^{(n)}(w_0) : \ f \in A^2_{mQ,n,w_0}, \ \|f\|_{mQ} \le 1\big\},$$

provided the maximum is positive, in which case the optimizer has norm $\|f\|_{mQ} = 1$. In the remaining case, the maximum equals 0, and either only $f = 0$ is possible, or there are several competing optimizers, simply because we may multiply the function by unimodular constants and obtain alternative optimizers. In both remaining instances we declare that $\mathrm{k}_{m,n,w_0} = 0$. When nontrivial, the root function of order n at w_0 is connected with the reproducing kernel K_{m,n,w_0}:

$$\mathrm{k}_{m,n,w_0}(z) = \lim_{\zeta \to w_0} K_{m,n,w_0}(\zeta, \zeta)^{-1/2} K_{m,n,w_0}(z, \zeta), \tag{1.1.3}$$

where the point ζ should approach w_0 not arbitrarily but in a fashion such that the limit exists and has positive n-th derivative at w_0. The root function k_{m,n,w_0} will play a key role in our analysis, similar to that of the orthogonal polynomials in the context of polynomial Bergman kernels. As a result of the relation (1.1.3), we may alternatively call the root function k_{m,n,w_0} a *normalized partial Bergman kernel*. In this context, we note that for $n = 0$, $\mathrm{k}_{m,0,w_0} = \mathrm{k}_{m,w_0}$ is the normalized Bergman kernel of (1.1.1). The root functions k_{m,n,w_0} all have norm equal to 1 in A^2_{mQ}, except when they are trivial and have norm 0. The spectral droplet associated to a family of partial Bergman kernels of the above type is defined in Subsection 2.1 in terms of an obstacle problem, and we briefly outline how this is done. For $0 \le \tau < +\infty$, let $\mathrm{SH}_{\tau,w_0}(\mathbb{C})$ denote the convex set

$$\mathrm{SH}_{\tau,w_0}(\mathbb{C}) = \big\{q \in \mathrm{SH}\,(\mathbb{C}) : q(z) \le \tau \log|z - w_0| + \mathrm{O}(1) \ \text{ as } \ z \to w_0\big\},$$

so that for $\tau = 0$ we recover $\mathrm{SH}(\mathbb{C})$. We consider the corresponding obstacle problem

$$\hat{Q}_{\tau,w_0}(z) = \sup\big\{q(z) \ : \ q \in \mathrm{SH}_{\tau,w_0}(\mathbb{C}),\, q \le Q \text{ on } \mathbb{C}\big\}, \tag{1.1.4}$$

and observe that $\tau \mapsto \hat{Q}_{\tau,w_0}$ is monotonically decreasing pointwise. We define a family of spectral droplets as the coincidence sets

$$\mathcal{S}_{\tau,w_0} = \big\{z \in \mathbb{C} : Q(z) = \hat{Q}_{\tau,w_0}(z)\big\}. \tag{1.1.5}$$

Due to the monotonicity, the droplets $\mathcal{S}_{\tau,w_0}$ get smaller as τ increases, starting from $\mathcal{S}_{0,w_0} = \mathcal{S}$ for $\tau = 0$. The *partial Bergman density*

$$\rho_{m,n,w_0}(z) := m^{-1} K_{m,n,w_0}(z,z)\,\mathrm{e}^{-2mQ(z)}, \qquad z \in \mathbb{C},$$

may be viewed as the normalized local dimension of the space $A^2_{mQ,n,w_0}(\mathbb{C})$, and, in addition, it has the interpretation as the intensity of a corresponding (possibly infinite) Coulomb gas. In the case $n = 0$ we omit the word "partial" and speak of the *Bergman density*. It is known that in the limit as $m, n \to +\infty$ with $n = m\tau$,

$$\rho_{m,n,w_0}(z) \to 2\Delta Q(z)\, 1_{\mathcal{S}_{\tau,w_0}}(z),$$

in the sense of convergence of distributions. In particular, $\Delta Q \ge 0$ holds a.e. on $\mathcal{S}$. Here, we write Δ for differential operator $\partial\bar{\partial}$, which is one quarter of the usual Laplacian. The above convergence reinforces our understanding of the droplets $\mathcal{S}_{\tau,w_0}$ as spectra, in the sense that the Coulomb gas may be thought to model eigenvalues (at least in the finite-dimensional case). The *bulk* of the spectral droplet $\mathcal{S}_{\tau,w_0}$ is the set

$$\{z \in \mathrm{int}(\mathcal{S}_{\tau,w_0}) : \Delta Q(z) > 0\},$$

where "int" stands for the operation of taking the interior.

Our motivation for the above setup originates with the theory of random matrices, specifically the *random normal matrix ensembles*. We should mention that an analogous situation occurs in the study of complex manifolds. The Bergman kernel then appears in the study of spaces of L^2-integrable global holomorphic sections of L^m, where L^m is a high tensor power of a holomorphic line bundle L over the manifold, endowed with an hermitian fiber metric h. If $\{e_i\}_i$ is a local basis for the fibers of L, then each section s may be written locally in the form $s_i\, e_i$ where the s_i are locally defined holomorphic functions. The pointwise norm of a section s may be then be written as $|s|_h^2 = |s|^2 \mathrm{e}^{-m\phi}$, for some smooth real-valued function ϕ Along with the base metric, this defines an L^2 space which shares many characteristics with the spaces considered here.

The asymptotic behaviour of Bergman kernels has been the subject of intense investigation. However, the understanding has largely been limited to the analysis of the kernel inside the bulk of the spectrum, in which case the kernel enjoys a full *local* asymptotic expansion. The pioneering work on Bergman kernel asymptotics begins with the efforts by Hörmander [13] and Fefferman [7]. Developing further

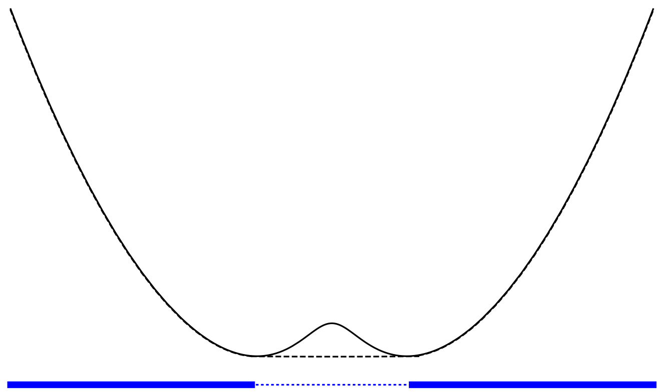

FIGURE 1.1. Illustration of the spectral droplet corresponding to the potential $Q(z) = |z|^2 - \log(a + |z|^2)$, with $a = 0.04$. The spectrum is illustrated with a thick line, and appears as the contact set between Q (solid) and the solution $\hat{Q}$ to the obstacle function (dashed).

the microlocal approach of Hörmander, Boutet de Monvel and Sjöstrand [5] obtain an expansion of the Bergman kernel near the diagonal and near the boundary. Later, in the context of Kähler geometry, the influential *peak section method* was introduced by Tian [18]. His results were refined further by Catlin and Zelditch [6, 19], while the connection with microlocal analysis was greatly simplified in the more recent work by Berman, Berndtsson, and Sjöstrand [4]. A key element of all these methods is that the kernel is determined by the local geometry around the given point. This feature is absent when we consider the kernel near an off-spectral or a boundary point.

In recent work [12], we analyze the boundary behaviour of polynomial Bergman kernels, for which the corresponding spectral droplet is compact, connected, and has a smooth Jordan curve as boundary. The analysis takes the path via a full asymptotic expansion of the orthogonal polynomials, valid off a sequence of increasing compacts which eventually fill the droplet. By expanding the polynomial kernel in the orthonormal basis provided by the orthogonal polynomials, the error function asymptotics emerges at the spectral boundary.

The appearance of an interface for partial Bergman kernels in higher dimensional settings and in the context of complex manifolds has been observed more than once, notably in the work by Shiffman and Zelditch [17] and by Pokorny and Singer [14]. That the error function governs the transition behaviour across the interface was observed later but only in a simplified geometric context. For instance, in [15], Ross and Singer investigate the partial Bergman kernels associated to spaces of holomorphic sections vanishing along a divisor, and obtain error function transition behaviour under the assumption that the set-up is invariant under a holomorphic S^1-action. More recently, Zelditch and Zhou [20] obtain the same transition for partial Bergman kernels defined in terms of Toeplitz quantization of a general smooth Hamiltonian.

1.2. **Off-spectral asymptotics of normalized Bergman kernels.** The main contribution of the present work, put in the planar context, is a non-local asymptotic expansion of normalized (partial) Bergman kernels rooted at an off-spectral point. For ease of exposition, we begin with a version that requires as few prerequisites as possible for the formulation. We denote by $Q(z)$ an *admissible potential*, by which we mean the following:

(i) $Q : \mathbb{C} \to \mathbb{R}$ is C^2-smooth, and has sufficient growth at infinity:

$$\tau_Q := \liminf_{|z|\to+\infty} \frac{Q(z)}{\log|z|} > 0.$$

(ii) Q is real-analytically smooth and strictly subharmonic in a neighbourhood of $\partial\mathcal{S}$,

(iii) there exists a bounded component Ω of the complement $\mathcal{S}^c = \mathbb{C} \setminus \mathcal{S}$ which is simply connected, and has real-analytically smooth Jordan curve boundary.

C^2-smooth potential which grows at least like $\tau_Q \log|z|$ as $|z| \to +\infty$, for some fixed positive real τ_Q. In addition, we assume that Q is strictly subharmonic (that is, $\Delta Q > 0$) and real-analytic in a neighbourhood of the spectral droplet $\mathcal{S}$ defined in (1.1.2). We consider the case when there exists a non-trivial off-spectral component Ω which is bounded and simply connected, with real-analytic boundary, and pick a "root point" $w_0 \in \Omega$. To be precise, by an *off-spectral component* we mean a connectivity component of the complement $\mathcal{S}^c$. This situation occurs, e.g., if the potential is strictly superharmonic in a portion of the plane, as is illustrated in Figure 1.1. In terms of the metric, this means that there is a region where the curvature is negative. To formulate our first main result, we need the function $\mathcal{Q}_{w_0}$, which is bounded and holomorphic in Ω and whose real part equals Q along the boundary $\partial\Omega$. To fix the imaginary part, we require that $\mathcal{Q}_{w_0}(w_0) \in \mathbb{R}$. In addition, we need the conformal mapping φ_{w_0} which takes Ω onto the unit disk $\mathbb{D}$ with $\varphi_{w_0}(w_0) = 0$ and $\varphi'_{w_0}(w_0) > 0$. Since the boundary $\partial\Omega$ is assumed to be a real-analytically smooth Jordan curve, the function $\mathcal{Q}_{w_0}$ extends holomorphically across $\partial\Omega$, and the conformal mapping φ_{w_0} extends conformally across $\partial\Omega$.

Theorem 1.2.1. *Assuming that Q is an admissible potential, we have the following. Given a positive integer κ and a positive real A, there exist a neighbourhood $\Omega^{\langle\kappa\rangle}$ of the closure of Ω and bounded holomorphic functions $\mathcal{B}_{j,w_0}$ on $\Omega^{\langle\kappa\rangle}$ for $j = 0, \ldots, \kappa$, as well as domains $\Omega_m = \Omega_{m,\kappa,A}$ with $\Omega \subset \Omega_m \subset \Omega^{\langle\kappa\rangle}$ which meet*

$$\mathrm{dist}_{\mathbb{C}}(\partial\Omega_m, \partial\Omega) \geq A\, m^{-\frac{1}{2}} (\log m)^{\frac{1}{2}},$$

such that the normalized Bergman kernel at the point w_0 enjoys the asymptotic expansion

$$\mathrm{k}_m(z, w_0) = m^{\frac{1}{4}} (\varphi'_{w_0}(z))^{\frac{1}{2}} \mathrm{e}^{m\mathcal{Q}_{w_0}(z)} \Bigg\{ \sum_{j=0}^{\kappa} m^{-j} \mathcal{B}_{j,w_0}(z) + \mathrm{O}\big(m^{-\kappa-1}\big) \Bigg\},$$

as $m \to +\infty$, where the error term is uniform on Ω_m. Here, the main term $\mathcal{B}_{0,w_0}$ is obtained as the unique zero-free holomorphic function on Ω which is smooth up to the boundary, positive at w_0, with prescribed modulus on the boundary

$$|\mathcal{B}_{0,w_0}(z)| = \pi^{-\frac{1}{4}} [\Delta Q(z)]^{\frac{1}{4}}, \qquad z \in \partial\Omega.$$

Remark 1.2.2. Using an approach based on Laplace's method, the functions $\mathcal{B}_{j,w_0}$ may be derived algorithmically, for $j = 1, 2, 3, \ldots$, see Theorem 3.2.2. The details of the algorithm are analogous with the case of the orthogonal polynomials presented in [12].

1.3. **Expansion of partial Bergman kernels in terms of root functions.** For a (τ, w_0)-admissible potential, the partial Bergman kernel K_{m,n,w_0} with the root point w_0 is well-defined and nontrivial. In analogy with Taylor's formula, it enjoys an expansion in terms of the root functions k_{m,n',w_0} for $n' \geq n$.

