SCALING LIMITS OF RANDOM NORMAL MATRIX PROCESSES AT SINGULAR BOUNDARY POINTS

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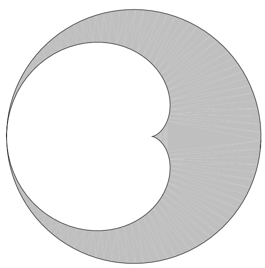
Abstract. We study scaling limits for eigenvalues of random normal matrices near certain types of singular points of the droplet. In particular, we prove existence of new types of determinantal point fields, which differ from those which can appear at a regular boundary point.

The method of rescaled Ward identities was introduced in the paper [3], where the main focus was on scaling limits at a regular boundary point of the droplet associated to a random normal matrix process. In this note, we will apply the same method to obtain results about scaling limits near *singular* boundary points, which may be either cusps or double points.



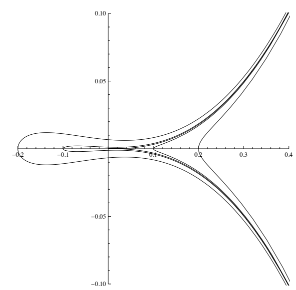


Figure 1. The left figure shows a droplet (shaded) exhibiting both kinds of singular points; a cusp and a double point. The picture on the right shows a $(5,2)$-cusp under Laplacian growth/Hele-Shaw evolution.

Recall that, in the random normal matrix model, we are given a suitable real-valued function Q, called the "potential". We consider configurations (or "systems") $\{\zeta_j\}_1^n$ of points in $\mathbb{C}$, having the interpretation of identical point charges subject to the external field Q. The energy of the system is defined to be

$$H_n\left(\zeta_1,\ldots,\zeta_n\right)=\sum_{j\neq k}\log\frac{1}{\mid\zeta_j-\zeta_k\mid}+n\sum_{j=1}^{n}Q\left(\zeta_j\right),$$

and we consider the ensemble of systems picked randomly with respect to the Boltzmann-Gibbs law,

$$d\mathbb{P}_n(\zeta)=\frac{1}{Z_n}\,e^{-H_n(\zeta)}\,dV_n(\zeta),\quad \zeta=(\zeta_1,\ldots,\zeta_n)\in\mathbb{C}^n. \tag{0.1}$$

Here dV_n is Lebesgue measure in $\mathbb{C}^n$ divided by π^n, and $Z_n=\int e^{-H_n}\,dV_n$.

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As $n \to \infty$, the random samples $\{\zeta_j\}_1^n$ tend to condensate on a compact set S known as the droplet, the boundary of which is a finite union of real-analytic arcs, possibly containing finitely many singular points where the arcs meet. The main problem considered in this note is to understand the microscopic properties of the system near such a singular boundary point $p \in \partial S$, as $n \to \infty$.

We prove that nothing of interest is to be found in the vicinity of the singular point itself, but instead we shall find new point fields located somewhat inside the droplet, close to the singular point. This is accomplished by zooming about a "moving point", which approaches the singular point at a proper rate. Cf. Figure 2.

1. Introduction and main results

Notational conventions. We write $D(p;r)$ for the open disk centered at p of radius r. $\mathbb{C}_+$ denotes the upper half-plane. The characteristic function of a set E will be denoted χ_E. We use the notation $\Delta = \partial\bar{\partial}$, so Δ is 1/4 of the usual Laplacian. We write $dA(z) = d^2z/\pi$ for two-dimensional Lebesgue measure in $\mathbb{C}$, normalized so that the unit disk has unit area. A continuous function $f : \mathbb{C}^2 \to \mathbb{C}$ is termed *Hermitian* if $f(z,w) = \overline{f(w,z)}$. We say that f is *Hermitian-entire* if f is Hermitian and entire as a function of z and $\bar{w}$. A Hermitian-entire function is uniquely determined by its diagonal values $f(z,z)$ by polarization. Finally, a Hermitian function c is called a *cocycle* if $c(z,w) = g(z)\overline{g(w)}$ for a continuous unimodular function g.

1.1. **External potential and droplet.** Our basic setup parallells that of [3]. In short, let $Q : \mathbb{C} \to \mathbb{R} \cup \{\infty\}$ is a suitable "external potential" of sufficient growth; requiring

$$\liminf_{\zeta \to \infty} \frac{Q(\zeta)}{\log |\,\zeta\,|^2} > 1$$

will do. If μ is a positive, compactly supported Borel measure we define its weighted logarithmic Q-energy by

$$I_Q[\mu] = \int_{\mathbb{C}} Q\, d\mu + \int_{\mathbb{C}^2} \log \frac{1}{|\,\zeta - \eta\,|}\, d\mu(\zeta)\, d\mu(\eta). \tag{1.1}$$

A theorem of Frostman asserts that there is a unique *equilibrium measure* σ of total mass 1, which minimizes $I_Q[\mu]$ over all compactly supported Borel probability measures μ on $\mathbb{C}$. The support of the measure σ is called the *droplet* in external field Q, and is denoted

$$S = S[Q] := \operatorname{supp} \sigma.$$

It is well-known (see e.g. [16]) that S is a compact set and σ is absolutely continuous and given by the formula

$$d\sigma(z) = \Delta Q(z)\, \chi_S(z)\, dA(z), \tag{1.2}$$

where we recall that we adhere to the convention that $\Delta := \partial\bar{\partial}$ is 1/4 of the standard Laplacian while $dA(z) := d^2z/\pi$ is two-dimensional Lebesgue measure divided by π.

We make the standing assumptions that there is a neighbourhood Ω of S such that

(1) Q is real-analytic in Ω,
(2) $\Delta Q > 0$ in Ω.

Under these conditions, the complement S^c has a local Schwarz function, and one can apply Sakai's regularity theorem [17] to conclude that the boundary points $p \in \partial S$ can be characterized as follows.

The most common type of boundary point is a *regular* boundary point. This is a point p such that there exists a neighbourhood $D = D(p;\epsilon)$ such that $D \setminus S$ is a Jordan domain and $D \cap (\partial S)$

is a simple real-analytic arc. By Sakai's theorem, all but finitely many boundary points of S are regular. The finitely many exceptional points are called *singular*.

When analyzing a singular point, we can without loss of generality assume that it is located on the *outer* boundary of S, i.e. on the boundary of the unbounded component U of $\hat{\mathbb{C}} \setminus S$. If there are other boundary components, they can be treated in the same way.

There are two kinds of singular boundary points. A point at the outer boundary $p \in \partial U$ is a (conformal) *cusp* if there is $D = D(p;\epsilon)$ such that $D \setminus S$ is a Jordan domain and every conformal map $\Phi : \mathbb{C}_+ \to D \setminus U$ with $\Phi(0) = p$ extends analytically to a neighbourhood of 0 and satisfies $\Phi'(0) = 0$; p is a *double point* if there is a disk D about p such that $D \setminus S$ is a union of two Jordan domains, and p is a regular boundary point of each of them. Note that the cusps which appear on the boundary of a droplet S always point out from S.

One can further classify singular points according to degrees of tangency; we briefly recall how this works for cusps.

