Random and optimal configurations in complex function theory

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Doctoral Thesis
Stockholm, Sweden 2018

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TRITA-SCI-FOU 2018:20
ISBN 978-91-7729-783-3

Akademisk avhandling som med tillstånd av Kungl Tekniska högskolan framlägges till offentlig granskning för avläggande av teknologie doktorsexamen i matematik Fredag den 8 juni 2018 kl 13.00 i sal E2, Kungl Tekniska högskolan, Lindstedtsvägen 3, Stockholm.

Tryck: Universitetsservice US AB

Abstract

This thesis consists of six articles spanning over several areas of mathematical analysis. The dominant theme is the study of random point processes and optimal point configurations, which may be thought of as systems of charged particles with mutual repulsion. We are predominantly occupied with questions of universality, a phenomenon that appears in the study of random complex systems where seemingly unrelated microscopic laws produce systems with striking similarities in various scaling limits. In particular, we obtain a complete asymptotic expansion of planar orthogonal polynomials with respect to exponentially varying weights, which yields universality for the microscopic boundary behavior in the random normal matrix (RNM) model (Paper A) as well as in the case of more general interfaces for Bergman kernels (Paper B). Still in the setting of RNM ensembles, we investigate properties of scaling limits near singular points of the boundary of the spectrum, including cusps points (Paper C). We also obtain a central limit theorem for fluctuations of linear statistics in the polyanalytic Ginibre ensemble, using a new representation of the polyanalytic correlation kernel in terms of algebraic differential operators acting on the classical Ginibre kernel (Paper D). Paper E is concerned with an extremal problem for analytic polynomials, which may heuristically be interpreted as an optimal packing problem for the corresponding zeros. The last article (Paper F) concerns a different theme, namely a sharp topological transition in an L^p-analogue of classical Carleman classes for $0 < p < 1$.

Sammanfattning

Denna avhandling utgörs av sex artiklar inom olika delar av matematisk analys. Temat är, med ett undantag, studiet av slumpmässiga punktprocesser och optimala punktkonfigurationer i planet, vilka kan tänkas på som system av laddade partiklar som repellerar varandra. Huvudsakligen undersöker vi något som kallas för universalitetsfenomenet – ett fenomen inom sannolikhetsteori och statistisk mekanik som syftar på observationen att stora klasser av komplexa slumpmässiga system, med till synes orelaterade definierande lagar, uppvisar slående likartat beteende. Specifikt lyckas vi hitta en asymptotisk utveckling av ortogonalpolynom med avseende på exponentiellt varierande vikter i planet, och med hjälp av detta resultat visar vi att randbeteendet hos egenvärden till normala slumpmatriser är universellt (Artikel A). Vi använder en vidareutveckling av dessa tekniker för att undersöka övergångsbeteendet för mer generella Bergmankärnor (Artikel B), inkluderande så kallade partiella kärnor. Artikel C handlar även den om skalningsgränser för normala slumpmatrisprocesser, men i detta fall behandlas singulära randpunker som spetsar. I artikel D visar vi att fluktuationer av lineära statistika i den polyanalytiska Ginibreensembeln uppfyller en central gränsvärdessats, genom att finna en representation av korrelationskärnan som en speciell differentialoperator verkande på den klassiska Ginibrekärnan. Artikel E behandlar en klass av extremalproblem för analytiska polynom, som kan tänkas på som packningsproblem för polynomens nollställen. Den sista artikeln (Artikel F) avviker från temat, och handlar om en skarp topologisk övergång för L^p-analoger till klassiska Carlemanklasser i fallet när $0 < p < 1$.

Contents

Acknowledgements

First and foremost, I want to thank my advisor Håkan Hedenmalm. I am deeply impressed by your insights and by the way in which you do mathematics. I am grateful for your generosity in sharing yours thoughts and ideas with me, and not least for encouraging me to come back to mathematics five years ago when I had my academic focus elsewhere.

I would like to thank my long-time office mate Simon Larson as well as Gustav Sædén Ståhl for great friendship and constant encouragement. I would also like to thank my colleagues in the math department at KTH, in particular Anders, Björn, Douglas, Eric, Erik, Gaultier, Gustav, Joakim, Julian, Michael and Tomas, who made this place a great environment during my years here. I also spent a semester at the department of mathematics at Lund University in 2015, and would like to express my gratitude towards the people there for contributing to a good research environment.

I want to thank my co-authors as well as previous mentors, in particular Antti Haimi, Anders Olofsson, Yacin Ameur, Anders Tingström, and Jonas Broms for rewarding collaborations, important advice and for being good friends. I also want to mention some people from the complex analysis community, including Alexandru Aleman, Avner Kiro, Adi Glücksam, Alon Nishry, Misha Sodin, Daniel Seco, Kristian Seip, Kalle Perfekt, Alan Sola and Quim Ortega-Cerdà, who have made me feel welcome in the world of mathematical research, no matter where we have met around the globe.

I want to thank my parents, siblings and grandparents: you are and have always been the best family imaginable. I am also grateful to my friends Rasmus, Joel, Elias, Sverker and Erik for always being there. Finally, to you Hanna, I want to express my deepest gratitude for your continued loving support and patience. My life is so much better and so much more fun with you in it.

Part I

Introduction and summary

1 Background

A few words to the reader

The aim of the present chapter is to supply the reader with some motivation as to why the topics studied in the thesis are of interest, and provide some general background on the tools and techniques used. The new developments of this thesis are encountered in Chapter 2, where the results of the six articles are summarized.

The universality phenomenon

In order to describe a random event, we often speak of its *probability density.* Put simply, the density may be thought of as the continuous analogue of a histogram: for each possible outcome, we plot how likely it is to occur in relation to other outcomes. Most people have probably seen plots of probability densities of real-valued random variables, the most famous of which is the *bell curve*, i.e. the density of the normal distribution. However, apart from being an integrable and non-negative function, there are no particular restrictions to what a density function could look like. One could easily construct a random process whose associated density function fits almost any given shape or form.

Why, then, is the bell curve so famous? The fantastic answer is that it is due to the fact that it is inherently more common than other distributions. In the world around us, there is a whole army of normally distributed random variables. Be it the error in a particular type of measurement, the variation in length within a large population, or almost any given homeostatic parameter (e.g. blood pressure or blood glucose levels), chances are that if you record the measurements and plot an approximation of the corresponding probability density function, the bell curve will appear[1].

The answer to the question as to *why the normal distribution shows up time and again* is still more fascinating and is of central importance to the field of probability theory. The following statement, a version of the *central limit theorem*, supplies

[1]This is a slight oversimplification; close to the mean value the variables alluded to are approximately normally distributed, but for non-negative random quantities the normal distribution should be replaced by a law known as the *log-normal distribution.*

the key to understand it, and is our first example of the phenomenon of *universality.*

Let $X_1, X_2, X_3, \dots$ *be a sequence of independent random variables with mean* 0 *and variance* 1*. If the sequence satisfies a mild condition known as* Lindeberg's condition*, then the sequence of random variables*

$$S_n = \frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}}, \qquad n = 1, 2, 3, \ldots$$

converges in distribution to a normally distributed random variable with mean 0 *and variance* 1*.*

There are a few technical terms that are not explained, but the take-home message is the following: if a random quantity S_n is a compound of a large number n of independent random quantities in the form above, then it is approximately normally distributed. It doesn't matter what the laws governing the individual contributions X_i are, as long as they are independent from each other: if we form their normalized sum S_n to produce a new random number, the bell curve should emerge, seemingly from nowhere. The reader who is interested in details about this version, as well as other versions of the central limit theorem (CLT), may find this in most probability textbooks, e.g. [63, Ch. III.4].

It is likely that the approximate normality of the distributions of the above-mentioned real-world quantities may be elucidated by the CLT. There are a large number of different determinants of the error of a measurement, for example, which do not correlate (too much) with each other, and hence the observed error should be approximately normally distributed.

That the central limit theorem is true is a fascinating fact in itself. However, the story doesn't end there. The special role played by the normal distribution among all possible distributions is one example of a phenomenon called *universality.* This refers to the observation that for many large (random) systems, as complexity increases towards the infinite, the limiting behavior of the system tends to admit a simple description in terms of a general law. In a wide range of different classes of random (or random-like) objects, the same distribution or limiting law stands out, much in the same way as the Gaussian does in the context of the CLT. As an example, we might mention the sine kernel $\frac{\sin(x-y)}{\pi(x-y)}$, which not only appears in a wide range of random matrix models, but which has also been shown to govern the behaviour of eigenvalues of certain Schrödinger equations on the half-line [54], as well as shifted moments of the Riemann zeta function [50].

As is pointed out by Deift [28], there is a physical intuition that may begin to explain these observations, or at least serves to illustrate the presence of the wider phenomenon. He argues that all physical systems in equilibrium must, at a macroscopic level, obey the laws of thermodynamics. At the same time, a physical system may be formed from different microscopic constituents with their own particular interactions in many different ways. The fact that the same laws of thermodynamics always arise is a manifestation of universality!

Coulomb gas ensembles

One of the main topics of this thesis is random point processes. In particular, we are interested in planar processes where the points experience mutual repulsion, in which case the points may be thought of as system of charged particles moving about in the complex plane $\mathbb{C}$. At a technical level, a random planar point process with n particles is described by a probability measure $\mathbb{P}_n$ on the configuration space $\Omega \subset \mathbb{C}^n$. We will encounter different classes of probability measures, but in all cases they will come in the form of *Gibbs measures.* Such a measure is based on an energy functional $\mathcal{E}$, which to each tuple $(z_1, \ldots, z_n) \in \Omega$ associates an energy $\mathcal{E}(z_1, \ldots, z_n) \in \mathbb{R} \cup \{+\infty\}$. Without exception, we will consider energies which decompose into two parts: the *particle interaction energy*, which encodes the mutual repulsion between particles, and the *field interaction energy* which describes the interaction of the charged particles with an external field. The field interaction typically only depends on the position of each individual particle, and furthermore decomposes as a sum, with a contribution for each point z_j. We may consequently write

$$\mathcal{E}(z_1, \ldots, z_n) = \mathcal{E}_{\text{int}}(z_1, \ldots, z_n) + \sum_{j=1}^{n} \mathcal{E}_{\text{field}}(z_i).$$

As the particles repel each other, the interaction energy $\mathcal{E}_{\text{int}}(z_1, \ldots, z_n)$ increases if the points are densely packed. We require the external field to balance this feature, which is accomplished by requiring that $\mathcal{E}_{\text{field}}(z)$ is large as $z \to \infty$.

The Gibbs probability distribution associated to such an energy functional is obtained by putting

$$\mathrm{d}\mathbb{P}_n(z_1, \ldots, z_n) = \frac{1}{Z_n} \mathrm{e}^{-\mathcal{E}(z_1, \ldots, z_n)} \mathrm{dA}^{\otimes n}(z_1, \ldots, z_n),$$

where Z_n is a normalizing constant which ensures that the probability of the tuple being somewhere in the configuration space Ω equals one, and $\mathrm{dA}^{\otimes n}$ denotes the standard volume form on $\mathbb{C}^n$, normalized by π^{-n} so that the unit polydisk has unit volume. At this point, the connection between the concepts of *optimal* and *random point configurations* becomes apparent: The optimal configurations $(z_1, \ldots, z_n)$ are the ones for which $\mathcal{E}(z_1, \ldots, z_n)$ is minimal, and one may immediately observe that the random configurations for which the energy is low are the most likely, while high-energy configurations tend to occur less often.

In two dimensions, the Coulomb interaction energy describes the electrostatic interaction between particles with equal charge at locations z and w, and is proportional to $\log|z - w|^{-1}$. A natural interaction energy $\mathcal{E}_{\text{int}}$ is obtained by adding the Coulomb interaction for all pairs of particles. If the field interaction energy is given by $\mathcal{E}_{\text{field}}(z) = 2mQ(z)$ for some smooth function Q and field strength parameter m, the energy functional takes the form

$$\mathcal{E}(z_1, \ldots, z_n) = \sum_{j<k} \log \frac{1}{|z_j - z_k|} + 2m \sum_{j=1}^{n} Q(z_j), \tag{1.0.1}$$

and the resulting Gibbs measure may be calculated as

$$\mathrm{d}\mathbb{P}_{n,m,Q}(z_1,\ldots,z_n) = \frac{1}{Z_n}\prod_{i<j}|z_i - z_j|^2 \mathrm{e}^{-2m\sum_{j=1}^n Q(z_j)}\mathrm{dA}^{\otimes n}(z_1,\ldots,z_n). \quad (1.0.2)$$

The random process $\Theta_{n,m,Q}$ obtained by picking a tuple $(z_1,\ldots,z_n)$ from the distribution induced by $\mathrm{d}\mathbb{P}_{n,m,Q}$ is known as *two dimensional Coulomb gas.*

It is known that as the number n of particles tends to infinity, proportional to the field strength $m = \tau^{-1}n$ for some positive parameter τ, the particles tend to condense to a compact set $\mathcal{S}_\tau$, called the *spectral droplet.* Moreover, if the potential Q satisfies some mild regularity conditions and is strictly subharmonic in a neighbourhood of $\mathcal{S}_\tau$, the limiting density is described by the *equilibrium measure* $\mathrm{d}\mu_{\mathrm{Q},\tau}$ for a weighted logarithmic potential theory problem. We will not go into details about this important connection with potential theory, but it deserves to be pointed out that the equilibrium measure $\mathrm{d}\mu_{\mathrm{Q},\tau}$ is the minimizer of the energy functional

$$I_Q[\mu] = \int_{\mathbb{C}^2} \log\frac{1}{|z-w|}\mathrm{d}\mu(z)\mathrm{d}\mu(w) + 2\int_{\mathbb{C}} Q(z)\mathrm{d}\mu(z) \quad (1.0.3)$$

among all positive Borel measures μ of total mass τ. We remark that it is in general difficult to find the equilibrium measure, but as soon as its support is known one may immediately recover it from the potential by the relation $\frac{\mathrm{d}\mu_{Q,\tau}}{\mathrm{dA}} = 2\Delta Q\, 1_{\mathcal{S}_\tau}$. We should remark that Δ is one quarter of the usual laplacian, which is attractive in view of the factorization

$$\Delta = \frac{1}{4}\left(\partial_x^2 + \partial_y^2\right) = \partial\bar{\partial},$$

where ∂ and $\bar{\partial}$ denotes the *Wirtinger derivatives*, given by

$$\partial = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y) \quad \text{and} \quad \bar{\partial} = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y),$$

respectively.

Intuitively, it is not surprising that the equilibrium measure describes the limiting distribution of the point configurations, as the energy $\mathcal{E}(z_1,\ldots,z_n)$ may be thought of as the energy I_Q of (1.0.3) associated to a discrete measure $\mu_\Theta = \mu_{\Theta,m}$ given by

$$\mu_\Theta = \frac{1}{m}\sum_{z\in\Theta}\delta_z,$$

for a point configuration $\Theta = (z_1,\ldots,z_n)$, where the notation $z\in\Theta$ indicates that $z = z_i$ for some i with $1\le i\le n$. In probabilistic terms, the above statement that the points condense to $\mathcal{S}_\tau$ may be stated as follows: the random empirical measure associated to a point configuration $\Theta_{n,m}$ drawn from $\mathbb{P}_{n,m,Q}$ converges to the equilibrium measure in the sense that for any bounded continuous function f, it holds that

$$\mathbb{P}_{n,m,Q}\Big\{\Big|\int_{\mathbb{C}} f(z)\mathrm{d}\mu_{\Theta_{n,m}}(z) - \int_{\mathbb{C}} f(z)\mathrm{d}\mu_{Q,\tau}\Big| = \mathrm{O}\big(m^{-\frac{1}{2}}\log m\big)\Big\} \geq 1 - \mathrm{e}^{-c_0 m},$$
$$n = \tau m + \mathrm{O}(1) \to \infty,$$

for some constant $c_0 > 0$, see e.g. [12]. For the details of the relation between Coulomb gas ensembles and potential theory, we refer to the work [39], and for a general account of weighted logarithmic potential theory the reader may consult the book [60].

We distinguish three regions based on this convergence: the interior $\mathcal{S}_\tau^\circ$ called *bulk* where the density is strictly positive, the *boundary* $\partial\mathcal{S}_\tau$ where the density has a steep decay, and the exterior, or the *off-spectral* region $\mathcal{S}_\tau^c$, where the density vanishes in the many-particle limit.

The determinantal property and the Bergman kernel

The Coulomb gas ensembles belong to a particularly important category, known as the class of *determinantal point processes*. For a random point configuration $\Theta = (z_1, \ldots, z_n)$ associated to a measure $\mathbb{P}$, we denote by $\#A$ the random quantity describing the number of particles in a set A, that is

$$\#A = |\Theta \cap A|,$$

where $|\cdot|$ denotes the size of a set. The *k-point correlation function* is then defined by

$$\varrho_k(z_1, \ldots, z_k) = \lim_{\epsilon \to 0^+} \frac{\mathbb{P}\big\{\#\mathbb{D}(z_j, \epsilon) \geq 1,\ j = 1, \ldots, k\big\}}{\epsilon^k},$$

and describe the intensities for finding particles located simultaneously at the points $z_1, \ldots, z_k$. Here and in the following, we use the notation $\mathbb{D}(z_0, r)$ for the open disk or radius r centered at z_0. If $r = 1$ and $z_0 = 0$, we simply write $\mathbb{D}$. A process is said to be determinantal if the correlation functions have a special structure: for each $k = 1, 2, \ldots, n$, the function ϱ_k is given by

$$\varrho_k(z_1, \ldots, z_k) = \det\big(\mathbf{K}_n(z_i, z_j)\big)_{1 \leq i,j \leq k} \tag{1.0.4}$$

for some function $\mathbf{K}_n$, called the *correlation kernel*.

The determinantal structure greatly decreases the complexity one needs to take into account when studying such a process. Indeed, a single function of two variables governs all correlation functions, and typically the correlation functions alone determine all characteristics of the underlying point process, as demonstrated by Soshnikov [66]. For this reason, once one has established the determinantal nature of a point process, the challenge becomes to understand the properties of the kernel, often by function theoretic rather than probabilistic means. Another key consequence of the determinantal structure is that the process is automatically repulsive. To see this property demonstrate itself, we merely need to note that if $z_{i_1} = z_{i_2}$ where $i_1 \neq i_2$, the correlation functions (1.0.4) are given by determinants of matrices with two identical rows, and consequently vanishes. The reader interested in further properties of determinantal point processes may consult the surveys of Borodin [17] and Hough, Krishnapur, Peres and Viràg [45].

Let us return to the Coulomb gas process (1.0.2). That these processes are determinantal may be seen as a consequence of the appearance of the squared modulus of the Vandermonde determinant

$$\prod_{i<j}|z_i - z_j|^2 = |\det V_n(z_1, \ldots, z_n)|^2$$

where $V_n(z_1, \ldots, z_n) = (z_i^j)_{1\le i,j\le n}$, which appears in the density. Indeed, by performing elementary row operations on the Vandermonde matrix, one may re-arrange it so that

$$\prod_{i<j}(z_i - z_j) = \det V_n(z_1, \ldots, z_n) = \det(z_i^j)_{1\le i,j\le n} = \det(\pi_j(z_i))_{1\le i,j\le n},$$

where $(\pi_j)_j$ is any sequence of monic polynomials such that $\deg \pi_j = j$. We shall require in addition that the polynomials $\pi_j(z) = \pi_{j,m}(z)$ are orthogonal with respect to the measure $\mathrm{e}^{-2mQ}\mathrm{dA}$, meaning that

$$\int_{\mathbb{C}} \pi_{j,m}\bar{\pi}_{k,m}\, \mathrm{e}^{-2mQ}\mathrm{dA} = 0, \qquad j \neq k.$$

As it is often more convenient to work with normalized polynomials we write $h_{j,m}$ for the squared norm of $\pi_{j,m}$ and set $P_{j,m} = h_{j,m}^{-\frac{1}{2}}\pi_{j,m}$, which becomes a normalized orthogonal polynomial. If we define a kernel $\mathbf{K}_{n,m}$ by

$$\mathbf{K}_{n,m}(z, w) = \sum_{j=0}^{n-1} P_{j,m}(z)\bar{P}_{j,m}(w)\mathrm{e}^{-mQ(z)-mQ(w)},$$

then it follows by the multiplicativity of the determinant that we have

$$\begin{aligned}\prod_{i<j}|z_i - z_j|^2\mathrm{e}^{-2m\sum_{j\le n}Q(z_j)} &= \det V_n(z)\det V_n^*(z)\mathrm{e}^{-2m\sum_{j\le n}Q(z_j)}\\ &\propto \det(\mathbf{K}_{n,m}(z_i, z_j)),\end{aligned}$$

where the symbol '$\propto$' indicates that the quantities differ only by a multiplicative constant. Hence the measure $\mathrm{d}\mathbb{P}_{n,m,Q}$ reduces to

$$\mathrm{d}\mathbb{P}_{n,m,Q}(z_1, \ldots, z_n) = \frac{1}{Z'_{n,m}}\det(\mathbf{K}_{n,m}(z_i, z_j))_{1\le i,j\le n}\mathrm{dA}^{\otimes n}(z_1, \ldots, z_n)$$

for some normalizing constant $Z'_{n,m}$. That the polynomials $P_{j,m}$ were orthogonal did mot matter up until this point, but the fact that they are allows us to integrate out any number of variables, to obtain the k-point correlation functions $\varrho_k = \varrho_{k,n,m}$ from the kernel by the formula

$$\varrho_k(z_1, \dots, z_k) = \det(\mathbf{K}_{n,m}(z_i, z_j))_{1 \le i,j \le k},$$

and hence, the Coulomb gas ensemble $\mathbb{P}_{n,m,Q}$ is determinantal. In particular, the correlation kernel, and by extension the orthogonal polynomials, determine the *density of particles*

$$R_{n,m}(z) = m^{-1} \varrho_{1,n,m}(z) = m^{-1} \mathbf{K}_{n,m}(z, z),$$

which may be seen as the main determinant of the global features of the ensembles.

The correlation kernel $\mathbf{K}_{n,m}$ factors as $K_{n,m}(z, w) \mathrm{e}^{-mQ(z) - mQ(w)}$, where $K_{n,m}$ is the reproducing kernel for the space $A^2_{mQ,n}$ of polynomials of degree at most $n-1$, endowed with the topology of $L^2(\mathbb{C}, \mathrm{e}^{-2mQ} \mathrm{dA})$. This means that for fixed $w \in \mathbb{C}$, the function $K_{n,m}(\cdot, w)$ is an element in the space, and for any $f \in A^2_{mQ,n}$ we have the reproducing property

$$f(w) = \int_{\mathbb{C}} f(z) K_{n,m}(w, z) \mathrm{e}^{-2mQ(z)} \mathrm{dA}(z)$$

This space is referred to as a *polynomial Bergman space*, and its kernel $K_{n,m}$ is the *polynomial Bergman kernel.*

The asymptotic study of Bergman kernels is a classical topic, originating with work by Hörmander [46] and Fefferman [33], followed by important contributions by Boutet de Monvel and Sjöstrand [18]. Among more recent contributions, we should mention the peak section method by Tian [68] and further developments by Zelditch [72] and Catlin [22], which yield complete asymptotic expansions of Bergman kernels in general settings. Though originally motivated by questions concerning the geometry of pseudoconvex domains, and later by questions in Kähler geometry, many of the developments in the planar context have arisen as a result of the connections to Coulomb gas ensembles. The works [5, 6, 8] as well as [14] provide references for some of the recent developments. Here, let us briefly mention one result, as it relates to the topics of Papers A through C. As the area of the droplet is a positive constant and since there are roughly n particles within this region, the typical inter-particle distance should be roughly $n^{-\frac{1}{2}}$. Hence, for a point $z_0 \in \mathbb{C}$ we look at the rescaled process $(\zeta_j)_j$, defined implicitly by

$$z_j = z_0 + \frac{\zeta_j}{\sqrt{2m\Delta Q(z_0)}}.$$

where $(z_1, \dots, z_n)$ is distributed according to $\mathbb{P}_{n,m,Q}$. It turns out that the microscopic behavior for z_0 a bulk point is universal, that is, it does not depend on the potential Q nor the point z_0. Bulk universality appears first to have been obtained by Berman [14] in the context of a plurisubharmonic potential on $\mathbb{C}^d$ and independently by Ameur, Hedenmalm and Makarov [5] in the planar case. The latter result may be phrased as follows:

Theorem 1.1. *Assume that Q is strictly subharmonic and C^2-smooth in a neighbourhood of $\mathcal{S}_\tau$. Let $z_0 \in \mathcal{S}_\tau^\circ$, and consider the associated rescaled process $(\zeta_j)_{j=1}^n$. Then, as $n = \tau m \to \infty$, the process $(\zeta_j)_j$ converges to a determinantal random point field associated to the kernel*

$$\mathbf{G}(\zeta, \eta) = \mathrm{e}^{\zeta\bar{\eta} - \frac{1}{2}(|\zeta|^2 + |\eta|^2)}.$$

The technical details of what is meant by convergence of random processes to a random point field is explained in e.g. [9, 8], but suffice it to say that the correlation functions converge uniformly on compact sets to the analogous objects defined in terms of the kernel $\mathbf{G}$.

If the point z_0 is instead located on the boundary $\partial\mathcal{S}_\tau$, the behavior is more subtle. One reason for why this question has offered considerable resistance is discussed below, in connection with $\bar{\partial}$-estimates.

The spectral droplet and an obstacle problem

The support $\mathcal{S}_\tau$ of the equilibrium measure has another interpretation which is often more useful to us. Recall that we require the potential to grow sufficiently fast near infinity, specifically the natural condition is

$$\liminf_{|z|\to\infty} \frac{Q(z)}{\log|z|} > \tau_0$$

for some positive number τ_0. In order to describe the obstacle problem, we introduce the notation SH_τ for the set of all subharmonic functions q in the plane which meet the growth restriction

$$q(z) \le \tau \log|z| + \mathrm{O}(1), \qquad |z| \to \infty.$$

For all $\tau \le \tau_0$, the elements of SH_τ grow strictly slower than the potential. For such τ, we denote by $\hat{Q}_\tau(z)$ the solution to the obstacle problem

$$\hat{Q}_\tau(z) = \sup\Big\{q(z) \,:\, q \in \mathrm{SH}_\tau, q \le Q\Big\},$$

and consider the contact set

$$\mathcal{S}_\tau^* = \big\{z \,:\, Q(z) = \hat{Q}_\tau(z)\big\}. \tag{1.0.5}$$

The notation is explained by the fact that in general, $\mathcal{S}_\tau^*$ and $\mathcal{S}_\tau$ differ at most by a small set, and under some à posteriori assumptions on regularity of the boundary $\partial\mathcal{S}_\tau^*$, we actually have equality $\mathcal{S}_\tau^* = \mathcal{S}_\tau$.

The function $\hat{Q}_\tau$ is subharmonic in the entire plane and is $C^{1,1}$-smooth, and hence $Q - \hat{Q}_\tau$ as well as the gradient $\nabla(Q - \hat{Q}_\tau)$ vanish on the boundary $\partial\mathcal{S}_\tau$. Outside the spectral droplet $\mathcal{S}_\tau$, the function $\hat{Q}_\tau$ is readily seen to be harmonic.

The reason that the obstacle problem has to do with condensation of the Coulomb gas to the spectral droplet may be seen as follows. For an element $f \in A^2_{mQ,n}$, the function $\frac{1}{m}\log|f|$ belongs to the class SH_τ. This observation can be used to prove growth bound

$$|f(z)|^2 \mathrm{e}^{-2mQ(z)} \leq Cm\mathrm{e}^{-2m(Q-\hat{Q}_\tau)(z)}\|f\|^2_{2mQ},$$

from which it also follows that the density of particles $R_{n,m}(z)$ tends to zero exponentially fast outside $\mathcal{S}_\tau$. This type of growth bound is essential to control the Bergman kernel pointwise, and appears in the work [13] by Berman in the context of several complex variables, and was obtained in the present context in [5] with precise control of the multiplicative constant.

$\bar{\partial}$-estimates and the non-locality of Bergman kernels at emergent interfaces

One of the difficulties when working with holomorphic functions, when compared to smooth functions on $\mathbb{R}^n$, is their inherent rigidity. Given a landscape of any shape, one can locally mimic every nook and cranny of it to arbitrary precision using the graph of a smooth function. This is far from true for holomorphic functions, and in particular there exist no non-trivial compactly supported holomorphic functions. In light of this, a scheme based on Hörmander's classical L^2-estimates for the $\bar{\partial}$-equation is invaluable, as it provides some possibilities to work around this rigidity. We proceed with a variant of Hörmander's classical result [46], adapted to the setting of varying exponential weights and spaces of functions with polynomial growth. We denote by Q a smooth potential with sufficient growth at infinity, which is strictly subharmonic in a neighbourhood of the spectral droplet $\mathcal{S}_\tau$.

Theorem 1.2. *For any $f \in L^\infty(\mathcal{S}_\tau)$, the $L^2(\mathrm{e}^{-2mQ})$-minimal solution u to the problem*

$$\bar{\partial}u = f$$

subject to the growth bound $|u(z)| \leq C|z|^n$ as $|z| \to +\infty$ satisfies the estimate

$$\int_{\mathbb{C}} |u|^2 \mathrm{e}^{-2mQ}\mathrm{dA} \leq \frac{1}{2m}\int_{\mathbb{C}} |f|^2 \frac{\mathrm{e}^{-2mQ}}{\Delta\hat{Q}_\tau}\mathrm{dA}.$$

A more elaborate version of this statement including the polynomial growth bound was obtained in [5].

One of the features that gives this theorem its strength is the remarkable uniformity with respect to f, Q and $\mathcal{S}_\tau$. We note here that the requirement that f is supported inside the droplet is crucial, as the denominator in the right-hand-side integrand vanishes outside $\mathcal{S}_\tau$.

As a consequence of this result, one may employ a localization scheme which allows us to 'mould' holomorphic functions in certain respects. The scheme is widely known, and a similar strategy was for instance employed by Tian [68] in his peak section method and by Berman, Berndtsson and Sjöstrand [15] to obtain Bergman kernel asymptotics, which apply in the more general setting of spaces of global holomorphic sections of powers of a positive hermitian line bundle L over a Kähler manifold. Let us briefly indicate how this scheme works in the context of the plane. In order to localize a Bergman kernel near a point $z_0 \in \mathcal{S}_\tau$ we multiply with a cut-off function χ, which is identically one in a disk of radius $m^{-\frac{1}{2}} \log m$ around z_0, and vanishes outside a slightly larger disk. Using direct methods (see e.g. [15] or [5]), one can constructively obtain a local approximately reproducing kernel $K_{n,m}^{\langle z_0 \rangle}(z, z_0)$ for the truncated weight $\chi \mathrm{e}^{-2mQ}$ of the form

$$K_{n,m}^{\langle z_0 \rangle}(z, z_0) = m\mathrm{e}^{2mQ(z,z_0)} \Big\{ b_0(z, z_0) + m^{-1} b_1(z, z_0) + \ldots + m^{-\kappa} b_\kappa(z, z_0) \Big\},$$

where $Q(z, w)$ is a polarization of $Q(z)$ and $b_j(z, w)$ are hermitian holomorphic functions (that is, holomorphic in the variable $(z, \bar{w})$ and subject to the hermitian symmetry condition $f(z, w) = \bar{f}(w, z)$). One may hope that this kernel provides a good approximation of the true kernel. In order to see that no substantial modifications are necessary to obtain the true kernel, one proceeds to solve the problem

$$\bar{\partial} u(z) = \bar{\partial}\big(\chi(z)\, K_{n,m}^{\langle z_0 \rangle}(z, z_0)\big),$$

with polynomial growth control, using Theorem 1.2. Then it follows from Liouville's theorem that the function

$$K_{n,m}^{\star}(z, z_0) = K_{n,m}^{\langle z_0 \rangle}(z, z_0) - u(z)$$

is a polynomial in the appropriate space. Moreover, the $\bar{\partial}$-estimate tells us that

$$\int_{\mathbb{C}} |u|^2 \mathrm{e}^{-2mQ} \mathrm{dA} \le \frac{1}{2m} \int_{\mathcal{S}_\tau} |\bar{\partial}\chi|^2 \, |K_{n,m}^{\langle z_0 \rangle}(z, z_0)|^2 \frac{\mathrm{e}^{-2mQ}}{\Delta \hat{Q}_\tau} \mathrm{dA},$$

so provided that $\bar{\partial}\chi$ is supported inside the spectral droplet, and that the constructed function

$$|K_{n,m}^{\langle z_0 \rangle}(z, z_0)|^2 \mathrm{e}^{-2mQ(z)}$$

is very small where the cut-off is non-constant, the correction u is negligible.

What is important here is that such a localization is possible only as long as we may cut out the point z_0 by a curve γ which does not meet the off-spectral region $\mathcal{S}_\tau^c$. In particular, if z_0 is a boundary point, any such surgery will leave the component of $\mathbb{C} \setminus \gamma$ containing z_0 unbounded, instead of very small and localized. This fact provides a true obstruction to applying classical approaches to the asymptotic expansion of Bergman kernels near the boundary of, or outside of, the spectral droplet.