Theorem 1.3.1. *Under the above assumption of (τ, w_0)-admissibility of Q, we have that*

$$K_{m,n,w_0}(z,w) = \sum_{n'=n}^{+\infty} \mathrm{k}_{m,n',w_0}(z)\overline{\mathrm{k}_{m,n',w_0}(w)}, \qquad (z,w) \in \mathbb{C} \times \mathbb{C}.$$

Proof. For $n', n'' \geq n$ with $n'' < n'$, the functions k_{m,n',w_0} and k_{m,n'',w_0} are orthogonal in A^2_{mQ}. If one of them is trivial, orthogonality is immediate, while if both are nontrivial, we argue as follows. Let $\zeta', \zeta'' \in \mathbb{C}$ be close to w_0, and calculate that

$$\begin{aligned} &K_{m,n',w_0}(\zeta',\zeta')^{-\frac{1}{2}} K_{m,n'',w_0}(\zeta'',\zeta'')^{-\frac{1}{2}} \big\langle K_{m,n',w_0}(\cdot,\zeta'), K_{m,n'',w_0}(\cdot,\zeta'')\big\rangle_{mQ} \\ &= K_{m,n',w_0}(\zeta',\zeta')^{-\frac{1}{2}} K_{m,n'',w_0}(\zeta'',\zeta'')^{-\frac{1}{2}} K_{m,n',w_0}(\zeta'',\zeta') = \mathrm{O}(|\zeta'' - w_0|^{n'-n''}), \end{aligned}$$

which tends to 0 as $\zeta'' \to w_0$. The claimed orthogonality follows. Moreover, since the root functions k_{m,n',w_0} have unit norm when nontrivial, the expression

$$\sum_{n'=n}^{+\infty} \mathrm{k}_{m,n',w_0}(z)\overline{\mathrm{k}_{m,n',w_0}(w)}$$

equals the reproducing kernel function for the Hilbert space with the norm of A^2_{mQ} spanned by the vectors k_{m,n',w_0} with $n' \geq n$. It remains to check that this is the whole partial Bergman space A^2_{mQ,n,w_0}. To this end, let $f \in A^2_{mQ,n,w_0}$ be orthogonal to all the the vectors k_{m,n',w_0} with $n' \geq n$. By the definition of the space A^2_{mQ,n,w_0}, this means that $f(z) = \mathrm{O}(|z - w_0|^n)$ near w_0. If f is notrivial, there exists an integer $N \geq n$ such that $f(z) = c\,(z - w_0)^N + \mathrm{O}(|z - w_0|^{N+1})$ near w_0, where $c \neq 0$ is complex. At the same time, the existance of such nontrivial f entails that the corresponding root functions k_{m,N,w_0} is nontrivial as well, and that $K_{m,N,w_0}(\zeta,\zeta) \asymp |\zeta - w_0|^{2N}$ for ζ near w_0. On the other hand, the orthogonality between f and k_{m,N,w_0} gives us that

$$\begin{aligned} 0 = \langle f, \mathrm{k}_{m,N,w_0}\rangle_{mQ} = \lim_{\zeta \to w_0} K_{m,N,w_0}(\zeta,\zeta)^{-\frac{1}{2}} \big\langle f, K_{m,N,w_0}(\cdot,\zeta)\big\rangle_{mQ} \\ = \lim_{\zeta \to w_0} K_{m,N,w_0}(\zeta,\zeta)^{-\frac{1}{2}} f(\zeta) \end{aligned}$$

where we approach w_0 only in an appropriate direction so that the limit exists. But this contradicts the given asymptotic behaviour of $f(\zeta)$ near w_0, since $c \neq 0$ tells us that any limit of the right-hand side would be nonzero. □

1.4. **Off-spectral asymptotics of partial Bergman kernels and interface transition.** Given a point $w_0 \in \mathbb{C}$ we recall the partial Bergman spaces A^2_{mQ,n,w_0}, and the associated spectral droplets $\mathcal{S}_{\tau,w_0}$ (see (1.1.5)), where we keep $n = \tau m$. Before we proceed with the formulation of the second result, let us fix some terminology.

FIGURE 1.2. Illustration of a compact spectral droplet (shaded) with two simply connected holes. In this case there are three off-spectral components: the two holes as well as the unbounded component.

Definition 1.4.1. A real-valued potential Q is said to be (τ, w_0)-*admissible* if the following conditions hold:

(i) $Q : \mathbb{C} \to \mathbb{R}$ is C^2-smooth and has sufficient growth at infinity:
$$\tau_Q := \liminf_{|z|\to+\infty} \frac{Q(z)}{\log|z|} > 0.$$
(ii) Q is real-analytically smooth and strictly subharmonic in a neighbourhood of $\partial\mathcal{S}_{\tau,w_0}$.
(iii) The point w_0 is an off-spectral point, i.e., $w_0 \notin \mathcal{S}_{\tau,w_0}$, and the component Ω_{τ,w_0} of the complement $\mathcal{S}^c_{\tau,w_0}$ containing the point w_0 is bounded and simply connected, with real-analytically smooth Jordan curve boundary.

If for an interval $I \subset [0, +\infty[$, the potential Q is (τ, w_0)-admissible for each $\tau \in I$ and $\{\Omega_{\tau,w_0}\}_{\tau\in I}$ is a smooth flow of domains, then Q is said to be (I, w_0)-admissible.

Generally speaking, off-spectral components may be unbounded. It is for reasons of simplicity that we focus on bounded off-spectral components in the above definition. We will assume in the sequel that Q is (I, w_0)-admissible for some non-trivial compact interval $I = I_0$. For an illustration of the situation, see Figure 1.2.

We let φ_{τ,w_0} denote the surjective Riemann mapping

$$\varphi_{\tau,w_0} : \Omega_{\tau,w_0} \to \mathbb{D}, \quad \varphi_{\tau,w_0}(0) = 0, \quad \varphi'_{\tau,w_0}(0) > 0, \tag{1.4.1}$$

which by our smoothness assumption on the boundary $\partial\Omega_{\tau,w_0}$ extends conformally across $\partial\Omega_{\tau,w_0}$. We denote by $\mathcal{Q}_{\tau,w_0}$ the bounded holomorphic function in Ω_{τ,w_0} whose real part equals Q on $\partial\Omega_{\tau,w_0}$ and is real-valued at w_0. It is tacitly assumed to extend holomorphically across the boundary $\partial\Omega_{\tau,w_0}$. We now turn to our second main result.

Theorem 1.4.2. *Assume that the potential Q is (I_0, w_0)-admissible, where the interval I_0 is compact. Given a positive integer κ and a positive real A, there exists* *a neighborhood* $\Omega^{\langle\kappa\rangle}_{\tau,w_0}$ *of the closure of* Ω_{τ,w_0} *and bounded holomorphic functions* $\mathcal{B}_{j,\tau,w_0}$ *on* $\Omega^{\langle\kappa\rangle}_{\tau,w_0}$*, as well as domains* $\Omega_{\tau,w_0,m}=\Omega_{\tau,w_0,m,\kappa,A}$ *with* $\Omega_{\tau,w_0}\subset\Omega_{\tau,w_0,m}\subset\Omega^{\langle\kappa\rangle}_{\tau,w_0}$ *which meet*

$$\mathrm{dist}_{\mathbb{C}}(\Omega^c_{\tau,w_0,m},\Omega_{\tau,w_0})\geq Am^{-\frac12}(\log m)^{\frac12},$$

such that the root function of order n *at* w_0 *enjoys the expansion*

$$\mathrm{k}_{m,n,w_0}(z)=m^{\frac14}(\varphi'_{\tau,w_0}(z))^{\frac12}[\varphi_{\tau,w_0}(z)]^n\mathrm{e}^{m\mathcal{Q}_{\tau,w_0}(z)}\Bigg\{\sum_{j=0}^{\kappa}m^{-j}\mathcal{B}_{j,\tau,w_0}(z)+\mathrm{O}(m^{-\kappa-1})\Bigg\}$$

on $\Omega_{\tau,m}$ *as* $n=\tau m\to+\infty$ *while* $\tau\in I_0$*, where the error term is uniform. Here, the main term* $\mathcal{B}_{0,\tau,w_0}$ *is zero-free and smooth up to the boundary on* Ω_{τ,w_0}*, positive at* w_0*, with prescribed modulus*

$$|\mathcal{B}_{0,\tau,w_0}(\zeta)|=\pi^{-\frac14}[\Delta Q(\zeta)]^{\frac14},\qquad \zeta\in\partial\Omega_\tau.$$

Remark 1.4.3. As in the case of the normalized Bergman kernels, the expressions $\mathcal{B}_{j,\tau,w_0}$ may be obtained algorithmically, for $j=1,2,3,\ldots$ (see Theorem 3.2.2).

As a consequence of Theorem 1.4.2, we obtain the transition behaviour of the (partial) Bergman densities at emergent interfaces. To explain how this works, we fix a bounded simply connected off-spectral component Ω with real-analytic Jordan curve boundary, associated to either a sequence of partial Bergman kernels K_{m,n,w_0} of the space A^2_{mQ,n,w_0} with the ratio $\tau=\frac{n}{m}$ fixed, or, alternatively, a sequence K_m of full Bergman kernels. We think of the latter as the parameter choice $\tau=0$. We assume that the potential Q is real-analytically smooth and strictly subharmonic near $\partial\Omega$. Let $z_0\in\partial\Omega$, and denote by $\nu\in\mathbb{T}$ the inward unit normal to $\partial\Omega$ at z_0. We define the rescaled density $\varrho_m=\varrho_{m,\tau,w_0,z_0}$ by

$$\varrho_m(\xi)=\frac{1}{2m\Delta Q(z_m(\xi))}K_{m,\tau m,w_0}\big(z_m(\xi),z_m(\xi)\big)\mathrm{e}^{-2mQ(z_m(\xi))},\tag{1.4.2}$$

where the rescaled variable is defined implicitly by

$$z_m(\xi)=z_0+\nu\frac{\xi}{\sqrt{2m\Delta Q(z_0)}}.$$

Corollary 1.4.4. *The rescaled density* ϱ_m *in* (1.4.2) *has the limit*

$$\lim_{m\to+\infty}\varrho_m(\xi)=\mathrm{erf}\,(2\,\mathrm{Re}\,\xi),$$

where the convergence is uniform on compact subsets.

1.5. **Comments on the exposition.** In Section 2, the main results, Theorems 1.2.1 and 1.4.2 as well as Corollary 1.4.4, are obtained. The analysis follows closely that of our recent work on the orthogonal polynomials. In particular, the asymptotic expansion is a consequence of Lemma 4.1.2 in [12]. In Section 3, we obtain a more general version of this lemma, which allows for a change of the base metric. This has several applications, including a stability result for the root functions and the orthogonal polynomials under a $\frac1m$-perturbation of the potential Q (Theorems 3.2.1 and 3.4.2).

2. Off-spectral expansions of normalized kernels

2.1. **A family of obstacle problems and evolution of the spectrum.** The spectral droplets (1.1.2) and the partial analogues (1.1.5) were defined earlier. From that point of view, the spectral droplet $\mathcal{S}$ is the instance $\tau = 0$ of the partial spectral droplets $\mathcal{S}_{\tau,w_0}$. We should like to point out here that the partial spectral droplet $\mathcal{S}_{\tau,w_0}$ emerges as the full spectrum under a perturbation of the potential Q. To see this, we consider the perturbed potential

$$\tilde{Q}(z) = \tilde{Q}_{\tau,w_0}(z) := Q(z) - \tau \log|z - w_0|,$$

and observe that the coincidence set $\tilde{\mathcal{S}}$ for $\tilde{Q}$ is the same as the partial spectral droplet $\mathcal{S}_{\tau,w_0}$.

The following proposition summarizes some basic properties of the function $\hat{Q}_{\tau,w_0}$ given by (1.1.4). We refer to [9] for the necessary details.

Proposition 2.1.1. *Assume that $Q \in C^2(\mathbb{C})$ is real-valued with the logarithmic growth of condition (i) of Definition 1.4.1. Then for each τ with $0 \le \tau < \tau_Q$ and for each point $w_0 \in \mathbb{C}$, the function $\hat{Q}_{\tau,w_0}$ is a subharmonic function in the plane $\mathbb{C}$ which is $C^{1,1}$-smooth off w_0, and harmonic on $\mathbb{C} \setminus (\mathcal{S}_{\tau,w_0} \cup \{w_0\})$. Near the point w_0 we have*

$$\hat{Q}_{\tau,w_0}(z) = \tau \log|z - w_0| + \mathrm{O}(1).$$

The evolution of the free boundaries $\partial\mathcal{S}_{\tau,w_0}$, which is of fundamental importance for our understanding of the properties of the normalized reproducing kernels, is summarized in the following.