Thus assume that ∂S has a cusp at the outer boundary ∂U at $p = 0$. We can assume that a conformal map $\Phi : \mathbb{C}_+ \to U$ satisfies

$$\Phi'(z) = z + a_2 z^2 + \cdots + (a_{\nu-1} + ib) z^{\nu-1} + \cdots$$

where a_j and b are real and $b \neq 0$. This means that

$$\Phi(z) = \frac{1}{2} z^2 + \frac{a_2}{3} z^3 + \cdots + \frac{a_{\nu-1} + ib}{\nu} z^{\nu} + \cdots .$$

If we write $\Phi = u + iv$, this gives

$$u(x) = \frac{1}{2} x^2 + \cdots , \quad v(x) = \frac{b}{\nu} x^{\nu} + \cdots , \quad (x \in \mathbb{R}).$$

By definition, this means that the cusp at 0 is of type $(\nu, 2)$.

Some cusps, in particular $(3, 2)$-cusps, which are generic in Sakai's theory can not appear on a free boundary such as those appearing in the present context. For technical reasons, the methods in this paper do not apply to $(3, 2)$-cusps, but we are able to treat $(\nu, 2)$ cusps where ν is odd and $\nu \geq 5$, provided that our standing assumptions (1) and (2) hold[1]. We will also obtain results for double points. For a brief discussion of these matters, which may be regarded folklore, see Subection 2.1 below.

Droplets with singular boundary points have been studied e.g. in the papers [7, 14, 21], the book [17], and the thesis [8]. We refer to those sources and the references there for more detailed information on various types of singular boundary points.

1.2. **Rescaled ensembles.** Let $\{\zeta_j\}_1^n$ be a random sample from the Boltzmann-Gibbs distribution (0.1). As in [3], we will denote by boldface characters the objects pertaining to this ensemble. We shall for example use the k-point function

$$\mathbf{R}_{n,k}(\eta_1, \ldots, \eta_k) = \lim_{\varepsilon \to 0} \left[\varepsilon^{-2k} \cdot \mathbb{P}_n \left(\cap_{j=1}^{k} \{ N_{D(\eta_j;\varepsilon)} \geq 1 \} \right) \right],$$

where N_B is the number of points ζ_j which fall in the set B. It is well-known that we can write

$$\mathbf{R}_{n,k}(\zeta_1, \ldots, \zeta_k) = \det(\mathbf{K}_n(\zeta_i, \zeta_j))_{i,j=1}^{k},$$

where $\mathbf{K}_n$ is a Hermitian function, which we call a correlation kernel of the process. The correlation kernel $\mathbf{K}_n$ is obtained as the reproducing kernel for the space of weghted polynomials

$$f(z) e^{-nQ(z)/2},$$

where f is a polynomial of degree at most $n - 1$, endowed with the topology of $L^2(\mathbb{C}, dA)$.

[1]In fact it is enough to require that they hold locally, near the singular point in question.

Now consider a moving point $p_n \in S$ and a sequence of angles $\theta_n \in \mathbb{R}$. We shall consider rescaled point processes $\Theta_n = \{z_j\}_1^n$ where

$$z_j = e^{-i\theta_n}\sqrt{n\Delta Q(p_n)}(\zeta_j - p_n). \tag{1.3}$$

The law of the process Θ_n is defined as the image of the Boltzmann-Gibbs law under the map $\{\zeta_j\}_1^n \mapsto \{z_j\}_1^n$.

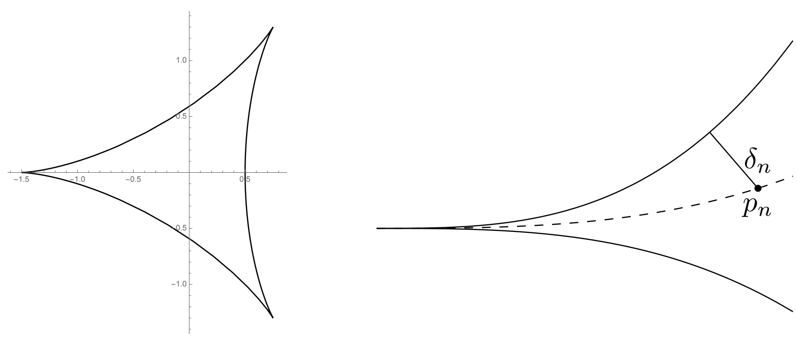

FIGURE 2. The deltoid (left) has three maximal $(3,2)$-cusps. The figure on the right shows a moving point p_n on the bisectrix of a cusp point. Here, $\delta_n = Tn^{-\frac{1}{2}}$ for some positive T.

Objects pertaining to the rescaled system $\{z_j\}_1^n$ are denoted by plain symbols. For example, the k-point function of the rescaled system will be written

$$R_{n,k}(z_1,\ldots,z_k) := \frac{1}{(n\Delta Q(p_n))^k}\mathbf{R}_{n,k}(\zeta_1,\ldots,\zeta_k).$$

The rescaled process is determinantal with correlation kernel

$$K_n(z,w) = \frac{1}{n\Delta Q(p_n)}\mathbf{K}_n(\zeta,\eta),$$

where

$$z = e^{-i\theta_n}\sqrt{n\Delta Q(p_n)}(\zeta - p_n), \quad w = e^{-i\theta_n}\sqrt{n\Delta Q(p_n)}(\eta - p_n).$$

We write $R_n(z) = K_n(z,z)$.

Lemma 1.1. *There is a sequence of cocycles c_n such that every subsequence of c_nK_n has a subsequence converging to $G\Psi$ where*

$$G(z,w) = e^{z\bar{w}-|z|^2/2-|w|^2/2}$$

is the Ginibre kernel and Ψ is a Hermitian entire function satisfying the mass-one inequality

$$\int_{\mathbb{C}} e^{-|z-w|^2}\left|\Psi(z,w)\right|^2\,dA(w) \le \Psi(z,z). \tag{1.4}$$

Further, the function $R(z) := K(z,z) = \Psi(z,z)$ is either identically zero, or else it is everywhere strictly positive.

Proof. The first statement follows from a normal families argument; see Theorem A in [3]. The last statement is proven in [3], Theorem B. $\square$

A limit point K in Lemma 1.1 will be called a *limiting kernel*, and

$$R(z) := K(z, z)$$

is the corresponding *limiting* 1-*point function*. The general theory of point fields implies that a limiting kernel K is the correlation kernel of a "limiting infinite point field" Θ with k-point function $R_k(z_1, \ldots, z_k) = \det(K(z_i, z_j))_{i,j=1}^k$. See [3] and the references there.

There is nothing which prevents the limiting kernel K from being trivial, i.e., we may well have $K = 0$. In this case the point process Θ_n degenerates to the trivial point field, all of whose k-point functions vanishes identically. As we shall see, this happens when we rescale about certain types of singularities. However, when we shift focus and rescale about a suitable moving point, approaching the singular point from the inside of the droplet at a suitable distance away, we will obtain new, non-degenerate, limiting point fields.