This apparent non-locality of Bergman kernels near the off-spectral region is not merely a chimera, as may be seen by elementary numerical investigations. For

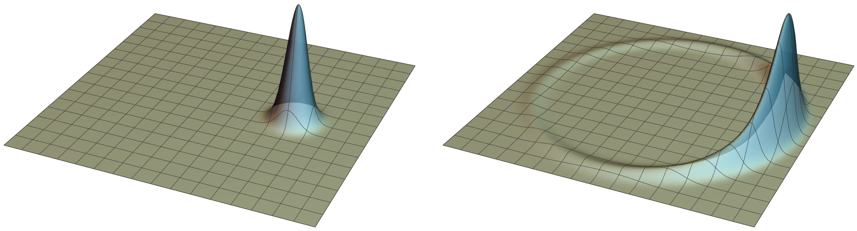

Figure 1.1: The normalized Berezin kernel $B_{m,m}(z,z_0)$ with $Q(z) = \frac{1}{2}|z|^2$ and $m = 100$ for the bulk point $z_0 = 0.6$ (left) and the off-spectral point $z_0 = 1.3$ (right).

instance, Figure 1.1 supplies an illustration of the Berezin density given by

$$B_{n,m,z_0}(z) = \frac{|K_{n,m}(z,z_0)|^2}{K_{n,m}(z_0,z_0)}\mathrm{e}^{-m|z|^2}$$

in the case of a quadratic potential $Q(z) = \frac{1}{2}|z|^2$ (corresponding to the spectral droplet $\mathcal{S}_1 = \mathbb{D}$) for the points $z_0 = 0.6$ and $z_0 = 1.3$.

This feature is not exclusive to the setting of polynomial Bergman kernels. Similar interfaces occur if one considers a general partial Bergman kernel (see Paper B) associated to the orthogonal projection onto a proper subspace of the full Bergman space, or they may appear as the result of a singularity in the metric. In each case, the necessary modifications for the $\bar{\partial}$-technique to produce objects with the desired properties (analogous to polynomial growth) will restrict the region where one may localize.

Despite these difficulties, some results have recently appeared on this topic. In recent work, Ross and Singer [58] studied such transition behavior in the setting of polarized Kähler manifolds. The kernel is then reproducing for the subspace of the full Bergman space consisting of all holomorphic sections that vanish to a prescribed order along a divisor. Assuming that the setup is invariant under a natural S^1-action, they obtain a description of the transition behavior in terms of the error function. Still more recently, Zelditch and Zhou obtained a similar result when the Bergman kernel is in a natural way associated to a Toeplitz quantized Hamiltonian [73]. Even in the context of the plane, however, the understanding has been limited to rather symmetric cases.

Planar orthogonal polynomials

As the polynomial Bergman kernel itself appears to be difficult to obtain near the boundary, it may be beneficial to investigate the properties of the individual building blocks, the orthogonal polynomials $P_{n,m}$. This has been done to great effect in the analogous theory of *random hermitian matrices.* There, the correlation kernel is expressed in terms of orthogonal polynomials on the real line, defined in terms of an inner product

$$\langle f, g\rangle_{L^2(\mathbb{R},\mathrm{e}^{-2mV})} = \int_{\mathbb{R}} f(x)g(x)\mathrm{e}^{-2mV(x)}\mathrm{d}x.$$

The study of such polynomials has a rich history with its roots in the theory of special functions, and include as special cases polynomials investigated by Hermite, Laguerre, Gegenbauer and Chebyshev, to mention a few.

There are two major difficulties in passing from the real to the complex orthogonal polynomials, relating to the loss of significant techniques. First, as a consequence of the fact that multiplication by the coordinate function is self-adjoint on $L^2(\mathbb{R}, \mathrm{e}^{-2mV})$ (see [64]), the corresponding monic orthogonal polynomials $\pi_{n,m}$ enjoy a recursion equation:

$$\pi_{n+1,m}(x) = (x - b_{n,m})\pi_{n,m}(x) + a_{n,m}\pi_{n-1,m}(x)$$

for some numbers $(a_{n,m})_n$ and $(b_{n,m})_n$. This imposes an extraneous rigidity which may be utilized to obtain information about the polynomials and associated objects. For instance, via the Christoffel-Darboux formula this recursion allows one to express the corresponding polynomial kernel in terms of the top-degree polynomials $P_{n,m}$ and $P_{n-1,m}$ alone. In the planar context, one should expect no such recurrence relations to exist except in highly symmetric cases, not even if one allow recurrences with any fixed (large) number of terms.

Second, that the orthogonality is taken with respect to the real line invites to the formulation of a natural matrix Riemann-Hilbert problem, where the jump occurs across the integration contour. This exciting connection was established by Fokas, Its and Kitaev [34], and techniques based on Riemann-Hilbert problems are nowadays central to the study of orthogonal polynomials on the real line. These developments were of great importance in obtaining universality for microscopic scaling limits of eigenvalue processes of random Hermitian matrices, see e.g. [29, 27].

Let us turn our attention to the case of planar orthogonal polynomials. These have been successfully studied in some contexts, originating with the work of Carleman [20]. Following initial insights by Szegő [67] on orthogonal polynomials on analytic curves, he studied Bergman orthogonal polynomials, defined in terms of the L^2-inner product with respect to planar area measure on a simply connected domain Ω with a real-analytic boundary. Carleman obtains a beautiful asymptotic description of the n:th normalized orthogonal polynomial P_n, in terms of the exterior conformal mapping

$$\varphi : \Omega^c \to \mathbb{D}_\mathrm{e}$$

with appropriate normalizations at infinity. As the degree n tends to infinity, the polynomials enjoy the asymptotics

$$P_n(z) = (n+1)^{\frac{1}{2}} \sqrt{\varphi'(z)} [\varphi(z)]^n \Big(1 + \mathrm{O}(\rho^n)\Big),$$

for some $\rho < 1$, where the expansion is valid in a neighbourhood of the entire exterior region. In the 1970's, Suetin extended Carleman's results to also cover the case when the inner product is taken with respect to a weight ω [65]. However, it should be pointed out that these results have no apparent bearing on the varying exponential weights needed for the purpose of random matrix theory.

The literature on planar orthogonal polynomials with respect to exponentially varying weights is rather limited. For some particular choices of Q one can, remarkably, still formulate and constructively solve a Riemann-Hilbert problem to obtain the asymptotics, see e.g. the works [16] by Bleher and Kuijlaars and [11] by Balogh, Bertola, Lee and McLaughlin [11]. In cases when the Riemann-Hilbert method can be applied, the asymptotic information gained is often fine enough to penetrate all the way down to the location of the zeros of the orthogonal polynomials, which is proven to be a delicate matter, see e.g. the work [53] by Lee and Yang. For general potentials, however, there appears to be no reason for such a strategy to work. To my knowledge, the only result pertaining to a general potential Q that existed before the developments in Paper A was obtained by Ameur, Hedenmalm and Makarov in [6], and maintains that the following convergence holds:

$$|P_{n,m}(z)|^2 \mathrm{e}^{-2mQ(z)} \mathrm{dA} \to \varpi(z, \infty, \mathcal{S}_\tau^c) \qquad \text{as } n = \tau m + \mathrm{O}(1) \to \infty \tag{1.0.6}$$

Here, the convergence is in the sense of measures, and $\varpi(z, \infty, \mathcal{S}_\tau^c)$ denotes the harmonic measure for the complement $\mathcal{S}_\tau^c$, evaluated at the point at infinity. Thus, at least, the probability distributions associated to the weighted polynomials $P_{n,m}\mathrm{e}^{-mQ}$ concentrate to the boundary $\partial\mathcal{S}_\tau$ in the L^2-sense. As may be seen by comparing the two densities $\mathbf{K}_{m+1,m}(z,z)$ and $\mathbf{K}_{m,m}(z,z)$, we may interpret the probability density in (1.0.6) as describing the net effect of adding a particle to an m-particle Coulomb gas with potential Q. The fact that it concentrates to the boundary reflects that that the spectral droplet is already occupied, and rather than increasing the density, the addition of another particle causes the droplet to grow in size. The convergence to harmonic measure is closely related to the fact that as the number of particles $n = \tau m$ grows, the resulting spectral droplets $\mathcal{S}_\tau$ evolve according to a law known as weighted *Laplacian growth*, or weighted *Hele-Shaw flow*. For a discussion of this topic, which is essential for the intuition behind the arguments in Papers A and B, we refer to the section on Hele-Shaw flow below.

β-ensembles and the Abrikosov conjecture

Although it is not treated in this thesis, it should be mentioned that two-dimensional Coulomb gas actually refers to a more general class of point processes, and to be precise $d\mathbb{P}_{n,m,Q}$ is the measure corresponding to a Coulomb gas *at inverse temperature* $\beta = 2$. The general Coulomb gas measure is obtained by

$$d\mathbb{P}^{\beta}_{n,m,Q} = \frac{1}{Z_{n,\beta}} e^{-\frac{\beta}{2}\mathcal{E}(z_1,\ldots,z_n)} dA^{\otimes n}(z_1,\ldots,z_1)$$

for $\beta > 0$, where $\mathcal{E}$ is the energy functional (1.0.1), as in the case $\beta = 2$. If not otherwise stated explicitly, the below results as well as the results in this thesis only apply to the particular case $\beta = 2$. It may seem odd that tweaking this parameter only so slightly would result in drastic changes. However, the reason for this breakdown is that only for the case $\beta = 2$ is the resulting process determinantal. Coulomb gas ensembles for general inverse temperatures β, known also as β-ensembles, are conjectured to demonstrate truly remarkable behavior as $\beta \to \infty$, where a rigidity phenomenon referred to as the *Abrikosov conjecture* is expected to manifest itself: The configurations should conjecturally freeze into a hexagonal lattice structure – the conjectured energy minimizer for the associated energy functional. Over recent years, several aspects of β-ensembles have attracted increasing attention, with works by Leblé and Serfaty [52] as well as Bauerschmidt, Bourgade, Nikula and Yau [12] establishing central limit theorems for fluctuations and asymptotic expansions of the partition function. The rigidity behavior, however, remains elusive.

It has even been suggested that there may be a finite-temperature phase transition, occuring around $\beta = 240$. Whether or not one should expect such a transition to occur seems not to be agreed upon in the physics community (see, e.g. numerical work in [19], as well as Leblé's informal but informative survey [51]), but it continues to demonstrate that this is a field rife with important questions left to resolve.

Random normal matrices and the Ginibre ensemble

The Coulomb gas ensemble associated to the measure $d\mathbb{P}_{n,m,Q}$ defined in (1.0.2) also has another interpretation, as eigenvalues of certain random matrices. In order not to obscure this introduction with technical matters, let us take for granted that the set of $n \times n$ non-zero normal matrices M, i.e. matrices which commute with their hermitian adjoint, form a manifold $\mathcal{N}_n$ which is canonically embedded into $\mathbb{C}^{n\times n}$, and that the standard metric on $\mathbb{C}^{n\times n}$ induces a (Riemannian) volume form on $\mathcal{N}_n$ which we denote by $d\mu_n$. For a smooth function Q with suitable integrability properties, one may pick a matrix M at random from the measure

$$d\mathcal{P}_n(M) = \frac{1}{\mathcal{Z}_n} e^{-2m \operatorname{Tr} Q(M)} d\mu_n(M),$$

where $Q(M)$ may be given a meaning in terms of functional calculus, and where $\mathcal{Z}_n$ is a normalizing constant. The random matrix M has n complex eigenvalues, $\lambda_1,\ldots,\lambda_n$, whose distribution may be determined and happens to coincide with

$\mathbb{P}_{n,m,Q}$. Even if the physical model is often more useful in terms of the intuition it supplies, it is a remarkable feature that eigenvalues of random matrices repel each other like charged particles. The random normal matrix model was introduced in [23] by Chau and Yu, and apart from the references mentioned above in connection with Coulomb gas ensembles, it has been studied extensively in the physics literature including works by Wiegmann and Zabrodin [71], Etingof and Ma [32], as well as Elbau and Felder [31].

In the special case when $Q(z) = \frac{|z|^2}{2}$, there is a further matrix model which has the same eigenvalue distribution. Indeed, if we form the matrix $\frac{1}{\sqrt{m}}(\xi_{i,j})_{1\le i,j\le n}$ where $\xi_{i,j} \in \mathcal{N}_{\mathbb{C}}(0,1)$ are independent complex normal random variables with mean 0 and variance 1, then the induced eigenvalue measure is $\mathbb{P}_{n,m,Q_0}$ where Q_0 is the quadratic potential $Q_0(z) = \frac{1}{2}|z|^2$. This eigenvalue process is known as the *Ginibre ensemble*. Due to the radial symmetry of the weight, the orthogonal polynomials are precicely properly weighted monomials, and we may obtain the correlation kernel $\mathbf{K}_n$ by the formula

$$\mathbf{K}_n(z,w) = m\sum_{j=0}^{n-1} \frac{(mz\bar{w})^j}{j!} \mathrm{e}^{-m|z|^2 - m|w|^2}.$$

For this reason, the Ginibre ensemble often serves as a model case when exploring for new features in random normal matrix ensembles.

Hele-Shaw flow with varying permeability

Hele-Shaw flow, also known as Laplacian growth, is a mathematical model with its roots in fluid dynamics. The model describes the movement of the interface between two phases of fluid confined to a thin space between two parallel plates as more fluid is injected into one of the phases.

To set it up rigorously, consider a bounded simply connected region Ω occupied by a viscous incompressible fluid, surrounded by air. At a point $p \in \Omega$, more fluid is injected at a constant rate. Denote by Ω_t the domain occupied by the fluid at time t, which we allow to assume positive as well as negative values. Under the simplifying assumptions that surface tension and air pressure vanish, the domains Ω_t may be characterised by the following property: If g_t denotes the Green function for the laplacian of Ω_t, then the velocity ν_{n} of the boundary in the outward normal direction n to $\partial\Omega_t$ is obtained by

$$\nu_{\mathrm{n}}(z) = \partial_{\mathrm{n}} g_t(z,p), \qquad z \in \partial\Omega_t.$$

If this is satisfied and $\Omega_0 = \Omega$, we say that the domains $\{\Omega_t\}_t$ evolve according to Hele-Shaw flow with injection at p and initial domain Ω.

Another characterization will often be useful to us. It may be shown that a smooth increasing family $\{\Omega_t\}_{-\epsilon\le t\le\epsilon}$ of domains such that $\Omega_0 = \Omega$ constitutes a

Hele-Shaw flow if, for all bounded harmonic functions on Ω_t, we have the mean-value property

$$\int_{\Omega_t\setminus\Omega_s} h(z)\mathrm{dA}(z) = (t-s)h(p), \qquad s,t\in(-\epsilon,\epsilon),\ s<t.$$

Hele-Shaw flows have been extensively studied, with notable early contributions by Hele-Shaw who introduced the concept in [44], as well as by Vinogradov and Kufarev who first established the short-time existence of solutions [69]. For an account of the early theory, as well as modern developments, we refer to the comprehensive book [35] by Gustafsson, Teodorescu and Vasil'ev.

If the space occupied between the two plates in the Hele-Shaw cell has a varying permeability, the model changes. In the latter characterization of the flow, this corresponds to introducing a weight in the integrals. If the permeability is described by a positive function $\kappa(z)$, we set $\omega(z)=\kappa(z)^{-1}$, and say that a smooth family of domains $\{\Omega_t\}_{-\epsilon\le t\le\epsilon}$ deform according to weighted Hele-Shaw flow with injection point p, initial domain Ω_0 and permeability κ if

$$\int_{\Omega_t\setminus\Omega_s} h(z)\omega(z)\mathrm{dA}(z) = (t-s)h(p), \qquad s,t\in(-\epsilon,\epsilon),\ s<t$$

for all bounded harmonic functions h on Ω_t. We remark that one may want to consider unbounded domains, in which case we prefer to think of them as domains on the Riemann sphere $\hat{\mathbb{C}}$. Short-time existence, both forward and backward in time, was obtained under the assumption that the weight ω is real analytic by Hedenmalm and Shimorin [40], and by Ross and Witt Nyström in [59] assuming only C^∞-smoothness of the weight.

It is sometimes useful to encode the flow $\{\Omega_t\}_t$ of simply connected domains (possibly on the Riemann sphere $\hat{\mathbb{C}}$) in terms of conformal mappings. Denote by φ_t the conformal mapping

$$\varphi_t:\mathbb{D}\to\Omega_t,$$

such that p is mapped to 0, with positive derivative. In terms of $\{\varphi_t\}_t$, the Hele-Shaw property leads to the *Polubarinova-Galin equations*, which reads in the weighted case

$$\mathrm{Re}\big\{\bar{\zeta}\partial_t\varphi_t(\zeta)\overline{\varphi_t'(z)}\big\}\omega(\varphi_t(\zeta))=1.$$

It may be useful to consider instead conformal mappings of the exterior disk, but due to limited space we defer such issues until they are needed in the articles.

As a further measure to simplify book-keeping, we may glue the conformal mappings together to form the *Hele-Shaw foliation mapping*, Ψ, which may be defined by

$$\Psi(z)=\varphi_{1+|z|}\Big(\frac{z}{|z|}\Big), \qquad z\in\mathbb{A}(1-\epsilon,1+\epsilon). \tag{1.0.7}$$

The Polubarinova-Galin equations take on an especially attractive form in this formulation, namely as a prescribed Jacobian equation

$$\omega\circ\Psi(z)\cdot J_\Psi(z)=(2|z|)^{-1}, \qquad z\in\mathbb{A}(1-\epsilon,1+\epsilon). \tag{1.0.8}$$

The literature on prescribed Jacobian equations is considerable, including early work by Dacorogna and Moser [26]. However, the existence of solutions of the form imposed by the requirement that the function Ψ obtained as a solution of (1.0.8) should encode a flow by the relation (1.0.7) seems not to have been investigated directly from this perspective.

For us, the interest in this type of deformation is two-fold. At the macroscopic level, the domains $\mathcal{S}_\tau$ associated to the random normal matrix ensemble grow according to weighted Laplacian growth (with reversed time) corresponding to the weight $2\Delta Q$ and injection at infinity. Moreover, the main tool in Papers A and B is the orthogonal foliation flow, which may be regarded as an approximate Laplacian growth with respect to the weight $|P_{n,m}|^2\mathrm{e}^{-2mQ}$. In this latter situation, one needs to determine not only the Hele-Shaw flow, but at the same time also the polynomial part of the weight, so that the flow exists for a sufficiently long time without encountering singularities.

Higher Landau levels: The polyanalytic Ginibre ensemble

The polyanalytic Ginibre ensemble is a generalization of the Ginibre ensemble, encountered above. To arrive at this generalization in a natural way, let us provide yet another interpretation of these random point configurations.

The (Landau) Hamiltonian describing a particle with charge e and mass m_0, moving around in a constant magnetic field A perpendicular to the plane may be expressed as

$$H = \frac{1}{2m_0}\Big(\mathrm{i}\hbar\nabla + \frac{e}{c}A\Big)^2,$$

understood in the quadratic form sense, where $\hbar$ denotes Planck's constant and c denotes the speed of light. We denote the strength of the magnetic field by B, and make a choice of gauge $A = \Big(-\frac{B}{2}y, \frac{B}{2}x\Big)$. If we for simplicity assume that $c = e = m_0 = \hbar = 1$, we obtain the representation

$$H_B = \frac{1}{2}\bigg(\Big(\mathrm{i}\partial_x - \frac{B}{2}y\Big)^2 + \Big(\mathrm{i}\partial_y + i\frac{B}{2}x\Big)^2\bigg).$$

This operator is densely defined on the space $L^2(\mathbb{C})$ of complex-valued functions, and its spectrum is purely discrete and consists of the eigenvalues

$$e_k^B = \Big(k + \frac{1}{2}\Big)B, \qquad k = 0, 1, 2, \ldots$$

The individual eigenvalues are known as *Landau levels*, and the eigenspace corresponding to the lowest Landau level is known to consist of all weighted holomorphic functions,

$$f(z)\mathrm{e}^{-\frac{B}{2}|z|^2}, \qquad f \in A^2_{\frac{B}{2}|z|^2}.$$

In physically relevant situations, the particles are confined to a fixed finite space, rather than allowed to spread over the entire plane. In order to fill as many particles as possible inside a given region, say a disk of a fixed radius, with particles in the lowest Landau level one should limit attention to the one-particle states spanned by

$$\psi_{j,B}(z) = \frac{B^{\frac{j+1}{2}}}{\sqrt{j!}} z^j \mathrm{e}^{-\frac{B}{2}|z|^2}, \qquad j = 0, \dots, n-1$$

for some n (depending on B). It then follows from Pauli's exclusion principle that the wave-function for the corresponding n-body system of free fermions in the lowest Landau level is given by the *Slater determinant*

$$\Psi_B(z_1, \dots, z_n) = \det\big(\psi_{j,B}(z_k)\big)_{1\le j,k\le n},$$

A computation shows that the probability density $|\Psi_B|^2$ is given by the density of the probability measure $\mathbb{P}_{n,B,Q_0}$ – the law for 2D Coulomb gas with Gaussian weight, or equivalently the density for the Ginibre ensemble with n particles. The omitted details in the above argument may be found in [1].

The *Polyanalytic Ginibre ensemble* appears if one considers $n \times q$ particles in the above model, distributed so that there are n particles in each of the first q Landau levels. In order to conform with the notation in earlier sections, let us rename the strength of the magnetic field to m. One may again construct a wave function in terms of a Slater determinant, but in this case we obtain that the orthonormal basis is

$$\Big\{e_{j,k}\mathrm{e}^{-\frac{n}{2}|z|^2} \;:\; 0 \le j \le n-1,\, 0 \le k \le q-1\Big\},$$

where the fuctions $e_{j,k}$ are given by

$$e_{j,k}(z) = \begin{cases} \sqrt{\frac{k!}{(k-j)!}} m^{\frac{j+1}{2}} z^j \, L_k^{(j)}(m|z|^2), & k \ge j \\ \sqrt{\frac{j!}{(j-k)!}} m^{\frac{k+1}{2}} \bar{z}^k \, L_j^{(k)}(m|z|^2), & j > k, \end{cases}$$

and $L_r^{(\alpha)}(x)$ denotes the generalized Laguerre polynomial

$$L_r^{(\alpha)}(x) = \sum_{k=0}^{r} (-1)^k \binom{r+\alpha}{r-k} \frac{x^k}{k!}. \tag{1.0.9}$$

In comparison with the previous case, the functions $e_{j,k}$ are no longer of the form $f\mathrm{e}^{-B\frac{|z|^2}{2}}$ where f is holomorphic. Rather, the coefficient function f is *polyanalytic*, which explains the name of the random process. We recall that an analytic function is one that solves $\bar{\partial} f = 0$, while a polyanalytic function (q-analytic) solves $\bar{\partial}^q f = 0$.

The polyanalytic Ginibre ensemble has been touched upon to different extent by several physicists, but was mathematically investigated first by Haimi and Hedenmalm [36], where a representation

$$\mathrm{d}\mathbb{P}_{n,m,q}(z_1, \dots, z_{nq}) = \frac{1}{Z_{n,q}} |\Delta_{n,q}(z_1, \dots, z_{nq})|^2 \mathrm{e}^{-m\sum_{j=1}^{nq}|z_j|^2} \mathrm{dA}^{\otimes n}(z_1, \dots, z_{nq}), \tag{1.0.10}$$

was obtained for the defining probability measure on the configuration space. Here $\Delta_{n,Q}$ is a certain generalization of the Vandermonde determinant $\Delta_n = \det V_n$. To mention one more foundational result, they established the circular law in the limit as $n = m \to \infty$ while the polyanalyticity degree q is kept fixed. Further recent developments include a proof of *hyperuniformity*, or anomalously small fluctuations in the density of particles, in the class of point processes known as the Weyl-Heisenberg ensembles, which encompass the polyanalytic Ginibre ensemble as a special case, see [2].

At present, much of the study of the polyanalytic Ginibre ensembles is focused on analysing further curious features of this point process, especially in the regime when n and q both tend to infinity. Remarkably, special functions from all around random matrix theory seem to appear in different scaling limits, including analogues of the sine kernel, the error function, and the semicircular law. For one example of this behavior, see the summary of results of Paper D where we sketch an unpublished result (obtained with Haimi and Hedenmalm) as an application of the tools developed therein.

Zero packing: discretizing a smooth density

Consider a positive smooth function $\omega(z)$ in the complex plane, with the property that the function

$$\kappa(z) = \Delta \log \omega(z)$$

is smooth and positive. How well may we approximate the function $\omega(z)$ with the modulus of polynomials? It is clearly impossible to obtain complete equality, even if we allow a general holomorphic function defined in a neighbourhood a point z_0. Indeed, the corresponding expression $\Delta \log|f|$ takes the form

$$\Delta \log|f| = \frac{1}{2} \sum_{z \in Z(f)} \delta_z,$$

where $Z(f)$ denotes the zero-set of the holomorphic function f, and such a sum is never smooth. The term *zero packing* refers to the attempt to, in a sense specified below, obtain a function f such that $|f|$ is as good as possible an approximant of ω. Heuristically, we imagine that we want to approximate the smooth density $\kappa(z)$ with $\Delta \log|f(z)|$ as well as possible, by placing the zeros of f evenly in the plane.

We will consider both *planar* and *hyperbolic* zero packing, which take place in the plane and the hyperbolic disk, respectively. Hyperbolic zero packing was introduced in recent work by Hedenmalm [38] in connection to work on the *universal asymptotic variance* of Bloch functions, which combines with work by Ivrii [47] to disprove a conjecture (appearing in work by Prause and Smirnov [56]) on the dimension of quasicircles. In contrast, an equivalent formulation of planar zero packing has its roots already in work by Abrikosov on superconductivity [3]. We defer further historic comments to Paper E, and proceed instead to define the problem.

The failure of approximation is measured by the size of certain *discrepancy functions*. In the planar case, we aim to approximate the function $\omega = \mathrm{e}^{|z|^2}$, for which the corresponding density $\Delta \log \omega$ is constant. The discrepancy function is in this case given by

$$\Psi_{\mathbb{C},f}(z,R) = \Big(|f(z)|\mathrm{e}^{-|z|^2} - 1_{\mathbb{D}(0,R)}(z)\Big)^2,$$

and measures the deviation of the quotient of $|f|$ and $\omega(z)$ from 1 on the given disk. Two associated discrepancy densities, $\rho_{\mathbb{C}}$ and $\rho_{\mathbb{C}}^*$, are then defined by

$$\rho_{\mathbb{C}} = \liminf_{R\to\infty} \inf_f \frac{1}{R^2} \int_{\mathbb{D}(0,R)} \Psi_{\mathbb{C},f}(z,R)\mathrm{dA}(z), \tag{1.0.11}$$

and

$$\rho_{\mathbb{C}}^* = \liminf_{R\to\infty} \inf_f \frac{1}{R^2} \int_{\mathbb{C}} \Psi_{\mathbb{C},f}(z,R)\mathrm{dA}(z), \tag{1.0.12}$$

respectively, where the minimization is done over the set of all holomorphic functions for which the integrals make sense. The difference between the two densities is that the latter, $\rho_{\mathbb{C}}^*$, adds an $L^2(\mathrm{e}^{-2|z|^2})$-punishment for f on the exterior disk $\mathbb{D}_\mathrm{e}(0,R)$, while the former only measures the failure of approximation. From this observation, it is clear that $\rho_{\mathbb{C}}^* \geq \rho_{\mathbb{C}}$.

The hyperbolic analogues, $\rho_{\mathbb{H}}$ and $\rho_{\mathbb{H}}^*$ are defined analogously as follows: We form the discrepancy function

$$\Psi_{\mathbb{H},f}(z,r) = \Big((1-|z|^2)|f(z)| - 1_{\mathbb{D}(0,r)}(z)\Big)^2, \qquad z \in \mathbb{D},\ r < 1,$$

which measures the failure of approximating the density of the hyperbolic metric $\omega(z) = (1-|z|^2)^{-1}$ with the modulus of holomorphic functions, and define the hyperbolic discrepancy $\rho_{\mathbb{H}}$ by

$$\rho_{\mathbb{H}} = \liminf_{r\to 1^-} \inf_f \frac{\int_{\mathbb{D}(0,r)} \Psi_{\mathbb{H},f}(z,r)\frac{\mathrm{dA}(z)}{(1-|z|^2)}}{\int_{\mathbb{D}(0,r)} \frac{\mathrm{dA}(z)}{(1-|z|^2)}} \tag{1.0.13}$$

and its analogue $\rho_{\mathbb{H}}^*$ by

$$\rho_{\mathbb{H}}^* = \liminf_{r\to 1^-} \inf_f \frac{\int_{\mathbb{D}} \Psi_{\mathbb{H},f}(z,r)\frac{\mathrm{dA}(z)}{(1-|z|^2)}}{\int_{\mathbb{D}(0,r)} \frac{\mathrm{dA}(z)}{(1-|z|^2)}}. \tag{1.0.14}$$

It may appear that the most natural choices of discrepancies for the approximation problems are $\rho_{\mathbb{C}}$ and $\rho_{\mathbb{H}}$, but particularly in the hyperbolic case it is the density $\rho_{\mathbb{H}}^*$ that plays the direct role in the function-theoretic applications indicated above. For this reason, it seems natural to seek to connect $\rho_{\mathbb{C}}$ with $\rho_{\mathbb{C}}^*$ as well as $\rho_{\mathbb{H}}$ with $\rho_{\mathbb{H}}^*$. Indeed, the equality of $\rho_{\mathbb{H}}$ and $\rho_{\mathbb{H}}^*$ was conjectured by Hedenmalm already when these quantities were introduced, [38].

Another interesting, and seemingly very difficult problem, is to obtain good descriptions of the minimizers f_R to the truncated problems

$$\rho_{\mathbb{C},R} = \inf_f \frac{1}{R^2} \int_{\mathbb{D}(0,R)} \Psi_{\mathbb{C},f}(z,R) \mathrm{dA}(z),$$

as well as stability results as $R \to \infty$. This has been sought after for some time, notably already Abrikosov believed that the minimizers should have zeros periodically placed on the vertices of the hexagonal lattice, see [4]. Hence, at least conjecturally, the minimizers to $\rho_{\mathbb{C},R}$ and to the optimal configurations for the Coulomb gas energy functional should have the same symmetries.

Carleman classes: Uniqueness properties of smooth functions

The class of analytic functions has the following uniqueness property: If f is known to be analytic it holds that

If f vanishes along with all its derivatives at a point, then f is necessarily the zero function.

A class $\mathcal{C}$ of smooth functions for which this uniqueness property holds is called *quasianalytic.*

This property does not hold for general classes of smooth functions, as the non-triviality of the space $C_0^\infty(\mathbb{R})$ shows. The problem of determining when, in a specified sense, a class of smooth functions is quasianalytic goes back to Hadamard. It is natural to formulate the problem for *Carleman classes* $\mathcal{C}_{\mathcal{M}}$, defined with respect to a weight sequence $\mathcal{M} = \{M_n\}_{n=0}^\infty$ as the class of all smooth functions f on an interval I which meet the estimate

$$\sup_{x \in I} |f^{(n)}(x)| \leq C_f A_f^n M_n, \tag{1.0.15}$$

for some constants C_f and A_f. After Denjoy [30] obtained partial results, the full solution to Hadamard's problem was obtained by Carleman [21] in 1926. Their collective answer has become known as the Denjoy-Carleman theorem, and is given below in a simple form for logarithmically convex sequences $\mathcal{M}$:

Theorem 1.3. *The class $\mathcal{C}_{\mathcal{M}}$ is quasianalytic if and only if*

$$\sum_{n \geq 0} \frac{M_n}{M_{n+1}} = +\infty.$$

For a simple proof of this Theorem, we refer the reader to [24]. The reader interested in Hadamard's uniqueness problem and the related problem of determining a function from its Taylor series at a point, may find further information in the recent work by Kiro [49] as well as in the references therein.

One may equally well study Carleman-type classes where the uniform norm in the left-hand-side of (1.0.15) is replaced by L^p-norms. For $p \geq 1$, the situation is well understood. In Paper F, we are led to study such classes in the small p-regime, i.e. when $0 < p < 1$. As it turns out, if the spaces are defined with respect to an appropriate class of test functions, there are two topological transitions in this case, instead of just the quasianalyticity barrier.

2 Results obtained in this thesis

Paper A: Planar orthogonal polynomials and boundary universality in the random normal matrix model

The first article [41] concerns Coulomb gas ensembles, or random normal matrix ensembles, defined by the measure (1.0.2) for a potential Q of sufficient growth at infinity. We are particularly interested in the behaviour the normalized orthogonal polynomials $P_{n,m}$, which we recall are uniquely determined by the orthogonality relation

$$\int_{\mathbb{C}} P_{n,m}(z)\bar{P}_{n',m}(z)\mathrm{e}^{-2mQ(z)}\mathrm{dA}(z) = \delta_{n,n'}.$$

along with the requirements that $P_{n,m}$ has degree n and positive leading coefficient.