Proposition 2.1.2. *The continuous chain of off-spectral components Ω_{τ,w_0} for $\tau \in I_0$ deform according to weighted Laplacian growth with weight $2\Delta Q$, that is, for $\tau, \tau' \in I_0$ with $\tau' < \tau$, and for any bounded harmonic function h on Ω_{τ,w_0}, we have that*

$$\int_{\Omega_{\tau,w_0}\setminus\Omega_{\tau',w_0}} h\, 2\Delta Q \mathrm{dA} = (\tau - \tau')h(w_0).$$

Fix a point $\zeta \in \partial\Omega_{\tau,w_0}$, and denote for real ε by ζ_{ε,w_0} the point closest to ζ in the intersection

$$(\zeta + \nu_\tau(\zeta)\mathbb{R}_+) \cap \partial\Omega_{\tau-\varepsilon,w_0},$$

where $\nu_\tau(\zeta) \in \mathbb{T}$ points in the inward normal direction at ζ with respect to Ω_{τ,w_0}. Then we have that

$$\zeta_\varepsilon = \zeta + \varepsilon\, \nu_\tau(\zeta) \frac{|\varphi'_{\tau,w_0}(\zeta)|}{4\Delta Q(\zeta)} + \mathrm{O}(\varepsilon^2), \quad \varepsilon \to 0,$$

and the outer normal $\mathrm{n}_{\tau-\varepsilon,w_0}(\zeta_\varepsilon)$ satisfies

$$\mathrm{n}_{\tau-\varepsilon,w_0}(\zeta_\varepsilon) = \mathrm{n}_{\tau,w_0}(\zeta) + \mathrm{O}(\varepsilon).$$

Proof. That the domains deform according to Hele-Shaw flow is a direct consequence of the relation of Ω_{τ,w_0} to the obstacle problem. To see how it follows, assume that h is harmonic on Ω_{τ,w_0} and C^2-smooth up to the boundary, and apply

Green's formula to obtain

$$\int_{\Omega_{\tau,w_0}} h(z)\Delta Q(z)\mathrm{dA}(z) = \frac{1}{4\pi}\int_{\partial\Omega_{\tau,w_0}} \Big(h(z)\partial_{\mathrm{n}}Q(z) - Q(z)\partial_{\mathrm{n}}h(z)\Big)|\mathrm{d}z|$$
$$= \frac{1}{4\pi}\int_{\partial\Omega_{\tau,w_0}} \Big(h(z)\partial_{\mathrm{n}}\hat{Q}_{\tau,w_0}(z) - \hat{Q}_{\tau,w_0}(z)\partial_{\mathrm{n}}h(z)\Big)|\mathrm{d}z|$$
$$= \int_{\Omega_{\tau,w_0}} h(z)\Delta\hat{Q}_{\tau,w_0}\mathrm{dA}(z),$$

where the latter integral is understood in the sense of distribution theory. As $\hat{Q}_{\tau,w_0}$ is a harmonic perturbation of τ times the Green function for Ω_{τ,w_0}, the result follows by writing the integral $\int_{\Omega_{\tau,w_0}\setminus\Omega_{\tau',w_0}} h\Delta Q\mathrm{dA}$ as the difference of two integrals of the above form, and by approximation of bounded harmonic functions by harmonic functions C^2-smooth up to the boundary.

The second part follows along the lines of [12, Lemma 2.3.1]. □

We turn next to an off-spectral growth bound for weighted holomorphic functions.

Proposition 2.1.3. *Assume that Q is admissible and denote by $\mathcal{K}_{\tau,w_0}$ a closed subset of the interior of $\mathcal{S}_{\tau,w_0}$. Then there exist constants c_0 and C_0 such that for any $f \in A^2_{mQ,n,w_0}(\mathbb{C})$ it holds that*

$$|f(z)| \le C_0\, m^{\frac{1}{2}}\mathrm{e}^{m\hat{Q}_{\tau,w_0}}\big\|1_{\mathcal{K}^{\mathrm{c}}_{\tau,w_0}} f\big\|_{mQ}, \qquad \mathrm{dist}(z,\mathcal{K}_{\tau,w_0}) \ge c_0 m^{-\frac{1}{2}}.$$

In case $\mathcal{K}_{\tau,w_0} = \emptyset$, the estimate holds globally.

Proof. This follows immediately by an application of the maximum principle, together with the result of Lemma 2.2.1 in [12], originating from [1]. □

2.2. **Some auxilliary functions.** There are a number of functions related to the potential Q that will be useful in the sequel. We denote by $\mathcal{Q}_{\tau,w_0}$ the bounded holomorphic function on Ω_{τ,w_0} whose real part on the boundary curve $\partial\Omega_{\tau,w_0}$ equals Q, uniquely determined by the requirement that $\operatorname{Im}\mathcal{Q}_{\tau,w_0}(w_0) = 0$. We also need the function $\breve{Q}_{\tau,w_0}$, which denotes the harmonic extension of $\hat{Q}_{\tau,w_0}$ across the boundary of the off-spectral component Ω_{τ,w_0}. These two functions are connected via

$$\breve{Q}_{\tau,w_0}(z) = \tau\log|\varphi_{\tau,w_0}(z)| + \operatorname{Re}\mathcal{Q}_{\tau,w_0}(z). \tag{2.2.1}$$

Since we work with (τ, w_0)-admissible potentials Q, the off-spectral component Ω_{τ,w_0} is a bounded simply connected domain with real-analytically smooth Jordan curve boundary. Without loss of generality, we may hence assume that $\mathcal{Q}_{\tau,w_0}$, $\breve{Q}_{\tau,w_0}$ as well as the conformal mapping φ_{τ,w_0} extend to a common domain Ω_0, containing $\bar{\Omega}_{\tau,w_0}$. By possibly shrinking the interval I_0, we may moreover choose the set Ω_0 to be independent of the parameter $\tau \in I_0$.

2.3. **Canonical positioning.** An elementary but important observation for the main result of [12] is that we may ignore a compact subset of the interior of the compact spectral droplet $\mathcal{S}_\tau$ associated to polynomial Bergman kernels when we study the asymptotic expansions of the orthogonal polynomials $P_{m,n}$ (with $\tau = \frac{n}{m}$). Indeed, only the behaviour in a small neighbourhood of the complement $\mathcal{S}^{\mathrm{c}}_\tau$ is of interest. The physical intuition behind this is the interpretation of the probability density $|P_{m,n}|^2\mathrm{e}^{-2mQ}$ as the net effect of adding one more particle to the system, and since the positions in the interior of the droplet are already occupied we would expect the net effect to occur near the boundary. The fact that we may restrict our attention to a simply connected proper subset of the Riemann sphere $\hat{\mathbb{C}}$ breaks up the rigidity and allows us to apply a conformal mapping to place ourselves in an appropriate model situation. At a technical level, this is accomplished by applying the *canonical positioning operator* $\mathbf{\Lambda}_{m,n}$ defined for functions f defined in a neighbourhood of the closure of the exterior disk

$$\mathbb{D}_{\mathrm{e}} := \{z \in \hat{\mathbb{C}} :\, 1 < |z| \le +\infty\}$$

in terms of the surjective conformal mapping $\phi_\tau : \mathcal{S}_\tau^c \to \mathbb{D}_{\mathrm{e}}$, which preserves the point at infinity and has $\phi_\tau'(\infty) > 0$, by

$$\mathbf{\Lambda}_{m,n}[f](z) = m^{\frac{1}{4}} \phi_\tau'(z)[\phi_\tau(z)]^n \mathrm{e}^{m\mathcal{Q}_\tau(z)} f \circ \phi_\tau(z).$$

We recall that for a compact set $\mathcal{K}$, a function F is called a *quasipolynomial of degree n on $\mathcal{K}^{\mathrm{c}}$* if F is holomorphic on $\mathcal{K}^{\mathrm{c}}$ and has the growth

$$|F(z)| \asymp |z|^n, \qquad |z| \to +\infty.$$

After canonical positioning, approximately orthogonal quasipolynomials with respect to the weight e^{-2mQ} are transformed into bounded holomorphic functions $f_{m,n}$ in a neighbourhood

$$\mathbb{D}_{\mathrm{e}}(0, \rho_0) = \{z \in \hat{\mathbb{C}} :\, \rho_0 < |z| \le +\infty\}$$

of the closed exterior disk, approximately orthogonal to all holomorphic functions on $\mathbb{D}_{\mathrm{e}}(0, \rho_0)$ vanishing at infinity with respect to an induced measure $\mathrm{e}^{-2mR_\tau}\mathrm{dA}$. The function R_τ is given by

$$R_\tau(z) = (Q - \breve{Q}_\tau) \circ \phi_\tau^{-1},$$

and is in particular quadratically flat along the unit circle $\mathbb{T}$, in the sense that both R_τ and its gradient ∇R_τ vanish on $\mathbb{T}$. The transformed problem is more tractable, and we perform the essential analysis after applying canonical positioning.

An important aspect of the analysis in [12] is the connection with a reproducing property. Indeed, after canonical positioning, the function $u_{m,n} = \mathbf{\Lambda}_{m,n}^{-1}[P_{m,n}]$ has the property of being essentially reproducing for the point at infinity. To see this, we observe that $u_{m,n}$ must be approximately orthogonal to the holomorphic functions that vanish at infinity, or, more accurately,

$$\int_{\mathbb{C}} \chi_1^2\, u_{m,n} \bar{q}\, \mathrm{e}^{-2mR_\tau} \mathrm{dA} = \frac{\bar{q}(\infty)}{u_{m,n}(\infty)} + \mathrm{O}\big(m^{-\kappa-1} \|\chi_1 q\|_{mR_\tau}\big),$$

for all bounded holomorphic q and any finite accuracy κ. Here, χ_1 is an appropriate cut-off function, which assumes the value 1 in a neighborhood of the closed exterior disk $\bar{\mathbb{D}}_{\mathrm{e}}$, and vanishes in the disk $\mathbb{D}(0, \rho_0)$ centered at 0 with radius ρ_0.

This point of view connects the orthogonal polynomials with the root functions at the point w_0. We let the canonical positioning operator for the point w_0 with respect to the off-spectral component Ω_{τ,w_0} be given by

$$\mathbf{\Lambda}_{m,n,w_0}[v] = \varphi_{\tau,w_0}'(z)[\varphi_{\tau,w_0}]^n \mathrm{e}^{m\mathcal{Q}_{\tau,w_0}} v \circ \varphi_{\tau,w_0}, \tag{2.3.1}$$

and make the corresponding ansatz $\mathrm{k}_{m,n,w_0} = \mathbf{\Lambda}_{m,n,w_0}[v_{m,n,w_0}]$. The assertion that k_{m,n,w_0} is the root function of order n at w_0 entails that v_{m,n,w_0} is approximately reproducing for the origin with respect to the corresponding weight R_{τ,w_0} given by

$$R_{\tau,w_0} := (Q - \breve{Q}_\tau) \circ \varphi_{\tau,w_0}^{-1}. \tag{2.3.2}$$

So, we should have that

$$\int_{\mathbb{C}} \chi_1^2 v_{m,n,w_0} \bar{q}\, \mathrm{e}^{-2mR_{\tau,w_0}} \mathrm{dA} = \frac{\bar{q}(0)}{v_{m,n,w_0}(0)} + \mathrm{O}\big(m^{-\kappa-1} \|\chi_1 q\|_{mR_{\tau,w_0}}\big), \tag{2.3.3}$$

where χ_1 an appropriate cut-off function, which assumes the value 1 in a neighborhood of the closed disk $\bar{\mathbb{D}}$, and vanishes in the exterior disk $\mathbb{D}_{\mathrm{e}}(0,\delta)$, for some $\delta > 1$.

This time around, we prefer to use the unit disk rather than the exterior disk as our model domain. The properties of the mapping $\mathbf{\Lambda}_{m,n,w_0}$ remain essentially the same, however, as summarized in the following proposition. For a potential V and a domain Ω, we denote by $A^2_{mV,n,w_0}(\Omega)$ the spaces of holomorphic functions on Ω which vanish to order n at $w_0 \in \Omega$, endowed with the topology of $L^2(\mathrm{e}^{-2mV}, \Omega)$. In case $n = 0$ we simply denote the space by $A^2_{mV}(\Omega)$.

Proposition 2.3.1. *Let Q be a (τ, w_0)-admissible potential, and let Ω_{τ,w_0} denote the corresponding off-spectral component. Moreover, let R_{τ,w_0} be given by* (2.3.2). *Then, for $\delta > 1$ sufficiently close to 1, the operator $\mathbf{\Lambda}_{m,n,w_0}$ defines an invertible isometry*

$$\mathbf{\Lambda}_{m,n,w_0} : A^2_{mR_{\tau,w_0}}\big(\mathbb{D}(0,\delta)\big) \to A^2_{mQ,n,w_0}(\Omega_0),$$

if Ω_0 is chosen to be of the form $\Omega_0 = \varphi^{-1}_{\tau,w_0}(\mathbb{D}(0,\delta))$. The isometry property remains valid in the context of weighted L^2-spaces as well.

Proof. The conclusion is immediate by the defining normalizations of the conformal mapping φ_{τ,w_0}. □

The following definition is an analogue of [12, Definition 3.1.2]. We denote by Ω_1 a domain containing the closure of the off-spectral component Ω_{τ,w_0}, and let $\chi_{0,\tau}$ denote a C^∞-smooth cut-off function which vanishes off Ω_1, and equals 1 in a neighbourhood of the closure of Ω_{τ,w_0}. As a remark before the formulation, we should point out that in (2.3.3), it is convenient to use one cut-off function for each of $v = v_{m,n,w_0}$ and q. However, if $\mathbf{\Lambda}_{m,n,w_0} q = f$ is an entire function, we can use the coordinates of the plane where f lives and make do with a single cut-off function.