If R is non-trivial, we can go on to define the Berezin kernel

$$B(z, w) = \frac{|K(z, w)|^2}{R(z)}$$

and its Cauchy transform

$$C(z) = \int_{\mathbb{C}} \frac{B(z, w)}{z - w}\, dA(w).$$

By Theorem B in [3], the function C is smooth and we have Ward's equation

$$\bar{\partial} C = R - 1 - \Delta \log R. \tag{1.5}$$

Since K (and therefore B and C) is uniquely determined by R by polarization, it makes sense to say that R *satisfies Ward's equation* if (1.5) holds.

1.3. **Main results.** Let p be a singular boundary point, i.e., a cusp or a double point.

Theorem I. *Let $e^{i\theta}$ be one of the normal directions to ∂S at p and rescale about p according to*

$$z_j = e^{-i\theta} \sqrt{n \Delta Q(p)}(\zeta_j - p), \quad j = 1, \ldots, n.$$

Then any limiting 1*-point function R vanishes identically. As a consequence, any limiting point field at p is trivial.*

Let us remark that the choice of angle θ above is merely made for convenience. Any other angle, or sequence θ_n of angles, would produce the same result.

Since Theorem I says that nothing of interest is to be found in the vicinity of the singular point p, we shift focus and look a bit to the inside of the droplet. We need to consider the case of a cusp and a double point separately.

Definition. Fix a positive parameter T.

(i) If ∂S has a cusp at p, we consider the moving point $p_n \in S$ of distance $T/\sqrt{n\Delta Q(p)}$ from the boundary ∂S, which is closest to the singular point p. (See Fig. 1.)

(ii) If S has a double point, there are instead two distinct points p_n', p_n'' in S of distance $T/\sqrt{n\Delta Q(p)}$ to ∂S, of minimal distance to p.

Theorem II. *Suppose that S has a $(\nu, 2)$-cusp at p where $\nu \geq 5$. Rescale about p_n according to*

$$z_j = e^{-i\theta_n} \sqrt{n \Delta Q(p_n)}(\zeta_j - p_n), \qquad j = 1, \ldots, n, \tag{1.6}$$

where the angle θ_n is chosen so that the image of the cusp, i.e. the point $e^{-i\theta_n}(p - p_n)$, is on the positive imaginary axis. Then, if T is sufficiently large, each limiting 1*-point function*

$R(z) = K(z,z)$ is everywhere positive and satisfies Ward's equation. Moreover, R satisfies the estimate

$$R(z) \le Ce^{-2(|x|-T)^2}, \qquad (x = \operatorname{Re} z). \tag{1.7}$$

The assumption that the parameter $T > 0$ be sufficiently large is made for technical reasons of the proof; it should really not be needed. See the concluding remarks for related comments.

Our result for double points is similar.

Theorem III. *If S has a double point at p, we rescale as in (1.6) with p_n replaced by p_n' or p_n''. The conclusions of Theorem II then hold also for the limiting 1-point function R rescaled about p_n' or p_n''.*

We remark that the limiting point fields, whose existence is guaranteed by Theorems II and III are necessarily different from those which can appear at a fixed regular boundary point. Indeed, as was observed in [3], a limiting 1-point function rescaled in the outer normal direction about a regular boundary point will satisfy the estimate

$$\left| R(z) - \chi_{(-\infty,0)}(x) \right| \le Ce^{-cx^2}$$

where c is some positive constant. The latter estimate is not consistent with (1.7), which is seen by letting $x \to -\infty$.

The concluding remarks in Section 5 contains further comments about the limiting point fields in theorems II and III.

1.4. **Organization of the paper.** Our approach uses Bergman space techniques and some rather careful estimates for the limiting 1-point function in the *exterior* of the droplet. It is convenient to start with the latter estimates.

In Section 2, we prove an estimate for the decay of a limiting 1-point function in the exterior of the droplet, near a cusp. In Section 3, we extend the estimate to rather general (moving) boundary points. In Section 4 we prove our main results, Theorems I, II, and III. Section 5 contains concluding remarks.

2. Exterior estimate for the 1-point function at a cusp

In this section, we assume that the droplet S has a cusp at the point $p \in \partial S$. To simplify the discussion, we will assume that p on the *outer* boundary of S.

Now let $e^{i\theta}$ be one of the normal directions to ∂S and rescale about p in the usual manner:

$$z_j = e^{-i\theta}\sqrt{n\Delta Q(p)}(\zeta_j - p), \qquad j = 1, \ldots, n.$$

Let $K = G\Psi$ denote a limiting kernel. Write $R(z) = K(z,z)$ and $R_n(z) = K_n(z,z)$. We have the following result.

Theorem 2.1. *With the above assumptions, we have*

$$R(z) \le Ce^{-2x^2}, \qquad (x = \operatorname{Re} z). \tag{2.1}$$

To prove the estimate (2.1) we shall use the well-known estimate (e.g. [3], Section 3)

$$\mathbf{R}_n(\zeta) \le Cne^{-n(Q-\check{Q})(\zeta)}. \tag{2.2}$$

Here $\check{Q}$ is the "obstacle function", i.e. the $C^{1,1}$-smooth function on $\mathbb{C}$ which coincides with Q on S and is harmonic in $\mathbb{C} \setminus S$ and increases like $\check{Q}(\zeta) = \log|\zeta|^2 + O(1)$ as $\zeta \to \infty$.

Now assume *w.l.o.g.* that $p = 0$ and that the cusp at p points in the positive real direction. The rescaling is then simply given by

$$z = i\sqrt{n\Delta Q(p)}\,\zeta, \tag{2.3}$$

and in the z-plane, the droplet appears as a narrow neighbourhood of the ray $(-i\infty, 0)$.

Write U for the unbounded component of $\mathbb{C} \setminus S$ and let $\Phi : \mathbb{C}_+ \to U$ be a conformal map such that $\Phi(0) = 0$ and $\Phi(i) = \infty$. Since 0 is a conformal cusp, Φ extends analytically to some neighbourhood N of the origin. Likewise, the harmonic function $\check{Q} \circ \Phi : \mathbb{C}_+ \to \mathbb{R}$ extends harmonically across $\mathbb{R}$ to a harmonic function V. Referring to (2.3), we shall put

$$\zeta = \Phi(\lambda).$$

We can assume that Φ' has the Taylor expansion

$$\Phi'(\lambda) = \lambda + a_2\lambda^2 + a_3\lambda^3 \ldots, \qquad (\lambda \in N).$$

We form the functions

$$Q_\Phi := Q \circ \Phi, \quad \check{Q}_\Phi := \check{Q} \circ \Phi.$$

The function $\check{Q}_\Phi$ is harmonic in $\mathbb{C}_+$ and extends across $\mathbb{R}$ to a harmonic function V. Write

$$M(\lambda) := (Q_\Phi - V)(\lambda), \qquad \lambda = \sigma + i\tau.$$

Thus $M = (Q - \check{Q}) \circ \Phi$ in $\mathbb{C}_+$.