The main theorem of Paper A is a full asymptotic expansion of Carleman-Szegő type of the polynomials $P_{n,m}$, under the assumption that the potential Q is 1-admissible in the sense of Definition 1.3.1 of Paper A. In particular this entails that Q is of at least logarithmic growth at infinity and that the spectral droplet $\mathcal{S}_\tau$ coincides with the coincidence set $\mathcal{S}_\tau^*$ of (1.0.5). Moreover, under this assumption the boundary $\partial\mathcal{S}_\tau$ of the associated spectral droplet $\mathcal{S}_\tau$ is a real-analytically smooth simple closed curve for $\tau \in [1-\epsilon_0, 1]$, for some $\epsilon_0 > 0$, and the potential Q is strictly subharmonic and real-analytically smooth in a neighbourhood of $\partial\mathcal{S}_\tau$. For the formulation of the main result, we denote by φ_τ the conformal mapping

$$\varphi_\tau : \mathcal{S}_\tau^c \to \mathbb{D}_\mathrm{e}$$

which fixes the point at infinity and has positive derivative there, and let $\mathcal{Q}_\tau$ denote the bounded holomorphic function on $\mathcal{S}_\tau^c$ whose real part equals Q on the boundary and which is real at infinity. Due to the smoothness of the boundary, these functions extend conformally and holomorphically, respectively, to a slightly larger domain.

Theorem 2.1. *Assume that Q is 1-admissible and suppose that an integer κ and a positive number A are given. Then there exist bounded holomorphic functions $\mathcal{B}_{j,\tau}$ defined in a fixed neighbourhood of $\mathcal{S}_\tau^c$, and compact subsets $\mathcal{K}_{m,\tau} \subset \mathcal{S}_\tau$ with*

$\mathrm{dist}_{\mathbb{C}}(\partial\mathcal{K}_{m,\tau}, \partial\mathcal{S}_\tau) \geq A(m^{-1}\log m)^{\frac{1}{2}}$, *such that the asymptotic formula*

$$P_{n,m}(z) = m^{\frac{1}{4}}[\varphi_\tau'(z)]^{\frac{1}{2}}[\varphi_\tau(z)]^n \mathrm{e}^{m\mathcal{Q}_\tau(z)}\left(\sum_{j=0}^{\kappa} m^{-j}\mathcal{B}_{j,\tau}(z) + \mathrm{O}\left(m^{-\kappa-1}\right)\right),$$

holds, where the error term is uniform for $z \in \mathcal{K}_{m,\tau}^c$ *as* $m, n \to \infty$, *while the ratio* $\tau = \frac{n}{m} \in [1-\epsilon_0, 1]$.

As a companion to this theorem, we supply a computational algorithm which yields the coefficients $\mathcal{B}_{j,\tau}$ in the above expansion (Theorem 1.3.5). In particular, the main term $\mathcal{B}_{0,\tau}$ is obtained as the unique bounded holomorphic function on $\mathcal{S}_\tau^c$ which is positive at infinity and has prescribed modulus

$$|\mathcal{B}_{0,\tau}|^2 = \pi^{-\frac{1}{2}}(\Delta Q)^{\frac{1}{2}}$$

at the boundary $\partial\mathcal{S}_\tau$, and the functions $\mathcal{B}_{j,\tau}$ are real at infinity. These results immediately demonstrate their strength by allowing us to deduce boundary universality in the random normal matrix model for the class of admissible potentials. For a point $z_0 \in \mathbb{C}$ and a direction $\nu \in \mathbb{T}$, we recall that the rescaled density $\rho_m(\xi)$ is defined by

$$\rho_m(\xi) = \frac{1}{2mQ(z_0)}\mathbf{K}_{m,m}(z_m(\xi), z_m(\xi)),$$

where $z_m(\xi)$ is given by

$$z_m(\xi) = z_0 + \nu\frac{\xi}{\sqrt{2mQ(z_0)}}.$$

Corollary 2.2. *For a* 1*-admissible potential* Q, *we fix a point* $z_0 \in \partial\mathcal{S}_1$, *and consider the rescaled density* $\rho_m(\xi)$ *in the outer normal direction at* z_0. *Then we have the locally uniform convergence*

$$\rho_m(\xi) \to \mathrm{erf}(2\mathrm{Re}\,\xi), \qquad m \to \infty.$$

In addition to these results, we obtain a novel qualitative property of planar orthogonal polynomials, which we proceed to describe. Consider first the simple case when the inner product is that of $L^2(\omega)$, where ω denotes a positive radial weight $\omega(z) = \omega(|z|)$:

$$\langle f, g\rangle_{L^2(\omega,\mathbb{C})} = \int_{\mathbb{C}} f(z)\bar{g}(z)\omega(|z|)\mathrm{dA}(z), \qquad \mathrm{dA}(x+\mathrm{i}\,y) = \frac{1}{\pi}\mathrm{d}x\mathrm{d}y$$

A small computation shows that in this case, the normalized orthogonal polynomials $P_{n,m}$ are given by properly normalized monomials $\kappa_{n,m}z^n$ for $n = 0, 1, 2, \ldots$. Although a trivial observation in itself, we note that this may be seen as a consequence of the fact that the complex exponentials $\mathrm{e}^{\mathrm{i}\theta}$ for $n \in \mathbb{Z}$ are orthogonal with respect to arc-length measure on the circle, as in computing the inner product between the monomials one can use polar coordinates to reduce the orthogonality to that of the exponentials.

Consider instead the sequence $P_{n,m}$ of orthogonal polynomials corresponding to a general potential. One may ask if there is an analogous way as before to disintegrate the orthogonality by reducing the problem by one dimension. The appropriate notion replacing the family of concentric circles is then a smooth flow of simple closed curves $\{\Gamma_t\}_t$, which cover a domain

$$\mathcal{D} = \bigcup_t \Gamma_t.$$

We may then replace area integrals over $\mathcal{D}$ by integration with respect to weighted arc-length measure along the curve flow, by

$$\int_{\mathcal{D}} f(z)\mathrm{e}^{-2mQ(z)}\mathrm{dA}(z) = 2\int_t \int_{\Gamma_t} f(z)\mathrm{e}^{-2mQ(z)}\nu(z)\mathrm{ds}\,\mathrm{d}t$$

where ν restricted to Γ_t gives the (scalar) velocity of the curve flow in the outward normal direction and ds denotes arc-length measure on the given curve. One of the main findings of Paper A is that for the given class of orthogonal polynomials, such a disintegration is always impossible in an approximate sense. The corresponding flow of curves is termed the *orthogonal foliation flow* (see Lemma 4.1.2). To each degree n and value of the parameter m, we describe a curve family $\Gamma_{n,m,t}$, such that $P_{n,m}$ is approximately orthogonal to all lower order polynomials with respect to the measure $\mathrm{e}^{-2mQ}\nu_{n,m}\mathrm{ds}$ on $\Gamma_{n,m,t}$, with the additional property that the domain $\mathcal{D} = \bigcup_t \Gamma_{n,m,t}$ covers the region where most of the mass of $P_{n,m}$ is located.

The orthogonal foliation flow is obtained as a certain Hele-Shaw flow, and the weight of the flow is given by the unknown function $|P_{n,m}|^2\mathrm{e}^{-2mQ}$. The algorithm that constructs the curves $\Gamma_{n,m,t}$ simultaneously obtains an approximation of $P_{n,m}$. In fact, one may think of this algorithm as constructing a *permeability landscape* of a prescribed form $|P|^2\mathrm{e}^{-2mQ}$ where P is a polynomial of degree at most n, with the property that the associated weighted Hele-Shaw flow does not encounter singularities too soon when continuing through the boundary $\partial\mathcal{S}_\tau$.

Paper B: Off-spectral analysis of Bergman kernels

The topic of Paper B [43] is the study of Bergman kernels on and outside emergent interfaces between the spectral droplet and an off-spectral component, or *forbidden region*.

While some of the impetus of this topic stems from an application to questions of universality at more general interfaces than considered in Paper A, our main motivation for this work was to explore a connection between normalized Bergman kernels rooted at off-spectral points and orthogonal polynomials, which enables a non-local description of these objects. For a reproducing kernel $K(z, w)$, we recall that the normalized reproducing kernel rooted at w_0 is the object

$$k_{w_0}(z) = \frac{K(z, w_0)}{\sqrt{K(w_0, w_0)}}.$$

The normalized reproducing kernel associated to the Bergman space A^2_{mQ} rooted at a point $w_0 \in \mathbb{C}$ is denoted by k_{m,w_0}. We denote by A^2_{mQ,n,w_0} the subspace of the Bergman space A^2_{mQ} consisting of functions which vanish to order n at w_0, and refer to this space as a *partial Bergman space*. Its reproducing kernel $K_{m,n,w_0}(z, w)$ is called the *partial Bergman kernel*. Since the kernel vanishes if one of the variables assumes the value w_0, we cannot define the corresponding normalized partial Bergman kernels at w_0 directly as above. However, recalling the extremal characterization of k_{w_0}, which says that k_{w_0} is the unique solution to the problem

$$\max \left\{ \operatorname{Re} f(w_0) : \ f \in A^2_{mQ}, \ \|f\|_{mQ} \le 1 \right\}$$

we proceed analogously to define the *root function* k_{m,n,w_0} as the unique solution to the extremal problem

$$\max \left\{ \operatorname{Re} f^{(n)}(w_0) : \ f \in A^2_{mQ,n,w_0}, \ \|f\|_{mQ} \le 1 \right\},$$

whenever this problem does not degenerate. An analysis shows that k_{m,n,w_0} may actually be computed as a limit

$$k_{m,n,w_0}(z) = \lim_{\zeta \to w_0} \frac{K_{m,n,w_0}(z, \zeta)}{\sqrt{K_{n,m,w_0}(\zeta, \zeta)}},$$

where the limit is taken along a ray such that the root function has positive derivative of order n at w_0, and for this reason we still refer to k_{m,n,w_0} as the normalized partial Bergman kernel rooted at w_0.

The spectral droplet associated to the sequence of Bergman spaces A^2_{mQ} is defined in terms of an obstacle problem as before:

$$\mathcal{S} = \left\{ z : \ Q(z) = \hat{Q}(z) \right\},$$

where

$$\hat{Q}(z) = \sup \left\{ q(z) : \ q \in \mathrm{SH}(\mathbb{C}), \ \ q \le Q \text{ on } \mathbb{C} \right\},$$

and will in general be unbounded. Let w_0 be a point in a bounded component of the complement $\mathcal{S}^c$ of the spectral droplet. We denote this component by Ω, and assume that it is simply connected with a real-analytically smooth simple closed boundary curve. We moreover assume that Q is real-analytically smooth and strictly subharmonic in a neighbourhood of $\partial\Omega$. The space A^2_{mQ,n,w_0} is in a natural way associated to a spectral droplet $\mathcal{S}_{\tau,w_0}$ where $n = \tau m$ via a corresponding obstacle problem (see equations (1.1.4)-(1.1.5) in [43]) and we denote the off-spectral component of $\mathcal{S}_{\tau,w_0}$ containing w_0 by Ω_{τ,w_0}. We define implicitly an interval $I_{w_0} \subset [0, \infty)$ by the requirement that Ω_{τ,w_0} is a simply connected domain with real-analytically smooth boundary for each $\tau \in I_{w_0}$, such that Q is real-analytically smooth and strictly subharmonic in a neighbourhood of each of the boundaries $\partial\Omega_{\tau,w_0}$.

We remark here that the parameter τ now encodes a prescribed order of vanishing rather than the degree of a polynomial. Still, as the parameter τ increases, the off-spectral component Ω_{τ,w_0} grows according to a Hele-Shaw flow with injection at the point w_0.

Denote by φ_{τ,w_0} the conformal mapping

$$\varphi_{\tau,w_0} : \Omega_{\tau,w_0} \to \mathbb{D}$$

which maps w_0 to 0 with positive derivative, and let $\mathcal{Q}_{\tau,w_0}$ denote the bounded holomorphic function on Ω_{τ,w_0} which equals Q on the boundary and whose imaginary part is zero at w_0. The following asymptotic expansion is one of the main theorems of Paper B.

Theorem 2.3. *Suppose that a positive integer κ and an arbitrary positive number A are given. Then there exist sets Ω_{m,τ,w_0} containing Ω_{τ,w_0} whose boundaries $\partial\Omega_{m,\tau,w_0}$ lie at a distance of at least $A(m^{-1}\log m)^{\frac{1}{2}}$ from Ω_{τ,w_0}, such that*

$$\begin{aligned} &k_{m,n,w_0}(z) \\ &\qquad = m^{\frac{1}{4}}(\varphi'_{\tau,w_0}(z))^{\frac{1}{2}}[\varphi_{\tau,w_0}(z)]^n \mathrm{e}^{m\mathcal{Q}_{\tau,w_0}}\Big(\sum_{j\le\kappa} m^{-j}\mathcal{B}_{j,\tau,w_0}(z) + \mathrm{O}\big(m^{-\kappa-1}\big)\Big) \end{aligned}$$

where the error term is uniform for $z \in \Omega_{m,\tau,w_0}$ while $\tau \in I_{w_0}$. Moreover, the coefficient functions $\mathcal{B}_{j,\tau,w_0}$ are real at w_0, and the main term $\mathcal{B}_{0,\tau,w_0}$ is the unique bounded holomorphic function on Ω_{τ,w_0} which is zero-free, positive at w_0 and has prescribed modulus

$$|\mathcal{B}_{0,\tau,w_0}(z)|^2 = \pi^{-\frac{1}{2}}\big(\Delta Q(z)\big)^{\frac{1}{2}}, \qquad z \in \partial\Omega_{\tau,w_0}.$$

As an application of the main result, we obtain error function transition behavior for the Bergman density at emergent interfaces in the plane. An important point is that these interfaces may occur either as a result of a singularity in the metric, a patch of negative curvature of the same, or when considering partial Bergman kernels. In any case, one may use the root functions (or normalized partial Bergman kernels) for the partial spaces A^2_{mQ,n,w_0} to describe the density locally at the transition.

The methods used in the proof share many similarities with the approach taken in Paper A. The main new insight is that the root functions k_{m,n,w_0} may be viewed as analogues of orthogonal polynomials. In particular, the above theorem is based on the orthogonal foliation flow, taken with respect to the potential

$$R = (Q - \breve{Q}_{\tau,w_0}) \circ \varphi^{-1}_{\tau,w_0},$$

where $\breve{Q}_{\tau,w_0}$ is the harmonic extension across $\partial\Omega_{\tau,w_0}$ of the solution $\check{Q}_{\tau,w_0}$ to the defining obstacle problem for the spectrum $\mathcal{S}_{\tau,w_0}$.

We also provide an extension of the foliation flow technique, allowing us to treat measures of the form $\mathrm{e}^{-2mQ(z)}V(z)\mathrm{dA}$, where V is a non-negative function which is real-analytic and zero-free near a given off-spectral interface. As a consequence, we derive perturbation results for root functions and orthogonal polynomials, which verify that the qualitative properties of these objects do not depend much upon the base metric on $\mathbb{C}$. Specifically, if we replace the standard area measure dA by VdA, the corresponding root functions $k_{m,n,w_0,V}$ enjoy a full asymptotic expansion like in Theorem 2.3. The only change lies in the structure of the coefficient functions $\mathcal{B}_{j,\tau,w_0} = \mathcal{B}_{j,\tau,w_0,V}$. For instance, the main term $\mathcal{B}_{0,\tau,w_0,V}$ is the unique zero-free holomorphic function on Ω_{τ,w_0} which is positive at w_0 and has prescribed modulus

$$|\mathcal{B}_{0,\tau,w_0,V}(z)|^2 = \pi^{-\frac{1}{2}}(\Delta Q(z))^{\frac{1}{2}}V(z)^{-1}$$

on the boundary $\partial\Omega_{\tau,w_0}$.

Paper C: Scaling limits of random normal matrix processes near singular boundary points

Denote by Q a potential with sufficient growth at infinity, and by $\mathcal{S} = \mathcal{S}_1$ the spectral droplet associated to the random normal matrix ensemble with potential Q. As a consequence of the fact that the spectral droplet appears as the solution to a free boundary problem, one may deduce from the theory of Sakai [61, 62] that the boundary $\partial\mathcal{S}$ enjoys considerable regularity. If the potential Q is real-analytically smooth near the boundary, the boundary itself must be real-analytic, except for a finite number of singular points. These points may be either cusps or double points, and may further be subdivided according to different degrees of tangency. Double points refer here to points where two parts of the smooth boundary touch, with formations of cusp-like structures on either side of the meeting point.

In Paper C [9] we explore some basic properties of scaling limits of random normal matrix ensembles near such singular points. Before describing the results, let us say a few words about one of the main tools. In [8], Ameur, Kang and Makarov studied an equation known as *Ward's equation* in the context of random normal matrix ensembles near regular boundary points. Ward's equation results as a scaling limit of identities known variously as loop equations, Ward identities or Dyson-Schwinger equations, which arise as a result of the reparameterization invariance of the partition function, that is, the normalizing constant $Z_{n,m,Q}$ in the definition of the measures $\mathbb{P}_{n,m,Q}$ in (1.0.2). In a wider context, loop equations are prevalent in the physics literature, and their use was put on solid mathematical ground for application to the study of eigenvalues of random (hermitian) matrices by Johansson in 1997 [48], and were further applied in the random normal matrix context by Ameur, Hedenmalm and Makarov [7]. Ward's equation states that that the limiting rescaled density $\rho(z) = \rho_1(z)$ and the corresponding two-point function

$\rho_2(z,w)$ are connected by

$$\bar{\partial}\int_{\mathbb{C}}\frac{B(z,w)}{z-w}\mathrm{dA}(w)=\rho(z)-1-\Delta\log\rho(z),$$

where B is the Berezin kernel

$$B(z,w)=\frac{\rho(z)\rho(w)-\rho_2(z,w)}{\rho(z)}.$$

Ward's equation derives its importance from the fact that it provides a non-trivial equation for ρ, owing to the fact that the kernel $\mathbf{k}$ and hence ρ_2 may be determined form ρ by polarization. The technique is applied in Paper C, which concerns the structure of limiting point fields associated to rescaled eigenvalue processes near a singular point of $\partial\mathcal{S}$. We formulate the main result below for cusp points alone, as the case of double points is completely analogous.

For the formulation of the main result (Theorem II), we let p denote a cusp point on the boundary of a spectral droplet $\mathcal{S}$, such that the potential Q is real-analytic and strictly subharmonic in a neighbourhood of p. Let p_n be the point on the bisectrix of the cusp inside $\mathcal{S}$, i.e. the curve of equal distance to each of the two arcs meeting at the cusp, which lies at a distance of $Tn^{-\frac{1}{2}}$ from the boundary $\partial\mathcal{S}$. When we talk about a rescaled coordinate system, we mean coordinates ξ defined implicitly by the relation

$$z_n(\xi)=p_n+\nu\frac{\xi}{\sqrt{2n\Delta Q(p_n)}},$$

where ν is chosen such that the cusp points in the positive imaginary direction in the ξ-coordinates. If $(z_1,\ldots,z_n)$ is drawn from the Coulomb gas measure $\mathbb{P}_{n,n,Q}$, this induces a determinantal point process $(\xi_1,\ldots,\xi_n)$, with associated rescaled correlation kernel $\mathbf{k}_n(\xi,\eta)$

$$\mathbf{k}_n(\xi,\eta)=\frac{1}{2n\Delta Q(p_n)}\mathbf{K}_{n,n}\big(z_n(\xi),z_n(\eta)\big).$$

It was shown in [8] that for any sequence $\mathcal{N}$ of positive numbers, there exists a subsequence $\mathcal{N}^*$ of $\mathcal{N}$, such that the sequence $\{\mathbf{k}_n\}_{n\in\mathcal{N}^*}$ converges uniformly on compact subsets of $\mathbb{C}^2$. The associated limiting function $\mathbf{k}$ is called a limiting kernel, and determines a limiting determinantal point field with correlation functions ρ_k for $k=1,2,3,\ldots$.

Theorem 2.4. *If T is large enough, then the limiting point field associated to any limiting rescaled kernel $\mathbf{k}$ is non-trivial, and the associated density function $\rho(z)=\mathbf{k}(z,z)$ enjoys the estimate*

$$\rho(z)\le C\mathrm{e}^{-2x^2},\qquad z=x+\mathrm{i}y,$$

for some constant C. Moreover, the limiting density satisfies Ward's equation.

That the parameter T must be large in order for the limiting point field to be non-trivial is most likely an artefact from our method of proof, which uses $\bar{\partial}$-estimates to construct elements in the unit ball of $A^2_{mQ,n}$ which after rescaling are bounded away from zero at p_n. A finer analysis could hopefully remove this requirement.

We obtain also a theorem (Theorem I) showing that any limiting point field obtained by rescaling directly around the cusp point p at scale $n^{-\frac{1}{2}}$ is trivial, that is the associated density $\rho(z)$ vanishes identically.

Furthermore, we explore a conditional theorem, concerning what happens if a certain structural assumption is made on the part of the limiting kernel. It is known [8] that any limiting kernel must be of the form $\mathbf{k} = \mathbf{G}\Psi_0$, where $\mathbf{G}$ denotes the Ginibre-∞ kernel of Theorem 1.1 of the introduction, and where Ψ_0 is a hermitian entire function. If the function Ψ_0 is furthermore assumed to have the property of translation invariance, in the sense that

$$\Psi_0(z + \mathrm{i}t, w - \mathrm{i}t) = \Psi_0(z, w)$$

is independent of t (c.f. the behavior at regular boundary points), then the kernel must necessarily be of the form $\mathbf{k} = \mathbf{G}(z, w)\Psi(z + \bar{w})$, where Ψ is a generalization of the error function, defined by

$$\Psi(z) = \frac{1}{\sqrt{2\pi}} \int_I \mathrm{e}^{-\frac{1}{2}(z-t)^2} \mathrm{d}t,$$

for some subset $I \subset [-2T, 2T]$.

Even if this observation offers a possible suggestion as to what limiting correlation kernels may look like, it seems unlikely that the translation invariance property could be proven by elementary means. What would be of considerable interest, however, is to explore if the asymptotics of the corresponding orthogonal polynomials could be obtained also in the case when the spectral droplet has a cusp. This should allow one to obtain a complete understanding of the microscopic behavior of the process, as is done in Paper A and B in various regular situations.

For the reader's convenience, we remark that in the article, the notation differs slightly from what we have used here. Mainly, this concerns conventions for use of bold-face symbols for certain classes of kernels, and the fact that in the article, we are working with the potential $2Q$ rather than Q. Consequently some associated objects are slightly changed. For instance, the equilibrium measure takes the form $\Delta Q\, 1_{\mathcal{S}} \mathrm{dA}$ rather than $2\Delta Q\, 1_{\mathcal{S}} \mathrm{dA}$.

Paper D: A Central Limit Theorem for the polyanalytic Ginibre ensemble

The main result of Paper D [37] is a central limit theorem for fluctuations of linear statistics in the polyanalytic Ginibre ensemble, the determinantal point process defined in (1.0.10). We work in the regime when the upper bound of the polyanalyticity degree q is held fixed while the analytic degree parameter n and the magnefic field strength m tend to infinity, while $n = m$. In this regime, Haimi and Hedenmalm in [36] obtained the circular law, which may be phrased in terms of the traces, or linear statistics, as

$$\frac{1}{nq}\mathbb{E}_{n,q}\mathrm{tr}_{n,q}(f) = \mathbb{E}_{n,q}(f) \to \int_{\mathbb{D}} f(z)\mathrm{dA}(z), \qquad n \to \infty,$$

where $\mathbb{E}_{n,q}$ denotes the expectation under $\mathbb{P}_{n,q} = \mathbb{P}_{n,n,q}$ and where the linear statistics, or traces $\mathrm{tr}_{n,q}(f)$, are defined for test functions f by

$$\mathrm{tr}_{n,q}(f) = \sum_{z\in\Theta_{n,q}} \delta_z.$$

The fluctuations of linear statistics are the random variables that describe the deviation from the mean, and are defined by

$$\mathrm{fluct}_{n,q}(f) = \mathrm{tr}_{n,q}(f) - \mathbb{E}_{n,q}\mathrm{tr}_{n,q}(f).$$

We denote by $\mathrm{fluct}_{\delta;\,n,q}$ the fluctuations associated to the random processes associated to the *pure* polyanalytic Bergman kernels, defined as the reproducing kernels for the *pure Polyanalytic Bergman spaces* $A^2_{n,q} \ominus A^2_{n,q-1}$, that is the orthogonal complement of the subspace $A^2_{n,q-1}$ in $A^2_{n,q}$.

Theorem 2.5. *As n tends to infinity while q is fixed, the fluctuations* $\mathrm{fluct}_{n,q}(f)$ *and* $\mathrm{fluct}_{\delta;n,q}(f)$ *tend to normal random variables, with mean* 0 *and variance* $\sigma_q(f)$ *and* $\sigma_{\delta;q}(f)$*, respectively, where*

$$\sigma_q(f) = q\|f\|^2_{H^1(\mathbb{D})} + \frac{q}{2}\|f\|^2_{H^{\frac{1}{2}}(\partial\mathbb{D})}$$

and

$$\sigma_{\delta;q}(f) = (2q-1)\|f\|^2_{H^1(\mathbb{D})} + \frac{1}{2}\|f\|^2_{H^{\frac{1}{2}}(\partial\mathbb{D})}.$$

Here $\|\cdot\|_{H^1(\mathbb{D})}$ denotes the Dirichlet seminorm of f, while $\|\cdot\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}$ is the $H^{\frac{1}{2}}(\mathbb{D})$-seminorm obtained by

$$\|f\|^2_{H^{\frac{1}{2}}(\partial\mathbb{D})} = \sum_{k\neq 0}|k|\,|\hat{f}(k)|^2.$$

We should point out that when forming the complete polyanalytic fluctuations (via adding the kernels) from the pure process, the bulk variance is averaged while the boundary variance is additive under this procedure. Apart from this peculiar feature, this central limit theorem parallells one that is known to hold in the context of the Ginibre ensemble [57]. It is not surprising that such a result is true, in light of the abundance of central limit theorems in probability theory.

Of perhaps more importance than the actual CLT is a technique developed in order to obtain the central limit theorem. Let us describe briefly these matters. The central limit theorem by Rider and Virag [57] was obtained with the use of the cumulant expansion method, developed for determinantal point processes by Costin and Lebowitz [25]. This method draws upon the fact that a random variable X is normally distributed if and only if all higher order cumulants, $C_k(X)$ for $k \geq 3$, vanish. Here, the cumulants $C_k(X)$ are implicitly defined by an expansion

$$\log \mathbb{E}\{\mathrm{e}^{tX}\} = \sum_{k\geq 1} \frac{t^k}{k!} C_k(X).$$

In the case at hand, it is known that the cumulants of the fluctuations may be expressed in terms of the reproducing kernels. For the pure fluctuations $\mathrm{fluct}_{\delta;n,q}(f)$ this reads

$$\int_{\mathbb{C}^k} G_k(z_1,\ldots,z_k) K_{\delta;n,q}(z_1,z_2)\ldots K_{\delta;n,q}(z_k,z_1)\mathrm{d}\mu_n^k(z)$$

where G_k is an explicit function defined in terms of the function f and where we write $\mathrm{d}\mu_n^k(z) = \mathrm{e}^{-n|z_1|^2-\ldots-n|z_k|^2}\mathrm{dA}^{\otimes k}(z_1,\ldots,z_k)$ for $z=(z_1,\ldots,z_k)$.

Although the strategy appeared to be an attractive starting point, the work by Rider and Virag employs involved combinatorial methods, which did not appear tractable to generalize to the polyanalytic case. Our approach bypasses this by relying directly on the main result of [57], which is accomplished by representing the pure polyanalytic kernels $K_{\delta;n,q}$ by differential operators (known as *raising operators*) acting on the Ginibre kernel. A partial integration procedure then shows that the cumulants associated to the pure fluctuations may be computed structurally as

$$C_k(\mathrm{fluct}_{\delta;n,q}(f)) = \int_{\mathbb{C}^k} \mathcal{D}_{k,q}[G_k](z) K_n(z_1,z_2)\cdots K_n(z_k,z_1)\mathrm{d}\mu_n^k(z),$$

where $\mathcal{D}_{k,q}$ are certain explicit differential operators expressed in terms of Laguerre polynomials. From this representation, it is possible to deduce the result by an application of Rider and Virag's theorem.

If one applies the same partial integration procedure, but replaces the function G_k by a function of the form $G(z_1,\ldots,z_k) = f(z_1)$ and integrates out the variables $z_2,\ldots,z_k$, a remarkable formula appears:

$$\int_{\mathbb{C}} f(z) K_{\delta;n,q}(z,z)\mathrm{e}^{-n|z|^2}\mathrm{dA}(z) = \int_{\mathbb{C}} f(z) L_{q-1}[-n^{-1}\Delta]\Big(K_n(z,z)\mathrm{e}^{-n|z|^2}\Big)\mathrm{dA}(z)$$

where L_{q-1} denotes the $q-1$:th Laguerre polynomial. This expresses an identity for the kernel $K_{\delta;n,q}$ in weak form, which may be strengthened by easily accessible a priori knowledge. By adding the different pure polyanalytic levels and using the summation rule $\sum_{r\leq q-1} L_r = L_{q-1}^{(1)}$ for the Laguerre polynomials (1.0.9), we obtain that

$$K_{n,q}(z,z)\mathrm{e}^{-n|z|^2} = L_{q-1}^{(1)}[-n^{-1}\Delta]\Big(K_n(z,z)\mathrm{e}^{-n|z|^2}\Big). \tag{2.0.1}$$

In unpublished joint work with Haimi and Hedenmalm, we have derived a limiting macroscopic law for the global limiting global density $\rho_t(z)$ given by

$$\rho_t(z) = \lim\,(nq)^{-1} K_{n,q}(z,z)\mathrm{e}^{-n|z|^2},$$

where the limit is taken as $n \to \infty$ and $q = tn + \mathrm{O}(1)$ for $t \in (0,1]$. Using the identity (2.0.1) and the properties of the Fourier transform, one may first derive that

$$\mathcal{F}\Big((nq)^{-1} K_{n,q}(z,z)\mathrm{e}^{-n|z|^2}\Big)(\xi) = \frac{L_{q-1}^{(1)}(n^{-1}|\xi|^2)}{q}\mathcal{F}\Big(n^{-1} K_n(z,z)\mathrm{e}^{-n|z|^2}\Big)(\xi),$$

and by classical asymptotic results on Laguerre polynomials and the convergence of the Ginibre kernel it follows by Fourier inversion that

$$\rho_t(z) = t^{-1} 1_{\mathbb{D}(0,\sqrt{t})} * 1_{\mathbb{D}(0,1)}(z)$$

where $*$ denotes convolution in $\mathbb{C} = \mathbb{R}^2$. The symmetric case $t = 1$ corresponds to an integrated semi-circular law, which interestingly enough also appears in a certain boundary scaling limit [36] in the regime when the parameter q is many order of magnitudes smaller than n. For $t < 1$, the limiting density is constant on a disk of positive radius (c.f. the Ginibre ensemble corresponding to $t = 0$).

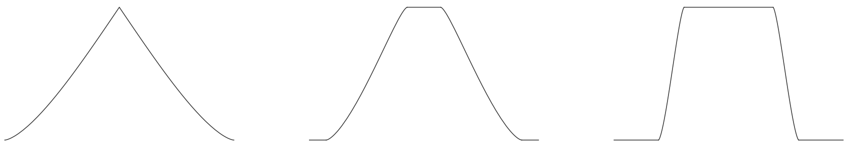

Figure 2.1: A radial plot of the density $\rho_t(z)$ for $t = 1$ (left), $t = 0.5$ (middle) and $t = 0.05$ (right).

At even finer scales, completely novel processes, which are given as Fourier transforms of disks intersected with half-planes (and thus provide natural generalizations to the sine kernel), appear naturally. In all, this class of point processes promise much in terms of possibilities for future developments.

Paper E: Discrepancy densities for planar and hyperbolic zero packing

In Paper E [70] we prove a result conjectured by Hedenmalm in [38], concerning the discrepancy densities for zero packing introduced above in the equations (1.0.11), (1.0.12), (1.0.13) and (1.0.14).