Definition 2.3.2. Let κ be a positive integer. A sequence $\{F_{m,n,w_0}\}_{m,n}$ of holomorphic functions on Ω_0 is called a *sequence of approximate root vectors of order n at w_0 of accuracy κ* for the space A^2_{mQ,n,w_0} if the following conditions are met as $m \to +\infty$ while $\tau = \frac{n}{m} \in I_{w_0}$:

(i) For all $f \in A^2_{mQ,n+1,w_0}$, we have the approximate orthogonality

$$\int_{\mathbb{C}} \chi_{0,\tau} F_{m,n,w_0} \bar{f}\, \mathrm{e}^{-2mQ} \mathrm{dA} = \mathrm{O}(m^{-\kappa-\frac{1}{3}} \|p\|_{mQ}).$$

(ii) The approximate root functions have norm approximately equal to 1,

$$\int_{\mathbb{C}} \chi_{0,\tau}^2 |F_{m,n,w_0}(z)|^2 \mathrm{e}^{-2mQ}(z) \mathrm{dA}(z) = 1 + \mathrm{O}(m^{-\kappa-\frac{1}{3}}).$$

(iii) The functions F_{m,n,w_0} are approximately real and positive at w_0, in the sense that the leading coefficient $a_{m,n,w_0} = \lim_{z\to w_0} (z-w_0)^{-n} F_{m,n,w_0}(z)$ satisfies $\operatorname{Re} a_{m,n,w_0} > 0$ and

$$\frac{\operatorname{Im} a_{m,n,w_0}}{\operatorname{Re} a_{m,n,w_0}} = \mathrm{O}(m^{-\kappa-\frac{1}{12}}).$$

We remark that the exponents in the above error terms are chosen for reasons of convenience, related to the correction scheme of Subsection 2.5.

2.4. **The orthogonal foliation flow.** The orthogonal foliation flow $\{\gamma_{m,n,t}\}_t$ is a smooth flow of closed curves near the unit circle $\mathbb{T}$, originally formulated in [12] in the context of orthogonal polynomials. The defining property is that $P_{m,n}$ should be approximately orthogonal to the lower degree polynomials along the curves $\Gamma_{m,n,t} = \phi_\tau^{-1}(\gamma_{m,n,t})$ with respect to the induced measure $\mathrm{e}^{-2mQ}\nu_{\mathrm{n}}\mathrm{ds}$, where ν_{n} denotes the normal velocity of the flow $\{\Gamma_{m,n,t}\}_t$ and ds denotes normalized arc length measure.

We recall a slight variant of Definition 4.2.1 in [12], which fixes the class of weights considered. For the formulation, we need the notion of polarization, which applies to real-analytically smooth functions. If $R(z)$ is real-analytic, there exists a function of two complex variables, denoted by $R(z,w)$, which is holomorphic in $(z,\bar{w})$, whose diagonal restriction equals $R(z,z) = R(z)$. The function $R(z,w)$ is called *polarization* of $R(z)$. If $R(z,w)$ is such a polarization of a function $R(z)$ which is real-analytically smooth near the circle $\mathbb{T}$ and quadratically flat there, then $R(z,w)$ factors as $R(z,w) = (1-z\bar{w})^2 R_0(z,w)$, where $R_0(z,w)$ is holomorphic in $(z,\bar{w})$ in a neigborhood of the diagonal where both variables are near $\mathbb{T}$. The function $R_0(z,w)$ may be viewed as the polarization of the function R_0 given by $R_0(z) = (1-|z|^2)^{-2}R(z)$.

Definition 2.4.1. For real numbers $\delta > 1$ and $0 < \sigma$, we denote by $\mathcal{W}(\delta,\sigma)$ the class of non-negative C^2-smooth functions R on $\mathbb{D}(0,\delta)$ such that R is quadratically flat on $\mathbb{T}$ with $\mathit{\Delta} R|_{\mathbb{T}} > 0$ which satisfy

$$\inf_{z\in\mathbb{D}(0,\delta)} R_0(z) = \frac{R(z)}{(1-|z|^2)^2} = \alpha(R) > 0,$$

while on the annulus

$$\mathbb{A}(\delta^{-1},\delta) := \{z \in \mathbb{C} :\, \delta^{-1} < |z| < \delta\}$$

R is real-analytically smooth, and has a polarization $R(z,w)$ which is holomorphic in $(z,\bar{w})$ on the fattened diagonal annulus

$$\hat{\mathbb{A}}(\delta,\sigma) = \big\{(z,w) \in \mathbb{A}(\delta^{-1},\delta)\times\mathbb{A}(\delta^{-1},\delta)\ :\ |z-w| \le 2\sigma\big\},$$

which factors as $R(z,w) = (1-z\bar{w})^2 R_0(z,w)$, where $R_0(z,w)$ is holomorphic $(z,\bar{w})$ on the set $\hat{\mathbb{A}}(\delta,\sigma)$, and bounded and bounded away from zero there. We say that a subset $S \subset \mathcal{W}(\delta,\sigma)$ is a *uniform family*, provided that for each $R \in S$, the corresponding $R_0(z,w)$ is uniformly bounded on $\hat{\mathbb{A}}(\delta,\sigma)$ while the $\alpha(R)$ is uniformly bounded away from 0.

The significance of the above definition is that it helps us encode uniformity properties in the parameter τ as well as the point w_0.

We recall from Proposition 4.2.2 of [12] that if $f(z,w)$ is holomorphic in $(z,\bar{w})$ on the 2σ-fattened diagonal annulus $\hat{\mathbb{A}}(\delta,\sigma)$, then the function $f_{\mathbb{T}}(z)$, which equals the diagonal restriction $f(z,z)$ for $z\in\mathbb{T}$, extends holomorphically to $\mathbb{A}((\delta')^{-1},\delta')$, where

$$\delta' := \min\{\delta, \sqrt{1+\sigma^2}+\sigma\} > 1 \tag{2.4.1}$$

depends only on δ and σ.

Lemma 2.4.2. *Let $\mathcal{K}$ be a compact subset of each of the domains Ω_{τ,w_0}, where $\tau \in I_0$. Then there exist constants $\delta > 1$ and $\sigma > 0$ such that the collection of weights R_{τ,w_0} with $w_0 \subset \mathcal{K}$ and $\tau \in I_0$ is a uniform family in $\mathcal{W}(\delta,\sigma)$.*

This is completely analogous to the corresponding claim in of [12], which was expressed in the context of an exterior conformal mapping.

The existence of the orthogonal foliation flow around the circle $\mathbb{T}$ and the asymptotic expansion of the root functions after canonical positioning are stated in the following lemma (compare with Lemma 4.1.2 in [12]).

Lemma 2.4.3. *Fix an accuracy parameter κ and let $R \in \mathcal{W}(\delta,\sigma)$. Then, if δ' is as in (2.4.1), there exist a radius δ'' with $1 < \delta'' < \delta'$, bounded holomorphic functions h_s on $\mathbb{D}(0,\delta')$ of the form*

$$h_s = \sum_{0\le j\le\kappa} s^j B_j, \qquad z \in \mathbb{D}(0,\delta'),$$

and conformal mappings $\psi_{s,t}$ from $\mathbb{D}_{\mathrm{e}}(0,\delta'')$ into the plane given by

$$\psi_{s,t} = \psi_{0,t} + \sum_{\substack{(j,l)\in\mathcal{I}_{2\kappa+1}\\ j\ge 1}} s^j t^l \hat{\psi}_{j,l}$$

such that for s,t small enough, the domains $\psi_{s,t}(\mathbb{D})$ grow with t, while they remain contained in $\mathbb{D}(0,\delta')$. Moreover, for $\zeta \in \mathbb{T}$ we have that

$$\begin{aligned}(2.4.2)\quad |h_s\circ\psi_{s,t}(\zeta)|^2\, \mathrm{e}^{-2s^{-1}R\circ\psi_{s,t}}\, \mathrm{Re}\,\big(\bar{\zeta}\partial_t\psi_{s,t}(\zeta)\overline{\psi'_{s,t}(\zeta)}\big) \\ = \mathrm{e}^{-s^{-1}t^2}\Big\{(4\pi)^{-\frac{1}{2}} + \mathrm{O}\big(|s|^{\kappa+\frac{1}{2}} + |t|^{2\kappa+1}\big)\Big\}.\end{aligned}$$

For small positive s, when t varies in the interval $[-\beta_s,\beta_s]$ with $\beta_s := s^{1/2}\log\frac{1}{s}$, the flow of loops $\{\psi_{s,t}(\mathbb{T})\}_t$ cover a neighborhood of the circle $\mathbb{T}$ of width proportional to β_s smoothly. In addition, the first term B_0 is zero-free, positive at the origin, and has modulus $|B_0| = \pi^{-\frac{1}{4}}(\Delta R)^{\frac{1}{4}}$ on $\mathbb{T}$. The other terms B_j are all real-valued at the origin. The implied constant in (2.4.2) is uniformly bounded, provided that R is confined to a uniform family of $\mathcal{W}(\delta,\sigma)$.

2.5. **$\bar\partial$-corrections and asymptotic expansions of root functions.** In this section, we supply a proof of the main result, Theorem 1.4.2. The proof consists of two parts. First, we construct a family of approximate root function of a given order and accuracy, after which we apply Hörmander-type $\bar\partial$-estimates to correct these approximate kernels to entire functions. The precise result needed for the correction scheme runs as follows.

Proposition 2.5.1. *Let $f \in L^\infty(\mathcal{S}_{\tau,w_0})$, where $\tau = \frac{n}{m}$, and denote by $u = u_{m,n,w_0}$ the norm-minimal solution in $L^2_{m\hat{Q}_\tau}$ to the problem*

$$\bar\partial u = f,$$

in the sense of distributions, which vanishes at w_0 to order n: $|u(z)| = \mathrm{O}(|z-w_0|^n)$ around w_0. Then u meets the bound

$$\int_{\mathbb{C}} |u|^2 \mathrm{e}^{-2m\hat{Q}_{\tau,w_0}}\mathrm{dA} \le \frac{1}{2m}\int_{\mathcal{S}_{\tau,w_0}} |f|^2 \frac{\mathrm{e}^{-2mQ}}{\Delta Q}\mathrm{dA}.$$

This is an immediate consequence of Corollary 2.4.2 in [12], and essentially amounts to Hörmander's classical bound for the $\bar\partial$-equation in the given setting.

We turn to the proof of Theorem 1.4.2.

Sketch of proof of Theorem 1.4.2. As the proof is analogous to that of Theorems 1.3.3 and 1.3.4 in [12], we supply only an outline of the proof.

THE CONSTRUCTION OF APPROXIMATE ROOT FUNCTIONS. We apply Lemma 2.4.3 with $s = m^{-1}$ and $R = R_{\tau,w_0}$ to obtain a smooth flow $\gamma_{s,t} = \gamma_{m,n,t,w_0}$ of curves, as well as bounded holomorphic functions $h_s = h^{\langle\kappa\rangle}_{m,n,w_0}$ such that the flow equation (2.4.2) is met. The sequence $\{B_j\}_j$ of bounded holomorphic functions produced by the lemma actually depend (smoothly) on the parameter τ and the root point w_0, so we put $B_j = B_{j,\tau,w_0}$ and define

$$h^{\langle\kappa\rangle}_{m,n,w_0} = \sum_{j=0}^{\kappa} m^{-j} B_{j,\tau,w_0}. \tag{2.5.1}$$

If we write

$$\mathcal{B}_{j,\tau,w_0} := (\varphi'_{\tau,w_0})^{\frac12} B_{j,\tau,w_0} \circ \varphi_{\tau,w_0},$$

it follows that

$$\mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0} := m^{\frac14} \mathbf{\Lambda}_{m,n,w_0}[h^{\langle\kappa\rangle}_{m,n,w_0}]$$

has the claimed form. It remains to show that $\mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0}$ is a family of approximate root functions of order n at w_0 with the stated uniformity property, and to show that it is close to the true normalizing reproducing kernel.

We denote by $\mathcal{D}_{m,n,w_0}$ the domain covered by the foliation flow, over the parameter range $-\delta_m \le t \le \delta_m$, where $\delta_m := m^{-\frac12}\log m$. Moreover, we define the domain $\mathcal{E}_{\tau,w_0}$ as the image of $\mathbb{D}(0,\delta'')$ under φ^{-1}_{τ,w_0}, and let $\chi_0 = \chi_{0,\tau,w_0}$ denote an an appropriately chosen smooth cut-off function, which takes the value 1 on a neighborhood of $\bar\Omega_{\tau,w_0}$ and vanishes off $\mathcal{E}_{\tau,w_0}$. If we let $\chi_1 := \chi_0 \circ \varphi^{-1}_{\tau,w_0}$ denote the corresponding cut-off extended to vanish off $\mathbb{D}(0,\delta'')$, we may show that

$$\int_{\mathbb{C}\setminus\mathcal{D}_{m,n,w_0}} \chi_1^2 |h^{\langle\kappa\rangle}_{m,n,w_0}|^2 \mathrm{e}^{-2mR_{\tau,w_0}} \mathrm{dA}(z) = \mathrm{O}(m^{-\alpha_0 \log m + \frac12}) \tag{2.5.2}$$

holds for some $\alpha_0 > 0$. Let $g \in A^2_{mQ,n,w_0}$ be given, and put $q = \mathbf{\Lambda}^{-1}_{m,n,w_0}[g]$. Then, by the isometric property of $\mathbf{\Lambda}_{m,n,w_0}$ and the estimate (2.5.2), it follows that

$$\begin{aligned}\int_{\mathbb{C}} \chi_0\, \mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0}(z)\bar g(z)\, \mathrm{e}^{-2mQ} \mathrm{dA}(z) &= m^{\frac14} \int_{\mathbb{C}} \chi_1 h^{\langle\kappa\rangle}_{m,n,w_0}(z)\bar q(z)\, \mathrm{e}^{-2mR_{\tau,w_0}} \mathrm{dA}(z) \\ = m^{\frac14} \int_{\mathcal{D}_{m,n,w_0}} \chi_1 h^{\langle\kappa\rangle}_{m,n,w_0}(z)\bar q(z)\, &\mathrm{e}^{-2mR_{\tau,w_0}} \mathrm{dA}(z) + \mathrm{O}\big(m^{-\frac{\alpha_0}{2}\log m + \frac34}\|g\|_{mQ}\big).\end{aligned}$$