Lemma 2.2. *For $\lambda = \sigma + i\tau$, we have*

$$M(\lambda) = 2\Delta Q(p)\tau^2\sigma^2 + O\left(|\,\lambda\,|^5\right), \qquad (\lambda \to 0). \tag{2.4}$$

Before we prove the lemma, we proceed to prove the estimate (2.1). To this end, note that the estimate (2.2) gives (with a new C depending on $\Delta Q(p)$)

$$R_n(z) \le Ce^{-nM(\lambda_n(z))}, \qquad \text{where} \quad \lambda_n(z) := \Phi^{-1}\left(-iz/\sqrt{n\Delta Q(p)}\right). \tag{2.5}$$

If $z = x + iy$, then, since $\Phi(\lambda) = \lambda^2/2 + O(\lambda^3)$ as $\lambda \to 0$,

$$-x = \sqrt{n\Delta Q(p)}\,\mathrm{Im}\left(\lambda^2/2 + O(\lambda^3)\right) = \sqrt{n\Delta Q(p)}\left(\sigma\tau + O(|\,\lambda\,|^3)\right), \quad (\lambda = \sigma + i\tau \to 0). \tag{2.6}$$

The estimates (2.4) and (2.6) now give that

$$nM\left(\lambda_n(z)\right) = 2x^2 + O\left(n\,|\,\lambda_n(z)\,|^5\right), \quad (n \to \infty).$$

Choosing, for example, $|\,z\,| \le \log n$, we see via (2.5) that the estimate (2.1) holds.

It remains to prove Lemma 2.2.

Recalling that $\check{Q}(\lambda) \sim \log|\,\lambda\,|^2$ as $\lambda \to \infty$, it is now seen that the Poisson representation of the harmonic function $\check{Q}_\Phi|_{\mathbb{C}_+}$ takes the form

$$\check{Q}_\Phi(\lambda) := \int_{\mathbb{R}} Q_\Phi(t)P(\lambda, t)\,dt + G(\lambda), \quad \lambda \in \mathbb{C}_+, \tag{2.7}$$

where $G(\lambda) = \log\left|\frac{\lambda+i}{\lambda-i}\right|^2$ is (twice) the Green's function for $\mathbb{C}_+$ with pole at i, and

$$P(\lambda, t) = \frac{1}{\pi}\frac{\tau}{(\sigma - t)^2 + \tau^2}, \quad \lambda = \sigma + i\tau$$

the Poisson kernel for $\mathbb{C}_+$.

Let us write C^ω for the real-analytic class. For a function $f \in C^\omega(\mathbb{R})$ we write $P_f(\lambda) = \int_{\mathbb{R}} f(t)P(\lambda,t)\,dt$ for the Poisson integral and $\mathrm{p.\,v.} = \mathrm{p.\,v.}^{(\sigma,\infty)}$ for the double principal value of integrals, defined by

$$\mathrm{p.\,v.} \int_{\mathbb{R}} f(t)dt = \lim_{\epsilon \downarrow 0} \int_{\epsilon < |\sigma - t| < \epsilon^{-1}} f(t)dt,$$

when this limit exists. If f is absolutely integrable near either the point σ or at infinity, then the principal value integral agrees with the usual (Lebesgue) integral. Noting that, for $\lambda = \sigma + i\tau \in \mathbb{C}_+$,

$$(\sigma - t)^2 P(\lambda,t) - \tau/\pi = -\tau^2 P(\lambda,t),$$

we compute

(2.8)

$$\begin{aligned}
P_f(\lambda) - f(\sigma) &= \mathrm{p.\,v.} \int_{\mathbb{R}} \left[f(t) - f(\sigma) - (t-\sigma)f'(\sigma)\right] P(\lambda,t)\,dt \\
&= \mathrm{p.\,v.} \int_{\mathbb{R}} \left[\frac{f(t) - f(\sigma) - (t-\sigma)f'(\sigma)}{(\sigma - t)^2}\right] (\sigma - t)^2 P(\lambda,t)\,dt \\
&= \frac{\tau}{\pi} \cdot \mathrm{p.\,v.} \int_{\mathbb{R}} \frac{f(t) - f(\sigma) - (t-\sigma)f'(\sigma)}{(\sigma - t)^2}\,dt - \tau^2 \int_{\mathbb{R}} \frac{f(t) - f(\sigma) - (t-\sigma)f'(\sigma)}{(\sigma - t)^2} P(\lambda,t)\,dt.
\end{aligned}$$

Note that the last integral is absolutely convergent, and approaches $\frac{1}{2}f''(\sigma)$ as $\tau \downarrow 0$.

Let us denote by $S_\sigma : C^\omega(\mathbb{R}) \to C^\omega(\mathbb{R})$ the backward shift by σ: $S_\sigma f(t) = \frac{f(t)-f(\sigma)}{t-\sigma}$. Write

$$I_f^k(\sigma) := \mathrm{p.\,v.} \int_{\mathbb{R}} S_\sigma^{2k} f(t)\,dt, \quad k = 1, 2, \ldots .$$

A repetition of the calculation in (2.8) gives that P_f has the asymptotic expansion

$$P_f(\sigma + i\tau) = f(\sigma) + I_f^1(\sigma) \cdot \tau - \frac{1}{2} f''(\sigma) \cdot \tau^2 - I_f^2(\sigma) \cdot \tau^3 + \frac{1}{4!} f^{(4)}(\sigma) \cdot \tau^4 + \ldots . \tag{2.9}$$

Choosing $f = Q_\Phi$ and using identity (2.7), we find

$$I_{Q_\Phi}^1(\sigma) = \partial_\tau P_{Q_\Phi}(\sigma) = \partial_\tau (Q_\Phi - G)(\sigma), \quad \sigma \in \mathbb{R}.$$

More generally, it is easy to verify by induction that

$$(2k-1)! \cdot I_{Q_\Phi}^k(\sigma) = \partial_\sigma^{2k-2} \partial_\tau (Q_\Phi - G)(\sigma). \tag{2.10}$$

Since $\partial \check{Q}_\Phi$ is continuous on $\mathrm{cl}\,\mathbb{C}_+$, while $Q_\Phi = \check{Q}_\Phi$ in the lower half plane, we can replace "Q_Φ" by "$\check{Q}_\Phi$" in the right side of (2.10), and $M = Q_\Phi - P_{Q_\Phi} - G$ satisfies $M = \partial_\tau M = 0$ on $\mathbb{R}$. Inserting the expansions (2.9) and (2.10), we thus find that

$$\begin{aligned}
M(\sigma + i\tau) &= \frac{1}{2}\partial_\tau^2 M(\sigma) \cdot \tau^2 + \frac{1}{3!}\partial_\tau^3 M(\sigma) \cdot \tau^3 + \frac{1}{4!}\partial_\tau^4 M(\sigma) \cdot \tau^4 + \cdots \\
&= \frac{1}{2}\partial_\tau^2 Q_\Phi(\sigma) \cdot \tau^2 + \frac{1}{3!}\partial_\tau^3 Q_\Phi(\sigma) \cdot \tau^3 + \frac{1}{4!}\partial_\tau^4 Q_\Phi(\sigma) \cdot \tau^4 + \cdots \\
&+ \frac{1}{2}\partial_\sigma^2 Q_\Phi(\sigma) \cdot \tau^2 + \frac{1}{3!}\partial_\sigma^2 \partial_\tau Q_\Phi(\sigma) \cdot \tau^3 - \frac{1}{4!}\partial_\sigma^4 Q_\Phi(\sigma) \cdot \tau^4 + \cdots \\
&- \frac{1}{2}\partial_\tau^2 G(\sigma) \cdot \tau^2 - \frac{1}{3!}\partial_\tau(\partial_\tau^2 + \partial_\sigma^2) G(\sigma) \cdot \tau^3 - \frac{1}{4!}\partial_\tau^4 G(\sigma) \cdot \tau^4 + \cdots .
\end{aligned}$$