Theorem 2.6. *We have the equalities*

$$\rho_{\mathbb{H}} = \rho_{\mathbb{H}}^* \qquad \textit{and} \qquad \rho_{\mathbb{C}} = \rho_{\mathbb{C}}^*.$$

The outline of the proof is rather straight-forward. As the planar case is similar, we discuss it for the hyperbolic case alone. First, as observed in the introduction one trivially has $\rho_{\mathbb{H}}^* \geq \rho_{\mathbb{H}}$, and hence we only need to demonstrate the opposite inequality. In order to do this, we pick a sequence of minimizers f_r of the functional $\rho_{\mathbb{H},r}(f)$, defined in the canonical way for finite r. What remains to be done is to show that these may be modified into admissible functions ν_r, such that

$$\rho_{\mathbb{H},r}^*(\nu_r) \leq \rho_{\mathbb{H}}(f_r) + \mathrm{o}(1), \qquad r \to 1^-.$$

The idea is to multiply f_r by a smooth cut-off function χ, which is identically one on a disk $\mathbb{D}(0,r)$ and vanishes outside a slightly larger disk $\mathbb{D}(0,r')$, where $0 < r < r' < 1$. By employing Hörmander-type $\bar{\partial}$-estimates, one can show that the norm-minimal solution to the equation

$$\bar{\partial} u_r = \bar{\partial}(f_r \chi) = f_r \bar{\partial} \chi$$

is small, so that $\nu_r := f_r - u_r$ should be close to f_r. Since moreover $\bar{\partial}\nu_r = 0$ by construction, and since ν_r grows at most polynomially, ν_r must itself be a polynomial by Liouville's theorem, and is hence admissible for the extremal problem.

The key difficulty lies in obtaining good control of f_r near the boundary, where $\bar{\partial}\chi$ is supported. This is completely indispensible both in the application of the $\bar{\partial}$-estimate, and in demonstrating that the functional $\rho_{\mathbb{H},r}^*(f_r)$ is bounded above by $\rho_{\mathbb{H},r}(f_r) + \mathrm{o}(1)$. The necessary control is accomplished by obtaining estimates of non-concentration type for the L^1- and L^2-norms of minimizers f_r, using variational techniques. In a nutshell, it is shown that if the minimizers grow near the boundary, then the delicate balance between the L^1 and L^2-norms appearing in the functional

$$\rho_{\mathbb{H},r}(f) = \frac{1}{\int_{\mathbb{D}(0,r)} \frac{\mathrm{dA}(z)}{1-|z|^2}} \Big\{ \int_{\mathbb{D}(0,r)} |f(z)|^2 (1-|z|^2) \mathrm{dA}(z) - 2 \int_{\mathbb{D}(0,r)} |f(z)| \mathrm{dA}(z) \Big\} + 1$$

enables one to say that a dilation would give us a function g_r for which $\rho_{\mathbb{H},r}(g_r) < \rho_{\mathbb{H},r}(f_r)$, contradicting minimality. Here, we have arrived at the above expression for $\rho_{\mathbb{H},r}(f)$ by expanding the square in the definition of the discrepancy function $\Psi_{\mathbb{H},f}(z,r)$.

As mentioned in the introduction, it is conjectured that the optimizers for the planar zero packing problem are functions whose zeros are periodically placed according to the Abrikosov lattice. I find it difficult to even imagine how a proof of such a theorem would look like, and to my knowledge not much progress has been made in this direction. What appears to be more tractable as a first step is to obtain some weaker equidistribution result, e.g. that a disc of a given radius is likely to contain roughly as many zeros as the area of the disk. As a small step in this direction, we obtain some control of the complexity of approximate minimizers: in order to obtain a sequence f_r such that

$$\rho_{\mathbb{H},r}(f_r) \to \rho_{\mathbb{H}},$$

it suffice to take f_r to be a polynomial whose degree is comparable to the hyperbolic area of $\mathbb{D}_r$.

Paper F: A critical topology for L^p-Carleman classes for $0 < p < 1$

The starting point of the work [42] is the following simple question. Assume that a positive number p with $0 < p < 1$ and a positive integer k are given, and define a Sobolev quasinorm on test functions on the real line by

$$\|f\|_{k,p} = \left(\sum_{j=0}^{k} \|f^{(j)}\|_p^p \right)^{1/p},$$

where $\|\cdot\|_p$ denotes the L^p quasinorm

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p \mathrm{d}x \right)^{1/p}.$$

If $W^{k,p}$ denotes the abstract completion of C_0^∞ in this quasinorm, what can be said about the space $W^{k,p}$?

We recall here that a quasinorm is defined similarly to a norm, only that the subadditivity condition is replaced by requiring that a quasi-triangle inequality of the form

$$\|f + g\| \le C\Big(\|f\| + \|g\|\Big)$$

holds for some constant C.

Initially, this looks almost like any other Sobolev space. However, an immediate observation tells us that care must be taken: given a function $f \in L^p$ for $0 < p < 1$, we cannot even be sure that its primitive exists. Indeed, with our limited information we may not infer that $f \in L^1_{\mathrm{loc}}$, and consequently we do not have the fundamental theorem of calculus at hand.

Jaak Peetre [55] studied this question in the 70's, and obtained the following remarkable property.

The canonical mapping

$$\boldsymbol{\pi}(f) = (f, f', f'', \ldots, f^{(k)}), \qquad f \in \mathrm{C}_0^\infty(\mathbb{R})$$

supplies an isometric isomorphism of the Sobolev space $W^{k,p}$ with the direct sum of $k+1$ copies of L^p:

$$W^{k,p} \cong L^p \oplus L^p \oplus \cdots \oplus L^p.$$

A natural interpretation of Peetre's theorem is that the derivatives of functions in the space $W^{k,p}$ disconnect: given any $(k+1)$-tuple $(f_0, \ldots, f_k)$ of L^p-functions, there exists a sequence of test functions g_j such that $\{g_j\}$ is Cauchy in the topology of $W^{k,p}(\mathbb{R})$, while $g_j^{(k)}$ converges to the prescribed element $f_k \in L^p(\mathbb{R})$ as $j \to \infty$.

In Paper F, we investigate similar issues of disconnection for classes defined with respect to stronger norms, taking derivatives of all order into account. The hope is that for an appropriate choice of test functions, this stronger norm prevents such a degeneration. To be precise, the test-classes are defined in terms of a mild à priori growth restriction, called *p-tameness*. A smooth function f is said to be p-tame if

$$\limsup_{n\to\infty}(1-p)^n \log\|f^{(n)}\|_\infty \le 0.$$

For a weight sequence $\mathcal{M} = \{M_n\}$, we define the L^p-Carleman space $W^p_{\mathcal{M}}$ as the abstract completion of the collection of all compactly supported p-tame functions f which respect to the quasi norm

$$\|f\|_{p,\mathcal{M}} = \sup_{n\ge 0} \frac{\|f^{(n)}\|_p}{M_n}.$$

This definition is reminiscent of the defining property for traditional Carleman classes, only the traditional supremum norms over an interval has been replaced by L^p-quasi norms and the dilation invariance of the topology is removed for convenience. Under some regularity conditions on the part of the weight sequence (p-regularity, see Definition 1.6 of Paper F), we find that these spaces undergo a sharp transition depending on the finiteness of a quantity $\kappa(p,\mathcal{M})$, defined by

$$\kappa(p,\mathcal{M}) = \lim_{n\to\infty}\sum_{j=0}^{n}(1-p)^j \log M_j.$$

The following is the main result.

Theorem 2.7. *If $\kappa(p,\mathcal{M})$ is finite, then $W^p_{\mathcal{M}}$ is a space of smooth functions, while if $\kappa(p,\mathcal{M})$ is infinite, the space $W^p_{\mathcal{M}}$ disconnects in Douady-Peetre sense under the canonical map:*

$$W^p_{\mathcal{M}} \cong L^p \oplus L^p \oplus \ldots \oplus L^p \oplus W^p_{\mathcal{M}_k},$$

where there are k copies of L^p and where $\mathcal{M}_k = \{M_{j+k}\}_j$ denotes the k-shifted sequence.

There is also a quasianalyticity barrier, in analogy to the classical Carleman classes. To formulate our result in this direction, we let $\mathcal{M}$ denote a weight sequence with $\kappa(\mathcal{M},p) < \infty$ and define the sequence $\mathcal{N} = \{N_n\}_n$ by

$$N_n = \prod_{j=1}^{\infty} M_{n+j}^{p(1-p)^{j-1}}.$$

Under some regularity assumptions on $\mathcal{M}$ (decay-regularity, see Definition 1.5 of Paper F), it turns out that the L^p-Carleman class $\mathcal{C}^p_{\mathcal{M}}$, obtained by taking all dilates of $f \in W^p_{\mathcal{M}}$, is quasianalytic if and oly if the classical Carleman class $\mathcal{C}_{\mathcal{N}}$ is. In view of the Denjoy-Carleman theorem, the status of the latter space may be determined by the growth properties of the sequence $\mathcal{N}$.

About the contributions of the author

Papers A, B, C, D and F are the result of joint works, and my role in each of these is indicated below.

The project that resulted in Paper A was initiated by Hedenmalm several years earlier. At the time of my joining the project, the approach based on Laplace's method (e.g., Theorem 1.3.7) was essentially understood, and the relevance of a foliation for the rigorous proof was felt at the intuitive level. My main contribution was to develop the orthogonal foliation flow algorithm (Lemma 4.1.2) of Section 4, which supplies the main tool in the proof of Theorem 1.3.6, which shows that the suggested expansion in Theorem 1.3.7 was indeed correct.

The issues studied in Paper C were originally considered by Ameur, Kang and Makarov (arXiv preprint 2014). After an oversight was observed in this manuscript, it was necessary to fix the problem. In particular, the triviality theorem (Theorem I) of Paper C originally claimed the existence of a non-trivial blow-up process, which turns out to only be possible if we blow up around a moving point near the singular point in question. These matters matters were explored jointly with Ameur.

Papers B, D and F are the results of joint work in the full sense of relatively equal contributions at all levels by all the listed co-authors.

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Part II

Scientific papers

Paper A

Planar orthogonal polynomials and boundary universality in the random normal matrix model
(joint with H. Hedenmalm)
Preprint: arXiv (2017), 61 pages.

PLANAR ORTHOGONAL POLYNOMIALS AND BOUNDARY UNIVERSALITY IN THE RANDOM NORMAL MATRIX MODEL

HAAKAN HEDENMALM AND ARON WENNMAN

Abstract. We show that the planar normalized orthogonal polynomials $P_{n,m}(z)$ of degree n with respect to an exponentially varying planar measure $\mathrm{e}^{-2mQ}\mathrm{dA}$ enjoy an asymptotic expansion

$$P_{n,m}(z) \sim m^{\frac{1}{4}}\sqrt{\phi_\tau'(z)}[\phi_\tau(z)]^n \mathrm{e}^{m\mathcal{Q}_\tau(z)}\left(\mathcal{B}_{0,\tau}(z)+\frac{1}{m}\mathcal{B}_{1,\tau}(z)+\frac{1}{m^2}\mathcal{B}_{2,\tau}(z)+\ldots\right),$$

as $n,m\to\infty$ while the ratio $\tau=\frac{n}{m}$ is fixed. Here $\mathcal{S}_\tau$ denotes the droplet, the boundary of which is assumed to be a smooth simple closed curve, and ϕ_τ is a normalized conformal mapping from the complement $\mathcal{S}_\tau^c$ to the exterior disk $\mathbb{D}_\mathrm{e}$. The functions $\mathcal{Q}_\tau$ and $\mathcal{B}_{j,\tau}(z)$ are bounded holomorphic functions which may be expressed in terms of Q and $\mathcal{S}_\tau$. We apply these results to obtain boundary universality in the random normal matrix model for smooth droplets, i.e., that the limiting rescaled process is the random process with correlation kernel

$$\mathrm{k}(\xi,\eta)=\mathrm{e}^{\xi\bar\eta-\frac{1}{2}(|\xi|^2+|\eta|^2)}\,\mathrm{erf}\,(\xi+\bar\eta).$$

A key ingredient in the proof of the asymptotic expansion of the orthogonal polynomials is the construction of an *orthogonal foliation* – a smooth flow of closed curves near $\partial\mathcal{S}_\tau$, on each of which $P_{n,m}$ is orthogonal to lower order polynomials, with respect to an induced measure. To compute the coefficient functions, we develop an algorithm which determines the coefficients $\mathcal{B}_{j,\tau}$ successively in terms of inhomogeneous Toeplitz kernel conditions. These inhomogeneous Toeplitz kernel conditions may be understood as scalar Riemann-Hilbert problems on the Schottky double of the complement of the droplet.

1. Introduction

1.1. **Historical comments on orthogonal polynomials.** The early 1920s in Berlin witnessed a rapid development of the understanding of orthogonal polynomials and related kernel functions. The pioneers were Gabor Szegő, Stefan Bergman, and Salomon Bochner. One of the early results is that of Szegő [42]. He obtained the first (main) term of an asymptotic expansion of the (analytic) orthogonal polynomials in $L^2(\Gamma,\mathrm{ds})$, where Γ is a real-analytically smooth Jordan curve in the complex plane $\mathbb{C}$ supplied with normalized arc length measure $\mathrm{ds}(z)=(2\pi)^{-1}|dz|$. Let $\mathbb{C}\setminus\Gamma=\Omega\cup\Omega_\mathrm{e}$ be the decomposition of the complement into disjoint connected components, where Ω is bounded and Ω_e is unbounded. Szegő's expansion involves the conformal mapping $\Omega_\mathrm{e}\to\mathbb{D}_\mathrm{e}$ which fixes the point at infinity, where $\mathbb{D}_\mathrm{e}$ is the exterior disk: $\mathbb{D}_\mathrm{e}:=\{z\in\mathbb{C}:|z|>1\}$. Moreover, the expansion is valid in Ω_e, with uniform control on compact subsets. Sligthly later, the Swedish mathematician Torsten Carleman [9] – inspired by the work of Szegő – considered instead the (analytic) orthogonal polynomials in $L^2(\Omega,\mathrm{dA})$, where $\mathrm{dA}(z)=\pi^{-1}\mathrm{d}x\mathrm{d}y$ (where $z=x+\mathrm{i}y$) is normalized area measure on the simply connected bounded domain Ω with real-analytic boundary curve Γ. He found an asymptotic formula for the orthogonal polynomials with a much smaller error (exponential decay) term than in Szegő's

Date: May 25, 2018.

case. Again, the asymptotic formula holds in Ω_e. It should be remarked that Carleman's asymptotic formula is actually valid in some fixed neighbourhood of the closure of Ω_e. Carleman's approach was to think of the $L^2(\Omega)$-norm of holomorphic functions as the Dirichlet norm of the primitive, and then to use Green's formula to switch the integration first to the boundary and second to the exterior domain.

In contrast with the above results, the study of orthogonal polynomials on the real line has a long history. Here, we should mention the classical orthogonal polynomials associated with the names of Hermite, Laguerre, Jacobi, Gegenbauer, Chebyshev, and Legendre. The study of more general orthogonal polynomials in weighted L^2-spaces on the line is associated with the names of Chebyshev, Markov, Stieltjes, Szegő, Bernstein, and Akhiezer, to just mention a few. The structure of general orthogonal polynomials on the line is rather rigid with the appearance of a three-term recursion relation, which has to do with the fact that multiplication by the independent variable is self-adjoint on the weighted L^2-space. This fact is important, and is used in many approaches to the asymptotics of orthogonal polynomials. If one considers measures supported on more general curves, however, there is no analogue. In fact, if the orthogonal polynomials with respect to arc length measure on an analytic curve Γ satisfy a three-term recursion formula, then Γ is necessarily an ellipse, including the symmetric and specialized cases of a circle a line, respectively [14]. For planar orthogonal polynomials, even finite term recursion formulas occur only rarely [34].

The above-mentioned results of Szegő and Carleman for the orthogonal polynomials in $L^2(\Gamma, \mathrm{ds})$ and $L^2(\Omega, \mathrm{dA})$ do not need any three-term recursion formula and lead to a different direction of development. Given the similarity with the work presented here, we wish to highlight some results in this direction. We first describe Szegő's result. Let $\{P_0, P_1, \ldots\}$ denote the sequence of orthogogonal polynomials in $L^2(\Gamma, \mathrm{ds})$, which have

$$\int_\Gamma P_n(z)\bar{P}_k(z)\mathrm{ds} = \delta_{n,k},$$

while $\deg P_n = n$ and $\lim_{z\to\infty} z^{-n}P_n(z) > 0$. Here, $\delta_{n,k}$ is the Kronecker delta symbol which equals 1 when $n = k$ and vanishes otherwise. Moreover, here and in the sequel, we use the notational convention that for a given function f, the expression $\bar{f}$ stands for the function whose values are the complex conjugates of those of f. Then, if $\phi\colon \Omega_e \to \mathbb{D}_e$ denotes the exterior conformal mapping, Szegő's theorem asserts that

$$P_n(z) = \sqrt{\phi'(z)}[\phi(z)]^n\,(1 + \mathrm{o}(1))\,, \qquad z \in \Omega_e, \tag{1.1.1}$$

where the error is uniform on compact subsets. In contrast, Carleman's work concerns the normalized orthogonal polynomials $\{P_0, P_1, P_2, \ldots\}$ in $L^2(\Omega)$, where Ω is a bounded simply connected domain with real-analytic boundary. Again, these are uniquely determined if we require that P_n has precise degree n and positive leading coefficient. Now let ϕ denote the conformal mapping $\phi\colon \Omega_e \to \mathbb{D}_e$, which fixes the point at infinity. Due to the real-analytically smooth boundary, ϕ extends to a conformal mapping of larger domain $\Omega_{\rho_0,e} \to \mathbb{D}_e(0, \rho_0)$. Here and in the sequel, we denote by $\mathbb{D}(z_0, r)$ the open disk centered at z_0 with radius r and by $\mathbb{D}_e(z_0, r)$ the complement of the closure of this set (which may be viewed as a disk on the Riemann sphere). In the special case when $z_0 = 0$ and $r = 1$, we drop the indication of these parameters and write $\mathbb{D}$ and $\mathbb{D}_e$. Then Carleman's formula states that

$$P_n(z) = (n+1)^{\frac{1}{2}}\,\phi'(z)[\phi(z)]^n\,(1 + \mathrm{O}(\rho^n))\,, \qquad z \in \Omega_{\rho_0,e}, \tag{1.1.2}$$

for $\rho_0 < \rho < 1$. As a consequence, the orthogonal polynomials may be thought of as push-forwards of the monomials to $A^2(\Omega)$.

One of the gems in the direction initiated by Carleman is the work of Suetin [41], which deals with domains whose boundary has a lower degree of smoothness, and the case when a weight function is present. Among many results, we recall the asymptotic formula concerning the system $\{P_n\}_n$ generated with respect to the inner product of $L^2(\Omega, \omega\,\mathrm{dA})$, where $\partial\Omega$ is real-analytically smooth, and where ω is non-negative and Hölder continuous with exponent α. In this setting, the orthogonal polynomials are asymptotically given by

$$P_n(z) = (n+1)^{\frac{1}{2}}\,\phi'(z)[\phi(z)]^n g(z)\left(1 + \mathrm{O}\left(\frac{\log n}{n}\right)^{\frac{\alpha}{2}}\right), \qquad z \in \Omega_e, \tag{1.1.3}$$

where the function $g(z)$ is holomorphic in the exterior domain Ω_e and satisfies $|g(z)|^2 = \omega^{-1}$ on $\partial\Omega_e$. Let us also remark that Szegő's result (1.1.1) extends to the setting of weights as well, see [43].

We might also mention the more recent work of Miña-Diaz [33] on an integral representation of the orthogonal polynomials in terms of a kernel which is associated with conformal mapping and the Beurling transform (see, e.g., [5]).

For an exposition of some of Szegő's work on orthogonal polynomials, we refer to the books by Simon [38, 39].

1.2. **Point processes on the line and plane from eigenvalues of random matrices.** In connection with the study of random Hermitian matrices and one-dimensional Coulomb gas, the orthogonal polynomials with respect to exponentially varying weights are central objects. The eigenvalue process associated to a random Hermitian matrix is goverened by a correlation kernel expressed in terms of the polynomial reproducing kernel

$$K_m(x,y) := \sum_{j=0}^{m-1} P_{j,m}(x)\bar{P}_{j,m}(y),$$

where the polynomials $P_{0,m}, P_{1,m}, P_{2,m}, \ldots$ are normalized and orthogonal in $L^2(\mathbb{R}, \mu_{2mQ})$, where $\mathrm{d}\mu_{2mQ}(x) = \mathrm{e}^{-2mQ(x)}\mathrm{d}x$, and are such that $P_{j,m}$ has precise degree j and positive leading coefficient $a_j > 0$. Here, Q is thought of as a confining potential, and the parameter m should be allowed to tend to infinity. The property characterizing the kernel $K_m(x,y)$ is the following. Let Pol_m denote the m-dimensional space of polynomials of degree $\le m-1$. Then we have for each $f \in \mathrm{Pol}_m$ that

$$\int_{\mathbb{R}} K_m(x,y) f(y)\mathrm{d}\mu_{2mQ}(y) = f(x), \qquad x \in \mathbb{R}.$$

For expositions on random Hermitian matrices and orthogonal polynomials, see, e.g., the books [32] and [12]. As a result of the rigid structure of the orthogonal polynomials on the line, there is a Christoffel-Darboux formula

$$K_m(x,y) = \sum_{j=0}^{m-1} P_{j,m}(x)\bar{P}_{j,m}(y) = \frac{a_{n-1}}{a_n}\,\frac{P_{m,m}(x)\bar{P}_{m-1,m}(y) - P_{m,m}(y)\bar{P}_{m-1,m}(x)}{x-y},$$

where it is implicit that $x, y \in \mathbb{R}$ with $x \neq y$. For an exposition of the Christoffel-Darboux formula, see [40]. The Christoffel-Darboux formula reduces the problem of analyzing the kernel to analyzing just the two highest degree polynomials, $P_{m-1,m}$ and $P_{m,m}$. By the efforts of e.g. Fokas, Its, Kitaev, and Deift and Zhou, the asymptotic behavior of the orthogonal polynomials is well understood in terms of solutions to matrix Riemann-Hilbert problems, see, e.g., [12, 13, 16, 17].

Recently, the properties of orthogonal polynomials with respect to exponentially varying planar measures of the form $\mathrm{e}^{-2mQ}\mathrm{dA}$ have been studied. The motivation comes from the theory of two-dimensional Coulomb gas, or Random Normal Matrix (RNM) ensembles. Indeed, the eigenvalues of a random normal matrix form a determinantal point process, and the correlation kernel is given by

$$K_m(z,w)\,\mathrm{e}^{-m(Q(z)+Q(w))} = \sum_{n=0}^{m-1} P_{n,m}(z)\bar{P}_{n,m}(w)\,\mathrm{e}^{-m(Q(z)+Q(w))}.$$

As a consequence of the lack of finite-term recursions, there is no Christoffel-Darboux formula, and instead we need to analyze the entire sequence $\{P_{n,m}\}_{n=0}^{m-1}$.

Macroscopically, the situation is well understood. It is known that the gas condensates to a certain compact set $\mathcal{S}_1$, called the *droplet*, see the discussion in Subsection 2.6. An interesting question is how the process behaves at the microscopic level. This can be studied via the rescaled density: for a point $z_0 \in \mathbb{C}$ and $\mathrm{n} \in \mathbb{T} = \partial\mathbb{D}$ we let

$$z_m(\xi) = z_0 + \mathrm{n}\frac{\xi}{\sqrt{2m\Delta Q(z_0)}} \tag{1.2.1}$$

consider

$$\rho_m(\xi) = \frac{1}{2m\Delta Q(z_0)} K_m(z_m(\xi), z_m(\xi))\, \mathrm{e}^{-2mQ(z_m(\xi))}. \tag{1.2.2}$$

Near a bulk point, $z_0 \in \mathcal{S}_1^\circ$, there exists a full asymptotic expansion of the kernel, see e.g. [2, 3]. In this case $\lim_m \rho_m(\xi) = 1$, uniformly on compact subsets. Away from the bulk, i.e. for $z_0 \in \mathcal{S}_1^c$ we instead have $\lim_m \rho_m(\xi) = 0$.

Let $z_0 \in \partial\mathcal{S}_1$, such that the boundary is real-analytically smooth near z_0, and let n be the outer normal to $\mathcal{S}_1$ at z_0. In this case, it is not known what the limit of the density ρ_m is. In the case when $Q(z) = \frac{1}{2}|z|^2$, the limit is known, and this limit is expected to be universal for regular boundary points.

Conjecture 1.2.1 (boundary universality)**.** Let $z_0 \in \partial\mathcal{S}_1$ and assume that $\partial\mathcal{S}_1$ is smooth in a neighbourhood of z_0. Then the density ρ_m converges as $m \to \infty$ to the limit

$$\rho(\xi) = \mathrm{erf}\,(2\,\mathrm{Re}\,\xi).$$

Here, we write erf for the *error function*

$$\mathrm{erf}\,(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty \mathrm{e}^{-t^2/2}\mathrm{d}t,$$

where is along a suitable contour from z to the origin and then from the origin to ∞ along the positive real line. This conjecture has been verified in some specific cases, and partial results have appeared recently. In connection with this we want to mention the work by Ameur, Kang and Makarov [4] who used a limiting form of Ward identities to show that if $\rho(\xi)$ is assumed to be translation invariant, then it must necessarily be as in Conjecture 1.2.1. Without simplifying assumptions, however, the full conjecture remains open. In the setting of Kähler manifolds, a similar problem appears in the context of partial Bergman kernels defined by vanishing to high order along a divisor. Under the assumption of S^1-invariance around the divisor, Ross and Singer [36] obtain the error function asymptotics near the emergent interface around the divisor (see also the work of Zelditch and Zhou [46]). In recent work, Zelditch and Zhou [47] find that this is a universal edge phenomenon along interfaces, which appears in the context of partial Bergman kernels defined in terms of a quantized Hamiltonian.

The standard approaches to the asymptotics of Bergman kernels are local in nature, both the peak section approach of Tian (see [44]) as well as the microlocal approach by Boutet de Monvel and Sjöstrand, as explained by Berman, Berndtsson, and Sjöstrand [6]. The same applies to older work of Hörmander, Diedrich, and Fefferman [15]. One reason to expect the boundary universality conjecture to be difficult is the non-locality of the correlation kernel. To illustrate this, we consider the Berezin density (associated with secondary quantization)

$$B_m^{\langle z_0\rangle}(z) = K_m(z_0, z_0)^{-1}|K_m(z, z_0)|^2\mathrm{e}^{-2mQ(z)}$$

studied in [1] and see that for boundary points $z_0 \in \partial\mathcal{S}_1$, this density develops a ridge along the whole boundary of the spectral droplet (see Figure 1.1). It is for this reason that we focus our analysis on the orthogonal polynomials, which share the nonlocal behavior (see Figure 3.1). The Berezin density and the Palm density add up to the density of states, so that the Berezin density may be understood as the repulsive influence of a particle placed at the given point z_0. Palm measures have been studied recently by Bufetov, Fan, and Qiu [8].

In the above context, we considered orthogonal polynomials of degree $< m$ with respect to the weight e^{-2mQ}. To be more general, we may separate the connection between the degree

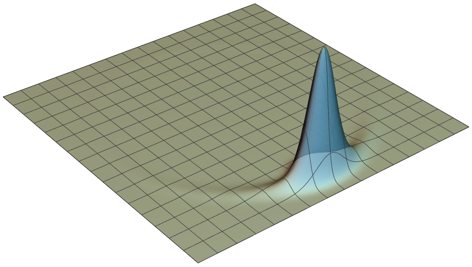

FIGURE 1.1. The Berezin density $K_m(z_0,z_0)^{-1}|K_m(z,z_0)|^2\mathrm{e}^{-2mQ(z)}$ with $Q(z)=\frac{1}{2}|z|^2$ for the boundary point $z_0=1$ and $m=120$.

of the polynomials and the quantization parameter m. When $n=m$, the probability density $|P_{n,m}|^2\mathrm{e}^{-2mQ}$ turns out to condensate along the boundary $\partial\mathcal{S}_1$. More generally, when $\tau=\frac{n}{m}$ is kept fixed, the condensation occurs along the boundary $\partial\mathcal{S}_\tau$ of another spectral droplet $\mathcal{S}_\tau$. For the special choice $Q(z)=\frac{1}{2}|z|^2+a\,\mathrm{Re}(z^2)$ where $a>0$, Lee and Riser [31] find explicitly the orthogonal polynomials, and verified Conjecture 1.2.1 in this case. Along the same lines, in [7], Balogh, Bertola, Lee and McLaughlin study the case of the potentials Q which are perturbations of the standard quadratic potential of the form

$$Q(z)=\frac{1}{2}|z|^2-c\log|z-a|^2,$$

for some $a\in\mathbb{R}$, $c>0$. For this choice of Q, they obtain an asymptotic expansion of the orthogonal polynomials. For choices of parameters such that the droplet $\mathcal{S}_\tau$ does not divide the plane, this is expressed in terms of the properly normalized conformal mapping of the complement $\mathcal{S}_\tau^c$ onto the exterior disk $\mathbb{D}_e$, denoted ϕ_τ. After some rewriting, their formula reads

$$P_{n,m}(z)=\left(\frac{m}{2\pi}\right)^{1/4}\sqrt{\phi'(z)}[\phi(z)]^n\mathrm{e}^{2m\mathcal{Q}(z)}\left(1+\mathrm{O}(m^{-1})\right),\quad z\in\mathcal{B}^c, \tag{1.2.3}$$

where $\frac{n}{m}=\tau+\mathrm{O}(m^{-1})$, $\mathcal{B}$ is some compact subset of the interior of $\mathcal{S}_\tau$, and $\mathcal{Q}$ the bounded holomorphic function on $\mathcal{B}^c$ with real part on $\partial\mathcal{S}_\tau$ equal to Q. Using (1.2.3), they verify Conjecture 1.2.1 for the given collection of potentials.

At the physical level, it is understood that the asymptotic formula (1.2.3) should hold for more general potentials, namely all those of the form $Q(z)=\frac{1}{2}|z|^2+H(z)$, where H is harmonic in a neighbourhood of the droplet (so called Hele-Shaw potentials) [45]. What the higher order correction terms should look like appears not to be understood even in this situation.

In another vein, for rather general potentials Q, the mean field approximation of the random normal matrix model [2, 3] supplies information regarding the individual orthogonal polynomials. Indeed, the convergence

$$|P_{n,m}|^2\mathrm{e}^{-2mQ}\to\varpi(\cdot,\widehat{\mathbb{C}}\setminus\mathcal{S}_1,\infty),$$

holds as $n,m\to\infty$ with $n=m+\mathrm{O}(1)$, in the sense of weak convergence of measures. Here, the left-hand side is interpreted as a probability measure, and the right-hand side denotes harmonic measure of the domain $\widehat{\mathbb{C}}\setminus\mathcal{S}_1$ evaluated at the point at infinity. We observe that harmonic measure is concentrated to the boundary, so that the above convergence can be interpreted as *boundary concentration*. Within the random normal matrix model, the addition of a new

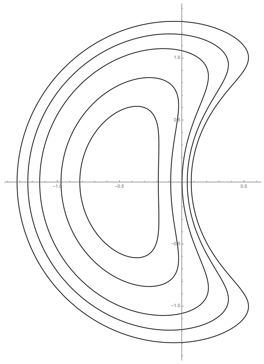

FIGURE 1.2. Laplacian growth of the compacts $\mathcal{S}_\tau$ for the potential $Q(z) = \frac{1}{2}|z|^2 - 2^{-\frac{1}{2}}\log|z-1|$ (boundary curves indicated).

particle has the net effect of adding a term $|P_{n,m}|^2\mathrm{e}^{-2mQ}$ of highest degree. This means that the net effect of adding a particle is felt along the boundary.

1.3. **Our contribution to the asymptotics of planar orthogonal polynomials.** Here, we shall study the (analytic) orthogonal polynomials in the complex plane $\mathbb{C}$ with respect to a rather general weight e^{-2mQ}. Specifically, we will treat potentials Q, that are admissible in the sense of the definition below. For $\tau \in (0,1]$, let

$$\mathcal{S}_\tau^\star = \mathcal{S}_{Q,\tau}^\star := \{z \in \mathbb{C} : \hat{Q}_\tau(z) = Q(z)\}$$

denote the coincidence set for the solution $\hat{Q}_\tau$ to the obstacle problem

$$\hat{Q}_\tau(z) = \sup\{q(z) : q \in \mathrm{Subh}_\tau(\mathbb{C}),\ q \le Q \text{ on } \mathbb{C}\},$$

where $\mathrm{Subh}_\tau(\mathbb{C})$ is the set of subharmonic functions in the plane which grow at most like $\tau\log|z|$ at infinity. The *droplet* $\mathcal{S}_\tau$ is the support set for the measure $1_{\mathcal{S}_\tau^\star}\Delta Q\mathrm{dA}$ as a distribution, which is a subset of $\mathcal{S}_\tau^\star$, and may differ from $\mathcal{S}_\tau^\star$ by a null set for the measure $|\Delta Q|\mathrm{dA}$. The droplet $\mathcal{S}_\tau$ grows with τ according to a weighted Darcy law, and the evolution is referred to as *weighted Laplacian growth*. See Figure 1.2 for a special case.

Definition 1.3.1. The potential $Q : \mathbb{C} \to \mathbb{R}$ is said to be τ-*admissible* if the following conditions are satisfied:

(i) Q is C^2-smooth in the entire complex plane,
(ii) Q is real-analytic and strictly subharmonic in a neighbourhood of $\mathcal{S}_\tau^\star$,
(iii) Q is grows sufficiently fast at infinity:

$$\liminf_{|z|\to+\infty} \frac{Q(z)}{\log|z|} > \tau. \tag{1.3.1}$$

(iv) The boundary $\partial\mathcal{S}_\tau^\star$ is a smooth, simple and closed curve.