The function $h^{\langle\kappa\rangle}_{m,n,w_0}$ may be assumed to be zero-free in $\mathcal{D}_{m,n}$, as the main term is bounded away from 0 in modulus, and consecutive terms are much smaller. For large enough m, $\chi_1 = 1$ holds on $\mathcal{D}_{m,n}$. We now introduce the function $q_{m,n} :=$

$q/h^{\langle\kappa\rangle}_{m,n}$, and integrate along the flow:

$$
\begin{aligned}
(2.5.3)\quad & m^{\frac14}\int_{\mathcal{D}_{m,n}} q_{m,n}(z)|h^{\langle\kappa\rangle}_{m,n,w_0}(z)|^2\mathrm{e}^{-2mR_{\tau,w_0}(z)}\mathrm{dA}(z)\\
&= 2m^{\frac14}\int_{-\delta_m}^{\delta_m}\int_{\mathbb{T}} q_{m,n}\circ\psi_{m,n,t}(\zeta)\big|h^{\langle\kappa\rangle}_{m,n,w_0}\circ\psi_{m,n,t}(\zeta)\big|^2\mathrm{e}^{-2m\,R_{\tau,w_0}\circ\psi_{m,n,t}(\zeta)}\\
&\qquad\qquad\times\operatorname{Re}\big\{\bar\zeta\partial_t\psi_{m,n,t}(\zeta)\overline{\psi'_{m,n,t}(\zeta)}\big\}\mathrm{ds}(\zeta)\mathrm{d}t\\
&= 2m^{\frac14}\int_{-\delta_m}^{\delta_m}\int_{\mathbb{T}} q_{m,n}\circ\psi_{m,n,t}(\zeta)\Big\{(4\pi)^{-\frac12}\mathrm{e}^{-mt^2}+\mathrm{O}\big(m^{-\kappa-\frac13}\mathrm{e}^{-mt^2}\big)\Big\}\mathrm{ds}(\zeta)\mathrm{d}t\\
&= m^{-\frac14}\frac{q(0)}{h^{\langle\kappa\rangle}_{m,n}(0)}\big(1+\mathrm{O}(m^{-\log m})\big)\\
&\quad+\mathrm{O}\bigg(m^{-\kappa-\frac{1}{12}}\int_{-\delta_m}^{\delta_m}\int_{\mathbb{T}}|q_{m,n}\circ\psi_{m,n,t}(\zeta)|\mathrm{ds}(\zeta)\,\mathrm{e}^{-mt^2}\mathrm{d}t\bigg)\\
&= m^{-\frac14}\frac{q(0)}{h^{\langle\kappa\rangle}_{m,n}(0)}+\mathrm{O}\big(m^{-\kappa-\frac13}\|g\|_{mQ}\big).
\end{aligned}
$$

Here, the crucial observation in the last step is that for fixed t, the composition $q_{m,n}\circ\psi_{m,n,t}$ is holomorphic, so that we may apply the mean value property:

$$\int_{\mathbb{T}} q_{m,n}\circ\psi_{m,n,t}\mathrm{ds} = q_{m,n}\circ\psi_{m,n,t}(0) = q_{m,n}(0) = \frac{q(0)}{h^{\langle\kappa\rangle}_{m,n}(0)}.$$

In particular, if $q(0)=0$, that is, if g vanishes to order $n+1$ or higher at w_0, then $\chi_0\mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0}$ and g are approximately orthogonal in L^2_{mQ}. Moreover, the same calculation also shows that up to a small error, $\chi_0\mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0}$ has norm 1 in L^2_{mQ}. The estimate in the error term comparing the integral in the flow parameters with the norm of g is analogous to the calculation for the orthogonal polynomials in Subsection 4.8 of [12].

The $\bar\partial$-correction scheme. The approximate normalized reproducing kernels are not globally defined, and are consequently not elements of our Bergman spaces of entire functions. However, by applying the Hörmander-type $\bar\partial$-estimate of Proposition 2.5.1, we obtain a solution $u=u_{m,n,w_0}$ to the equation

$$\bar\partial u = \mathrm{k}^{\langle\kappa\rangle}_{m,n,q_0}\bar\partial\chi_0.$$

By the proposition, it vanishes to order n at the root point w_0, and has exponentially small norm in $L^2_{mQ}=L^2(\mathbb{C},\mathrm{e}^{-2mQ})$. The function

$$\mathrm{k}^{\star}_{m,n,w_0} := \chi_0\mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0} - u_{m,n,w_0}$$

is also an approximate normalized partial Bergman reproducing kernel of the correct accuracy, but this time it at least is an element of the right space,

$$\mathrm{k}^{\star}_{m,n,w_0}\in A^2_{mQ,n,w_0}.$$

We denote by $\mathbf{P}_{m,n+1,w_0}$ the orthogonal projection onto the subspace $A^2_{mQ,n+1,w_0}$ of functions vanishing to order at least $n+1$, and put

$$\tilde{\mathrm{k}}_{m,n,w_0} = \mathrm{k}^{\star}_{m,n,w_0} - \mathbf{P}_{m,n+1,w_0}\mathrm{k}^{\star}_{m,n,w_0}.$$

By construction, $\mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0}$ vanishes precisely to the order n at the root point w_0, and the same is true for $\mathrm{k}^{\star}_{m,n,w_0}$ as well because the perturbation is very small. It follows that $\tilde{\mathrm{k}}_{m,n,w_0}$ inherits this property, that is to say,

$$\tilde{\mathrm{k}}_{m,n,w_0}(z) = C(z-w_0)^n + \mathrm{O}(|z-w_0|^{n+1}),$$

holds near w_0 for some complex constant $C \neq 0$, which would be positive from Lemma 2.4.3 except that we correct with the small $\bar\partial$-solution u_{m,n,w_0}. This gives that $C = (1+\mathrm{O}(\mathrm{e}^{-\alpha_1 m}))\,C_1$ where $C_1 > 0$ may depend on all the parameters but $\alpha_1 > 0$ is a uniform constant. Since the function $\tilde{\mathrm{k}}_{m,n,w_0}$ is orthogonal to all functions in A^2_{mQ} that vanish to order at least $n+1$ at the root point w_0, it follows that $\tilde{\mathrm{k}}_{m,n,w_0}$ equals a scalar multiple of the true root function k_{m,n,w_0}:

$$\tilde{\mathrm{k}}_{m,n,w_0} = c\,\mathrm{k}_{m,n,w_0},$$

for some complex constant $c \neq 0$. In view of the above, we conclude that $c = (1+\mathrm{O}(\mathrm{e}^{-\alpha_1 m}))\,c_1$, where $c_1 > 0$ may depend on all the parameters. As $\tilde{\mathrm{k}}_{m,n,w_0}(w_0)$ is approximately real, it follows that $c = c'\gamma$, where c' is real and positive, while $\gamma = 1+\mathrm{O}(m^{-\kappa-\frac12})$. From the approximate orthogonality discussed earlier, and the smallness of u_{m,n,w_0}, we find that the orthogonal projection $\mathbf{P}_{m,n+1,w_0}\mathrm{k}^{\star}_{m,n,w_0}$ has small norm:

$$\|\mathbf{P}_{m,n+1,w_0}\mathrm{k}^{\star}_{m,n,w_0}\|_{mQ} = \mathrm{O}(m^{-\kappa-\frac{1}{12}}).$$

This leads to the estimate

$$\|\tilde{\mathrm{k}}_{m,n,w_0} - \chi_0\mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0}\|_{mQ} = \mathrm{O}(m^{-\kappa-\frac{1}{12}})$$

and since

$$\|\chi_0\mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0}\|_{mQ} = 1+\mathrm{O}(m^{-\kappa-\frac13})$$

as a consequence of the computation in (2.5.3), we obtain that positive constant c_1 has the asymptotics $c_1 = 1+\mathrm{O}(m^{-\kappa-\frac{1}{12}})$, which allows to say that $\tilde{\mathrm{k}}_{m,n,w_0}$ and the true root function k_{m,n,w_0}, which differ by a multiplicative constant, are very close. It now follows that

$$\|\mathrm{k}_{m,n,w_0} - \chi_0\mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0}\|_{mQ} = \mathrm{O}(m^{-\kappa-\frac{1}{12}}),$$

so that k_{m,n,w_0} has the desired asymptotic expansion in norm. In view of Proposition 2.1.3, the pointwise expansion is essentially immediate, at least in the region $\Omega_{\tau,w_0,m}$ where

$$\mathrm{dist}_{\mathbb{C}}(z,\Omega_{\tau,w_0}) \le A\,m^{-\frac12}(\log m)^{\frac12},$$

which is where the functions $\check{Q}_{\tau,w_0}$ and $\hat{Q}_{\tau,w_0}$ are comparable in the sense that

$$0 \le m(\hat{Q}_{\tau,w_0} - \check{Q}_{\tau,w_0}) \le A^2 D\log m$$

for some fixed positive constant D depending only on Q. The only remaining issue is that the error terms are slightly worse than claimed. However, by replacing κ with $\kappa+2+A^2D$ and deriving the expansion with the indicated higher accuracy, we conclude that they hold as well. This completes the outline of the proof. □

2.6. **Interface asymptotics of the Bergman density.** In this section we show how to obtain the error function transition behaviour of Bergman densities at interfaces, where the interface may occurs as a result of a region of negative curvature (understood as where $\Delta < 0$ holds in terms of the potential Q) or as a consequence of dealing with partial Bergman kernels. Here, we focus on the the partial Bergman kernel analysis. In fact, we may think of the first instance of the full Bergman kernel as a special case and maintain that it is covered by the presented material.

The following Corollary of the main theorem summarizes the asymptotics of normalized off-spectral partial Bergman kernels in a suitable form. The domains $\Omega_{\tau,w_0,m}$ are as in Theorem 1.4.2, for a given positive parameter A chosen suitably large.

Corollary 2.6.1. *Under the assumptions of Theorem 1.4.2, we have the asymptotics*

$$|\mathrm{k}_{m,n,w_0}(z)|^2 \mathrm{e}^{-2mQ(z)} \\ = \pi^{-\frac{1}{2}} m^{\frac{1}{2}} |\varphi'_{\tau,w_0}(z)|\, \mathrm{e}^{-2m(Q-\breve{Q}_{\tau,w_0})(z)} \big\{ \mathrm{e}^{2\mathrm{Re}\,\mathcal{H}_{Q,\tau,w_0}(z)} + \mathrm{O}(m^{-1}) \big\},$$

on the domain $\Omega_{\tau,w_0,m}$, as $n = \tau m \to +\infty$ while $\tau \in I_0$, where $\mathcal{H}_{Q,\tau,w_0}$ is the bounded holomorphic function on Ω_{τ,w_0} whose real part equals $\frac{1}{4}\log(2\Delta Q)$ on the boundary, and is real-valued at the root point w_0.

Proof. In view of the decomposition (2.2.1), this is just the assertion of Theorem 1.4.2 with accuracy $\kappa = 1$. □

We proceed with a sketch of the error function asymptotics at interfaces, in particular we point out why we may proceed exactly as is done in the proof of Theorem 1.4.1 of [12].

Proof sketch of Corollary 1.4.4. We expand the partial Bergman kernel K_{m,n,w_0} along the diagonal in terms of the root functions k_{m,n',w_0}, for $n' \geq n$. We keep $\tau = \frac{n}{m}$ throughout. In view of Theorem 1.3.1, we have

(2.6.1)
$$K_{m,n,w_0}(z_m(\xi), z_m(\xi))\, \mathrm{e}^{-2mQ(z_m(\xi))} = \sum_{n'=n}^{+\infty} |\mathrm{k}_{m,n',w_0}(z_m(\xi))|^2 \mathrm{e}^{-2mQ(z_m(\xi))},$$

where $z_0 \in \partial\Omega_{\tau,w_0}$ and where $z_m(\xi)$ gives the rescaled coordinate implicitly by

$$z_m(\xi) = z_0 + \nu \frac{\xi}{\sqrt{2m\Delta Q(z_0)}}.$$

The rescaled Bergman density is then obtained by

$$\rho_m(\xi) = \frac{1}{2m\Delta Q(z_0)} \sum_{n \geq \epsilon m} |\mathrm{k}_{m,n,w_0}(z_m(\xi))|^2 \mathrm{e}^{-2mQ(z_m(\xi))}.$$

In view of the assumed (I_0, w_0)-admissibility, we may apply the asymptotic expansion in the main result, specifically in the form of Corollary 2.6.1. Since Proposition 2.1.2 tells us how the smooth Jordan curves $\partial\Omega_{\tau,w_0}$ propagate, a Taylor series expansion of the function $Q - \breve{Q}_{\tau,w_0}$ allows us to write the partial Bergman density approximately as a sum of translated Gaussians

$$\rho_m(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{j \geq 0} \frac{\gamma_0}{\sqrt{m}} \mathrm{e}^{-\frac{1}{2}(2\,\mathrm{Re}\,\xi + j\frac{\gamma_0}{\sqrt{m}})^2} + \mathrm{O}\big(m^{-\frac{1}{2}}(\log m)^3\big), \tag{2.6.2}$$

where $\gamma_0 = \gamma_{z_0,w_0,Q}$ is a positive constant. As in the proof of Theorem 1.4.1 of [12], we proceed to interpret the above sum (2.6.2) as a Riemann sum for the integral formula for the error function:

$$\operatorname{erf}(2\mathrm{Re}\,\xi) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-\frac{1}{2}(2\mathrm{Re}\,(\xi)+t)^2} \mathrm{d}t.$$