The last line vanishes because all coefficients are derivatives of ΔG evaluated at σ. This implies that

$$M(\sigma+i\tau)=2\Delta Q_\Phi(\sigma)\cdot\tau^{2}+\frac{4}{3!}\partial_\tau\Delta Q_\Phi(\sigma)\cdot\tau^{3}+\frac{4}{4!}(\partial_\tau^{2}-\partial_\sigma^{2})\Delta Q_\Phi(\sigma)\cdot\tau^{4}+\frac{4}{5!}(\partial_\tau^{3}-\partial_\tau\partial_\sigma^{2})\Delta Q_\Phi(\sigma)\tau^{5}+\cdots. \tag{2.11}$$

But $\Delta Q_\Phi(\sigma+i\tau)=\Delta Q(\Phi(\sigma+i\tau))\,|\,\Phi'(\sigma+i\tau)\,|^{2}=\Delta Q(\Phi(\sigma+i\tau))\cdot(\sigma^{2}+\tau^{2}+\ldots)$. We have shown that

$$M(\sigma+i\tau)\geq 2\Delta Q(0)\cdot\sigma^{2}\tau^{2}+O\left(|\,\lambda\,|^{5}\right),\qquad(\lambda=\sigma+i\tau\to 0).$$

The proof of Lemma 2.2 is complete. $\square$

2.1. **Classification of cusps.** Consider a droplet $\mathcal{S}$ for which a cusp appears at $p\in\partial\mathcal{S}$. Since Q is assumed to be real-analytically smooth and strictly subharmonic in a neighbourhood of $\mathcal{S}$, it follows from Sakai's regularity theorem that the conformal mapping $\Phi:\mathbb{H}^{+}\to\mathcal{S}^{c}$ which maps the origin to p extends holomorphically across $\mathbb{R}$, and hence admits a power series expansion at p. After possibly applying an affine transformation, we may deduce that the boundary admits a local approximate parameterization

$$\partial\mathcal{S}\cap\mathbb{D}(0,\delta)=\left\{x+iy\,:\,x=-\frac{1}{2}t^{2}+O(t^{3}),\;\;y=c_\nu t^{\nu}+O(t^{\nu+1}),\;\;t\in(-\epsilon,\epsilon)\right\}, \tag{2.12}$$

for some positive numbers ϵ and δ and some constant c_ν. Here, we may immediately observe that the above parameterization says that $y=x^{\frac{\nu}{2}}+O(x^{\frac{\nu}{2}+\frac{1}{2}})$ on the boundary near the cusp point, and from this it follows that ν is necessarily odd. Indeed, otherwise (2.12) parameterizes a curve which is differentiable across the cusp.

However, more is true. Only a cusp where $\nu=4k+1$ for some $k=1,2,3,\ldots$ may appear on a free boundary such as $\mathcal{S}$. Indeed, it may be shown that if $\nu=4k+3$ for some k, then it must necessarily hold that $Q-\check{Q}$ assumes negative values arbitrarily close to p, see e.g. [13, pp. 388-390]. It appears to be difficult to find a good reference for this general classification result, which is perhaps to be regarded as folklore. For us, it is of importance to rule out the occurrence of $(3,2)$-cusps, and we present proof of this fact below for completeness.

Proposition 2.3. *A cusp of type* $(3,2)$ *cannot occur on the boundary of the droplet* $\mathcal{S}$.

Proof. Assume without loss of generality that a cusp of type $(3,2)$ occurs at the origin, and moreover that it points in the positive real direction. In order to reach a contradiction, we intend to compute $M(i\tau)$ using (2.11), and show that $M(i\tau)$ must take on negative values arbitrarily close to 0. Since the cusp is assumed to be of type $(3,2)$ the conformal mapping Φ takes the form

$$\Phi(z)=-\frac{1}{2}z^{2}+\frac{a+ib}{3}z^{3}+O(|z|^{4})$$

where $b\neq 0$, from which it follows that

$$\Delta Q_\Phi(z)=\Delta Q(\Phi(z))\;|\Phi'(z)|^{2}=\sigma^{2}+\tau^{2}+2a(\sigma^{3}+\sigma\tau^{2})-2b(\tau^{3}+\sigma^{2}\tau)+O(|z|^{4})$$

when $z=\sigma+i\tau\to 0$. A small computation using (2.11) shows that

$$M(i\tau)=-\frac{32b}{5!}\tau^{5}+O(\tau^{6}),\qquad y\to 0$$

from which the assertion follows. $\square$

3. A general exterior estimate for the 1-point function

In this section, we consider a (possibly moving) boundary point p and rescale about p in the outer normal direction $e^{i\theta}$ in the usual way,

$$z_j = e^{-i\theta}\sqrt{n\Delta Q(p)}(\zeta_j - p), \qquad j = 1, \ldots, n.$$

We can then form the rescaled 1-point function $R_{n,p}(z)$, and also sequential limiting 1-point functions

$$R_p(z) = \lim_{k\to\infty} R_{n_k,p}(z), \qquad z \in \mathbb{C}.$$

In the situations relevant to this paper, we find that the decay of $R_p(z)$ is at least like e^{-2x^2} where $x = \operatorname{Re} z$. Our estimates are however not quite uniform; there is a critical case of boundary points at distance about $n^{-1/3}$ to the closest singular boundary point when our method is inconclusive. Since such points play no role to our analysis anyway, we will simply disregard them here.

Theorem 3.1. *Let δ_n be a sequence of positive numbers with $\delta_n \to 0$ and $n^{1/3}\delta_n \to \infty$ as $n \to \infty$. Then there is a constant C such that if the distance from a boundary point p to the nearest singular point is at least δ_n, then $R_{n,p}(z) \le Ce^{-2x^2}$, when $|z| \le \log n$, $x = \operatorname{Re} z$. As a consequence, any limiting 1-point function R_p satisfies the estimate*

$$R_p(z) \le Ce^{-2x^2}.$$

As a simple corollary, we will obtain the estimate (1.7) in Theorem II.

The proof will be carried out in steps, in the following three paragraphs. We will first treat the simple case of an "ordinary" fixed boundary point p. After that we consider cases of moving points near singular boundary points.

3.1. **Fixed boundary points.** Let p be an arbitrary fixed boundary point of S such that $\Delta Q(p) > 0$. We rescale about p according to

$$z = \sqrt{n\Delta Q(p)}e^{-i\theta}(\zeta - p)$$

where $e^{i\theta}$ is chosen as an outer normal direction to ∂S. The case when p is a cusp was already treated in Theorem 2.1. Double points can be treated similarly to regular points, so we give complete details only in the regular case.