Note that under these conditions, $\mathcal{S}_\tau^\star = \mathcal{S}_\tau$. In the sequel, we will focus on $\tau = 1$, and *assume that Q is 1-admissible*. We recall that the norm in the Hilbert space $L^2(\mathbb{C}, \mathrm{e}^{-2mQ})$ is given by

$$\|f\|_{2mQ} := \left(\int_{\mathbb{C}} |f|^2\mathrm{e}^{-2mQ}\mathrm{dA}\right)^{\frac{1}{2}}. \tag{1.3.2}$$

By applying the Gram-Schmidt procedure to the monomial sequence $1, z, z^2, \ldots$ in this space, we obtain the sequence of normalized orthogonal polynomials

$$P_{0,m}, P_{1,m}, P_{2,m}, \ldots,$$

where $P_{n,m}$ has degree n. Depending on the growth of potential Q, this may or may not be a finite sequence. The condition (1.3.1) with $\tau = 1$ guarantees that all polynomials of degree strictly less than $(1+\epsilon)m-1$ belong to the space $L^2(\mathbb{C}, \mathrm{e}^{-2mQ})$, for some fixed small $\epsilon > 0$. As Q is assumed 1-admissible, the curve $\partial\mathcal{S}_1$ is smooth, simple and closed. This assumption actually has as a consequence that the same holds for the boundaries $\partial\mathcal{S}_\tau$ for $\tau \in I_{\epsilon_0} := [1-\epsilon_0, 1+\epsilon_0]$ for some $\epsilon_0 > 0$ (this fact is non-trivial, and has to do with local properties of the weighted Laplacian growth flow (a variant of Hele-Shaw flow), and the fact that the coincidence is set as regular as assumed, c.f. [25, 22]). By considering a smaller ϵ_0, we can make sure this property holds on the larger interval $I_{2\epsilon_0}$ as well. Moreover, the assumptions of admissibility entail that the smooth curves $\partial\mathcal{S}_\tau$ are actually real-analytically smooth for $\tau \in I_{\epsilon_0}$. This follows from the work of Sakai [37] on boundaries with a one-sided Schwarz function, as observed in [25]. Under these assumptions, we find that the orthogonal polynomials have a full asymptotic expansion which is valid in a neighbourhood of $\mathcal{S}_\tau^c$, with uniformly bounded error terms for $\tau \in I_{\epsilon_0}$. To set things up, we denote by ϕ_τ the conformal mapping

$$\phi_\tau : \widehat{\mathbb{C}} \setminus \mathcal{S}_\tau^c \to \mathbb{D}_\mathrm{e}.$$

As a consequence of this, ϕ_τ extends to a conformal mapping $\mathcal{K}_{0,\tau} \to \mathbb{D}_\mathrm{e}(0, \rho_0)$, for some compact subset $\mathcal{K}_{0,\tau}$ of $\mathcal{S}_\tau$, and some ρ_0 with $0 < \rho_0 < 1$. We let $\mathcal{Q}_\tau$ be the bounded holomorphic function on $\mathcal{S}_\tau^c$ whose real part equals Q on $\partial\mathcal{S}_\tau$ and whose imaginary part vanishes at infinity. We need the following notion.

Definition 1.3.2. If $\mathcal{K}$ and $\mathcal{S}$ are compact sets in the plane with $\mathcal{K} \subset \mathcal{S}$ and

$$\mathrm{dist}_\mathbb{C}(\mathcal{K}, \mathcal{S}^c) = \varepsilon,$$

we say that a compact set $\mathcal{X}$ is *intermediate* between $\mathcal{K}$ and $\mathcal{S}$ if $\mathcal{K} \subset \mathcal{X} \subset \mathcal{S}$ with

$$\mathrm{dist}_\mathbb{C}(\mathcal{K}, \mathcal{X}^c) \geq \frac{\varepsilon}{10} \quad \text{and} \quad \mathrm{dist}_\mathbb{C}(\mathcal{X}, \mathcal{S}^c) \geq \frac{\varepsilon}{10}.$$

We recall the notation $I_{\epsilon_0} = [1-\epsilon_0, 1+\epsilon_0]$, where ϵ_0 is fixed and positive, with the property that the curves $\partial\mathcal{S}_\tau$ form a smooth flow of simple loops for $\tau \in I_{\epsilon_0}$.

Theorem 1.3.3. *Assume that Q is 1-admissible, and fix the precision κ which is a positive integer. Then, for each $\tau \in I_{\epsilon_0}$ there exists a compact subset $\mathcal{K}_\tau \subset \mathcal{S}_\tau$ with $\mathrm{dist}_\mathbb{C}(\mathcal{K}_\tau, \partial\mathcal{S}_\tau) \geq \varepsilon$ for some positive real number ε, such that the following holds. On the complement $\mathcal{K}_\tau^c$, there are bounded holomorphic functions $\mathcal{B}_{j,\tau}$ such that the associated function*

$$F_{n,m}^{\langle\kappa\rangle} = m^{\frac{1}{4}} \sqrt{\phi_\tau'}\, [\phi_\tau]^n \mathrm{e}^{m\mathcal{Q}_\tau} \sum_{j=0}^{\kappa} m^{-j} \mathcal{B}_{j,\tau},$$

approximates well the normalized orthogonal polynomials $P_{n,m}$ in the sense that we have the norm control

$$\big\| P_{n,m} - \chi_{0,\tau} F_{n,m}^{\langle\kappa\rangle} \big\|_{2mQ} = \mathrm{O}(m^{-\kappa-1})$$

as $n, m \to \infty$ while $\tau = \frac{n}{m} \in I_{\epsilon_0}$. Here, $\chi_{0,\tau}$ denotes a smooth cut-off function with $0 \leq \chi_{0,\tau} \leq 1$ and uniformly bounded gradient. In addition the function $\chi_{0,\tau}$ vanishes on $\mathcal{K}_\tau$, and equals 1 on the set $\mathcal{X}_\tau^c$ where $\mathcal{X}_\tau$ is an intermediate set between $\mathcal{K}_\tau$ and $\mathcal{S}_\tau$. In the above estimate, the implicit constant is uniform for $\tau \in I_{\epsilon_0}$.

In the above theorem, the products $\chi_{0,\tau} F_{n,m}^{\langle\kappa\rangle}$ are thought to vanish on the set $\mathcal{K}_\tau$, where $F_{n,m}$ may be undefined.

Remark 1.3.4. (a) We mention that the compact sets $\mathcal{K}_\tau$ as well as $\mathcal{X}_\tau$ may be obtained, e.g., as the complements of the conformal images under ϕ_τ^{-1} of the exterior disks $\mathbb{D}_\mathrm{e}(0, \rho)$ with $\rho = \rho_0$ and $\rho = \rho_0'$, respectively, where $0 < \rho_0 < \rho_0' < 1$. By inserting a further family of intermediate sets, we can make sure that the cut-off function $\chi_{0,\tau}$ vanishes not only on $\mathcal{K}_\tau$, but on an intermediate set between $\mathcal{K}_\tau$ and $\mathcal{X}_\tau$.

(b) Without loss of generality, we may assume that the cut-off function $\chi_{0,\tau}$ is uniformly smooth in the sense that for any given positive integer k the $C^k(\mathbb{C})$-norm of $\chi_{0,\tau}$ is uniformly bounded for $\tau \in I_{\epsilon_0}$.
(c) Our approach to the proof involves Toeplitz kernel problems and the construction of an approximate orthogonal foliation flow of loops. The underlying idea is inspired by an approach to the local expansion of Bergman kernels, which involves a flow of loops emanating from the point of expansion [19].

Turning to pointwise control, it can be shown that Theorem 1.3.3 has the following consequence.

Theorem 1.3.5. *Assume that Q is 1-admissible, and let κ be an arbitrary positive integer. Then there exist bounded holomorphic functions $\mathcal{B}_{j,\tau}$ defined in a fixed neighbourhood of $\mathcal{S}_\tau^c$, and compact subsets $\mathcal{K}_{m,\tau} \subset \mathcal{S}_\tau$ with $\mathrm{dist}_{\mathbb{C}}(\partial\mathcal{K}_{m,\tau}, \partial\mathcal{S}_\tau) \geq (m^{-1}\log\log m)^{\frac{1}{2}}$, such that the asymptotic formula*

$$P_{n,m}(z) = m^{\frac{1}{4}}[\phi_\tau'(z)]^{\frac{1}{2}}[\phi_\tau(z)]^n \mathrm{e}^{m\mathcal{Q}_\tau(z)}\bigg(\sum_{j=0}^{\kappa} m^{-j}\mathcal{B}_{j,\tau}(z) + \mathrm{O}\left(m^{-\kappa-1}\right)\bigg),$$

holds, where the error term is uniform for $z \in \mathcal{K}_{m,\tau}^c$, as $m, n \to \infty$ and the ratio $\tau = \frac{n}{m} \in I_{\epsilon_0}$.

In other words, the orthogonal polynomials $P_{n,m}$ have an asymptotic expansion

$$P_{n,m}(z) \sim m^{\frac{1}{4}}[\phi_\tau'(z)]^{\frac{1}{2}}[\phi_\tau(z)]^n \mathrm{e}^{m\mathcal{Q}_\tau(z)}\Big(\mathcal{B}_{0,\tau}(z) + \frac{1}{m}\mathcal{B}_{1,\tau}(z) + \dots\Big), \quad z \in \mathcal{K}_{m,\tau}^c$$

as $n = \tau m \to \infty$ and $\tau \in I_{\epsilon_0}$.

Remark 1.3.6. (a) It is possible to obtain the asymptotics of Theorem 1.3.5 in the complement of the smaller set $\mathcal{K}_{m,\tau}' \subset \mathcal{K}_{m,\tau} \subset \mathcal{S}_\tau$ with the property $\mathrm{dist}_{\mathbb{C}}(\partial\mathcal{K}_{m,\tau}', \partial\mathcal{S}_\tau) \geq A(m^{-1}\log m)^{\frac{1}{2}}$, for any given positive constant A. The difference in the proof amounts to working with m^{A^2D} instead of with $(\log m)^D$. As the positive constants A and D are fixed, this can be managed by the standard trick of expanding further, replacing the precision κ by κ', where $\kappa' \geq \kappa + A^2D$.

(b) Theorem 1.3.5 is a consequence of applying a certain maximum principle (Proposition 2.2.2) to the asymptotic expansion of Theorem 1.3.3. Although most of the weighted L^2-mass of $P_{n,m}$ is concentrated near the boundary $\partial\mathcal{S}_\tau$, we obtain good pointwise asymptotics in the whole exterior of $\mathcal{S}_\tau$, in particular at infinity. As a consequence, the κ-abschnitt $\mathfrak{f}_{n,m}^{\langle\kappa\rangle}$ given by $\mathfrak{f}_{n,m}^{\langle\kappa\rangle} = \sum_{j=0}^{\kappa} m^{-j}\mathcal{B}_{j,\tau}$ must satisfy

$$\mathrm{Im}\,\mathfrak{f}_{n,m}^{\langle\kappa\rangle}(\infty) = \mathrm{O}(m^{-\kappa-1}).$$

This normalization is important to determine the coefficient functions in the expansion. The notion of "κ-abschnitt" is from German and was mentioned in [21].

In order for Theorems 1.3.3 and 1.3.5 to be useful, we need to find a way to calculate the coefficient functions $\mathcal{B}_{j,\tau}$. This is explained in the following theorem. For the formulation, we need the Szegő projection $\mathbf{P}_{H^2_{-,0}}$ of $L^2(\mathbb{T})$ onto the conjugate Hardy space $H^2_{-,0} = L^2(\mathbb{T}) \ominus H^2$ (see Section 2.5 for details). In addition, we need the *modified weight* R_τ defined in a neighbourhood of $\bar{\mathbb{D}}_{\mathrm{e}}$ by

$$R_\tau = (Q - \breve{Q}_\tau) \circ \phi_\tau^{-1}, \tag{1.3.3}$$

where the function $\breve{Q}_\tau$ is the harmonic extension of the restriction to $\hat{Q}_\tau|_{\mathcal{S}_\tau^c}$ across the boundary $\partial\mathcal{S}_\tau$. That such a harmonic extension exists is a consequence of the real-analyticity of $\partial\mathcal{S}_\tau$ which in its turn follows the 1-admissibility of Q, at least for $\tau \in I_{\epsilon_0}$.

Theorem 1.3.7. *In the asymptotic expansion of Theorem 1.3.3, we have $\mathcal{B}_{0,\tau} = \pi^{-\frac{1}{4}}\mathrm{e}^{H_{Q,\tau}}$, where $H_{Q,\tau}$ is bounded and holomorphic in a neighbourhood of $\mathcal{S}_\tau^c$ and satisfies $\mathrm{Im}\,H_{Q,\tau}(\infty) = 0$ and*

$$\mathrm{Re}\,H_{Q,\tau} = \frac{1}{4}\log\Delta Q, \quad \text{on} \quad \partial\mathcal{S}_\tau.$$

Moreover, if H_{R_τ} denotes the bounded holomorphic function on $\mathbb{D}_{\mathrm{e}}$ with

$$\operatorname{Re} H_{R_\tau} = \frac{1}{4}\log(4\Delta R_\tau) \quad \text{on } \mathbb{T},$$

and $\operatorname{Im} H_{R_\tau}(\infty) = 0$, then for $j = 1, 2, 3, \ldots$, the coefficients $\mathcal{B}_{j,\tau}$ have the form

$$\mathcal{B}_{j,\tau} = [\phi_\tau']^{\frac{1}{2}}\, B_{j,\tau} \circ \phi_\tau,$$

where the functions $B_{j,\tau}$ are bounded and holomorphic in a neighbourhood of $\bar{\mathbb{D}}_{\mathrm{e}}$, and their restrictions to $\mathbb{D}_{\mathrm{e}}$ are given by

$$B_{j,\tau} = c_{j,\tau}\, \mathrm{e}^{H_{R_\tau}} - e^{H_{R_\tau}} \mathbf{P}_{H^2_{0,-}}[\mathrm{e}^{\bar{H}_{R_\tau}} F_{j,\tau}]$$

for some real-analytic functions $F_{j,\tau}$ on the circle $\mathbb{T}$ and constants $c_{j,\tau} \in \mathbb{R}$. The functions $F_{j,\tau}$ as well as the constants $c_{j,\tau}$ may be computed algorithmically in terms of the potential R_τ and the functions $B_{0,\tau}, \ldots, B_{j-1,\tau}$, where $B_{0,\tau} = (4\pi)^{-\frac{1}{4}} \mathrm{e}^{H_{R_\tau}}$.

We point out that Theorem 1.3.3 and Theorem 1.3.7 together with Remark 1.3.6(a) imply that for large enough m, and for $\tau = \frac{n}{m} \in I_{\epsilon_0}$, all the zeros of the polynomial $P_{n,m}(z)$ lie in the compact subset $\mathcal{K}'_{m,\tau}$ of the interior $\mathcal{S}_\tau^\circ$ of $\mathcal{S}_\tau$ defined in Remark 1.3.6(a).

In view of the way we defined the functions $H_{Q,\tau}$ and $H_{\mathbb{R}_\tau}$, it is immediate that

$$H_{R_\tau} \circ \phi_\tau = \frac{1}{2}\log(2\phi_\tau') + H_{Q,\tau}.$$

While Theorem 1.3.7 gives the asymptotic structure of the orthogonal polynomials, it remains to specify how to obtain the real-analytic functions $F_{j,\tau}$ and the constants $c_{j,\tau}$, for $j = 0, 1, 2, \ldots$. For $k = 0, 1, 2, \ldots$, let $\mathbf{L}_k$ be given by

$$\mathbf{L}_k[f] = \sum_{\nu=k}^{3k} \frac{(-1)^{\nu-k} 2^{-\nu}}{\nu!(\nu-k)![\partial_r^2 R_\tau(r\mathrm{e}^{\mathrm{i}\theta})]^\nu} \partial_r^{2\nu}\Big(\Big[R_\tau - \frac{1}{2}(r-1)^2 \partial_r^2 R_\tau(\mathrm{e}^{\mathrm{i}\theta})\Big]^{\nu-k} f\Big). \tag{1.3.4}$$

This is a differential operator of order $6k$, acting on a smooth function f defined in a neighbourhood of the unit circle. Note that the restriction $\mathbf{L}_k[f](r\mathrm{e}^{\mathrm{i}\theta})\big|_{r=1}$ only involves derivatives of order at most $2k$. This operator results from the asymptotical analysis of integrals using Laplace's method, see Proposition 2.7.1 below. Later on, in Lemma 3.2.1, we show the existence of certain differential operators $\mathbf{M}_k$ with the property that

$$\int_{\mathbb{T}} e^{il\theta}\big(\partial_r^2 R_\tau(r\mathrm{e}^{\mathrm{i}\theta})\big)^{-\frac{1}{2}} \mathbf{L}_k[r^{1-l} f(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1} \mathrm{d}\theta = \int_{\mathbb{T}} \mathrm{e}^{il\theta} \mathbf{M}_k[f(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1} \mathrm{d}\theta,$$

for $l = 1, 2, 3, \ldots$. Operators similar to $\mathbf{M}_k$ arise in the theory of pseudodifferential operators. In this context, we use them to rid the left-hand side of any unwanted dependence on the radial contribution r^{1-l}. In terms of the operators $\mathbf{L}_k$ and $\mathbf{M}_k$, we may now express $F_{j,\tau}$ and $c_{j,\tau}$ as follows:

$$F_{j,\tau}(\theta) = \sum_{k=1}^{j} \mathbf{M}_k[B_{j-k,\tau}](\mathrm{e}^{\mathrm{i}\theta}), \qquad j \geq 1, \tag{1.3.5}$$

and the real constants $c_{j,\tau}$ are given by $c_{0,\tau} = (4\pi)^{-1/4}$ while

$$c_{j,\tau} = -\frac{1}{2}(4\pi)^{\frac{1}{4}} \sum_{(i,k,l)\in\mathcal{J}_j} \int_{\mathbb{T}} \big(4\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta})\big)^{-\frac{1}{2}} \mathbf{L}_k[r B_{i,\tau}(r\mathrm{e}^{\mathrm{i}\theta}) \bar{B}_{l,\tau}(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1} \mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}) \tag{1.3.6}$$

for $j = 1, 2, 3, \ldots$, where $\mathcal{J}_j = \{(i,k,l) \in \mathbb{Z}^3 \,:\, i, l < j,\ k \geq 0,\ i + k + l = j\}$.

1.4. **Our contribution to boundary universality for random normal matrices.** As a direct consequence of Theorem 1.3.3 and Theorem 1.3.7, we manage to resolve the boundary universality conjecture (Conjecture 1.2.1) for admissible potentials.

Theorem 1.4.1. *If Q is* 1*-admissible and $z_0 \in \partial\mathcal{S}_1$ is a boundary point, then if ρ_m is the blow-up density given by* (1.2.1) *and* (1.2.2)*, we have the convergence*

$$\lim_{m\to\infty} \rho_m(\zeta) = \operatorname{erf}(2\zeta),$$

locally uniformly on $\mathbb{C}$.

By polarization, it follows that a corresponding result holds for the rescaled correlation kernels

$$\mathrm{k}_m(\xi,\eta) = \frac{1}{2m\Delta Q(z_0)} K_m(z_m(\xi), z_m(\eta)) \mathrm{e}^{-mQ(z_m(\xi))-mQ(z_m(\eta))}.$$

Determinantal point processes are determined by a correlation kernel, which is unique up to cocycles. Here, cocycles are understood as the multiplication by a function $c(\xi)\bar{c}(\eta)$, for some continuous unimodular function $c:\mathbb{C}\to\mathbb{T}$. Modulo such cocycles, it is known that the correlation kernels converge uniformly on compact subsets of $\mathbb{C}^2$, to a function of the form

$$\mathrm{k}(\xi,\eta) = \mathrm{e}^{\xi\bar{\eta}-\frac{1}{2}(|\xi|^2+|\eta|^2)} F(\xi,\eta),$$

where $F(\xi,\eta)$ is a positive definite kernel, which depends holomorphically on ξ and conjugate-holomorphically on η. Since it follows from our theorem that $F(\xi,\xi) = \operatorname{erf}(2\operatorname{Re}\xi)$, it follows that its polarization is $F(\xi,\eta) = \operatorname{erf}(\xi+\bar{\eta})$. As a consequence, we obtain the convergence

$$\lim_{m\to\infty} c_m(\xi)\bar{c}_m(\eta)\,\mathrm{k}_m(\xi,\eta) = \mathrm{e}^{\xi\bar{\eta}-\frac{1}{2}(|\xi|^2+|\eta|^2)} \operatorname{erf}(\xi+\bar{\eta}),$$

where $c_m:\mathbb{C}\to\mathbb{T}$ is a sequence of continuous unimodular functions. We formalize this in the following corollary.

Corollary 1.4.2. *Denote by $\Phi_m = \{z_{j,m}\}_{j=1}^m$ the eigenvalues of a random normal $m\times m$ matrix associated to the weight $\exp(-2mQ)$, and define a rescaled process $\Psi_{m,z_0} = \{\zeta_{j,m}\}_{j=1}^m$ by*

$$\zeta_{j,m} = \bar{\mathrm{n}}\sqrt{2m\Delta Q(z_0)}(z_{j,m}-z_0), \qquad j = 1,\ldots m,$$

where $z_0 \in \partial\mathcal{S}_1$ and n is the unit outer normal to $\mathcal{S}_1$ at z_0, interpreted as a complex number. Then Ψ_{m,z_0} converges to a determinantal point field with correlation kernel

$$\mathrm{k}(\xi,\eta) = \mathrm{e}^{\xi\bar{\eta}-\frac{1}{2}(|\xi|^2+|\eta|^2)} \operatorname{erf}(\xi+\bar{\eta}),$$

in the sense of locally uniform convergence of correlation kernels on $\mathbb{C}^2$.

The necessary details are supplied in Sections 5.3, 2.6 (see also the paper [4]).

To complement the present exposition on planar orthogonal polynomials, we explain in [27] how the method of canonical positioning and orthogonal foliation flow also applies to give a full asymptotic expansion of the Bergman kernel for exponentially varying weights when one of the variables is away from the spectrum. From the physical point of view, this may be interpreted as an instance of bosonization.

In a separate work [28], we intend to explore further the implications of Theorem 1.3.3 and 1.3.7 for the theory of random normal matrices. In particular, we will study what we call *contractible potentials* Q, which are strictly subharmonic, $\mathbb{C}^\omega$-smooth and have the additional property that $\{\partial\mathcal{S}_\tau\}_{0<\tau\le 1}$ is a flow of simple, closed C^ω-smooth curves which shrink down to a point. Such potentials will necessarily have a unique minimum point. For these contractible potentials we analyze the asymptotics of the free energy $\log\mathcal{Z}_{m,Q}$, where $\mathcal{Z}_{m,Q}$ denotes the partition function of the ensemble, and relate this analysis to the planar analogue of the classical Szegő limit theorem.

It has been suggested that, in analogy with the one-dimensional case, the key to the asymptotics of orthogonal polynomials ought to be Riemann-Hilbert problem techniques. For instance, in the work of Its and Takhtajan [30] a natural *Thick Riemann-Hilbert problem*, or matrix $\bar{\partial}$-problem, is set up. Moreover, it is shown that the 2×2 matrix built with the orthogonal polynomials with respect to the measure $\mathrm{e}^{-2mQ}\mathrm{dA}$ is the unique solution to this problem. However, it is not clear how to implement this approach as it is not obvious how to constructively solve these thick Riemann-Hilbert problems. Our orthogonal foliation flow approach allows reduction to one-dimensional problems, which could be understood as ordinary Riemann-Hilbert problems along the curve family in the flow. In this presentation, however, we find a more direct approach to derive the asymptotics. We supply a brief discussion of this matter in Section 6.

1.5. **Comments on the exposition.** In Section 2, we supply some preliminary material which will be needed later on. In particular, we discuss some aspects of weighted logarithmic potential theory and obstacle problems, and introduce weighted Laplacian growth. In addition, we collect some results on Hörmander type L^2-estimates for the $\bar{\partial}$-operator, and asymptotic analysis of integrals based on Laplace's method.

In Section 3, we show how the algorithm of Theorem 1.3.7 appears, assuming that Theorem 1.3.3 is valid. The proof is based on Laplace's method for asymptotic integrals, which lets us collapse planar integrals to integrals over curves. The collapsed orthogonality conditions reduces to inhomogeneous Toeplitz kernel conditions, which gives rise to the algorithm.

In Section 4, we prove the existence of the approximate orthogonal foliation flow, which then allows us to obtain Theorem 1.3.3 using Hörmander type $\bar{\partial}$-methods. A visualization of the orthogonal foliation flow is supplied in Figure 2.1.

In Section 5, we supply proofs of Theorem 1.4.1 and Corollary 1.4.2 on boundary universality in the random normal matrix model for 1-admissible potentials.

Finally, in Section 6, we connect our orthogonal foliation flow with the Its and Takhtajan approach involving 2×2 matrix $\bar{\partial}$-problems.

2. Preliminaries

2.1. **An obstacle problem and logarithmic potential theory.** In this section, we follow the presentation of [22]. For a positive real parameter τ, let $\mathrm{Subh}_\tau(\mathbb{C})$ denote the convex set of all functions $q:\mathbb{C}\to\mathbb{R}\cup\{-\infty\}$ which are subharmonic in $\mathbb{C}$ and have the growth bound

$$q(z)\le \tau\log|z|+\mathrm{O}(1)$$

as $|z|\to\infty$. For lower semicontinuous potentials Q, subject to the growth condition (1.3.1), we let $\hat{Q}_\tau$ be the solution to the obstacle problem

$$\hat{Q}_\tau(z):=\sup\big\{q(z):\, q\in\mathrm{Subh}_\tau(\mathbb{C})\ \text{ and }\ q\le Q\ \text{ on }\ \mathbb{C}\big\}, \tag{2.1.1}$$

and observe that trivially $\hat{Q}_\tau\le Q$, and if we regularize $\hat{Q}_\tau$ on a set of logarithmic capacity 0 (and keep the same notation for the regularized function) then $\hat{Q}_\tau\in\mathrm{Subh}_\tau(\mathbb{C})$ holds. Suppose now that Q is C^2-smooth. Standard regularity results then give that $\hat{Q}_\tau$ is $C^{1,1}$-smooth, so that the partial derivatives of order 2 of $\hat{Q}_\tau$ are locally bounded (in the sense of distribution theory). As a consequence of the growth condition (1.3.1) on Q, the *coincidence set* defined by

$$\mathcal{S}_\tau^\star:=\{z\in\mathbb{C}:\,\hat{Q}_\tau(z)=Q(z)\}.$$

is compact, and moreover, it follows from the smoothness that $\varDelta\hat{Q}_\tau=1_{\mathcal{S}_\tau^\star}\varDelta Q$ holds in the sense of distribution theory. The above obstacle problem has an important connection to potential theory. The weighted logarithmic energy, with respect to a continuous weight function $V:\mathbb{C}\to\mathbb{R}$, of a compactly supported finite real-valued Borel measure μ is defined as

$$I_V[\mu]=\int_{\mathbb{C}\times\mathbb{C}}\log\frac{1}{|z-w|}\mathrm{d}\mu(z)\mathrm{d}\mu(w)+2\int_{\mathbb{C}}V(z)\mathrm{d}\mu(w).$$

With $V=\tau^{-1}Q$, we set out to minimize the energy $I_{\tau^{-1}Q}[\mu]$ over the all compactly supported Borel probability measures μ. There exists a unique minimizer, called the *equilibrium measure*, which we denote by μ_τ. The connection with the obstacle problem is via the following relationship:

$$\frac{\tau}{2}\,\mathrm{d}\mu_\tau(z) = \varDelta\hat{Q}_\tau\mathrm{dA} = 1_{\mathcal{S}_\tau^\star}\Delta Q(z)\mathrm{dA}. \tag{2.1.2}$$

Traditionally we call the support (as a distribution) of the equilibrium measure μ_τ the *droplet*, and denote it by $\mathcal{S}_\tau$. In general this is a subset of the coincidence (or contact) set $\mathcal{S}_\tau^\star$. However, difference set $\mathcal{S}_\tau^\star \setminus \mathcal{S}_\tau$ is small, in the sense that it is a null set with respect to the weighted area measure $|\varDelta Q|\mathrm{dA}$. In this presentation, we will assume throughout that the potential Q is 1-admissible. Under this assumption, we have the equality $\mathcal{S}_\tau = \mathcal{S}_\tau^\star$ for $\tau \in I_{\epsilon_0} := [1-\epsilon_0, 1+\epsilon_0]$ for some small but positive ϵ_0, and from this point onward we will use the notation $\mathcal{S}_\tau$ and think of it as the support of the equlibrium measure and the coincidence set at the same time.

2.2. **Bounds on polynomials.** The significance of the set $\mathcal{S}_\tau$ in relation to orthogonal polynomials is highlighted by Proposition 2.2.2 below. We begin with a useful lemma taken from [1], see Lemma 3.2.

Lemma 2.2.1. *Let u be holomorphic in a disk $\mathbb{D}(z, m^{-1/2}\delta)$ Then*

$$|u(z)|^2\mathrm{e}^{-2mQ(z)} \le \frac{m\mathrm{e}^{A\delta^2}}{\delta^2}\int_{\mathbb{D}(z,m^{-1/2})}|u|^2\mathrm{e}^{-2mQ}\mathrm{dA},$$

where A denotes the essential supremum of ΔQ on $\mathbb{D}(z, m^{-1/2}\delta)$.

This lemma is used in [1] to obtain growth bounds on the entire plane $\mathbb{C}$ for polynomials of degree at most n. In fact, that the function is polynomial is non-essential, as long as the polynomial growth control at infinity is retained.

Proposition 2.2.2. *Let $\tau = n/m$, and suppose $\mathcal{K}$ is a compact subset of the interior of $\mathcal{S}_\tau$. Then there exists a constant C such that for any $u \in A^2_{2mQ}(\mathcal{K}^c)$ with $|u(z)| = \mathrm{O}(|z|^n)$ as $|z| \to \infty$, we have that*

$$|u(z)| \le Cm^{\frac{1}{2}}\|u\|_{L^2(\mathbb{C}\setminus\mathcal{K},\mathrm{e}^{-2mQ})}\mathrm{e}^{m\hat{Q}_\tau(z)}, \qquad \mathrm{dist}_{\mathbb{C}}(z,\mathcal{K}) \ge \delta m^{-1/2}.$$

Sketch of proof. Assume that $z \in \mathcal{S}_\tau \setminus \mathcal{K}$ lies at a distance of at least $m^{-1/2}\delta$ from $\mathcal{K}_\tau$. By Proposition 2.2.1, we have the estimate

$$|u(z)|^2 \le \frac{\mathrm{e}^{A\delta^2}}{\delta^2}me^{2mQ(z)}\|u\|^2_{L^2(\mathbb{C}\setminus\mathcal{K},\mathrm{e}^{-2mQ})},$$

which yields the claim for $z \in \mathcal{S}_\tau \setminus \mathcal{K}$ with the constant $C = C_\delta = \delta^{-2}e^{A\delta^2}$. Now, suppose u has norm equal to 1, and let $q(z)$ be the subharmonic function

$$q(z) = \frac{1}{2m}\log\frac{|u(z)|^2}{mC_\delta}, \qquad z \in \mathcal{K}^c.$$

It follows from the above estimate on $|u(z)|^2$ that $q(z) \le Q$ for $z \in \mathcal{S}_\tau \setminus \mathcal{K}$, and the growth bound on $|u(z)|$ as $|z| \to \infty$ shows that $q(z) \le \tau\log|z| + \mathrm{O}(1)$ as $|z| \to \infty$. By applying the maximum principle for subharmonic functions to the function

$$q(z) - \tau\log|\phi_\tau(z)|, \qquad z \in \mathcal{S}_\tau^c,$$

it follows that $q(z) \le \hat{Q}_\tau(z)$ for $z \in \mathcal{S}_\tau^c$, which completes the proof. □

In particular, we observe that Proposition 2.2.2 shows that $|P_{n,m}(z)|^2\mathrm{e}^{-2mQ}$ decays exponentially outside the set $\mathcal{S}_\tau$ if $\tau = n/m$. As alluded to in the introduction, it is possible to further localize the mass of $|P_{n,m}(z)|^2\mathrm{e}^{-2mQ(z)}$. The following is from [2]. Denote by $\varpi(\cdot, \widehat{\mathbb{C}} \setminus \mathcal{S}_t, \infty)$ the harmonic measure of $\widehat{\mathbb{C}} \setminus \mathcal{S}_t$ relative to the point at infinity.