This proof is complete. □

3. The foliation flow for a twisted base metric

3.1. **Twists of the base metric and inversion invariance.** It will be desirable to obtain some flexibility on the part of the weight e^{-2mQ} in the expansion of Theorem 1.4.2. In particular, in the following subsections we will discuss various situations in which one needs asymptotics for root functions and orthogonal polynomials with respect to measures

$$\mathrm{e}^{-2mQ} V \,\mathrm{dA},$$

where V is non-negative C^2-smooth function which is real-analytic and non-zero in a neighbourhood of the fixed smooth spectral interface of interest, which meet the polynomial growth bound

$$V(z) = \mathrm{O}(|z|^N), \qquad |z| \to +\infty, \tag{3.1.1}$$

for some number $N < +\infty$. In particular, this covers working with the spherical area measure $\mathrm{dA}_{\mathbb{S}}(z) := (1 + |z|^2)^{-2}\mathrm{dA}(z)$ in place of planar area measure simply by putting $V(z) = (1 + |z|^2)^{-2}$. Working with the spherical area measure has the advantage of invariance with respect to rotations and inversion. For a more general twist V, we factor $V\mathrm{dA} = V_{\mathbb{S}}\mathrm{dA}_{\mathbb{S}}$, where $V_{\mathbb{S}}(z) = (1 + |z|^2)^2 V(z)$, and see that our weighted measure is

$$\mathrm{e}^{-2mQ} V_{\mathbb{S}} \,\mathrm{dA}_{\mathbb{S}},$$

which has a more invariant appearance. If we write $\iota(z) = z^{-1}$, the spaces of polynomials of degree at most n with respect to the L^2-space with measure $\mathrm{e}^{-2mQ} V_{\mathbb{S}} \,\mathrm{dA}_{\mathbb{S}}$ becomes isometrically isomorphic to the L^2-space of rational functions on the sphere $\mathbb{S}$ with a simple pole of order at most n at the origin, with respect to the L^2-space with measure $\mathrm{e}^{-2mQ\circ\iota} V_{\mathbb{S}} \circ \iota \,\mathrm{dA}_{\mathbb{S}}$. This provides an extension of the scale of root functions to zeros of negative order (i.e. poles), and the apparent similarities between orthogonal polynomials and root functions may be viewed in this light. This analogy goes even deeper than that. Assuming that 0 is an off-spectral point for the weighted L^2-space with measure $\mathrm{e}^{-2mQ\circ\iota} V_{\mathbb{S}} \circ \iota \,\mathrm{dA}_{\mathbb{S}}$, we may multiply by a suitable power of the conformal mapping from the off-spectral region to the unit disk $\mathbb{D}$, which preserves the origin, to obtain a space of functions holomorphic in a neighborhood of the off-spectral region. Hörmander-type estimates for the $\bar\partial$-equation then permit us to correct the functions so that they are entire, with small cost in norm.

We note that twists appear naturally from working with perturbations of the potential Q. Indeed, if we consider $\tilde{Q} = Q - m^{-1}h$ for some smooth function h of modest growth, we have that

$$\mathrm{e}^{-2m\tilde{Q}}\mathrm{dA} = \mathrm{e}^{-2mQ}\mathrm{e}^{2h}\mathrm{dA},$$

which corresponds precisely to the twist weight $V = \mathrm{e}^{2h}$.

3.2. **The asymptotics of root functions and orthogonal polynomials for twisted base metrics.** Our analysis will show that the root function asymptotics of Theorem 1.4.2 holds also in the context of a twisted base metric, with only a slight change in the structure of the coefficients $\mathcal{B}_{j,\tau,w_0}$. Let $A^2_{mQ,V}$ denote the weighted Bergman space of entire functions with respect to the Hilbert space norm

$$\|f\|^2_{mQ,V} := \int_{\mathbb{C}} |f|^2 \mathrm{e}^{-2mQ} V \mathrm{dA} < +\infty.$$

The corresponding Bergman kernel is denoted by $K_{m,V}$. We also need the partial Bergman spaces A^2_{mQ,V,n,w_0}, consisting of the functions in $A^2_{mQ,V}$ that vanish at w_0 to order n or higher. These are closed subspaces of $A^2_{mQ,V}$ which get smaller as n increases: $A^2_{mQ,V,n+1,w_0} \subset A^2_{mQ,V,n,w_0}$. The successive difference spaces $A^2_{mQ,V,n,w_0} \ominus A^2_{mQ,V,n+1,w_0}$ have dimension at most 1. If the dimension equals 1, we single out an element $\mathrm{k}_{m,n,w_0,V} \in A^2_{mQ,V,n,w_0} \ominus A^2_{mQ,V,n+1,w_0}$ of norm 1, which has positive derivative of order n at w_0. In the remaining case when the dimension equals 0 we put $\mathrm{k}_{m,n,w_0,V} = 0$. We call $\mathrm{k}_{m,n,w_0,V}$ *root functions*, and observe that these are the same objects as defined previously for $V = 1$ in terms of an extremal problem.

Theorem 3.2.1. *Under the assumptions of Theorem 1.4.2 and the above-mentioned assumptions on V, with respect to the interface $\partial\Omega_{\tau,w_0}$, we have, using the notation of the same theorem, for fixed accuracy and a given positive real A, the asymptotic expansion of the root function*

$$\mathrm{k}_{m,n,w_0,V}(z) \\ = m^{\frac14}(\varphi'_{\tau,w_0}(z))^{\frac12}(\varphi_{\tau,w_0}(z))^n \mathrm{e}^{m\mathcal{Q}_{\tau,w_0}} \Big\{ \sum_{j=0}^{\kappa} m^{-j}\mathcal{B}_{j,\tau,w_0,V}(z) + \mathrm{O}\big(m^{-\kappa-1}\big) \Big\},$$

on the domain $\Omega_{\tau,w_0,m}$ which depends on A, where $\tau = \frac{n}{m}$, and the implied constant is uniform. Here, the main term $\mathcal{B}_{0,\tau,w_0}$ is zero-free and smooth up to the boundary on Ω_{τ,w_0}, positive at w_0, with prescribed modulus

$$[\mathcal{B}_{0,\tau,w_0,V}(z)| = \pi^{-\frac14}(\Delta Q(z))^{\frac14} V(z)^{-\frac12}, \qquad z \in \partial\Omega_{\tau,w_0}.$$

The proof of this theorem is analogous to that of Theorem 1.4.2, given that we have explained how to modify the orthogonal foliation flow with respect to the twist in Lemma 3.3.2. The lemma is applied with $s = m^{-1}$. We omit the necessary details.

We turn next to the computation of the coefficients $\mathcal{B}_{j,\tau,w_0,W}$ in the above expansion. We recall that R_{τ,w_0} is the potential induced by Q in the canonical positioning procedure, and we put analogously

$$W_{\tau,w_0}(z) = V \circ \varphi^{-1}_{\tau,w_0}(z), \qquad z \in \mathbb{D}(0,\delta).$$

For the formulation, we need the orthogonal projection $\mathbf{P}_{H^2_0}$ of $L^2(\mathbb{T})$ onto the Hardy space H^2_0 of functions f in the Hardy space H^2 that vanish at the origin.

Theorem 3.2.2. *In the asymptotic expansion of root functions in Theorem 3.2.1, the coefficient functions $\mathcal{B}_{j,\tau,w_0,V}$ are obtained by*

$$\mathcal{B}_{j,\tau,w_0,V} = (\varphi'_{\tau,w_0})^{\frac12} B_{j,\tau,w_0,V} \circ \varphi_{\tau,w_0}, \qquad j = 1, 2, 3, \ldots.$$

If $H_{\tau,w_0,V}$ denotes the unique bounded holomorphic function on $\mathbb{D}$, whose real part meets

$$\operatorname{Re} H_{\tau,w_0,V} = \frac{1}{4}\log(4\Delta R_{\tau,w_0}) + \frac{1}{2}\log(W_{\tau,w_0}), \qquad \text{on } \mathbb{T},$$

with $\operatorname{Im} H_{\tau,w_0,V}(0) = 0$, the functions $B_{j,\tau,w_0,V}$ may be obtained algorithmically as

$$B_{j,\tau,w_0,V} = c_j \mathrm{e}^{H_{\tau,w_0,V}} - \mathrm{e}^{H_{\tau,w_0,V}} \mathbf{P}_{H^2_0}\big[\mathrm{e}^{\bar{H}_{\tau,w_0,V}} F_j\big]$$

for some real constants $c_j = c_{j,\tau,w_0,V}$ and real-analytically smooth functions $F_j = F_{j,\tau,w_0,V}$ on the unit circle $\mathbb{T}$. Here, both the constants c_j and the functions F_j may be computed iteratively in terms of $B_{0,\tau,w_0,V}, \ldots B_{j-1,\tau,w_0,V}$.

One may further derive concrete expressions for the constants c_j and the real-analytic functions F_j in the above result, in terms of the rather complicated explicit differential operators $\mathbf{L}_k$ and $\mathbf{M}_k$ as defined in equation (1.3.4) and Lemma 3.2.1 in [12]. We should mention that the definition of the operator $\mathbf{M}_k$ contains a parameter l, which is allowed to assume only non-negative values. However, the same definition works also for $l < 0$, which is necessary for the present application. In terms of the operators $\mathbf{M}_k$ and $\mathbf{L}_k$, we have

$$F_j(\theta) = \sum_{k=1}^{j} \mathbf{M}_k[B_{j-k,\tau,w_0,V} W_{\tau,w_0}]$$

and

$$c_j = -\frac{(4\pi)^{-\frac{1}{4}}}{2} \sum_{(i,k,l)\in\mathcal{J}_j} \int_{\mathbb{T}} \frac{\mathbf{L}_k[rB_{i,\tau,w_0,V}(r\mathrm{e}^{\mathrm{i}\theta})\bar{B}_{l,\tau,w_0,V}(r\mathrm{e}^{\mathrm{i}\theta})W_{\tau,w_0}(r\mathrm{e}^{\mathrm{i}\theta})]}{(4\Delta R_{\tau,w_0}(r\mathrm{e}^{\mathrm{i}\theta}))^{\frac{1}{2}}}\Big|_{r=1} \mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}).$$

Here, the index set $\mathcal{J}_j$ is defined as

$$\mathcal{J}_j = \{(i,k,l) \in \mathbb{N}^3 : i,l < j, \;\; i+k+l = j\},$$

where we use the convention that the natural numbers $\mathbb{N}$ includes 0. This theorem is obtained in the same fashion as Theorem 1.3.7 in [12] in the context of orthogonal polynomials, and we do not write down a proof here.

3.3. **The flow modified by a twist.** We proceed first to modify the book-keeping slightly by formulating an analogue of Definition 2.4.1, which applies to weights after canonical positioning.

Definition 3.3.1. Let δ and σ be given positive numbers, with $\delta > 1$. A pair (R, W) of non-negative C^2-smooth weights defined on $\mathbb{D}(0,\delta)$ is said to belong to the class $\mathcal{W}_1(\delta,\sigma)$ if $R \in \mathcal{W}(\delta,\sigma)$ and if the weight W meets the following conditions:

- (i) W is real-analytic and zero-free in the neighbourhood $\mathbb{A}(\delta^{-1},\delta)$ of the unit circle $\mathbb{T}$,
- (ii) The polarization $W(z,w)$ of W extends to a bounded holomorphic function of $(z,\bar{w})$ on the 2σ-fattened diagonal annulus $\hat{\mathbb{A}}(\sigma,\delta)$, which is also bounded away from 0.

A collection S of pairs (R, W) is said to be a uniform family in $\mathcal{W}_1(\delta,\sigma)$ if the weights R with $(R,W) \in S$ are confined to a uniform family in $\mathcal{W}(\delta,\sigma)$, while $W(z,w)$ is uniformly bounded and bounded away from 0 in $\hat{\mathbb{A}}(\delta,\sigma)$.

In connection with this definition, we recall the number $\delta' = \delta'(\delta, \sigma)$ given by

$$\delta' = \min\left\{\delta, \sqrt{1+\sigma^2} + \sigma\right\} > 1,$$

obtained from the defining property that if $f(z,w)$ is holomorphic in $(z,\bar{w})$ on the set $\hat{\mathbb{A}}(\sigma,\delta)$, then the function $f_{\mathbb{T}}(z) = f(z, \bar{z}^{-1})$ may be continued holomorphically to the annulus $\mathbb{A}((\delta')^{-1}, \delta')$.

We proceed with the main result of this section.