Thus suppose that p is a regular boundary point at distance at least δ from all singular boundary points, where $\delta > 0$ is independent of n. It will suffice to prove that there is a constant $C = C(\delta)$ such that

$$\zeta \in S^c, \quad |z| \le \log n \quad \Rightarrow \quad R_n(z) \le Ce^{-2x^2}, \qquad (z = x + iy). \tag{3.1}$$

As before, we shall use the estimate $\mathbf{R}_n(\zeta) \le Cne^{-n(Q-\check{Q})(\zeta)}$. Let N_p be the outer normal direction at p and let V be the harmonic continuation of $\check{Q}|_{S^c}$ to a neighbourhood of p. We write $\delta_n = C\log n/\sqrt{n}$.

By Taylor's formula we have, for $x > 0$, with $M := Q - V$,

$$M\left(p + \frac{x}{\sqrt{n\Delta Q(p)}}N_p\right) = \frac{1}{2n\Delta Q(p)}\frac{\partial^2 M}{\partial n^2}(p)\,x^{\,2} + \frac{1}{6(n\Delta Q(p))^{3/2}}\frac{\partial^3 M}{\partial n^3}(p+\theta)\,x^{\,3},$$

where "$\partial/\partial n$" is exterior normal derivative and $\theta = \theta(n,p)$ is some number between 0 and δ_n. However, since p is a regular point and $Q = V$ on ∂S we have $\frac{\partial^2 M}{\partial s^2}(p) = 0$ where $\partial/\partial s$ denotes differentiation in the tangential direction. Adding this to the above Taylor expansion, using that $(\partial_s^2+\partial_n^2)M = 4\Delta M = 4\Delta Q$, we obtain, when $|z|\le C\log n$,

$$nM\left(p+\frac{z}{\sqrt{n\Delta Q(p)}}N_p\right) = 2\,x^2 + O\left(\frac{\log^3 n}{\sqrt{n}}\right),\qquad 0\le x\le \delta_n. \tag{3.2}$$

This proves the desired implication (3.1).

3.2. **Boundary points near cusps.** Assume now that the droplet S has a $(\nu,2)$-cusp at the origin, pointing in the positive real direction. Here $\nu\ge 5$ is an odd integer.

Fix $T>0$ and a large integer n. Let p_n be the unique point such that the disk

$$D_n := D\left(p_n; T/\sqrt{n\Delta Q(0)}\right)$$

is inside S and is tangent to ∂S at two points; one on each arc terminating in the cusp. Let q_n be the point in $(\partial D_n)\cap(\partial S)$ in the upper half-plane. It is easy to check that $|\,p_n\,|\sim n^{-1/\nu}$ and $|\,q_n\,|\sim n^{-1/\nu}$. (Here "$a_n\sim b_n$" means that there is a constant $C>0$ such that $C^{-1}a_n\le b_n\le Ca_n$.)

We rescale about q_n as follows. Let $e^{i\theta_n}$ be the outer normal to ∂S at q_n and put

$$z = e^{-i\theta_n}\sqrt{n\Delta Q(0)}\,(\zeta-q_n)\,. \tag{3.3}$$

The rescaled droplet near q_n looks roughly like the infinite strip

$$0<\operatorname{Re} z<2T,$$

and the image of the cusp is far away on the positive imaginary axis at a distance $\sim n^{1/2-1/\nu}$ from the origin.

Let Φ be a conformal map $\mathbb{C}_+\to U$ where $\mathbb{C}_+$ is the upper half-plane and U is the complement of the component of S containing 0. We assume that $\Phi(0)=0$ and

$$\Phi(\lambda) = -\frac{1}{2}\lambda^2 + O(\lambda^3),\qquad \lambda\to 0.$$

Since the cusp is conformal, Φ extends analytically across $\mathbb{R}$.

Now let σ_n be the point on $\mathbb{R}$ such that $\Phi(\sigma_n)=q_n$ and let $\varepsilon_n=\alpha_n+i\beta_n$ be the point in $\mathbb{C}$ such that

$$\Phi(\sigma_n+\varepsilon_n) = q_n + \frac{ze^{i\theta_n}}{\sqrt{n\Delta Q(0)}}.$$

We will be dealing with the Taylor expansion about σ_n given by

$$\Phi(\sigma_n+\varepsilon_n) = q_n + \Phi'(\sigma_n)\varepsilon_n + \Phi''(\sigma_n)\frac{\varepsilon_n^2}{2}+\cdots.$$

We now note the simple approximation

$$|\,\Phi'(\sigma_n)\,|\asymp|\,\sigma_n\,|\asymp|\,q_n\,|^{1/2}\asymp n^{-1/2\nu}, \tag{3.4}$$

which implies that the condition $|z|\le\log n$ in (3.3) corresponds to

$$|\,\varepsilon_n\,|\le C_0\frac{\log n}{n^{1/2-1/(2\nu)}}. \tag{3.5}$$

Observe first that it follows from (3.5) that

$$n\,|\,\varepsilon_n\,|^3\le C_0 n\frac{\log^3 n}{n^{3/2-3/(2\nu)}}\le C_0\frac{\log^3 n}{n^{(\nu-3)/(2\nu)}}\to 0,\quad (n\to\infty) \tag{3.6}$$

where the last assertion holds true since $\nu\ge 5$.

After these observations, we can prove the required decay about the moving point q_n.

Lemma 3.2. *Let $R_n(z)$ be the rescaled one-point function according to the rescaling* (3.3)*. There is then a constant C such that $R_n(z) \le Ce^{-2x^2}$ when $|z| \le \log n$ and $x = \operatorname{Re} z < 0$.*

Proof. We know that $\mathbf{R}_n(\zeta) \le Cne^{-n(Q-\check{Q})(\zeta)}$. We will compare $2x^2$ with $n(Q-\check{Q})(\zeta)$. We shall be done if we can prove that

$$2\,x^2 = n(Q_\Phi - \check{Q}_\Phi)(\sigma_n + \varepsilon_n) + o(1). \tag{3.7}$$

However, by the estimate (2.11) we have

$$n(Q_\Phi - \check{Q}_\Phi)(\sigma_n + \varepsilon_n) = 2n\Delta Q(\Phi(\sigma_n + \alpha_n))|\Phi'(\sigma_n + \alpha_n)|^2 \beta_n^2 + O(n\beta_n^3).$$

It follows from (3.4) – (3.6) that

$$n(Q_\Phi - \check{Q}_\Phi)(\sigma_n + \varepsilon_n) = 2n\Delta Q(0)\,|\Phi'(\sigma_n)|^2\,\beta_n^2 + o(1).$$

Indeed, by (3.4) and (3.5), we have

$$n\Delta Q(\Phi(\sigma_n + \alpha_n))|\Phi'(\sigma_n + \alpha_n)|^2\beta_n^2 - n\Delta Q(0)\,|\Phi'(\sigma_n + \alpha_n)|^2\,\beta_n^2 = O(n|\sigma_n + \alpha_n|^3\beta_n^2) = o(1)$$

and

$$n\Delta Q(0)\,|\Phi'(\sigma_n + \alpha_n)|^2\,\beta_n^2 - n\Delta Q(0)\,|\Phi'(\sigma_n)|^2\,\beta_n^2 = O(n|\sigma_n\alpha_n|\beta_n^2) = o(1).$$