Theorem 2.2.3. *As $m, n \to \infty$ with $\tau = \frac{n}{m} = \tau_0 + \mathrm{O}(m^{-1})$ for some τ_0 with $0 < \tau_0 \le 1$, we have the convergence*

$$|P_{n,m}|^2 \mathrm{e}^{-2mQ} \to \varpi(\cdot, \widehat{\mathbb{C}} \setminus \mathcal{S}_{\tau_0}, \infty),$$

in the sense of weak convergence of measures.

We will not need this theorem per se, but merely draw intuition from it. It is clear that in the sense of $L^2(\mathrm{e}^{-2mQ})$, the orthogonal polynomial is concentrated near $\partial\mathcal{S}_\tau$. In light of this, performing a cut-off by removing a compact subset set $\mathcal{K}_\tau$ from the interior of $\mathcal{S}_\tau$ should matter little. See Figure 3.1 for an illustration of the norm concentration.

2.3. **Weighted Laplacian growth.** Weighted Laplacian growth (or weighted Hele-Shaw flow) describes the movement of the boundary of a viscous fluid droplet in a porous medium, as fluid is injected into the droplet, where the weight appears as a result of the variable permeability of the medium, or, alternatively, as a result of curved geometry. For the mathematical formulation, consider a simply connected domain Ω_0 on the Riemann sphere $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and an injection point $\alpha \in \Omega_0$. A smoothly increasing family $\{\Omega_t\}_t$ of domains is said to be a Hele-Shaw flow with weight ω, relative to the injection point α if the infinitesimal change of the measure $1_{\Omega_\tau}\omega(z)\mathrm{dA}$ equals harmonic measure (the derivative is as usual taken in the sense of distribution theory):

$$\partial_t(1_{\Omega_t}\omega\mathrm{dA}) = \mathrm{d}\varpi(\cdot, \Omega_t, \alpha). \tag{2.3.1}$$

A basic reference on Hele-Shaw flow is the book [20] by Gustafsson, Teodorescu and Vasili'ev. The weighted Hele-Shaw flow problem appears to have been treated first in the paper [25] by Hedenmalm and Shimorin, where the weight was interpreted as a Riemannian metric, motivated by considerations in the potential theory of clamped plates [26]. This line of work is continued by [24], [23]. In this connection, we mention the recent work [35] by Ross and Witt-Nyström, which deals with a slightly less regular situation than the aforementioned work.

We recall that we assume that the potential Q is 1-admissible (see Definition 1.3.1). The reason why we are interested in weighted Laplacian growth is twofold. The first reason is that the compact sets $\mathcal{S}_\tau$ evolve according to weighted Laplacian growth, which is a Hele-Shaw flow for the complements $\mathcal{S}_\tau^c$ in the backward time direction as τ increases. More precisely, in [22] it is shown that in the present context the compacts $\mathcal{S}_\tau$, $0 < \tau \le 1$, undergo Laplacian growth with the weight $2\Delta Q$. This also carries over to the slightly larger parameter range $0 < \tau \le 1+\epsilon$. The second reason connects with the weak formulation of weighted Laplacian growth. In terms of weighted Hele-Shaw flow, the flow of domains $\{\Omega_t\}_t$ constitutes a weak weighted Hele-Shaw flow solution with injection point α and weight ω if

$$\int_{\Omega_t \setminus \Omega_s} h\,\omega\mathrm{dA} = (t-s)h(\alpha), \quad s < t,$$

for all bounded harmonic functions h on Ω_t, for some smooth increasing real-valued function c. We shall be interested in the choices $\alpha = \infty$ and $\Omega_0 = \mathcal{S}_\tau^c$ for fixed τ, while the *orthogonal foliation flow* defined by the loops $\partial\Omega_t$ will cover a region close to the curve $\partial\mathcal{S}_\tau$ for the relevant range of parameter values t. Let p be a polynomial of degree n, whose zeros are confined to a fixed compact subset of the interior of $\mathcal{S}_\tau$, where we keep $\tau = \frac{n}{m}$. For the moment, we consider the weighted Laplacian growth with respect to the weight $\omega = |p|^2\mathrm{e}^{-2mQ}$ and recall that the injection point is $\alpha = \infty$. Since the polynomial p has zeros confined to a fixed compact subset of $\mathcal{S}_\tau$, p is automatically orthogonal to all lower order polynomials on each curve $\partial\Omega_t$ with respect to the measure $|p|^{-2}\mathrm{d}\varpi(\cdot, \Omega_t, \alpha)$. Hence, by the weighted Laplacian growth equation (2.3.1), the integral of $|p|^{-2}\mathrm{d}\varpi(\cdot, \Omega_t, \alpha)$ with respect to t, taken over an appropriate range of parameter $t_1 < t < t_2$, equals the measure

$$1_{\Omega_{t_2}\setminus\Omega_{t_1}} \frac{\omega}{|p|^2}\mathrm{dA} = 1_{\Omega_{t_2}\setminus\Omega_{t_1}} \mathrm{e}^{-2mQ}\mathrm{dA}, \tag{2.3.2}$$

and consequently, p is orthogonal to all lower order polynomials with respect to the measure (2.3.2). We remark that for a rather generic such polynomial p the weighted Laplacian growth will typically run into trouble with the appearance of cusps or double points and a successful

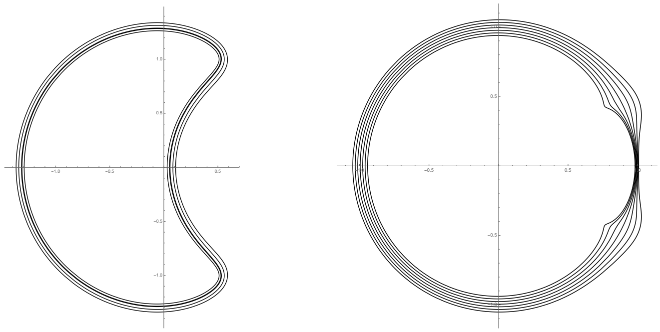

FIGURE 2.1. Approximate orthogonal foliation flow, near $\partial\mathcal{S}_1$ (left) and near $\mathbb{T}$ (right), associated with the potential $Q(z) = \frac{1}{2}|z|^2 - 2^{-\frac{1}{2}}\log|z-1|$.

flow over a substantial range of parameters would require special properties of p. The above-mentioned orthogonal foliation flow is crucial for our proof of the main theorem (Theorem 1.3.3). An illustration of this flow is supplied in Figure 2.1. In practice, we will work with the corresponding flow around the unit circle $\mathbb{T}$, illustrated in the same figure.

We will need the following lemma, about the movement of the loops $\partial\mathcal{S}_\tau$ as τ varies.

Lemma 2.3.1. *Fix $\tau \in I_{\epsilon_0} = [1-\epsilon_0, 1+\epsilon_0]$. Denote by $\mathrm{n}_\tau(\zeta)$ the outer unit normal to $\partial\mathcal{S}_\tau$ at a point $\zeta \in \partial\mathcal{S}_\tau$, and let $\mathrm{n}_\tau(\zeta)\mathbb{R}$ denote the straight line which contains $\mathrm{n}_\tau(\zeta)$ and the origin. Then, if for real ε the point ζ_ε is closest to ζ in the intersection*

$$(\zeta - \mathrm{n}_\tau(\zeta)\mathbb{R}) \cap \partial\mathcal{S}_{\tau-\varepsilon},$$

we have as $\varepsilon \to 0$ that

$$\zeta_\varepsilon = \zeta - \varepsilon\mathrm{n}_\tau(\zeta)\frac{|\phi'_\tau(\zeta)|}{4\Delta Q(\zeta)} + \mathrm{O}(\varepsilon^2)$$

and the outer normal $\mathrm{n}_{\tau-\varepsilon}(\zeta_\varepsilon)$ satisfies

$$\mathrm{n}_{\tau-\varepsilon}(\zeta_\varepsilon) = \mathrm{n}_\tau(\zeta) + \mathrm{O}(\varepsilon).$$

Proof. We recall that the compact sets $\mathcal{S}_\tau$ evolve according to weighted Laplacian growth with respect to the weight $2\varDelta Q$, see (2.1.2). If we compare with (2.3.1), we see that this means that

$$\partial_\tau(1_{\mathcal{S}_\tau}2\varDelta Q\mathrm{dA}) = \mathrm{d}\varpi(\cdot, \mathcal{S}_\tau^c, \infty) = |\phi'_\tau|\mathrm{ds},$$

where we recall that ϕ_τ is the (surjective) conformal mapping $\mathcal{S}_\tau^c \to \mathbb{D}_\mathrm{e}$. This means that the boundary $\partial\mathcal{S}_\tau$ moves at local speed $(4\varDelta Q)^{-1}|\phi'_\tau|$ in the exterior normal direction, where the number 4 appears instead of 2 as a result of the different normalizations associated with ds and dA. Since the loops $\partial\mathcal{S}_\tau$ deform smoothly, the claimed assertions follow from e.g. Taylor's formula. □

2.4. **Polynomial $\bar\partial$-methods.** Let ϕ be a strictly subharmonic function on $\mathbb{C}$. Hörmander's classical result states that the inhomogeneous $\bar\partial$-equation

$$\bar\partial u = f$$

can be solved for any datum $f \in L^2_{\mathrm{loc}}(\mathbb{C})$ with the estimate

$$\int_\mathbb{C}|u|^2\mathrm{e}^{-\phi}\mathrm{dA} \le \int_\mathbb{C}|f|^2\frac{\mathrm{e}^{-\phi}}{\Delta\phi}\mathrm{dA}.$$

Taking this a starting point, in [1], Ameur, Hedenmalm, and Makarov investigate the case when the solution u is constrained by an additional polynomial growth condition at infinity. We now describe this result. To this end, let $L^2_{\phi,n}(\mathbb{C})$ denote the subspace of $L^2_\phi(\mathbb{C})$ subject to the growth restraint

$$f(z) = \mathrm{O}(|z|^{n-1})$$

near infinity. The polynomial growth Bergman space $A^2_{\phi,n}$ is analogously defined.

The following is Theorem 4.1 in [1].

Theorem 2.4.1. *Let $\mathcal{T}$ be a compact subset of $\mathbb{C}$, and assume that ϕ, $\hat{\phi}$ and ϱ are real-valued $C^{1,1}$-functions such that*

(i) $L^2_{\hat{\phi}}$ contains the function $z \mapsto (1+|z|)^{-1}$, and there exist numbers a and b such that

$$\hat{\phi} \le \phi + a \text{ on } \mathbb{C} \quad \text{and} \quad \phi \le \hat{\phi} + b \text{ on } \mathcal{T}$$

(ii) $\Delta(\hat{\phi} + \varrho) > 0$ on $\mathbb{C}$,

(iii) $A^2_{\hat{\phi}} \subset A^2_{\phi,n}$,

(iv) $\nabla \varrho = 0$ on $\mathbb{C} \setminus \mathcal{T}$,

(v) there exists a number $\kappa \in (0,1)$ such that

$$\frac{|\bar{\partial}\varrho|^2}{4\Delta(\hat{\phi}+\varrho)} \le \frac{\kappa^2}{\mathrm{e}^{a+b}}, \quad \text{a.e. on } \mathcal{T}.$$

Then for any $f \in L^\infty(\mathcal{T})$ the $L^2_{\phi,n}$ minimal solution to $\bar{\partial} u_{0,n} = f$ satisfies

$$\int_{\mathbb{C}} |u_{0,n}|^2 \mathrm{e}^{\varrho-\phi} \mathrm{dA} \le \frac{\mathrm{e}^{a+b}}{(1-\kappa)^2} \int_{\mathcal{T}} |f|^2 \frac{\mathrm{e}^{\varrho-\phi}}{\Delta(\hat{\phi}+\varrho)} \mathrm{dA}.$$

We will require the following specialization of Theorem 2.4.1 to our needs.

Corollary 2.4.2. *Let $f \in L^\infty(\mathcal{S}_\tau)$. Then the $L^2_{2mQ,n}$-minimal solution $u_{0,n}$ to the problem*

$$\bar{\partial} u_{0,n} = f$$

satisfies

$$\int_{\mathbb{C}} |u_{0,n}|^2 \mathrm{e}^{-2mQ} \mathrm{dA} \le \frac{1}{2m} \int_{\mathcal{S}_\tau} |f|^2 \frac{\mathrm{e}^{-2mQ}}{\Delta Q} \mathrm{dA} \tag{2.4.1}$$

Proof. We apply Theorem 2.4.1 with $\mathcal{T} = \mathcal{S}_\tau$, $\phi = 2mQ$, $\varrho = 0$, and

$$\hat{\phi} = 2m\big(1 - \frac{\varepsilon}{\tau}\big)\hat{Q}_\tau + \varepsilon m \log(1+|z|^2).$$

Then all conditions except (iii) are trivially satisfied with $a, b = \mathrm{o}(1)$ as $\varepsilon \to 0^+$. To see why (iii) holds, it is enough to observe that

$$\hat{\phi}(z) = 2m\tau\big(1 - \frac{\varepsilon}{\tau}\big)\log|z| + 2\varepsilon m \log|z| + \mathrm{O}(1) = \log(|z|^{2n}) + \mathrm{O}(1)$$

as $|z| \to \infty$. Hence the inclusion $A^2_{\hat{\phi}} \subset \mathrm{Pol}_n$ follows. Letting $\varepsilon \to 0^+$ for fixed (m,n) completes the proof. □

Remark 2.4.3. We mention Theorem 2.4.1 because it helps us to build intuition on the behavior of the space of polynomials Pol_n equipped with the inner product of L^2_ϕ. In the setting of [1], the theorem is applied (up to inessential modifications) using $\phi = 2mQ$, $\hat{\phi} = 2m\hat{Q}$ and $\mathcal{T} = \mathcal{S}_\tau$. In Theorem 2.4.1 the function ϱ, which modifies the weight, illustrates the amount of flexibility we can achieve. If $\mathcal{S}_\tau$ does not divide the plane, then ϱ is necessarily constant in the exterior, and is allowed to deviate only in the interior direction of $\mathcal{S}_\tau$. This tells us that the exterior is rigid, while the interior is more flexible. Note that in the corollary we use the trivial modifying function $\varrho = 0$. This particular instance may be obtained more directly using Hörmander's classical $\bar{\partial}$-estimate.

2.5. **Holomorphic boundary value problems and Toeplitz operators.** For the reader's convenience, we include some elementary facts from the theory of Herglotz kernels and Hardy spaces. Let f be holomorphic in the unit disk $\mathbb{D}$ with continuous extension to the boundary. The classical Schwarz Integral Formula [18, pp. 45] states that that

$$f(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \operatorname{Re}(f(\zeta)) \, \mathrm{ds}(\zeta) + \operatorname{Im}(f(0)), \qquad z \in \mathbb{D}.$$

Thus, if $F \in L^1(\mathbb{T})$ is real-valued, this allows us to solve the boundary value problem

$$\operatorname{Re} f\big|_{\mathbb{T}} = F$$

where f is holomorphic in the disk by the integral formula

$$f(z) = \mathbf{H}_{\mathbb{D}}[F](z) := \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} F(\zeta) \, \mathrm{ds}(\zeta), \quad z \in \mathbb{D}.$$

Moreover, the solution is unique up to an additive imaginary constant. For us, it is more natural to work in the exterior disk. By reflection in the unit circle, we obtain the formula

$$f(z) = \mathbf{H}_{\mathbb{D}_{\mathrm{e}}}[F](z) := \int_{\mathbb{T}} \frac{z + \zeta}{z - \zeta} F(\zeta) \, \mathrm{ds}(\zeta), \quad z \in \mathbb{D}_{\mathrm{e}},$$

which we refer to as the *Herglotz transform* of F. If F is only $L^2(\mathbb{T})$-integrable, its Herglotz transform is in the Hardy space H^2. If we assume slightly more smoothness, e.g. that F is C^1-smooth, then its Herglotz transform is continuous and bounded in the closed exterior disk $\bar{\mathbb{D}}_{\mathrm{e}}$. Analogously, if we have a lot of smoothness, e.g. F is C^ω-smooth, then its Herglotz transform extends to a bounded analytic function on a slightly bigger exterior disk $\mathbb{D}_{\mathrm{e}}(0,\rho)$ with $\rho < 1$. As the Hardy space $H^2 = H^2(\mathbb{D})$ was just mentioned, we recall the precise definition. A function f is in H^2 if it is holomorphic in $\mathbb{D}$ with

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 \mathrm{ds}(\zeta) < +\infty.$$

Alternatively, in terms of the boundary values it is a closed subspace of $L^2(\mathbb{T})$ defined by the property that the Fourier coefficients with negative index vanish. The conjugate Hardy space H^2_- consists of all functions of the form $\bar{f}$, where $f \in H^2$, which may also be viewed as Hardy space on the exterior disk $\mathbb{D}_{\mathrm{e}}$. In a similar fashion, the standard H^p-spaces can be defined. For $p = \infty$ the space H^∞ consists of the bounded holomorphic functions in the unit disk $\mathbb{D}$ equipped with the supremum norm.

Associated with the Hardy and conjugate Hardy subspaces of $L^2(\mathbb{T})$ there are the orthogonal projections $\mathbf{P}_{H^2} : L^2(\mathbb{T}) \to H^2$ and $\mathbf{P}_{H^2_-} : L^2(\mathbb{T}) \to H^2_-$. These are associated with the Szegő integral kernel:

$$\mathbf{P}_{H^2} f(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - z\bar{\zeta}} \mathrm{ds}(\zeta), \qquad z \in \mathbb{D},$$

and

$$\mathbf{P}_{H^2_-} f(z) = \int_{\mathbb{T}} \frac{z f(\zeta)}{z - \zeta} \mathrm{ds}(\zeta), \qquad z \in \mathbb{D}_{\mathrm{e}}.$$

We will also be interested in the subspace $H^2_{-,0}$ of H^2_- consisting of all functions that vanish at infinity (or equivalently, have average 0 on the unit circle). The associated projection is

$$\mathbf{P}_{H^2_{-,0}} f(z) = \int_{\mathbb{T}} \frac{\zeta f(\zeta)}{z - \zeta} \mathrm{ds}(\zeta), \qquad z \in \mathbb{D}_{\mathrm{e}}.$$

It is clear from the above concrete formulae that the Herglotz transform $\mathbf{H}_{\mathbb{D}_{\mathrm{e}}}$ can be expressed in terms of projections: $\mathbf{H}_{\mathbb{D}_{\mathrm{e}}} = \mathbf{P}_{H^2_-} + \mathbf{P}_{H^2_{-,0}}$. For an $L^\infty(\mathbb{T})$-function Θ, we define the (exterior) Toeplitz operator $\mathbf{T}_\Theta : H^2_- \to H^2_-$ by

$$\mathbf{T}_\Theta f = \mathbf{P}_{H^2_-}[\Theta f], \qquad f \in H^2_-.$$

The kernel of this operator consists of all solutions in H^2_- to $\mathbf{T}_\Theta f = 0$. Assuming that Θ is non-zero almost everywhere on the circle $\mathbb{T}$, it follows that the condition that f belongs to the kernel is equivalent to $f \in H^2_- \cap \Theta^{-1} H^2_0$, where H^2_0 consists of the functions in H^2 with mean 0. If we implicitly define the function ϑ by $\Theta(z) = z\vartheta(z)$, we may rephrase this condition as

$$f \in H^2_- \cap \vartheta^{-1} H^2, \tag{2.5.1}$$

which we refer to as a *homogeneous (exterior) Toeplitz kernel condition.* For a function F defined on the circle $\mathbb{T}$, we also consider the related condition

$$f \in H^2_- \cap \vartheta^{-1}(-F + H^2), \tag{2.5.2}$$

which we refer to as an *inhomogeneous Toeplitz kernel condition.* In terms of Toeplitz operators, this condition may be written as $\mathrm{T}_{z\vartheta} f + \mathbf{P}_{H^2_-}[zF] = 0$. The following proposition provides the structure of solutions to the homogeneous and inhomogeneous Toeplitz kernel conditions for sufficiently regular ϑ.

Proposition 2.5.1. *Suppose that ϑ can be written in the form $\vartheta = \mathrm{e}^{u+\bar{v}}$, where u and v are in H^∞, and let F be a function in $L^\infty(\mathbb{T})$. Then f solves*

$$f \in H^2_- \cap \vartheta^{-1}(-F + H^2)$$

if and only if

$$f = C\mathrm{e}^{-\bar{v}} - \mathrm{e}^{-\bar{v}} \mathbf{P}_{H^2_{-,0}}[\mathrm{e}^{-u} F],$$

for some constant C.

Proof. That $f \in H^2_- \cap \vartheta^{-1}(-F + H^2)$ is equivalent to having

$$\mathrm{e}^{\bar{v}} f \in \mathrm{e}^{\bar{v}} H^2_- \cap (-\mathrm{e}^{-u} F + \mathrm{e}^{-u} H^2) = H^2_- \cap (-\mathrm{e}^{-u} F + H^2). \tag{2.5.3}$$

Since $\mathrm{e}^{\bar{v}} f \in H^2_-$, an application of the projection $\mathbf{P}_{H^2_{-,0}}$ gives

$$\mathbf{P}_{H^2_{-,0}}[\mathrm{e}^{\bar{v}} f] = \mathrm{e}^{\bar{v}} f - C$$

for some constant C. On the other hand, since $\mathrm{e}^{\bar{v}} f \in -e^{-u} F + H^2$, it is immediate that

$$\mathbf{P}_{H^2_{-,0}}[\mathrm{e}^{\bar{v}} f] = -\mathbf{P}_{H^2_{-,0}}[\mathrm{e}^{-u} F],$$

since H^2 projects to $\{0\}$. It follows that

$$\mathrm{e}^{\bar{v}} f = C + \mathbf{P}_{H^2_{-,0}}[\mathrm{e}^{\bar{v}} f] = C - \mathbf{P}_{H^2_{-,0}}[\mathrm{e}^{-u} F],$$

as claimed. ☐

Remark 2.5.2. The Toeplitz kernel equation (2.5.3) may be viewed as a scalar Riemann-Hilbert problem with jump from the inside $\mathbb{D}$ to the outside $\mathbb{D}_\mathrm{e}$ equal to $\mathrm{e}^{-u} F$. Later, we will use the conformal mapping from the complement $\mathcal{S}^c_\tau$ to the interior, and the interpretation of the Toeplitz kernel equation in that context is as a scalar Riemann-Hilbert problem on the Schottky double of $\mathcal{S}^c_\tau$.

2.6. **The random normal matrix model.** For extensive treatments of the random normal matrix ensembles, see e.g. [22, 2, 3, 4, 45]. Here we only briefly discuss the topic, in order to fix the notation and recall some basic concepts. Let M be a matrix, picked with respect to the probability measure ("tr" stands for trace)

$$\mathrm{d}\mu_m(M) = \frac{1}{Z_{m,Q}} \mathrm{e}^{-2m\,\mathrm{tr}(Q(M))} \mathrm{d}M,$$

where $\mathrm{d}M$ denotes the measure induced from Lebesgue measure on the manifold of normal $m \times m$ matrices embedded canonically into $\mathbb{C}^{m^2}$, and where $Z_{m,Q}$ is a normalizing constant. Such a matrix M has a set of m random eigenvalues, which we denote by $\Phi_m = \{z_{1,m}, \ldots, z_{m,m}\}$. It is known that the eigenvalues follow the law

$$\mathrm{d}\mathbb{P}_m(z_1, \ldots, z_m) = \frac{1}{\mathcal{Z}_{m,Q}} \left[\prod_{j<k} |z_j - z_k|^2 \right] \mathrm{e}^{-2m \sum_{j=1}^m Q(z_j)} \mathrm{d}A^{\otimes n}(z_1, \ldots, z_m), \tag{2.6.1}$$

where $\mathcal{Z}_{m,Q}$ is a normalizing constant, known as the *partition function* of the ensemble. Here, $\mathrm{dA}^{\otimes n}$ stands for volume measure in $\mathbb{C}^n$ normalized by the factor π^{-n}. We recognize this as the law for Coulomb gas with m particles at the inverse temperature $\beta = 2$ in the external field Q. Courtesy of the fact that the product expression in (2.6.1) may be written as the square modulus of a Vandermondian determinant, these ensembles are determinantal. That is, if the k-point intensities $R_{k,m}(z_1,\ldots z_k)$ are defined as the intensities associated to finding points simultaneously at the locations $z_1,\ldots,z_k$, then we may compute $R_{k,m}$ by

$$R_{k,m}(z_1,\ldots,z_k) = \det\left(\mathrm{K}_m(z_j,z_l)\right)_{1\le j,l\le k}. \tag{2.6.2}$$

Here K_m is the *correlation kernel*

$$\mathrm{K}_m(z,w) = K_m(z,w)\,\mathrm{e}^{-m(Q(z)+Q(w))}, \qquad z,w\in\mathbb{C}$$

where K_m is the reproducing kernel for the space Pol_m, supplied with the inner product of the space $L^2_{2mQ}(\mathbb{C})$. We remark that the correlation kernel K_m is not uniquely determined by the above-mentioned intensities, since any kernel modified by a cocycle

$$\mathrm{K}^c_m(z,w) = c(z)\bar{c}(w)\mathrm{K}_m(z,w),$$

will generate the same point process by the determinantal formula (2.6.2). Here, the cocycle is associated with a continuous unimodular function $c:\mathbb{C}\to\mathbb{T}$. This means that in terms of convergence of point processes, we need only correlation kernel convergence modulo cocycles. It is known (see [22, 45]) that Φ_m condensates to the droplet $\mathcal{S}_1$ as $m\to\infty$. Indeed, if ν_m denotes the empirical measure

$$\nu_m = \frac{1}{m}\sum_{z\in\Phi_m}\delta_z,$$

then almost surely, ν_m converges weakly to μ_{2Q}, the equilibrium measure, the support of which equals $\mathcal{S}_1$. We will be interested in rescaling the point process near a boundary point z_0 in the outer normal direction n, in order to understand the microscopic behavior of the point process. To rescale around the point z_0 we use the linear transformation

$$z_m(\zeta) := z_0 + \mathrm{n}\frac{\zeta}{\sqrt{2m\Delta Q(z_0)}}.$$

We recall that $\Phi_m = \{z_{j,m}\}_j$ denotes the original point process, and introduce the rescaled local process around the point z_0 by $\Psi_m = \{\zeta_{j,m}\}_j$, where

$$z_{j,m} = z_m(\zeta_{j,m}), \qquad j=1,\ldots,m.$$

Similarly, we denote by k_m the rescaled correlation kernel

$$\mathrm{k}_m(\xi,\eta) = \frac{1}{2m\Delta Q(z_0)}\mathrm{K}_m(z_m(\xi),z_m(\eta)).$$

The following is from [4].

Theorem 2.6.1. *To a given sequence of positive integers $\mathcal{N}$, there exist a subsequence $\mathcal{N}^*\subset\mathcal{N}$ and an Hermitian entire function $F(\xi,\eta)$ such that*

$$\lim_{\mathcal{N}^*\ni m\to\infty} c_m(\xi)\bar{c}_m(\eta)\,\mathrm{k}_m(z_m(\xi),z_m(\eta)) = \mathrm{e}^{\xi\bar\eta-\frac12(|\xi|^2+|\eta|^2)}F(\xi,\eta),$$

for some sequence of continuous unimodular functions $c_m:\mathbb{C}\to\mathbb{T}$.

2.7. **Steepest descent analysis.** When describing our computational algorithm in Section 3.3, we will find need for the following result ([29], p. 220, Theorem 7.7.5). For the formulation, we need some notation. For an open subset Ω of $\mathbb{R}$, we let $C^k(\Omega)$ denote the space of k times differentiable functions on Ω, and for a compact subset K of $\mathbb{R}$, we let $C^k_0(K)$ denote the space

k times differentiable, compactly supported functions on $\mathbb{R}$ whose support is contained in K. The norm in the space $C^k(\Omega)$ is defined as

$$\|u\|_{C^k(\Omega)} = \sum_{j=0}^{k} \|u^{(j)}\|_{L^\infty(\Omega)},$$

and the norm in $C_0^k(K)$ is analogously defined.

Proposition 2.7.1. *Let* $\mathrm{K} \subset \mathbb{R}$ *be a compact interval,* Ω *an open neighbourhood of* K*, and* k *a positive integer. If* $u \in C_0^{2k}(\mathrm{K})$*,* $V \in C^{3k+1}(\Omega)$ *and* $V \geq 0$ *in* X*,* $V'(x_0) = 0$*,* $V''(x_0) > 0$*, and* $V' \neq 0$ *in* $\mathrm{K} \setminus \{x_0\}$*, then, for* $\omega > 0$*, we have*

$$\left| \mathrm{e}^{\omega V(x_0)} \int_{\mathrm{K}} u(x) \mathrm{e}^{-\omega V(x)} \mathrm{d}x - \left(\frac{2\pi}{\omega V''(x_0)} \right)^{\frac{1}{2}} \sum_{j=0}^{k-1} \omega^{-j} \mathbf{L}_j u(x_0) \right| \leq C \omega^{-k} \|u\|_{C^{2k}(\mathrm{K})}. \tag{2.7.1}$$

Here, C *is bounded when* V *stays in a bounded set in* $C^{3k+1}(\Omega)$*, and* $|x - x_0|/|V'(x)|$ *has a uniform bound. With*

$$W_{x_0}(x) := V(x) - V(x_0) - \frac{1}{2}(x - x_0)^2 V''(x_0),$$

we have

$$\mathbf{L}_j u(x) := \sum_{(k,l): l-k=j,\, 2l \geq 3k} \frac{(-1)^k 2^{-l}}{k! l! [V''(x_0)]^l} \partial_x^{2l} (W_{x_0}^k u)(x).$$

In the definition of the above differential operator $\mathbf{L}_j$, it is implicit that the summation takes place over non-negative integers k and l.

The following proposition is tailored to our needs, based on Proposition 2.7.1.

Proposition 2.7.2. *Let three reals* ρ_0, ρ_1, ρ_2 *be given, with* $0 < \rho_0 < 1 < \rho_1 < \rho_2$*. Assume that* $V : [\rho_0, \infty) \to \mathbb{R}$ *is* C^{3k+1}*-smooth, and that* V *has a unique minimum at* 1*, with* $V(1) = V'(1) = 0$*. Suppose furthermore that we have*
(a) *the convexity bound* $V'' \geq \alpha$ *on* (ρ_0, ρ_2) *for some real* $\alpha > 0$*,*
(b) *and that* V *has a bound from below of the form* $V(x) \geq \vartheta \log x$ *on the interval* $[\rho_1, \infty)$*, for some real constant* $\vartheta > 0$*.*
If the function $u : (\rho_0, \infty) \to \mathbb{C}$ *is bounded and continuous throughout, and in addition* u *is* C^{2k}*-smooth on the interval* $[0, \rho_2]$ *and vanishes on* $[0, \rho_0]$*, then we have*

$$\int_{\rho_0}^{\infty} u(x) \mathrm{e}^{-\omega V(x)} \mathrm{d}x = \left(\frac{2\pi}{\omega V''(1)} \right)^{\frac{1}{2}} \sum_{j=0}^{k-1} \omega^{-j} \mathbf{L}_j[u](1) + E,$$

where the error term $E = E(\omega, k, u, \vartheta, \rho_0, \rho_1, \rho_2)$ *enjoys the bound*

$$|E| \leq C_1 \omega^{-k} \|u\|_{C^{2k}([\rho_0, \rho_2])} + \|u\|_{L^\infty([\rho_1, \infty))} \rho_1^{-\omega\vartheta + 1},$$

provided that $\omega > \frac{2}{\vartheta}$*, where* C_1 *remains uniformly bounded when* V *stays in a bounded set of* $C^{3k+1}([\rho_0, \rho_2])$*.*

Sketch of proof. Let χ be a smooth cut-off function with $0 \leq \chi \leq 1$ throughout, which equals 1 on the interval $[\rho_0, \rho_1]$, and vanishes on $[\rho_2, \infty)$. We use the cut-off function to split the integral

$$\int_{\rho_0}^{\infty} u(x) \mathrm{e}^{-\omega V(x)} \mathrm{d}x = \int_{\rho_0}^{\rho_2} \chi(x) u(x) \mathrm{e}^{-\omega V(x)} \mathrm{d}x + \int_{\rho_1}^{\infty} (1 - \chi(x)) u(x) \mathrm{e}^{-\omega V(x)} \mathrm{d}x.$$

The first integral gives the main contribution, which is estimated using Proposition 2.7.1. The other two integrals are estimated using the given bounds from below on V. The details are omitted. $\square$

2.8. **Notation.** For the convenience of the reader, we supply a list of commonly used notation.