Lemma 3.3.2. *Fix an accuracy parameter κ and let $(R, W) \in \mathcal{W}_1(\delta,\sigma)$. Then there exist a radius δ'' with $\delta' > \delta'' > 1$, bounded holomorphic functions h_s on $\mathbb{D}(0,\delta')$ of the form*

$$h_s = \sum_{j=0}^{\kappa} s^j B_j, \qquad z \in \mathbb{D}(0, \delta')$$

and normalized conformal mappings $\psi_{s,t}$ on $\mathbb{D}(0,\delta'')$ given by

$$\psi_{s,t} = \psi_{0,t} + \sum_{\substack{(j,l)\in\mathcal{I}_{2\kappa+1} \\ j\geq 1}} s^j t^l \hat{\psi}_{j,l}$$

such that for s, t small enough it holds that the domains $\psi_{s,t}(\mathbb{D})$ increase with t, while they remain contained in $\mathbb{D}(0,\delta')$. Moreover, for $\zeta \in \mathbb{T}$, we have

$$\begin{aligned} (3.3.1) \quad & |h_s \circ \psi_{s,t}(\zeta)|^2 \,\mathrm{e}^{-2s^{-1} R\circ\psi_{s,t}} \operatorname{Re}\left(\bar{\zeta}\partial_t\psi_{s,t}(\zeta)\overline{\psi'_{s,t}(\zeta)}\right) W \circ \psi_{s,t}(\zeta) \\ & \qquad = \mathrm{e}^{-s^{-1}t^2}\left\{(4\pi)^{-\frac{1}{2}} + \mathrm{O}\left(|s|^{\kappa+\frac{1}{2}} + |t|^{2\kappa+1}\right)\right\}. \end{aligned}$$

For small positive s, when t varies in the interval $[-\beta_s, \beta_s]$ with $\beta_s := s^{1/2}\log\frac{1}{s}$, the flow of loops $\{\psi_{s,t}(\mathbb{T})\}_t$ cover a neighborhood of the circle $\mathbb{T}$ of width proportional to β_s smoothly. In addition, the main term B_0 is zero-free, positive at the origin, and has modulus $|B_0| = \pi^{-\frac{1}{4}}(\varDelta R)^{\frac{1}{4}} W^{-\frac{1}{2}}$ on $\mathbb{T}$, and the other terms B_j are all real-valued at the origin. The implied constant in (3.3.1) *is uniformly bounded, provided that (R, W) is confined to a uniform family of $\mathcal{W}_1(\delta,\sigma)$*

In order to obtain this lemma, we need to modify the algorithm which gives the original result. We proceed to sketch the outlines of this modification. The omitted details are avaiable in [12], and we intend to guide the reader for easy reading.

We recall the following index sets from [12]. For an integer n, we introduce

$$\mathcal{I}_n = \left\{(j,l) \in \mathbb{N}^2 \,:\, 2j + l \leq n\right\}. \tag{3.3.2}$$

We endow the set $\mathcal{I}_n$ with the ordering $\prec$ induced by the lexicographic ordering, so we agree that $(j,l) \prec (a,b)$ if $j < a$ or if $j = a$ and $l < b$.

Proof of Lemma 3.3.2. The conformal mappings $\psi_{s,t}$ are assumed to have the form

$$\psi_{s,t} = \psi_{0,t} + \sum_{\substack{(j,l)\in\mathcal{I}_{2\kappa+1} \\ j\geq 1}} s^j t^l \hat{\psi}_{j,l}$$

for some bounded holomorphic coefficients $\hat{\psi}_{j,l}$ and a conformal mapping

$$\psi_{0,t} = \sum_{l=0}^{+\infty} t^l \hat{\psi}_{0,l}.$$

We make the following initial observation. In the limit case $s = 0$, the flow equation (3.3.1) of the lemma forces $\psi_{0,t}$ to be a mapping from $\mathbb{D}$ onto the interior of suitably chosen level curves of R, and from this we may obtain the coefficients $\hat{\psi}_{0,l}$. Indeed, if we take logarithms of both sides of the equation and multiply by s we obtain

$$s\log|h_s\circ\psi_{s,t}|^2 - 2R\circ\psi_{s,t} + s\log(1-t)\log J_\Psi + s\log W\circ\psi_{s,t} = -t^2 + \mathrm{O}(s), \tag{3.3.3}$$

where the J_Ψ denotes a Jacobian, of the form

$$J_{\Psi_s}((1+t)\zeta) = \mathrm{Re}\left(\bar\zeta\partial_t\psi_{s,t}\overline{\psi'_{s,t}(\zeta)}\right), \qquad \zeta\in\mathbb{T}.$$

Assuming some reasonable stability with respect to the variable s as $s\to 0$ in (3.3.3), we obtain in the limit

$$2R\circ\psi_{0,t}(\zeta) = t^2, \qquad \zeta\in\mathbb{T}. \tag{3.3.4}$$

It follows that the loop $\psi_{0,t}(\mathbb{T})$ is a part of the level set where $R = t^2/2$. This level set consists of two disjoint simple closed curves, one on either side of $\mathbb{T}$, at least for small enough t. For $t > 0$, we choose the curve outside the unit circle, while for $t < 0$ we choose the other one. We normalize the mapping $\psi_{0,t}$ so that it preserves the origin and has positive derivative there. In this fashion, the coefficients $\hat{\psi}_{0,l}$ are now uniquely determined by the level set condition. The smoothness of the level curves was worked out in some detail in Proposition 4.2.5 in [12]. The details of the determination of the coefficients $\hat{\psi}_{0,l}$ are given in terms of Herglotz integrals as in Proposition 4.6.1 of [12].

Our next task is to obtain iteratively the coefficients B_j for $j = 0,1,2,\ldots$ and the higher order corrections to the conformal mapping, given in terms of the coefficients $\hat{\psi}_{j,l}$ for $j = 1,\ldots,\kappa$. These coefficient functions are obtained by differentiating the flow equation (3.3.1). To set things up correctly, we write

$$\omega_{s,t}(\zeta) = |h_s\circ\psi_{s,t}|^2\mathrm{e}^{-2s^{-1}(R\circ\psi_{s,t}-\frac{t^2}{2})}W\circ\psi_{s,t}\,\mathrm{Re}\left(\bar\zeta\partial_t\psi_{s,t}\overline{\psi'_{s,t}(\zeta)}\right)$$

for $\zeta\in\mathbb{T}$, where we have used that $R\circ\psi_{0,t} = \frac{t^2}{2}$. We need to show that

$$\omega_{s,t}(\zeta) = (4\pi)^{-\frac12} + \mathrm{O}\big(|s|^{\kappa+\frac12} + |t|^{2\kappa+1}\big). \tag{3.3.5}$$

Since we believe that $\omega_{s,t}$ should be smooth in both parameters s and t, we aim to deduce (3.3.5) from demonstrating the solvability of the system of equations

$$\partial_s^j\partial_t^l\omega_{s,t}(\zeta)\big|_{s=t=0} = \begin{cases}(4\pi)^{-\frac12} & \text{for } \zeta\in\mathbb{T} \text{ and } (j,l) = (0,0),\\ 0 & \text{for } \zeta\in\mathbb{T} \text{ and } (j,l)\in\mathcal{I}_{2\kappa}\setminus(0,0),\end{cases} \tag{3.3.6}$$

in terms of the unknown coefficients. If, in addition, we can show that the functions B_j and $\hat{\psi}_{j,l}$ remain holomorphic and uniformly bounded in the appropriate domains provided that (R,W) remains confined to a uniform family of $\mathcal{W}_1(\delta,\sigma)$, the result follows.

To proceed to solve the system (3.3.6), we express the partial derivatives of $\omega_{s,t}$ in terms of our unknowns. We determine these unknowns by an iterative procedure. At the given step in the iteration, some of the coefficient functions will be already found. We split the equation (3.3.6) into a term containing an unknown coefficient function and a second term which contains only already determined coefficient functions. Here, we refrain from giving a complete account, which is available with minor modifications in [12]. The higher order partial derivatives of the function $\omega_{s,t}$ with respect of s and t are given by

$$(3.3.7)\quad 0 = \frac{1}{(j-1)!l!}\partial_s^{j-1}\partial_t^l\omega_{s,t}(\zeta)\big|_{s=t=0}$$
$$= 4(4\pi)^{-\frac{1}{2}}\varDelta R(\zeta)\,\mathrm{Re}(\bar\zeta\hat\psi_{j,l-1}(\zeta))\,\mathrm{Re}(\bar\zeta\hat\psi_{0,1}(\zeta))W(\zeta) + \mathfrak{F}_{j-1,l,W}(\zeta), \qquad \zeta\in\mathbb{T},$$

when $j,l>0$, and

(3.3.8)
$$0 = \frac{1}{j!}\partial_s^j\omega_{s,t}(\zeta)\big|_{s=t=0} = 2\,\mathrm{Re}\left(\bar B_0(\zeta)B_j(\zeta)\right)W(\zeta)(4\varDelta R(\zeta))^{-\frac{1}{2}} + \mathfrak{F}_{j,0,W}(\zeta), \qquad \zeta\in\mathbb{T},$$

when $l=0$ and $j>0$. Here, we have inserted the equation (3.3.6) for convenience. The expressions $\mathfrak{F}_{j,l,W}$ are real-valued real-analytic functions which are uniformly bounded while (R,W) remains in a uniform family in $\mathcal{W}_1(\delta,\sigma)$, and may be explicitly written down using the Faà di Bruno's formula. The crucial point for us is the dependence structure of the functions $\mathfrak{F}_{j,l,W}$, which remains the same as in the algorithm for the orthogonal polynomials:

The function $\mathfrak{F}_{j-1,l,W}$, $l\ge 1$ *is an expression in terms of* $B_0,\dots,B_{j-1,0}$ *and* $\hat\psi_{p,q}$ *for indices* $(p,q)\in\mathcal{I}_{2\kappa+1}$ *with* $(p,q)\prec(j,l-1)$

and

The function $\mathfrak{F}_{j,0,W}$ *depends only on* $B_0,\dots,B_{j-1}$ *and* $\hat\psi_{p,q}$ *with* $(p,q)\prec(j+1,0)$.

That this is so follows immediately from noticing that Propositions 4.2.5 and 4.6.1 in [12] and remain unchanged, while Proposition 4.6.2-4.6.4 in [12] require the obvious modifications related to our replacing the weight e^{-2mR} on the exterior disk $\mathbb{D}_{\mathrm{e}}(0,\rho)$ by a weight $\mathrm{e}^{-2mR}W$ on the disk $\mathbb{D}(0,\delta)$. In particular, it is paramount that $W(z)$ is strictly positive in a fixed neighbourhood of the unit circle $\mathbb{T}$. A natural approach to the computations is to write

$$\omega_{s,t}^I = |h_s\circ\psi_{s,t}|^2\mathrm{e}^{-2s^{-1}(R\circ\psi_{s,t}-\frac{t^2}{2})}\,\mathrm{Re}\left(\bar\zeta\partial_t\psi_{s,t}\overline{\psi_{s,t}'(\zeta)},\right)$$

which is essentially the expression which gets expanded in [12], and introduce the modification $\omega_{s,t}^{II} = W\circ\psi_{s,t}$, since then $\omega_{s,t} = \omega_{s,t}^I\omega_{s,t}^{II}$. By Leibnitz' formula, it is now easy to express the partial derivatives of $\omega_{s,t}$ in terms of derivatives of $\omega_{s,t}^I$ and of $\omega_{s,t}^{II}$. The partial derivatives of $\omega_{s,t}^I$ are as in Proposition 4.6.4 of [12], while the latter may be computed explicitly using the Faà di Bruno formula. The leading behaviour of $\partial_s^j\partial_t^l\omega_{s,t}$ comes from the contribution of $\partial_s^j\partial_t^l\omega_{s,t}^I$, which is as in [12]. It is easily verified that the remainders $\mathfrak{F}_{j,l,W}$ have the indicated properties.

We sketch the solution algorithm below, and indicate where the necessary modifications to the corresponding steps in [12] are required. We need the notation Γ_t for the component curve of the level set where

$$R(z) = \tfrac{t^2}{2}$$

chosen as before to expand as t increases.

Step 1. Take as above $\psi_{0,t}$ to be the conformal mapping $\psi_{0,t}:\mathbb{D}\to D_t$, where $\psi_{0,t}(0)=0$ and $\psi_{0,t}'(0)>0$, where D_t denotes the domain bounded by the curve

Γ_t. It follows from the smoothness of the flow of the level curves Γ_t (for details see Proposition 4.2.5 in [12]) that we have an expansion

$$\psi_{0,t} = \sum_{l=0}^{+\infty} t^l \hat{\psi}_{0,l}$$

of $\psi_{0,t}$, which determines the coefficient functions $\hat{\psi}_{0,l}$ for all $l = 0, 1, 2, \ldots$ (see Proposition 4.6.1 [12]). In particular, we have

$$\mathrm{Re}(\bar{\zeta}\hat{\psi}_{0,1}(\zeta)) = [4\mathit{\Delta} R(\zeta)]^{-\frac{1}{2}}, \qquad \zeta \in \mathbb{T}.$$

Step 2. The equation (3.3.6) for $(j, l) = (0, 0)$ gives that

$$|B_0(\zeta)|^2 \,\mathrm{Re}(\bar{\zeta}\hat{\psi}_{0,1}(\zeta))\, W(\zeta) = (4\pi)^{-\frac{1}{2}}, \qquad \zeta \in \mathbb{T},$$

which in its turn determines B_0. Indeed, the only outer function which is positive at the origin and meets this equation is

$$B_0(\zeta) = (4\pi)^{-\frac{1}{2}} \exp\Big\{\frac{1}{2}\mathbf{H}_{\mathbb{D}}\Big[\log[4\mathit{\Delta} R]^{\frac{1}{2}} - \log W(\zeta)\Big]\Big\},$$

where the Herglotz operator $\mathbf{H}_{\mathbb{D}}$ is given by

$$\mathbf{H}_{\mathbb{D}} f(z) := \int_{\mathbb{T}} \frac{1 + \bar{w}z}{1 - \bar{w}z}\, f(w)\, \mathrm{ds}(w), \qquad z \in \mathbb{D},$$

extended to the boundary via nontangential boundary values, and, when possible, also by analytic continuation.