Inserting here the Taylor expansion of Φ about σ_n we find that

$$z = e^{-i\theta_n}\sqrt{n\Delta Q(0)}\left(\Phi'(\sigma_n)(\alpha_n + i\beta_n) + O(|\varepsilon_n|^2)\right), \qquad e^{i\theta_n} = i\frac{\Phi'(\sigma_n)}{|\Phi'(\sigma_n)|}.$$

It follows from (3.4) – (3.6) that

$$2x^2 = 2n\Delta Q(0)\,|\Phi'(\sigma_n)|^2\,\beta_n^2 + o(1),$$

which leads to

$$2x^2 - n(Q-\check{Q})\Big(q_n + \frac{ze^{i\theta_n}}{\sqrt{n\Delta Q(0)}}\Big) = o(1).$$

The proof of the lemma is finished. □

Let us now change the rescaling so that p_n is mapped to the origin,

$$z = e^{-i\theta_n}\sqrt{n\Delta Q(0)}(\zeta - p_n),$$

and write R_n for the rescaled 1-point function. Since

$$|\,p_n - q_n\,| = \operatorname{Re}(q_n + p_n) + o(1) = T/\sqrt{n\Delta Q(0)}$$

we obtain the estimate

$$R_n(z) \le Ce^{-2(\,|\,x\,|\,-T\,)^2}, \qquad |z| \le \log n, \quad x = \operatorname{Re} z. \tag{3.8}$$

3.3. **Boundary points near double points.** The case of double points is a rather routine application of (2.11), which applies also in the case when the conformal mapping Φ extends univalently to a neighbourhood of the origin.

Of course, in this situation one could perform a more direct analysis using the fact that M extends real-analytically across each of the two arcs of $\partial\mathcal{S}$ which meet at the double point. Hence, one may apply Taylor's formula directly at the point q_n of rescaling and proceed as in the case of a regular point.

We have now completely proved Theorem 3.1. Since the estimate (3.8) holds also for the case of double points, we obtain as a corollary the estimate (1.7) in Theorem II. q.e.d.

4. Proofs of the main results

4.1. **The triviality theorem.** We now prove Theorem I. Suppose that p is either a double point or a cusp of type $(\nu, 2)$ where $\nu > 3$ and that $\Delta Q(p) > 0$ and rescale about p according to

$$z = e^{-i\theta}\sqrt{n\Delta Q(p)}(\zeta - p)$$

where $e^{i\theta}$ is one of the normal directions to ∂S at p.

Let $K = G\Psi$ be a limiting kernel. We must prove that the limiting 1-point function $R(z) = K(z, z) = \Psi(z, z)$ vanishes identically. To this end, we shall use the corresponding *holomorphic kernel*

$$L(z, w) = e^{z\bar{w}}\Psi(z, w).$$

We can now finish the proof of Theorem I.

The function $S(z) := |z|^2 + \log R(z)$ is subharmonic, see e.g. Lemma 4.3 in [3] for an argument. Next recall the estimate $R(z) \le Ce^{-2x^2}$ for some constant C, obtained in Theorem 2.1 for cusps and in §3.1 in the case of double points. This gives the bound

$$S(z) \le \log C + y^2 - x^2.$$

But $y^2 - x^2$ is harmonic, so the function $\tilde{S} = S - (y^2 - x^2)$ is subharmonic and bounded above by $\log C$. Hence it is constant, i.e.,

$$R(z) = Ce^{-2x^2}$$

for a (new) constant C. If R is nontrivial we can assume that $C = 1$. By polarization, then

$$\Psi(z, w) = e^{-(z+\bar{w})^2/2},$$

so the kernel $L(z, w) = e^{z\bar{w}}\Psi(z, w)$ must satisfy

$$\int |L(0, w)|^2 e^{-|w|^2} dA(w) = \int |\Psi(0, w)|^2 e^{-|w|^2} dA(w) = \int e^{-2x^2} dA = \infty.$$

This contradicts Lemma 4.9 in [3]. The contradiction shows that we must have $C = 0$. q.e.d.

We are grateful to Håkan Hedenmalm for helpful communication in connection with the above proof, [11].

4.2. **Proof of the existence theorems.** We now prove the existence theorems, Theorem II and III.

Let p be either a $(\nu, 2)$-cusp with $\nu \ge 5$ or a double point. In both cases we assume that $\Delta Q(p) > 0$. Also fix a number $T > 0$. For a given $n \in \mathbb{Z}_+$, we let p_n be a point in S whose distance to the boundary is $T/\sqrt{n\Delta Q(p)}$ and whose distance to p is minimal.

We rescale about p_n,

$$z_j = e^{-i\theta_n}\sqrt{n\Delta Q(p)}(\zeta_j - p_n), \qquad j = 1, \ldots, n,$$

where the angle θ_n is chosen so that the image of the point p lies on the positive imaginary axis.

Note that as $n \to \infty$, the image of S near p_n looks approximately like the strip

$$\Sigma_T : \quad -T < \operatorname{Re} z < T. \tag{4.1}$$

Let K_n be the kernel of the rescaled system $\Theta_n = \{z_j\}_1^n$. We write $R_n(z) = K_n(z, z)$. By Lemma 1.1 we know that there is a sequence of cocycles c_n such that every subsequence of $c_n K_n$ has a subsequence converging to $G\Psi$ where Ψ is some Hermitian entire function. It remains only to show that the function $R(z) = \Psi(z, z)$ does not vanish identically if T is large enough.

To this end, we shall use the estimate in [3], Theorem 5.4,

$$|\mathbf{R}_n(\zeta) - n\Delta Q(\zeta)| \le C\left(1 + ne^{-n\ell\Delta Q(\zeta)\cdot\delta(\zeta)^2}\right), \quad \zeta \in S,$$

where ℓ is a positive constant and $\delta(\zeta) = \operatorname{dist}(\zeta, \partial S)$. If we choose $\zeta = p_n$ where $\delta(p_n) = T/\sqrt{n\Delta Q(p_n)}$, we obtain for the rescaled 1-point function R_n that

$$|\, R_n(0) - 1\,| \le Ce^{-\ell T^2}.$$

choosing T sufficiently large that the right hand side is < 1, we obtain that $R(0) > 0$. An application on Lemma 1.1 now shows that $R(z) > 0$ for all $z \in \mathbb{C}$. q.e.d.

5. Concluding remarks

We shall here discuss a family of natural candidates of the limiting point fields whose existence is guaranteed by theorems II and III.

Let p be a singular point; for definiteness, let us say it is a cusp.

For a "large" T, consider the point $p_n \in S$ at distance $T/\sqrt{n\Delta Q(p)}$ from ∂S being closest to p and rescale about p_n as in Theorem II. Let $K = G\Psi$ be a limiting kernel in Lemma 1.1; then K is non-trivial.

Recall that a Hermitian-entire function Ψ is called translation invariant (in short: *t.i.*) if it takes the form $\Psi(z, w) = \Phi(z + \bar{w})$ for some entire function Φ, which entails that $\Psi(z + it, w - it)$ is independent of $t \in \mathbb{R}$. We do not know that a limiting kernel must be *t.i.* but it seems to be a reasonable assumption, since the rescaled droplet looks like the strip

$$\Sigma_T = \{z;\, -T \le \operatorname{Re} z \le T\}.$$

In any case, we shall now use theory from [3] to narrow down the set of possible limiting kernels, under the extra hypothesis of translation invariance.