$\mathbb{C}$, $\mathbb{D}$, $\mathbb{T}$	Complex plane, open unit disk and unit circle, respectively.
$\partial_z, \bar\partial_z$	Wirtinger derivatives, given by $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar\partial_z = \frac{1}{2}(\partial_x + i\partial_y)$, where $z = x + iy$.
X^c	Complement of the set X, understood as $\mathbb{C} \setminus X$ unless specified otherwise.
Δ	Laplacian, which factorizes as $\Delta = \partial\bar\partial$. N.B.: this equals one-quarter of the usual laplacian.
Q	Subharmonic, smooth real-valued function with superlogarithmic growth.
$\hat{Q}_\tau$	Solution to obstacle problem – maximal subharmonic function with $\hat{Q}_\tau \le Q$. and $\hat{Q}_\tau(z) = \tau \log\lvert z\rvert + \mathrm{O}(1)$ as $z \to \infty$.
$\mathcal{S}_\tau, \mathcal{S}_\tau^\star$	The droplet and the coincidence set, which coincide under our assumptions.
ϕ_τ	Conformal mapping $\phi_\tau : \mathcal{S}_\tau^c \to \mathbb{D}_\mathrm{e}$ with $\phi_\tau'(\infty) > 0$.
$\breve{Q}_\tau$	Harmonic extension of $\hat{Q}_\tau\big|_{\mathcal{S}_\tau^c}$ across $\partial\mathcal{S}_\tau$.
$Q_\tau^\circledast$	Bounded harmonic extension of $Q\big|_{\partial\mathcal{S}_\tau}$ to $\mathcal{S}_\tau^c$.
$\mathcal{Q}_\tau$	Holomorphic function on $\mathcal{S}_\tau^c$ with $\operatorname{Re}\mathcal{Q}_\tau = Q_\tau^\circledast$.
χ_0, χ_1	Various smooth cut-off functions.
$\varpi(E, \Omega, z_0)$	Harmonic measure of E relative to (Ω, z_0).
$\mathbf{H}_\Omega$	The Herglotz operator for a domain Ω containing the point at infinity.
$\mathbf{P}_{H^2}, \mathbf{P}_{H^2_-}$	Orthogonal projection onto Hardy spaces.
$\mathcal{I}_n$	Indexing set $\{(j, l) \in \mathbb{N}^2 :\ 2j + l \le n\}$.

3. Heuristic Algorithm

3.1. **Introduction of quasipolynomials and a renormalizing ansatz.** In the present section, we discuss the *approximate orthogonal quasipolynomials* $F_{n,m}$, by which we mean certain functions which behave like orthogonal polynomials in $L^2(\mathbb{C}, \mathrm{e}^{-2mQ})$, in a sense specified below. Let $\mathcal{K}_\tau$ be an appropriately chosen compact subset of the droplet $\mathcal{S}_\tau$, which lies at a fixed positive distance from $\partial\mathcal{S}_\tau$. Moreover, we require that the conformal mapping $\phi_\tau : \mathcal{S}_\tau \to \mathbb{D}_\mathrm{e}$ extends to a (surjective) conformal mapping

$$\phi_\tau \colon \mathcal{K}_\tau^c \to \mathbb{D}_\mathrm{e}(0, \rho_0), \qquad \tau \in I_{\epsilon_0},$$

for some ρ_0 with $0 < \rho_0 < 1$. In what follows, we will disregard the behavior on the compact set $\mathcal{K}_\tau$. We expect this to have little effect, and will be justified a posteriori, since the orthogonal polynomials are highly concentrated near $\partial\mathcal{S}_\tau$ (see Figure 3.1 for an illustration of the behavior of $\lvert P_{n,m}\rvert^2 \mathrm{e}^{-2mQ(z)}$ in a special case).

In the context of the following definition, we use the standard notation that $A \asymp B$ provided that $A = \mathrm{O}(B)$ and $B = \mathrm{O}(A)$ in some limiting procedure involving positive quantities A and B.

Definition 3.1.1. We say that a function F is a *quasipolynomial* on $\mathcal{K}_\tau^c$ of degree n if it is defined and holomorphic on $\mathbb{C} \setminus \mathcal{K}_\tau$, with polynomial growth near infinity: $\lvert F(z)\rvert \asymp \lvert z\rvert^n$ as $\lvert z\rvert \to \infty$.

In the context of this definition, a quasipolynomial F of degree n has $F(z) = az^n + \mathrm{O}(\lvert z\rvert^{n-1})$ near infinity, for some complex number $a \neq 0$. We refer to the number a as the *leading coefficient* of the quasipolynomial F.

We now fix a positive integer κ, which we think of as an accuracy parameter. Moreover, we denote by $\chi_{0,\tau}$ a smooth cut-off function that vanishes on $\mathcal{K}_\tau$ and equals 1 on $\mathcal{X}_\tau^c$, where $\mathcal{X}_\tau$ denotes an intermediate set between $\mathcal{K}_\tau$ and $\mathcal{S}_\tau$. In addition, we shall require that the $C^{2(\kappa+1)}$-norm of $\chi_{0,\tau}$ remains uniformly bounded for $\tau \in I_{\epsilon_0}$.

Definition 3.1.2. We say that a sequence $\{F_{n,m}\}_{n,m}$ of quasipolynomials of degree n on $\mathcal{K}_\tau^c$ is *normalized and approximately orthogonal (of accuracy κ)* if the following asymptotic conditions (i)-(iii) are met as $m \to \infty$ while $\tau = \frac{n}{m} \in I_{\epsilon_0}$:

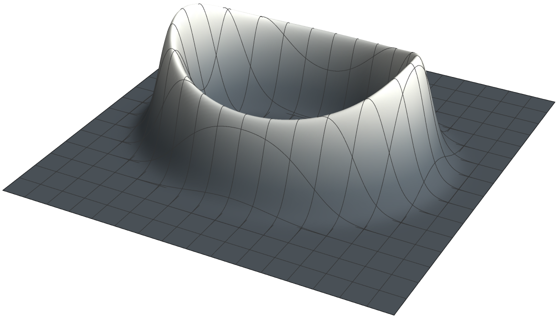

FIGURE 3.1. The orthogonal polynomial density $|P_{n,m}(z)|^2 e^{-2mQ(z)}$ for $n = 6, m = 20$ and $Q(z) = \frac{1}{2}|z|^2 - \mathrm{Re}(tz^2)$, where $t = 0.4$.

(i) we have the approximate orthogonality

$$\forall p \in \mathrm{Pol}_n : \quad \int_{\mathbb{C}} \chi_{0,\tau} F_{n,m}(z) \bar{p}(z)\, e^{-2mQ(z)} \mathrm{dA}(z) = \mathrm{O}\left(m^{-\kappa-\frac{1}{3}} \|p\|_{2mQ}\right),$$

(ii) the quasipolynomials $F_{n,m}$ have approximately unit norm,

$$\int_{\mathbb{C}} \chi^2_{0,\tau}(z) |F_{n,m}(z)|^2 e^{-2mQ(z)} \mathrm{dA}(z) = 1 + \mathrm{O}(m^{-\kappa-\frac{1}{3}}),$$

(iii) and the quasipolynomial $F_{n,m}$ has leading coefficient $a_{n,m}$ at infinity which is approximately real and positive, in the sense that

$$\frac{\mathrm{Im}\, a_{n,m}}{\mathrm{Re}\, a_{n,m}} = \mathrm{O}(m^{-\kappa-\frac{1}{12}})$$

where all the implied constants are uniform.

In terms of the above definition, Theorem 1.3.3 implies in particular that $F^{\langle\kappa\rangle}_{n,m}$ is a sequence of approximately orthogonal quasipolynomials with accuracy κ. The fraction $\frac{1}{3}$ which appears in the definition is convenient in our calculations. The concept would be meaningful even if this number were replaced by e.g. $\frac{1}{5}$.

We proceed to list some functions related to Q.

(i) $\hat{Q}_\tau$ is the solution to the obstacle problem (2.1.1). Recall that $\hat{Q}_\tau$ is harmonic function on $\mathcal{S}_\tau^c$ which equals Q on $\partial\mathcal{S}_\tau$, and grows like $\tau \log|z|$ as $|z| \to \infty$.
(ii) $\breve{Q}_\tau$ is defined as the harmonic extension of the restriction of $\hat{Q}_\tau$ to $\mathbb{C} \setminus \mathcal{S}_\tau$ across $\partial\mathcal{S}_\tau$.
(iii) $Q^{\circledast}_\tau$ is the bounded harmonic harmonic function on $\mathcal{S}_\tau^c$ which equals Q on $\partial\mathcal{S}_\tau$.
(iv) $\mathcal{Q}_\tau$ is a bounded holomorphic function in $\mathcal{S}_\tau^c$ such that $\mathrm{Re}\, \mathcal{Q}_\tau = Q^{\circledast}_\tau$ on $\mathcal{S}_\tau^c$.

Since Q is assumed 1-admissible, the curves $\Gamma := \partial\mathcal{S}_\tau$ ramain real-analytically smooth and simple for $\tau \in I_{\epsilon_0} = [1-\epsilon_0, 1+\epsilon_0]$. By possibly making $\mathcal{K}_\tau$ a little bigger, we may assume that the functions $Q^{\circledast}_\tau$ and $\breve{Q}_\tau$ are harmonic, and that $\mathcal{Q}_\tau$ is holomorphic, in the domain $\mathbb{C} \setminus \mathcal{K}_\tau$. We define the operator $\mathbf{\Lambda}_{n,m}$ by

$$\mathbf{\Lambda}_{n,m}[f](z) := \phi'_\tau(z)\, [\phi_\tau(z)]^n e^{m\mathcal{Q}_\tau(z)}\, (f \circ \phi_\tau)(z), \qquad \tau = \frac{n}{m}. \tag{3.1.1}$$

If f is well-defined in a neighbourhood of $\bar{\mathbb{D}}_{\mathrm{e}}(0,\rho_0)$, then $\mathbf{\Lambda}_{n,m}[f]$ is well-defined in a neighbourhood of $\mathbb{C}\setminus\mathcal{K}_\tau$. Observe that

$$\text{(3.1.2)}\qquad \int_{\mathcal{K}_\tau^c}\mathbf{\Lambda}_{n,m}[f]\,\overline{\mathbf{\Lambda}_{n,m}[g]}\mathrm{e}^{-2mQ}\mathrm{dA}=\int_{\mathcal{K}_\tau^c} f\circ\phi_\tau\bar{g}\circ\phi_\tau\mathrm{e}^{-2m(Q-\log|\phi_\tau|-\mathrm{Re}\,\mathcal{Q}_\tau)}|\phi_\tau'|^2\mathrm{dA}$$
$$=\int_{\mathbb{D}_{\mathrm{e}}(0,\rho_0)} f\,\bar{g}\,\mathrm{e}^{-2mR_\tau}\mathrm{dA},$$

where we write

$$R_\tau:=(Q-\breve{Q}_\tau)\circ\phi_\tau^{-1},$$

and the first equality holds since the Green function for $\mathcal{S}_\tau^{\mathrm{c}}$ with pole at infinity is given by $G_{\infty,\tau}=\log|\phi_\tau|$, so that we may decompose

$$\breve{Q}_\tau=Q_\tau^{\circledast}+\tau\log|\phi_\tau|.$$

The function R_τ is a central object, and we turn to some of its basic properties.

Proposition 3.1.3. *The function R_τ is defined on $\mathbb{D}_{\mathrm{e}}(0,\rho_0)$, and is real-analytic in a neighbourhood of $\mathbb{T}$. Moreover, near the unit circle R_τ satisfies*

$$R_\tau(r\mathrm{e}^{\mathrm{i}\theta})=2\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta})(1-r)^2+\mathrm{O}((1-r)^3),\qquad r\to1,$$

where the implied constant is uniform for $\mathrm{e}^{\mathrm{i}\theta}\in\mathbb{T}$ and $\tau\in I_{\epsilon_0}$. Moreover, it has the growth bound from below

$$R_\tau(z)\geq\vartheta\log|z|,\qquad z\in\mathbb{D}_{\mathrm{e}}(0,\rho_1),$$

for some real parameters $\vartheta>0$ and $\rho_1>1$, which do not depend on $\tau\in I_{\epsilon_0}$.

Sketch of proof. The assertion on the local behaviour near the circle $\mathbb{T}$ results from an application of Taylor's formula, using that along the boundary $\partial\mathcal{S}_\tau$ we have $Q=\breve{Q}_\tau$, $\nabla Q=\nabla\breve{Q}_\tau$ while

$$\partial_{\mathrm{n}}^2(Q-\breve{Q}_\tau)=(\partial_{\mathrm{n}}^2+\partial_{\mathrm{t}}^2)(Q-\breve{Q}_\tau)=4\Delta Q.$$

Here we recall that ∂_{n} and ∂_{t} denote the normal and tangential derivatives, respectively. We turn to the global estimate from below on R_τ. By the assumption (1.3.1) with $\tau=1$ on the growth of Q near infinity, and the growth control

$$\breve{Q}_\tau(z)=\hat{Q}_\tau(z)=\tau\log|z|+\mathrm{O}(1),\qquad\text{as }\ |z|\to\infty,$$

it follows that

$$\liminf_{z\to\infty}\frac{(Q-\breve{Q}_\tau)(z)}{\log|z|}>0$$

for $\tau\in I_{\epsilon_0}$, provided that ϵ_0 is small enough. Since $|\phi_\tau^{-1}(z)|\asymp|z|$ near infinity, we see that

$$\lim_{|z|\to\infty}\frac{R_\tau(z)}{\log|z|}>0.$$

There is no point in $\mathbb{D}_e$ where R_τ vanishes, since the coincidence set (where $\hat{Q}_\tau$ and Q coincide) equals $\mathcal{S}_\tau$ (see Definition 1.3.1). We may conclude that the ratio $\frac{R_\tau(z)}{\log|z|}$ is bounded below by a positive constant ϑ on the exterior disk $\mathbb{D}_{\mathrm{e}}(0,\rho_1)$. A careful analysis of this argument shows that we may assume that ϑ does not depend on τ, as long as $\tau\in I_{\epsilon_0}$. □

Remark 3.1.4. Since $\breve{Q}_\tau$ is harmonic on $\mathcal{K}_\tau^c$, we find that

$$\Delta R_\tau=\Delta(Q-\breve{Q}_\tau)\circ\phi_\tau^{-1}=|(\phi_\tau^{-1})'|^2\,(\Delta Q)\circ\phi_\tau^{-1},$$

which shows that near the circle $\mathbb{T}$, we have uniform bound of ΔR_τ from below by a positive constant. As a consequence the same holds for $\partial_r^2R_\tau(r\mathrm{e}^{\mathrm{i}\theta})$ for r close to 1 (cf. Proposition 2.7.2).

Proposition 3.1.3 explains why near the unit circle, the function e^{-2mR_τ} may be thought of as a Gaussian wave around the unit circle $\mathbb{T}$.

We return to the operator $\mathbf{\Lambda}_{n,m}$, defined in (3.1.1). It renormalizes the weight, and transports holomorphic functions in $\mathcal{K}_\tau^c$ to the exterior disk $\mathbb{D}_\mathrm{e}(0,\rho_0)$. In the sequel, we will refer to $\mathbf{\Lambda}_{n,m}$ as *the canonical positioning operator*. Its basic properties are summarized in the following proposition.

Proposition 3.1.5. *The canonical positioning operator $\mathbf{\Lambda}_{n,m}$ is an isometric isomorphism $L^2_{2mR_\tau}(\mathbb{D}_\mathrm{e}(0,\rho_0)) \to L^2_{2mQ}(\mathcal{K}_\tau^c)$, and the inverse operator is given by*

$$\mathbf{\Lambda}_{n,m}^{-1}[g](z) = z^{-n}[\phi_\tau^{-1}]'(z)\,\mathrm{e}^{-m(\mathcal{Q}_\tau\circ\phi_\tau^{-1})(z)}(g\circ\phi_\tau^{-1})(z), \qquad g \in L^2_{2mQ}(\mathcal{K}_\tau^c).$$

Moreover, the operator $\mathbf{\Lambda}_{n,m}$ preserves holomorphicity, and in addition, it maps the subspace $A^2_{2mR_\tau,0}(\mathbb{D}_\mathrm{e}(0,\rho_0))$ onto $A^2_{2mQ,n}(\mathcal{K}_\tau^c)$.

Proof. As direct consequence of the (3.1.2), we see that $L^2_{2mR_\tau}(\mathbb{D}_\mathrm{e}(0,\rho_0))$ is mapped isometrically into $L^2_{2mQ}(\mathcal{K}_\tau^c)$, and moreover if $\mathbf{\Lambda}_{n,m}^{-1}$ is given by the above formula, we see that it is actually the inverse to $\mathbf{\Lambda}_{n,m}$. By definition, $\mathbf{\Lambda}_{n,m}[f]$ is holomorphic in $\mathcal{K}_\tau^c$, if f is holomorphic in $\mathbb{D}_\mathrm{e}(0,\rho_0)$. It follows that $\mathbf{\Lambda}_{n,m}$ is actually an isometric isomorphism $A^2_{2mR_\tau}(\mathbb{D}_\mathrm{e}(0,\rho_0)) \to A^2_{2mQ}(\mathcal{K}_\tau^c)$. It remains to note that $\mathbf{\Lambda}_{m,n}$ maps bijectively

$$A^2_{2mR_\tau,0}(\mathbb{D}_\mathrm{e}(0,\rho_0)) \to A^2_{2mQ,n}(\mathcal{K}_\tau^c),$$

which is a direct consequence of the fact that $|\phi_\tau(z)| \asymp |z|$ as $|z|\to\infty$. $\square$

The canonical positioning operator is useful to our finding the asymptotic expansion formula for the orthogonal polynomials.

3.2. **Implementation of the radial Laplace method.** We turn to the algorithm of Theorem 1.3.7. To proceed, we need two families of differential operators. We recall the differential operators $\mathbf{L}_k$ appearing in the application of Laplace's method in Proposition 2.7.1. We need to apply these operators to functions defined in a neighbourhood of the unit circle, and we apply them in the radial direction. So, for functions $f(r\mathrm{e}^{\mathrm{i}\theta})$, we put

$$\mathbf{L}_k[f](r\mathrm{e}^{\mathrm{i}\theta}) = \sum_{\nu=k}^{3k} \frac{(-1)^{\nu-k}2^{-\nu}}{\nu!(\nu-k)![\partial_r^2 R_\tau(r\mathrm{e}^{\mathrm{i}\theta})]^\nu}\partial_r^{2\nu}\Big(\big[W_\tau(r\mathrm{e}^{\mathrm{i}\theta})\big]^{\nu-k}f(r\mathrm{e}^{\mathrm{i}\theta})\Big),$$

where

$$W_\tau(r\mathrm{e}^{\mathrm{i}\theta}) = R_\tau(r\mathrm{e}^{\mathrm{i}\theta}) - \frac{1}{2}(r-1)^2\partial_x^2 R_\tau(x\mathrm{e}^{\mathrm{i}\theta})\Big|_{x=1}.$$

The second family of operators is defined implicitly in the following Lemma, which is inspired by the theory of pseudodifferential operators.

Lemma 3.2.1. *Let k be a non-negative integer. Then there exist partial differential operators $\mathbf{M}_k$ of order $2k$ with real-analytic coefficients, such that for any integer $l\geq 0$ and any function smooth function f defined in a neighbourhood of $\mathbb{T}$, we have that*

$$\int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}l\theta}\big(\partial_r^2 R_\tau(r\mathrm{e}^{\mathrm{i}\theta})\big)^{-\frac{1}{2}}\mathbf{L}_k[r^{1-l}f(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1}\mathrm{d}\theta = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}l\theta}\mathbf{M}_k[f](\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta.$$

Proof. We first observe that by integration by parts, multiplication by l corresponds to applying the differential operator $\mathrm{i}\partial_\theta$ inside the integral:

$$l\int_{\mathbb{T}} f(\theta)\mathrm{e}^{\mathrm{i}l\theta}\mathrm{d}\theta = \int_{\mathbb{T}} \mathrm{i}\partial_\theta f(\theta)\mathrm{e}^{\mathrm{i}l\theta}\mathrm{d}\theta.$$

From this it is immediate that the formula

$$p(l)\int_{\mathbb{T}} f(\theta)\mathrm{e}^{\mathrm{i}l\theta}\mathrm{d}\theta = \int_{\mathbb{T}} p(i\partial_\theta)f(\theta)\mathrm{e}^{\mathrm{i}l\theta}\mathrm{d}\theta \tag{3.2.1}$$

holds for polynomials p. Structurally, $\mathbf{L}_k[r^{1-l}f(r\mathrm{e}^{\mathrm{i}\theta})]$ can be written as

$$\mathbf{L}_k[r^{1-l}f(r\mathrm{e}^{\mathrm{i}\theta})]=\sum_{\nu=k}^{3k}b_\nu(r\mathrm{e}^{\mathrm{i}\theta})\partial_r^{2\nu}\big[[W_\tau(r\mathrm{e}^{\mathrm{i}\theta})]^{\nu-k}r^{1-l}f(r\mathrm{e}^{\mathrm{i}\theta})\big],\tag{3.2.2}$$

where b_ν is the real-analytic function given by

$$b_\nu(r\mathrm{e}^{\mathrm{i}\theta})=\frac{(-1)^{\nu-k}2^{-\nu}}{\nu!(\nu-k)![\partial_r^2R_\tau(r\mathrm{e}^{\mathrm{i}\theta})]^\nu}.$$

We observe that by the Leibniz rule

$$\begin{aligned}\partial_r^j(r^{1-l}f(r\mathrm{e}^{\mathrm{i}\theta}))\Big|_{r=1}&=\sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}(l-1)_{j-i}r^{1-l-j+i}\partial_r^if(r\mathrm{e}^{\mathrm{i}\theta})\Big|_{r=1}\\&=\sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}(l-1)_{j-i}\partial_r^if(r\mathrm{e}^{\mathrm{i}\theta})\Big|_{r=1},\end{aligned}\tag{3.2.3}$$

where $(x)_i=x(x+1)\cdots(x+i-1)$ denotes the standard Pochhammer symbol. We return to the formula (3.2.2) for $\mathbf{L}_k$. Again by the Leibniz formula we have that

$$\begin{aligned}\partial_r^{2\nu}[W_\tau^{\nu-k}(\mathrm{e}^{\mathrm{i}\theta})r^{1-l}f(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1}&=\sum_{j=0}^{2\nu}\binom{2\nu}{j}\partial_r^{2\nu-j}\Big([W_\tau(r\mathrm{e}^{\mathrm{i}\theta})]^{\nu-k}\Big)\partial_r^j\Big(r^{1-l}f(r\mathrm{e}^{\mathrm{i}\theta})\Big)\Big|_{r=1}\\&=\sum_{j=0}^{3k-\nu}\binom{2\nu}{j}\partial_r^{2\nu-j}\Big([W_\tau(r\mathrm{e}^{\mathrm{i}\theta})]^{\nu-k}\Big)\partial_r^j\Big(r^{1-l}f(r\mathrm{e}^{\mathrm{i}\theta})\Big)\Big|_{r=1}\\&=\sum_{j=0}^{3k-\nu}\sum_{i=0}^{j}(-1)^{j-i}\binom{2\nu}{j}\binom{j}{i}(l-1)_{j-i}\partial_r^{2\nu-j}\Big([W_\tau(r\mathrm{e}^{\mathrm{i}\theta})]^{\nu-k}\Big)\partial_r^if(r\mathrm{e}^{\mathrm{i}\theta})\Big|_{r=1},\end{aligned}$$

where the truncation of the sum follows from an application of the flatness of W_τ near the unit circle $\mathbb{T}$, and the last equality is due to (3.2.3). We write the expression for $\mathbf{L}_k[r^{1-l}f(r\mathrm{e}^{\mathrm{i}\theta})]$ as

$$\mathbf{L}_k[r^{1-l}f(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1}=\sum_{\nu=k}^{3k}\sum_{j=0}^{3k-\nu}\sum_{i=0}^{j}(l-1)_{j-1}c_{i,j,\nu}(\mathrm{e}^{\mathrm{i}\theta})\partial_r^if(r\mathrm{e}^{\mathrm{i}\theta})\Big|_{r=1},$$

where

$$c_{i,j,\nu}(\mathrm{e}^{\mathrm{i}\theta})=(-1)^{j-i}\binom{2\nu}{j}\binom{j}{i}(l-1)_{j-i}b_\nu(\mathrm{e}^{\mathrm{i}\theta})\partial_r^{2\nu-j}\Big([W_\tau(r\mathrm{e}^{\mathrm{i}\theta})]^{\nu-k}\Big)\Big|_{r=1}.$$

Changing the order of summation, we arrive at

$$\begin{aligned}&\Big(\partial_r^2R_\tau(r\mathrm{e}^{\mathrm{i}\theta})\Big)^{-\frac12}\mathbf{L}_k[r^{1-l}f(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1}\\&\qquad=\sum_{i=0}^{2k}\sum_{j=i}^{2k}(-1)^{j-i}\binom{j}{i}(l-1)_{i-j}\Big(\partial_r^2R_\tau(r\mathrm{e}^{\mathrm{i}\theta})\Big)^{-\frac12}d_j(\mathrm{e}^{\mathrm{i}\theta})\partial_r^if(r\mathrm{e}^{\mathrm{i}\theta})\Big|_{r=1},\end{aligned}$$

where

$$d_j(\mathrm{e}^{\mathrm{i}\theta})=\sum_{\nu=k}^{3k-j}\binom{2\nu}{j}b_\nu(\mathrm{e}^{\mathrm{i}\theta})\partial_r^{2\nu-j}\Big([W_\tau(r\mathrm{e}^{\mathrm{i}\theta})]^{\nu-k}\Big)\Big|_{r=1}.$$

It follows from (3.2.1) that the asserted identity holds with $\mathbf{M}_k$ given by

$$\mathbf{M}_k[f](\mathrm{e}^{\mathrm{i}\theta})=\sum_{i=0}^{2k}\sum_{j=i}^{2k}(-1)^{j-i}\binom{j}{i}(\mathrm{i}\partial_\theta-1)_{i-j}\Big[(\partial_r^2R_\tau(r\mathrm{e}^{\mathrm{i}\theta}))^{-\frac12}d_j(\mathrm{e}^{\mathrm{i}\theta})\partial_r^if(r\mathrm{e}^{\mathrm{i}\theta})\Big]\Big|_{r=1}.$$

The proof of the lemma is complete. $\square$

3.3. **Algorithmic computation of the coefficients in the asymptotic expansion.** In this section we supply the proof of Theorem 1.3.7, and explain the underlying computational algorithm. The main point is that we show how to iteratively obtain the coefficients, given that an asymptotic expansion exists, as formulated in Theorem 1.3.3. The proof of Theorem 1.3.3 is then supplied later on in Section 4.

Proof of Theorem 1.3.7. Fix the precision κ to be a positive integer. Let $F_{n,m}^{\langle\kappa\rangle}$ be the approximate orthogonal quasipolynomials from Theorem 1.3.3 with the expansion

$$F_{n,m}^{\langle\kappa\rangle}(z)=\left(\frac{m}{2\pi}\right)^{\frac14}\sqrt{\phi_\tau'(z)}[\phi_\tau(z)]^n\mathrm{e}^{m\mathcal{Q}_\tau(z)}\sum_{j=0}^{\kappa}m^{-j}\mathcal{B}_{j,\tau}(z),$$

where the functions $\mathcal{B}_{j,\tau}$ are bounded and holomorphic on $\mathcal{K}_\tau^c$ for some compact subset $\mathcal{K}_\tau$ of $\mathcal{S}_\tau^\circ$, which we may assume to be the conformal image of the exterior disk $\mathbb{D}_\mathrm{e}(0,\rho_0)$ under the mapping ϕ_τ^{-1}. If we make the ansatz

$$\mathcal{B}_{j,\tau}(z)=(2\pi)^{1/4}\sqrt{\phi_\tau'(z)}(B_{j,\tau}\circ\phi_\tau)(z),$$

we may express $F_{n,m}$ using the canonical positioning operator as $F_{n,m}^{\langle\kappa\rangle}=m^{\frac14}\mathbf{\Lambda}_{n,m}[f_{n,m}^{\langle\kappa\rangle}]$, where

$$f_{n,m}^{\langle\kappa\rangle}(z)=\sum_{j=0}^{\kappa}m^{-j}B_{j,\tau}(z),\qquad z\in\mathbb{D}_\mathrm{e}(0,\rho_0). \tag{3.3.1}$$

According to Theorem 1.3.3, the functions $F_{n,m}^{\langle\kappa\rangle}$ have the approximate orthogonality property

$$\int_{\mathbb{C}}\chi_{0,\tau}F_{n,m}^{\langle\kappa\rangle}\bar{p}\,\mathrm{e}^{-2mQ}\mathrm{dA}=\mathrm{O}(m^{-\kappa-1}\|p\|_{2mQ}),\qquad p\in\mathrm{Pol}_n. \tag{3.3.2}$$