As W is real-analytic and positive in a neighbourhood of $\mathbb{T}$, and moreover meets the regularity requirements of Definition 3.3.1, nothing is essentially different from the situation described in [12]. For instance, we may conclude that B_0 extends as a bounded, zero-free holomorphic function to $\mathbb{D}(0, \delta')$ where $\delta' > 1$.

We proceed to Step 3 with $j_0 = 1$.

Step 3. At the outset, we have an integer $j_0 \geq 1$ such that we have already successfully determined the coefficient functions $B_0, \ldots, B_{j_0-1}$ as well as $\hat{\psi}_{j,l}$ for all $(j, l) \prec (j_0, 0)$. In this step, we intend to determine all the coefficient functions $\hat{\psi}_{j_0,l}$ such that $(j_0, l) \in \mathcal{I}_{2\kappa+1}$. This is done in inductively in the parameter l, starting with $\hat{\psi}_{j_0,0}$. Assume that it has been carried out for all $l = 0, \ldots, l_0 - 1$. The coefficient $\hat{\psi}_{j_0,l_0}$ which we are looking for appears as the leading term in the equation (3.3.7) corresponding to $j = j_0$ and $l = l_0 + 1$. We solve for the coefficient function $\hat{\psi}_{j_0,l_0}$ in terms of the Herglotz operator:

$$\hat{\psi}_{j_0,l_0}(\zeta) = -(4\pi)^{\frac{1}{4}}\zeta\mathbf{H}_{\mathbb{D}}\Big[\frac{\mathfrak{F}_{j_0-1,l_0+1,W}}{(4\mathit{\Delta} R)^{\frac{1}{2}} W}\Big](\zeta).$$

The first step $l_0 = 0$ of the induction is carried out in a similar fashion. This completes Step 3.

Step 4. After completing Step 3, we find ourselves in the following situation: the coefficients B_j are known for $j < j_0$, while $\hat{\psi}_{j,l}$ are known for all $(j, l) \in \mathcal{I}_{2\kappa+1}$ with $(j, l) \prec (j_0 + 1, 0)$. We proceed to determine B_{j_0} using the equation (3.3.8) with index $(j, l) = (j_0, 0)$. The equation says that on $\mathbb{T}$,

$$2W(4\mathit{\Delta} R)^{-\frac{1}{2}}\,\mathrm{Re}\big(\bar{B}_0 B_{j_0}\big) + \mathfrak{F}_{j_0,0,W} = 0,$$

where $\mathfrak{F}_{j_0,0,W}$ depends on the known data. So we may simply solve for B_{j_0} using the formula

$$B_{j_0} = -\tfrac{1}{2} B_0\, \mathbf{H}_{\mathbb{D}_e}\Big[\frac{(4\varDelta R)^{\frac{1}{2}}\mathfrak{F}_{j_0,0,W}}{|B_0|^2 W}\Big] = -\pi^{\frac{1}{2}}\mathbf{H}_{\mathbb{D}}\big[\mathfrak{F}_{j_0,0,W}\big],$$

where we have used that $|B_0|^2 W = (4\pi)^{-\frac{1}{2}}[\mathrm{Re}(\bar\zeta\hat\psi_{0,1})]^{-1}$ and that $[\mathrm{Re}(\bar\zeta\hat\psi_{0,1})]^{-1} = (4\varDelta R)^{\frac{1}{2}}$ on $\mathbb{T}$. This completes Step 4, and we have extended the set of known data so that we may proceed to Step 3 with j_0 replaced with j_0+1.

The above algorithm continues until all the unknowns have been determined, up to the point where the whole index set $\mathcal{I}_{2\kappa+1}$ has been exhausted. In the process, we have in fact solved the equations (3.3.6) for all $(j,l)\in\mathcal{I}_{2\kappa}$. This means that if we form h_s and $\psi_{s,t}$ in terms of the functions B_j and $\hat\psi_{j,l}$ obtained with the above algorithm, a Taylor series expansion of $\omega_{s,t}$ in the parameters s and t along with the equations (3.3.6) shows that (3.3.5) holds. This completes the sketch of the proof. □

3.4. **Changing the base metric for orthogonal polynomials.** The results concerning twisting of the base metric for root functions apply also to the setting of orthogonal polynomials. Although many things are pretty much the same, we make an effort to explain what the precise result is in this context.

We need an appropriate notion of admissibility of the potential Q which applies to the setting of orthogonal polynomials. We recall that the spectral droplet $\mathcal{S}_\tau$ is the contact set

$$\mathcal{S}_\tau = \{z\in\mathbb{C}:\, \hat Q_\tau(z) = Q(z)\},$$

which is typically compact, where $\hat Q_\tau$ is the function

$$\hat Q_\tau(z) := \sup\big\{q(z):\, q\in\mathrm{SH}(\mathbb{C}),\ q\le Q \ \text{ on }\ \mathbb{C},\ \ q(z)\le \tau\log(|z|+1)+\mathrm{O}(1)\big\}.$$

Definition 3.4.1. We say that the potential Q is τ-admissible if the following conditions are met:

(i) $Q:\mathbb{C}\to\mathbb{R}$ is C^2-smooth,

(ii) Q meets the growth bound

$$\tau_Q := \liminf_{|z|\to+\infty}\frac{Q(z)}{\log|z|} > \tau > 0,$$

(iii) The unbounded component Ω_τ of the complement of the spectral droplet $\mathcal{S}_\tau$ is simply connected on the Riemann sphere $\hat{\mathbb{C}}$, with real-analytic Jordan curve boundary,

(iv) Q is strictly subharmonic and real-analytically smooth in a neighbourhood of the boundary $\partial\Omega_\tau$.

As before, we consider measures $\mathrm{e}^{-2mQ}V\mathrm{dA}$, where the twist function V is assumed to be nonnegative, positive near the curve $\partial\Omega_\tau$, and real-analytically smooth in a neighbourhood of $\partial\Omega_\tau$ with at most polynomial growth at infinity (3.1.1). We denote by ϕ_τ the surjective conformal mapping

$$\phi_\tau:\Omega_\tau\to\mathbb{D}_e,$$

which preserves the point at infinity and has $\phi_\tau'(\infty)>0$. The function $\mathcal{Q}_\tau$ is defined as the bounded holomorphic function on Ω_τ whose real part equals Q on the boundary $\partial\Omega_\tau$, and whose imaginary part vanishes at infinity.

The orthogonal polynomials $P_{m,n,V}$ have degree n, positive leading coefficient, and unit norm in $A^2_{mQ,V}$. They have the additional property that

$$\langle P_{m,n,V}, P_{m,n',V}\rangle_{mQ,V} = 0, \qquad n \neq n'.$$

We will work with $\tau = \frac{n}{m}$.

Theorem 3.4.2. *Suppose Q is τ-admissible for $\tau \in I_0$, where I_0 is a compact interval of the positive half-axis. Suppose in addition that V meets the above regularity requirements. Given a positive integer κ and a positive real A, there exists a neighborhood $\Omega_\tau^{\langle\kappa\rangle}$ of the closure of Ω_τ and bounded holomorphic functions $\mathcal{B}_{j,\tau,V}$ on $\Omega_\tau^{\langle\kappa\rangle}$, as well as domains $\Omega_{\tau,m} = \Omega_{\tau,m,\kappa,A}$ with $\Omega_\tau \subset \Omega_{\tau,m} \subset \Omega_\tau^{\langle\kappa\rangle}$ which meet*

$$\mathrm{dist}_{\mathbb{C}}(\Omega^c_{\tau,m}, \Omega_\tau) \geq A m^{-\frac{1}{2}}(\log m)^{\frac{1}{2}},$$

such that the orthogonal polynomials enjoy the expansion

$$P_{m,n,V}(z) = m^{\frac{1}{4}}(\phi'_\tau(z))^{\frac{1}{2}}(\phi_\tau(z))^n \mathrm{e}^{m\mathcal{Q}_\tau}\Big\{\sum_{j=0}^{\kappa} m^{-j}\mathcal{B}_{j,\tau,V}(z) + \mathrm{O}\big(m^{-\kappa-1}\big)\Big\},$$

on $\Omega_{\tau,m}$ as $n = \tau m \to +\infty$ while $\tau \in I_0$, where the error term is uniform. Here, the main term $\mathcal{B}_{0,\tau,V}$ is zero-free and smooth up to the boundary on Ω_τ, positive at infinity, with prescribed modulus

$$|\mathcal{B}_{0,\tau,V}(\zeta)| = \pi^{-\frac{1}{4}}[\varDelta Q(\zeta)]^{\frac{1}{4}} V^{-\frac{1}{2}}, \qquad \zeta \in \partial\Omega_\tau.$$

In view of Lemma 3.3.2, the construction of approximately orthogonal quasipolynomials may be carried out in the same way as in the case when $V = 1$. The same can be said of the $\bar\partial$-correction scheme, since the solution of the corresponding $\bar\partial$-problem only requires properties of the weight in a neighborhood of the closure of the off-spectral component. For instance, if $F_{m,n,V}$ denotes an appropriate sequence of approximately orthogonal quasipolynomials, we may obtain a solution u to the problem

$$\bar\partial u = F_{m,n,W}\bar\partial\chi_0,$$

with polynomial growth $|u(z)| = \mathrm{O}(|z|^n)$ which enjoys the estimate

$$\int_{\mathbb{C}} |u|^2 \mathrm{e}^{-2mQ} V \mathrm{dA} \leq \frac{1}{2m}\int_{\mathrm{supp}\,\bar\partial\chi_0} |F_{m,n,W}|^2\, |\bar\partial\chi_0|^2 \frac{\mathrm{e}^{-2mQ}V}{\varDelta Q - \frac{1}{2m}\varDelta \log V}\mathrm{dA},$$

where we point out that $\bar\partial\chi_0$ ought to be supported in the region where V is real-analytic and non-vanishing, and in addition where $\varDelta\hat{Q}_\tau - \frac{1}{2m}\varDelta\log V > 0$. From this it is clear how to proceed as we did without the twist.

References

[1] Ameur, Y., Hedenmalm, H., Makarov, N., *Berezin transform in polynomial Bergman spaces.* Comm. Pure Appl. Math. **63** (2010), no. 12, 1533-1584.

[2] Ameur, Y., Hedenmalm, H., Makarov, N., *Fluctuations of random normal matrices.* Duke Math. J. **159** (2011), 31-81.

[3] Balogh, F., Bertola, M., Lee., S-Y., McLaughlin, K. D., *Strong Asymptotics of Orthogonal Polynomials with Respect to a Measure Supported on the Plane.* Comm. Pure Appl. Math. **68** (2015), no. 1, 112-172.

[4] Berman, R., Berndtsson, B. Sjöstrand, J., *A direct approach to Bergman kernel asymptotics for positive line bundles*, Ark. Mat. **46** (2008), 197-217.

[5] Boutet de Monvel, L., Sjöstrand, J., *Sur la singularité des noyaux de Bergman et de Szegő.* Journées: Équations aux Dérivées Partielles de Rennes (1975), pp. 123–164. Astérisque, No. **34-35**, Soc. Math. France, Paris, 1976.

[6] Catlin, D., *The Bergman kernel and a theorem of Tian*, in Analysis and Geometry in Several Complex Variables (Katata, 1997), Trends Math., pp. 1-23.
[7] Fefferman, C., *The Bergman kernel and biholomorphic mappings of pseudoconvex domains.* Invent. Math. **26** (1974), 1-65.
[8] Gustafsson, B., Vasil'ev, A., Teodorescu, R., *Classical and Stochastic Laplacian Growth.* Advances in Mathematical Fluid Mechanics, Birkhäuser, Springer International Publishing, 2014.
[9] Hedenmalm, H., Makarov, N., *Coulomb gas ensembles and Laplacian growth.* Proc. London Math. Soc. (3) **106** (2013), 859-907.
[10] Hedenmalm, H., Olofsson, A., *Hele-Shaw flow on weakly hyperbolic surfaces.* Indiana Univ. Math. J. **54** (2005), no. 4, 1161-1180.
[11] Hedenmalm, H., Shimorin, S., *Hele-Shaw flow on hyperbolic surfaces.* J. Math. Pures Appl. **81** (2002), 187-222.
[12] Hedenmalm, H., Wennman, A. *Planar orthogonal polynomials and boundary universality in the random normal matrix model.* Preprint (2017).
[13] Hörmander, L., *L^2-estimates and existence theorems for the $\bar{\partial}$-operator.* Acta Math. **113** (1965), 89-152.
[14] Pokorny, F., and Singer, M., *Toric partial density functions and stability of toric varieties.* Math. Ann. **358** (2014), no.3-4, 879-923.
[15] Ross, J., Singer, M., *Asymptotics of partial density functions for divisors.* J. Geom. Anal. **27** (2017) 1803-1854.
[16] Ross, J., Witt Nyström, D., *The Hele-Shaw flow and moduli of holomorphic discs.* Compos. Math. **151**, (2015) 2301-2328.
[17] Shiffman, B., and Zelditch, S., *Random polynomials with prescribed Newton polytope.* J. Am. Math. Soc. **17**, (2004), 49-108.
[18] Tian, G., *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geometry **32** (1990), 99-130.
[19] Zelditch, S., *Szegő kernels and a theorem of Tian*, Int. Math. Res. Notices 1998 (1998), 317-331.
[20] Zelditch, S., Zhou, P., *Central limit theorem for spectral partial Bergman kernels.* arXiv:1708.09267

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