First of all, Theorem E in [3] implies that a *t.i.* limiting kernel $K(z, w) = G(z, w)\Phi(z + \bar{w})$ has the "Gaussian representation"

$$\Phi(z) = \gamma * f(z) = \int_{-\infty}^{+\infty} \gamma(z - t) f(t)\, dt,$$

where $\gamma(z) = (2\pi)^{-1/2} e^{-z^2/2}$ is the Gaussian kernel and $f(t)$ is some Borel function with $0 \le f \le 1$. Secondly, Theorem F says that the function Φ above gives rise to a solution to Ward's equation (1.5) if and only if $f = \chi_I$ is the characteristic function of some *interval* I.

Finally, by the estimate (1.7), we know that the function $R(z) = \Phi(z + \bar{z})$ must satisfy

$$\Phi(2x) \le Ce^{-2(|x| - T)^2}, \qquad x \in \mathbb{R}.$$

In order for this to be consistent with the identity

$$\Phi(x) = \gamma * \chi_I(x) = \frac{1}{\sqrt{2\pi}} \int_I e^{-(x-t)^2/2}\, dt,$$

we must have $I \subset [-2T, 2T]$.

Thus if there exists a limiting *t.i.* kernel, it necessarily has the structure $\Phi = \gamma * \chi_I$ where I is an interval contained in $[-2T, 2T]$.

For reasons on symmetry, it is natural to assume that an interval I as above can be chosen to be symmetric, i.e. $I = [-s/2, s/2]$ for some number $s = S(T)$ between 0 and $4T$.

Let us denote by Φ_s the function

$$\Phi_s(z) := \gamma * \chi_{(-s/2, s/2)}(z) = \frac{1}{\sqrt{2\pi}} \int_{-s/2}^{s/2} e^{-(z-t)^2/2}\, dt.$$

We know from Lemma 7.7 in [3] that the kernel $K_s(z, w) := G(z, w)\Phi_s(z + \bar{w})$ appears as the correlation kernel of a unique point field.

It is close at hand to guess that one of the K_s $(0 < s \leq 4T)$ will appear as a limiting kernel in Theorem II.

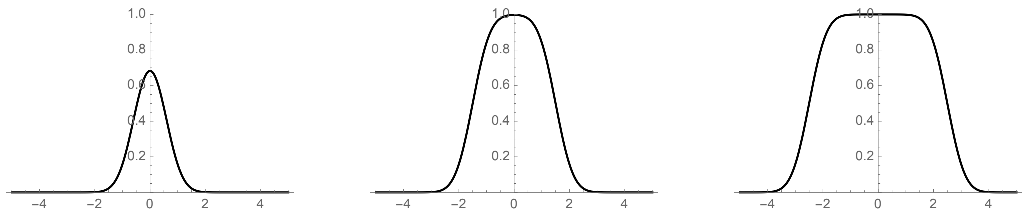

FIGURE 3. The graph of $R_s(x) := \gamma * \chi_{(-s/2,s/2)}(2x)$ for $s = 2$, $s = 5$, and $s = 8$.

To further corroborate our point, we may observe that the kernel K_s interpolates in a natural way between the trivial kernel $K_0 = 0$ (at the singular point) and the Ginibre kernel $K_\infty = G$ (the bulk regime). See Figure 3.

Now fix a large enough number T. The conditions shown for a limiting 1-point function R, i.e., that it be non-trivial, satisfy Ward's equation, and have the decay $R(x) \leq Ce^{-2(|x|-T)^2}$, are not enough to fix a solution uniquely, even under the assumption of translation invariance.

In the paper [3], the corresponding question was solved for regular boundary points, by using the 1/8-formula ([3], Theorem D)

$$\int_{\mathbb{R}} t \cdot (\chi_{(-\infty,0)}(t) - R(t))\, dt = \frac{1}{8}. \tag{5.1}$$

This suggests that, in order to single out a unique solution, one could seek some additional condition similar to (5.1), which is to hold for a limiting kernel near a singular boundary point. One of the main challenges with this is that the boundary fluctuation theorem from [2], which was used to prove the condition (5.1), is so far only known for connected droplets with everywhere smooth boundary. Thus, it would seem natural to try to extend the methods in [2] to domains with singular boundary points.

We also mention that the reason that we had to insist that the parameter T be sufficiently large is that we have used the technique of Hörmander estimates to ensure non-triviality of the 1-point function, and Hörmander estimates require a certain distance to the boundary. On the other hand, if one could prove a generalized boundary fluctuation theorem, as suggested above, one would probably obtain non-triviality also for small $T > 0$.

There is a parallel "hard-edge" theory, where one confines the system $\{\zeta_j\}_1^n$ to the droplet by redefining Q outside the droplet to be $+\infty$ there. Similar results to those studied in this paper (the "free boundary" case) can be obtained; cf. [3] and [4]. In the following we will freely use results from the forthcoming paper [4].

If one rescales near a singular boundary point p in the hard-edge case, at distance $T/\sqrt{n\Delta Q(p)}$ from the boundary, one obtains limiting kernels of the form

$$K(z, w) = G(z, w)\, \Psi(z, w)\, \chi_{(-T,T)}(\operatorname{Re} z)\, \chi_{(-T,T)}(\operatorname{Re} w)$$

where Ψ is some Hermitian-entire function. In the *t.i.* case $\Psi(z, w) = \Phi(z + \bar{w})$, we anticipate that Φ necessarily has a representation

$$\Phi = \gamma * \left(\frac{\chi_I}{F_T}\right)$$

where $F_T = \gamma * \chi_{(-2T,2T)}$ and I is some interval. In this case, the natural candidates for limiting point fields again correspond to symmetric intervals $I = (-2T, 2T)$ (or perhaps $I = (-s, s)$ for some suitable value of $s = s(T)$). We are thus led to study the functions of the type

$$H_T(z) := \frac{1}{\sqrt{2\pi}} \int_{-2T}^{2T} \frac{e^{-(z-t)^2/2}}{F_T(t)}\, dt.$$

The corresponding "1-point function" is then $R_T^h(z) := H_T(z + \bar{z})\chi_{(-T,T)}(\operatorname{Re} z)$.

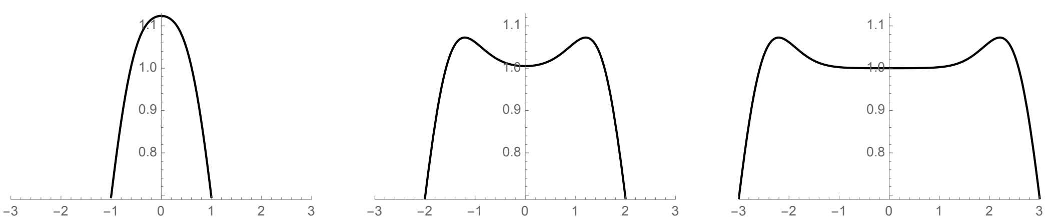

FIGURE 4. The graph of R_T^h restricted to the reals, for $T = 2$, $T = 5$, and $T = 8$.

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