The function $\chi_{0,\tau}$ is a cut-off function with $0\le\chi\le1$ throughout $\mathbb{C}$, such that $\chi_{0,\tau}$ vanishes on $\mathcal{K}_\tau$ and equals 1 on $\mathcal{X}_\tau^c$, where $\mathcal{K}_\tau$ lies at a fixed positive distance from $\partial\mathcal{S}_\tau$, and $\mathcal{X}_\tau$ is an intermediate set between them (cf. Definiton 1.3.2). We consider the associated cut-off function $\chi_{1,\tau}=\chi_{0,\tau}\circ\phi_\tau^{-1}$, tacitly extended to vanish where it is undefined. Without loss of generality, we may assume that $\chi_{1,\tau}$ is radial. By Remark 1.3.4, we may assume that $\chi_{1,\tau}$ vanishes on $\mathbb{D}(0,\rho_0')$ for some number ρ_0' with $\rho_0<\rho_0'<1$. In order to compute the functions $B_{j,\tau}$, we would like to apply equation (3.3.2) to

$$q(z)=\mathbf{\Lambda}_{n,m}[z^{-l}]=\phi_\tau'(z)[\phi_\tau(z)]^{n-l}\mathrm{e}^{m\mathcal{Q}_\tau(z)}$$

for a positive integer l, but this function is not a polynomial! To fix this, we consider the $L^2_{2mQ,n}$-minimal solution v to the $\bar\partial$-problem

$$\bar\partial v=\bar\partial(\chi_{0,\tau}q)=q\,\bar\partial\chi_{0,\tau}.$$

If v is the solution, then the difference $\chi_{0,\tau}q-v$ will be an entire function with the polynomial growth bound $\mathrm{O}(|z|^{n-1})$ at infinity, and hence a polynomial of degree less than or equal to $n-1$. By the estimate of Corollary 2.4.2, we have the norm control

$$\int_{\mathbb{C}}|v|^2\mathrm{e}^{-2mQ}\mathrm{dA}\le\frac{1}{2m}\int_{\mathbb{C}}|q|^2|\bar\partial\chi_{0,\tau}|^2\frac{\mathrm{e}^{-2mQ}}{\Delta Q}\mathrm{dA}\le\frac{A^2}{2m\alpha_1}\int_{\mathcal{X}_\tau\setminus\mathcal{K}_\tau}|q|^2\mathrm{e}^{-2mQ}\mathrm{dA},$$

where we have used that there exists a positive real α_1 such that $\Delta Q\ge\alpha_1$ holds on $\mathcal{S}_\tau$, which contains the support of $\bar\partial\chi_{0,\tau}$, and that we have the bound $|\bar\partial\chi_{0,\tau}|\le A$. Since the support of $\bar\partial\chi_{0,\tau}$ lies in $\mathcal{K}_\tau^c$, we may use the structure of q as $q=\mathbf{\Lambda}_{n,m}[z^{-l}]$ and Proposition 3.1.5

$$\int_{\mathcal{X}_\tau\setminus\mathcal{K}_\tau}|q|^2\mathrm{e}^{-2mQ}\mathrm{dA}=\int_{\rho_0\le|z|\le\rho_0''}|z|^{-2l}\mathrm{e}^{-2mR_\tau(z)}\mathrm{dA}(z),$$

where ρ_0'' is associated with a natural choice of the intermediate set $\mathcal{X}_\tau$ as the image of an exterior disk under ϕ_τ^{-1}, and satisfies $\rho_0<\rho_0'<\rho_0''<1$. Due to Proposition 3.1.3, this immediately gives that for any fixed positive integer l

$$\int_{\mathbb{C}}|v|^2\mathrm{e}^{-2mQ}\mathrm{dA}=\mathrm{O}(\mathrm{e}^{-\epsilon_1 m})$$

as m,n tend to infinity while $\tau=\frac{n}{m}\in I_{\epsilon_0}$, for some positive real ϵ_1. This means that for a fixed positive integer l, we have for $q=\mathbf{\Lambda}[z^{-l}]$ the approximate orthogonality

$$\int_{\mathbb{C}}\chi_{0,\tau}^2F_{n,m}^{\langle\kappa\rangle}\bar{q}\,\mathrm{e}^{-2mQ}\mathrm{dA}=\mathrm{O}(m^{-\kappa-1}),\tag{3.3.3}$$

where we have used that $\chi_{0,\tau}q-v$ is a polynomial of degree at most $n-1$, and the above smallness of v. If we use the canonical positioning operator as in Proposition 3.1.5 in polarized form, (3.3.3) reads in polar coordinates

$$m^{\frac14}\int_{\mathbb{T}}\mathrm{e}^{\mathrm{i}l\theta}\int_{\rho_0}^{\infty}r^{1-l}\chi_{1,\tau}^2(r)f_{n,m}^{\langle\kappa\rangle}(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-2mR_\tau(r\mathrm{e}^{\mathrm{i}\theta})}\mathrm{d}r\mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta})=\mathrm{O}\left(m^{-\kappa-1}\right),\tag{3.3.4}$$

for fixed l. We now apply proposition 2.7.2 to the radial integral, with $V(r)=2R_\tau(r\mathrm{e}^{\mathrm{i}\theta})$. Note that $\partial_r^2R_\tau(r\mathrm{e}^{i\theta})|_{r=1}=4\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta})$. As a consequence, the inner integral in (3.3.4) has an expansion

$$\int_{\rho_0}^{\infty}r^{1-l}\chi_{1,\tau}^2(r)f_{n,m}^{\langle\kappa\rangle}(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-2mR_\tau(r\mathrm{e}^{\mathrm{i}\theta})}\mathrm{d}r=\left(\frac{\pi}{4m\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta})}\right)^{\frac12}\sum_{j=0}^{\kappa}m^{-j}\mathbf{L}_j[r^{1-l}f_{m,n}^{\langle\kappa\rangle}(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1}$$
$$+\mathrm{O}\Big(m^{-\kappa-1}\big\|r^{1-l}\chi_{1,\tau}^2f_{n,m,\theta}^{\langle\kappa\rangle}\big\|_{C^{2(\kappa+1)}([\rho_0,\rho_2])}+\big\|r^{1-l}\chi_{1,\tau}^2f_{n,m,\theta}^{\langle\kappa\rangle}\big\|_{L^\infty([\rho_1,\infty))}\rho_1^{-m\vartheta+1}\Big),$$

where we to simplify the notation we use the subscript θ to denote the radial restriction $f_\theta(r)=f(r\mathrm{e}^{\mathrm{i}\theta})$. Here, ϑ,α and ρ_1 are some real numbers with $\vartheta>0$, $\alpha>0$ and $1<\rho_1<\rho_2$, which are independent of $\tau\in I_{\epsilon_0}$. By applying the standard Cauchy estimates to the functions $f_{m,n}$, and by Remark 1.3.4 (both part (a) and (b) are needed) we have uniform control on the norms

$$\big\|r^{1-l}\chi_{1,\tau}^2f_{n,m,\theta}^{\langle\kappa\rangle}\big\|_{C^{2(\kappa+1)}([\rho_0,\rho_2])}\quad\text{and}\quad\big\|r^{1-l}\chi_{1,\tau}^2f_{n,m,\theta}^{\langle\kappa\rangle}\big\|_{L^\infty([\rho_1,\infty))}$$

provided that l is fixed, and that $f_{m,n}^{\langle\kappa\rangle}$ are uniformly bounded. For fixed l, it follows that

$$\begin{aligned}\int_{\rho_0}^{\infty}&r^{1-l}\chi_{1,\tau}^2(r)f_{n,m}^{\langle\kappa\rangle}(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-2mR_\tau(r\mathrm{e}^{\mathrm{i}\theta})}\mathrm{d}r\\&=\left(\frac{\pi}{4m\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta})}\right)^{\frac12}\sum_{j=0}^{\kappa}m^{-j}\mathbf{L}_j[r^{1-l}f_{m,n}^{\langle\kappa\rangle}(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1}+\mathrm{O}\big(m^{-\kappa-1}\big),\end{aligned}\tag{3.3.5}$$

where the implied constant is uniformly bounded as long as $f_{n,m}^{\langle\kappa\rangle}$ is uniformly bounded on $\mathbb{D}_{\mathrm{e}}(0,\rho_0)$. By expanding the expression (3.3.1) for $f_{n,m}^{\langle\kappa\rangle}$, it follows from (3.3.5) that

$$\begin{aligned}\int_{\rho_0}^{\infty}&r^{1-l}\chi_{1,\tau}^2(r)f_{n,m}^{\langle\kappa\rangle}(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-2mR_\tau(r\mathrm{e}^{\mathrm{i}\theta})}\mathrm{d}r\\&=\left(\frac{\pi}{4m\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta})}\right)^{\frac12}\sum_{k=0}^{\kappa}m^{-k}\mathbf{L}_k[r^{1-l}f_{n,m}^{\langle\kappa\rangle}(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1}+\mathrm{O}(m^{-\kappa-1})\\&=\left(\frac{\pi}{4m\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta})}\right)^{\frac12}\sum_{j=0}^{\kappa}m^{-j}\sum_{k=0}^{j}\mathbf{L}_k[r^{1-l}B_{j-k,\tau}(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1}+\mathrm{O}(m^{-\kappa-1}),\end{aligned}\tag{3.3.6}$$

as $m\to\infty$. We multiply the expression (3.3.6) by $\mathrm{e}^{\mathrm{i}l\theta}$ and integrate with respect to θ to get

$$\begin{aligned}m^{\frac14}&\int_{\mathbb{T}}\mathrm{e}^{\mathrm{i}l\theta}\int_{\rho_0}^{\infty}r^{1-l}\chi_{1,\tau}^2(r)f_{n,m}^{\langle\kappa\rangle}(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-2mR_\tau(r\mathrm{e}^{\mathrm{i}\theta})}\mathrm{d}r\mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta})\\&=\sum_{j=0}^{\kappa}m^{-j-\frac14}\int_{\mathbb{T}}\mathrm{e}^{\mathrm{i}l\theta}\left(\frac{\pi}{4\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta})}\right)^{\frac12}\sum_{k=0}^{j}\mathbf{L}_k[r^{1-l}B_{j-k,\tau}(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1}\mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta})+\mathrm{O}(m^{-\kappa-\frac34}),\end{aligned}$$

as $m \to \infty$. This is an asymptotic series, and so is (3.3.4), only that all the coefficients vanish in the latter, and only the error term remains. Since two asymptotic series coincide only if they coincide term by term, we find that for integers $j = 0, \ldots, \kappa$,

$$\int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}l\theta} \big(4\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta})\big)^{-\frac{1}{2}} \sum_{k=0}^{j} \mathbf{L}_k[r^{1-l} B_{j-k,\tau}(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1} \mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}) = 0, \qquad l = 1, 2, 3, \ldots.$$

This condition looks like the standard condition membership in the Hardy space H^2. The problem with this is that the functions unfortunately depend on the parameter l, so the criterion does not apply. To remedy this, we apply Lemma 3.2.1, which gives

$$\int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}l\theta} \sum_{k=0}^{j} \mathbf{M}_k[B_{j-k,\tau}](\mathrm{e}^{\mathrm{i}\theta}) \mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}) = 0, \qquad l = 1, 2, 3, \ldots, \tag{3.3.7}$$

which is now of the desired form. So, by the standard Fourier analytic characterization of the Hardy space, the equation (3.3.7) is equivalent to having

$$\sum_{k=0}^{j} \mathbf{M}_k[B_{j-k,\tau}]\Big|_{\mathbb{T}} \in H^2, \qquad j = 0, \ldots, \kappa. \tag{3.3.8}$$

We look at the case $j = 0$ first. Then (3.3.8) says that $\mathbf{M}_0[B_{0,\tau}]\big|_{\mathbb{T}} \in H^2$. The operator $\mathbf{M}_0$, with the defining property given by Lemma 3.2.1, has the form

$$\mathbf{M}_0[f](\mathrm{e}^{\mathrm{i}\theta}) = (4\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta}))^{-\frac{1}{2}} f(\mathrm{e}^{\mathrm{i}\theta}). \tag{3.3.9}$$

We recall that it is given that $B_{0,\tau}$ is bounded and holomorphic in a neighbourhood of the closed exterior disk $\bar{\mathbb{D}}_{\mathrm{e}}$, so that in particular $B_{0,\tau}\big|_{\mathbb{T}} \in H^2_-$. If we combine this with the observation that $\mathbf{M}_0[B_{0,\tau}]\big|_{\mathbb{T}} \in H^2$ together with the explicit expression (3.3.9) for $\mathbf{M}_0$, we arrive at

$$B_{0,\tau}\big|_{\mathbb{T}} \in (4\Delta R_\tau)^{\frac{1}{2}} H^2 \cap H^2_-. \tag{3.3.10}$$

Let H_{R_τ} be a bounded holomorphic function in $\mathbb{D}_{\mathrm{e}}$ such that

$$\operatorname{Re} H_{R_\tau} = \frac{1}{2} \log(4\Delta R_\tau)^{\frac{1}{2}} = \frac{1}{4} \log(4\Delta R_\tau), \qquad \text{on } \mathbb{T}. \tag{3.3.11}$$

It follows from the given regularity of R_τ that H_{R_τ} is a bounded holomorphic function in the exterior disk, which extends holomorphically to a neighbourhood of $\bar{\mathbb{D}}_{\mathrm{e}}$. We may rewrite (3.3.10) in the form

$$B_{0,\tau}\Big|_{\mathbb{T}} \in \mathrm{e}^{2 \operatorname{Re} H_{R_\tau}} H^2 \cap H^2_-.$$

By Proposition 2.5.1 applied with $u = v = -\bar{H}_{R_\tau}$ and $F = 0$, it follows that $B_{0,\tau}$ is of the form

$$B_{0,\tau} = c_{0,\tau} \mathrm{e}^{H_{R_\tau}} \tag{3.3.12}$$

for some constant $c_{0,\tau}$.

We now proceed to consider more generally $j = 1, 2, 3, \ldots$. If we separate out the term corresponding to $k = 0$ from equation (3.3.8), we find that

$$\frac{B_{j,\tau}}{(4\Delta R_\tau)^{\frac{1}{2}}} + \sum_{k=1}^{j} \mathbf{M}_k[B_{j-k,\tau}]\Big|_{\mathbb{T}} \in H^2, \qquad j = 1, \ldots, \kappa. \tag{3.3.13}$$

This equation allows us to compute $B_{j,\tau}$, given that we have already obtained the functions $B_{0,\tau}, \ldots, B_{j-1,\tau}$. Indeed, if we put

$$F_{j,\tau} = \sum_{k=1}^{j} \mathbf{M}_k[B_{j-k,\tau}],$$

which involves only the functions $B_{0,\tau}, \ldots, B_{j-1,\tau}$, we may write (3.3.13) in the form

$$B_{j,\tau}\big|_{\mathbb{T}} \in H^2_- \cap (4\Delta R_\tau)^{\frac{1}{2}}(-F_{j,\tau} + H^2) = H^2_- \cap \mathrm{e}^{2 \operatorname{Re} H_{R_\tau}}(-F_{j,\tau} + H^2),$$

which by Proposition 2.5.1 has the solution

$$B_{j,\tau} = c_{j,\tau}\mathrm{e}^{H_{R_\tau}} - \mathrm{e}^{H_{R_\tau}}\mathbf{P}_{H^2_{0,-}}[\mathrm{e}^{\bar{H}_{R_\tau}}F_{j,\tau}], \tag{3.3.14}$$

for some constant $c_{j,\tau}$. Since $B_{0,\tau}$ is known up to a constant multiple, this allows us to iteratively derive $B_{j,\tau}$ for $j = 1, \ldots, \kappa$. The only remaining freedom is the choice of the constants $c_{j,\tau}$ for $j = 0, \ldots, \kappa$. We proceed to determine them. Since the orthogonal polynomials $P_{n,m}$ are normalized, it follows from Theorem 1.3.3 together with the triangle inequality that

$$\left\|\chi_{0,\tau}F^{\langle\kappa\rangle}_{n,m}\right\|_{2mQ} = 1 + \mathrm{O}(m^{-\kappa-1})$$

as $m \to \infty$. Since $\chi_{0,\tau}F^{\langle\kappa\rangle}_{n,m} = m^{\frac{1}{4}}\mathbf{\Lambda}_{n,m}[\chi_{1,\tau}f^{\langle\kappa\rangle}_{n,m}]$, it follows from the isometric property described in Proposition 3.1.5 that

$$m^{\frac{1}{2}}\int_{\mathbb{C}}\chi^2_{1,\tau}|f^{\langle\kappa\rangle}_{n,m}|^2\mathrm{e}^{-2mR_\tau}\mathrm{dA} = \int_{\mathbb{C}}\chi^2_{0,\tau}|F^{\langle\kappa\rangle}_{n,m}|^2\mathrm{e}^{-2mQ}\mathrm{dA} = 1 + \mathrm{O}(m^{-\kappa-1}). \tag{3.3.15}$$

Here, the integrals are over the whole plane, although the isometry is only over the the complements of certain compact subsets. However, since we interpret the products with the cut-off functions as vanishing where the cut-off function vanishes itself, this is of no concern to us. We now expand $f^{\langle\kappa\rangle}_{n,m}$ according to (3.3.1), so that by equation (3.3.15),

$$\begin{aligned} 2m^{\frac{1}{2}}\sum_{j,k=0}^{\kappa} m^{-(j+k)}\int_{\mathbb{T}}\int_{\rho_0}^{\infty}\chi^2_{1,\tau}(r)B_{j,\tau}(r\mathrm{e}^{\mathrm{i}\theta})\bar{B}_{k,\tau}(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-2mR_\tau(r\mathrm{e}^{\mathrm{i}\theta})}r\mathrm{d}r\mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}) \\ = 1 + \mathrm{O}(m^{-\kappa-1}), \end{aligned} \tag{3.3.16}$$

where the factor 2 appears as a result of our normalizations. This equation is what will give us the values of the constants $c_{j,\tau}$. We turn first to the case $j = 0$. By a trivial version of Proposition 2.7.2, for any integers j, k with $0 \le j, k \le \kappa$ we have the rough estimate

$$\int_{\rho_0}^{\infty}\chi^2_{1,\tau}(r)B_{j,\tau}(r\mathrm{e}^{\mathrm{i}\theta})\bar{B}_{k,\tau}(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-2mR_\tau(r\mathrm{e}^{\mathrm{i}\theta})}r\mathrm{d}r\mathrm{ds}(r\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{O}(m^{-\frac{1}{2}}),$$

where the implicit constant is uniform for $\tau \in I_{\epsilon_0}$. If we disregard all the contributions in (3.3.16) which are of order $\mathrm{O}(m^{-\frac{1}{2}})$, we see that only $j = k = 0$ gives a nontrivial contribution. The term corresponding to $j = k = 0$ in (3.3.16) can be expanded using the Laplace method of Proposition 2.7.2 (recall the formula (3.3.12) for $B_{0,\tau}$), to give

$$\begin{aligned} 2m^{\frac{1}{2}}&\int_{\mathbb{T}}\int_{\rho_0}^{\infty}\chi^2_{1,\tau}(r)|B_{0,\tau}(r\mathrm{e}^{\mathrm{i}\theta})|^2\mathrm{e}^{-2mR_\tau(r\mathrm{e}^{\mathrm{i}\theta})}r\mathrm{d}r\mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}) \\ &= 2m^{\frac{1}{2}}|c_{0,\tau}|^2\int_{\mathbb{T}}\Big(\frac{\pi}{4m\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta})}\Big)^{\frac{1}{2}}\mathbf{L}_0[r\mathrm{e}^{2\,\mathrm{Re}\,H_{R_\tau}(r\mathrm{e}^{\mathrm{i}\theta})}]\Big|_{r=1}\mathrm{ds} + \mathrm{O}(m^{-\frac{1}{2}}). \end{aligned}$$

Since in general, for a smooth function f we have that $\mathbf{L}_0[f(r)]\big|_{r=1} = f(1)$, the leading contribution simplifies to (recall the definition (3.3.11) of H_{R_τ}),

$$\begin{aligned} 2m^{\frac{1}{2}}|c_{0,\tau}|^2&\int_{\mathbb{T}}\Big(\frac{\pi}{4m\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta})}\Big)^{\frac{1}{2}}\mathbf{L}_0[r\mathrm{e}^{2\,\mathrm{Re}\,H_{R_\tau}(r\mathrm{e}^{\mathrm{i}\theta})}]\Big|_{r=1}\mathrm{ds} \\ &= 2\pi^{\frac{1}{2}}|c_{0,\tau}|^2\int_{\mathbb{T}}\big(4\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta})\big)^{-\frac{1}{2}}\mathrm{e}^{2\,\mathrm{Re}\,H_{R_\tau}(\mathrm{e}^{\mathrm{i}\theta})}\mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}) \\ &= 2\pi^{\frac{1}{2}}|c_{0,\tau}|^2\int_{\mathbb{T}}\mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}) = 2\pi^{\frac{1}{2}}|c_{0,\tau}|^2. \end{aligned}$$

As this is the leading contribution to (3.3.16), we must have $2\pi^{\frac{1}{2}}|c_{0,\tau}|^2 = 1$. This determines the constant $c_{0,\tau}$ up to a unimodular factor, and we choose $c_{0,\tau} = (4\pi)^{-\frac{1}{4}}$.

We turn to the remaining coefficients $c_{j,\tau}$, for $j = 1, \ldots, \kappa$. By applying the Laplace method of Proposition 2.7.1 to the radial integral in the formula (3.3.16), we arrive at

$$2\pi^{\frac{1}{2}} \sum_{j=0}^{\kappa} m^{-j} \sum_{(i,k,l)\in\mathcal{J}_j^\star} \int_{\mathbb{T}} (4\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta}))^{-\frac{1}{2}} \mathbf{L}_k[rB_{i,\tau}(r\mathrm{e}^{\mathrm{i}\theta})\bar{B}_{l,\tau}(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1} \mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}) = 1 + \mathrm{O}(m^{-\kappa-\frac{1}{2}}),$$

where $\mathcal{J}_j^\star$ denotes the index set $\mathcal{J}_j^\star = \big\{(i,k,l) \in \mathbb{N}^3 \; : \; i+k+l = j\big\}$, and $\mathbb{N}$ stands for the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. As this represents an equality of asymptotic series, we may identify term by term. The term with $j = 0$ was already analyzed, and it follows that for $j = 1, \ldots, \kappa$ we have

$$\begin{aligned}
(3.3.17) \qquad & \sum_{(i,k,l)\in\mathcal{J}_j^\star} \int_{\mathbb{T}} (4\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta}))^{-\frac{1}{2}} \mathbf{L}_k[rB_{i,\tau}(r\mathrm{e}^{\mathrm{i}\theta})\bar{B}_{l,\tau}(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1} \mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}) \\
&= 2\,\mathrm{Re} \int_{\mathbb{T}} (4\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta}))^{-\frac{1}{2}} \mathbf{L}_0[rB_{j,\tau}(r\mathrm{e}^{\mathrm{i}\theta})\bar{B}_{0,\tau}(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1} \mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}) \\
&\quad + \sum_{(i,k,l)\in\mathcal{J}_j} \int_{\mathbb{T}} (4\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta}))^{-\frac{1}{2}} \mathbf{L}_k[rB_{i,\tau}(r\mathrm{e}^{\mathrm{i}\theta})\bar{B}_{l,\tau}(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1} \mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}) = 0,
\end{aligned}$$

where $\mathcal{J}_j$ denotes the restricted index set $\mathcal{J}_j = \mathcal{J}_j^\star \cap \big\{(i,k,l) \in \mathcal{J}_j \; : \; i,l < j\big\}$, and where we separate out the terms involving the leading term $B_{j,\tau}$. We successfully resolve the first term on the right-hand side of (3.3.17), while the second term is much more complicated. However, we may observe that it only depends on the functions $B_{\nu,\tau}$ with $\nu = 0, \ldots, j-1$, and hence only on the constants $c_{\nu,\tau}$ with $\nu = 0, \ldots, j-1$. This allows us to algorithmically determine these constants, albeit with increasing degree of complexity. As for the first term on the right-hand side, we observe that the operator $\mathbf{L}_0\big|_{r=1}$ only evaluates at $r = 1$. Using the structure of $B_{j,\tau}$ as given by (3.3.14), we find that

$$\begin{aligned}
&\int_{\mathbb{T}} (4\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta}))^{-\frac{1}{2}} \mathbf{L}_0[rB_{j,\tau}(r\mathrm{e}^{\mathrm{i}\theta})\bar{B}_{0,\tau}(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1} \mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}) \\
&\qquad = \int_{\mathbb{T}} \big(4\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta})\big)^{-\frac{1}{2}} B_{j,\tau}(\mathrm{e}^{\mathrm{i}\theta})\bar{B}_{0,\tau}(\mathrm{e}^{\mathrm{i}\theta}) \mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}) \\
&\qquad = c_{0,\tau} \int_{\mathbb{T}} \big(4\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta})\big)^{-\frac{1}{2}} \mathrm{e}^{2\,\mathrm{Re}\, H_{R_\tau}(r\mathrm{e}^{\mathrm{i}\theta})} \big(c_{j,\tau} - \mathbf{P}_{H^2_{-,0}}[\mathrm{e}^{\bar{H}_{R_\tau}} F_{j,\tau}](\mathrm{e}^{\mathrm{i}\theta})\big) \mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}) \\
&\qquad = c_{0,\tau} \int_{\mathbb{T}} \big(c_{j,\tau} - \mathbf{P}_{H^2_{-,0}}[\mathrm{e}^{\bar{H}_{R_\tau}} F_{j,\tau}](\mathrm{e}^{\mathrm{i}\theta})\big) \mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}) = c_{0,\tau} c_{j,\tau}.
\end{aligned}$$

Here we use the definition (3.3.11) of H_{R_τ} and the fact that the projection $\mathbf{P}_{H^2_{-,0}}$ maps into a subspace of functions with mean 0. Assume now that j is given, and that we have determined $c_{k,\tau}$ for $k = 0, \ldots, j-1$. The above equality together with (3.3.17) then gives that

$$2\,\mathrm{Re}\, c_{j,\tau} c_{0,\tau} = - \sum_{(i,k,l)\in\mathcal{J}_j} \int_{\mathbb{T}} \big(4\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta})\big)^{-\frac{1}{2}} \mathbf{L}_k[rB_{i,\tau}(r\mathrm{e}^{\mathrm{i}\theta})\bar{B}_{l,\tau}(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1} \mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}).$$

Since $c_{0,\tau} = (4\pi)^{-\frac{1}{4}}$, we obtain that

$$\mathrm{Re}\, c_{j,\tau} = -\frac{1}{2}(4\pi)^{\frac{1}{4}} \sum_{(i,k,l)\in\mathcal{J}_j} \int_{\mathbb{T}} \big(4\Delta R_\tau(\mathrm{e}^{\mathrm{i}\theta})\big)^{-\frac{1}{2}} \mathbf{L}_k[rB_{i,\tau}(r\mathrm{e}^{\mathrm{i}\theta})\bar{B}_{l,\tau}(r\mathrm{e}^{\mathrm{i}\theta})]\Big|_{r=1} \mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}).$$

This completes the proof. □

4. The existence of asymptotic expansions

4.1. **Approximately orthogonal quasipolynomials and orthogonal foliations.** We recall from Section 3.1 the canonical positioning operator $\mathbf{\Lambda}_{n,m}$

$$\mathbf{\Lambda}_{n,m}[f](z) = \phi_\tau'(z)[\phi_\tau(z)]^n \mathrm{e}^{m\mathcal{Q}_\tau(z)}(f \circ \phi_\tau)(z), \qquad z \in \mathcal{K}_\tau^c,$$

defined for f holomorphic in $\mathbb{D}_{\mathrm{e}}(0,\rho_0)$. We intend to show the existence of a sequence of *approximately orthogonal quasipolynomials* with an asymptotic expansion. Specifically, in Section 4.8, we will prove the following result.

Proposition 4.1.1. *Let $\kappa\in\mathbb{N}$ be given. There exists a sequence of normalized approximately orthogonal quasipolynomials $F^{\langle\kappa\rangle}_{n,m}$ to accuracy κ in the sense of Definition 3.1.2, of the form*

$$F^{\langle\kappa\rangle}_{n,m}(z)=m^{\frac14}\phi_\tau'(z)[\phi_\tau(z)]^n\mathrm{e}^{m\mathcal{Q}_\tau(z)}(f^{\langle\kappa\rangle}_{n,m}\circ\phi_\tau)(z)$$

where

$$f^{\langle\kappa\rangle}_{n,m}(z)=\sum_{j=0}^{\kappa}m^{-j}B_{j,\tau}(z),$$

and $B_{j,\tau}$ are holomorphic functions in $\mathbb{D}_{\mathrm{e}}(0,\rho_0)$, which are uniformly bounded for $\tau\in I_{\epsilon_0}$.

We will obtain Proposition 4.1.1 as a consequence of the existence of what we call the *approximate orthogonal foliation flow of simple loops* $\Gamma_{n,m,t}$, parameterized by the parameter t. For convenience of notation, let δ_m be the number

$$\delta_m=m^{-1/2}\log m.$$

A conformal mapping ψ of the exterior disk $\mathbb{D}_{\mathrm{e}}$ onto a domain containing the point at infinity is said to be *normalized* if it maps ∞ to ∞, and has $\psi'(\infty)>0$. Given a smooth family ψ_t of normalized conformal mappings on the exterior disk, indexed by a real parameter t close to 0, such that the image domains $\Omega_t:=\psi_t(\mathbb{D}_{\mathrm{e}})$ increase increase with t, we may form the *foliation mapping* Ψ by the formula

$$\Psi(z)=\psi_{1-|z|}\Big(\frac{z}{|z|}\Big),$$

for z in some annulus $\mathbb{A}$ containing the unit circle. The foliation mapping Ψ maps $\mathbb{A}$ onto the domain $\mathcal{D}$ covered by the boundaries

$$\mathcal{D}=\bigcup_t\psi_t(\mathbb{T}).$$

Moreover, the Jacobian J_Ψ is given by

$$J_\Psi(z)=-\operatorname{Re}\Big\{\frac{\bar z}{|z|^2}\partial_t\psi_t\Big(\frac{z}{|z|}\Big)\overline{\psi_t'\Big(\frac{z}{|z|}\Big)}\Big\}\Big|_{t=1-|z|}. \tag{4.1.1}$$

Since the loops $\Gamma_{n,m,t}$ of the orthogonal foliation flow are Jordan curves, they each divide the plane into two components. We denote the normalized conformal mapping of the exterior disk $\mathbb{D}_{\mathrm{e}}$ onto the unbounded component of $\mathbb{C}\setminus\Gamma_{n,m,t}$ by $\psi_{n,m,t}$. Denote the corresponding foliation mapping by $\Psi_{n,m}$. We may integrate over a flow, encoded by a foliation mapping Ψ as follows: If we denote by $\mathbb{A}_\epsilon$ the annulus $\mathbb{A}_\epsilon=\mathbb{D}(0,1+\epsilon)\setminus\bar{\mathbb{D}}(0,1-\epsilon)$, we have for integrable f,

$$\begin{aligned}\int_{\Psi(\mathbb{A}_\epsilon)}f\mathrm{dA}&=\int_{\mathbb{A}_\epsilon}f\circ\Psi\ J_\Psi\mathrm{dA}\\&=2\int_{-\epsilon}^{\epsilon}\int_{\mathbb{T}}f\circ\psi_t(\zeta)(1-t)J_\Psi\big((1-t)\zeta\big)\mathrm{ds}(\zeta)\mathrm{d}t.\end{aligned}\tag{4.1.2}$$

The existence of the foliation flow may be phrased as follows.

Lemma 4.1.2. *Fix the precision parameter κ to be a positive integer. For $\tau=\frac{n}{m}\in I_{\epsilon_0}$, there exist compact subsets $\mathcal{K}_\tau\subset\mathcal{S}_\tau$ with*

$$\inf_{\tau\in I_{\epsilon_0}}\operatorname{dist}_{\mathbb{C}}(\mathcal{K}_\tau,\partial\mathcal{S}_\tau)>0,$$

and bounded holomorphic functions $B_{j,\tau}$ on $\mathcal{K}_\tau^c$ for $j=1\ldots,\kappa$ and a smooth family of normalized conformal mappings $\{\psi_{n,m,t}\}_{n,m,t}$ on $\mathbb{D}_{\mathrm{e}}$, such that if $f^{\langle\kappa\rangle}_{n,m}$ is given as a κ-abschnitt of an *asymptotic expansion* (3.3.1)*, then for* $\zeta \in \mathbb{T}$*,*

$$m^{\frac12}\left|f_{n,m}\circ\psi_{n,m,t}(\zeta)\right|^2 \mathrm{e}^{-2m(R_\tau\circ\psi_{n,m,t})(\zeta)}(1-t)J_{\Psi_{n,m}}((1-t)\zeta) = \frac{m^{\frac12}}{(4\pi)^{\frac12}}\mathrm{e}^{-mt^2} + \mathrm{O}\left(m^{-\kappa}\mathrm{e}^{-2mR\circ\psi_{n,m,t}}\right), \tag{4.1.3}$$

where the implicit constant is uniform for $|t| \le \delta_m$ *while* $\tau \in I_{\epsilon_0}$*. Moreover, if* $\mathcal{D}_{n,m} = \bigcup_{|t|\le\delta_m}\psi_{n,m,t}(\mathbb{T})$*, then* $\mathrm{dist}_{\mathbb{C}}(\mathcal{D}^c_{n,m}, \partial\mathcal{S}_\tau) \ge c_0\delta_m$ *for some constant* $c_0 > 0$ *so it holds that*

$$\int_{\mathcal{D}_{n,m}} m^{\frac12}|f^{\langle\kappa\rangle}_{n,m}|^2\mathrm{e}^{-2mR_\tau}\mathrm{d}A = 1 + \mathrm{O}(\delta_m^{2\kappa+1}) = 1 + \mathrm{O}(m^{-\kappa-\frac13}).$$

Remark 4.1.3. The equation (4.1.3) may be understood as an approximate *weighted Polubarinova-Galin equation* with weight $|f^{\langle\kappa\rangle}_{n,m}|^2\mathrm{e}^{-2mR_\tau}$, and variable speed of expansion. Indeed, we should compare with equation (6.11) in [25], which states in a similar context that along concentric circles,

$$J_\Psi = \omega^{-1}\circ\Psi,$$

where Ψ is a foliation mapping, and ω denotes a weight. In comparison, our factor $(4\pi)^{-\frac12}\mathrm{e}^{-mt^2}$ appears as consequence of the variable speed.

4.2. **The orthogonal foliation flow I. Renormalization.** In this section we present the outlines of the construction of the approximate orthogonal foliation flow, and the first step in the algorithm that produces it. In order to proceed with less obscuring notation, we consider a smooth family of bounded holomorphic functions $f_s(z)$, a smooth family of normalized conformal mappings $\psi_{s,t}$. Moreover, we denote by R a weight whose properties mirror those of R_τ, captured in the following definition. We denote by $\mathbb{A}(\rho_1,\rho_2)$ the annulus

$$\mathbb{A}(\rho_1,\rho_2) := \mathbb{D}(0,\rho_2)\setminus\bar{\mathbb{D}}(0,\rho_1),$$

for positive real numbers ρ_1 and ρ_2 with $\rho_1 < \rho_2$.

For a real-analytic function R there exists a *polarization* $R(z,w)$, which is holomorphic in $(z,\bar w)$ and has $R(z,z) = R(z)$. This is easy to see using convergent local Taylor series expansions of $R(z)$ in the coordinates which are the real and imaginary parts, $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$. By replacing $\mathrm{Re}\, z$ by $\frac12(z+w)$ and $\mathrm{Im}\, z$ by $\frac{1}{2\mathrm{i}}(z-w)$ in this expansion, we obtain the polarization $R(z,w)$. We observe that if $R(z,w)$ is such a polarization of a function $R(z)$ which is real-analytically smooth near the circle $\mathbb{T}$ and in addition is quadratically flat there, then $R(z,w)$ factors as $R(z,w) = (1-z\bar w)^2R_0(z,w)$, where $R_0(z,w)$ is holomorphic in $(z,\bar w)$ in a neigborhood of the diagonal where both variables are near $\mathbb{T}$.

Definition 4.2.1. For positive real numbers ρ_0,σ_0 where $\rho_0 < 1$, we denote by $\mathcal{W}(\rho_0,\sigma_0)$ the class of C^2-smooth functions $R : \mathbb{D}_{\mathrm{e}}(0,\rho_0) \to \mathbb{R}_{+,0} := \mathbb{R}_+\cup\{0\}$, such that R and ∇R both vanish on $\mathbb{T}$, while $\varDelta R > 0$ holds on $\mathbb{T}$, and in addition, R is real-analytic in the annulus $\mathbb{A}(\rho,\rho_0^{-1})$ with polarization $R(z,w)$ which is holomorphic in $(z,\bar w)$ on the $2\sigma_0$-fattened diagonal annulus

$$\hat{\mathbb{A}}(\rho_0,\sigma_0) := \{(z,w)\in\mathbb{A}(\rho_0,\rho_0^{-1})\times\mathbb{A}(\rho_0,\rho_0^{-1})\ :\ |z-w|\le 2\sigma_0\},$$

and factors as $R(z,w) = (1-z\bar w)^2R_0(z,w)$, where $R_0(z,w)$ is holomorphic in $(z,\bar w)$ and bounded and bounded away from 0 on the set $\hat{\mathbb{A}}(\rho,\sigma)$, while, in addition,

$$R_0(z,z) \ge \alpha(R) > 0, \qquad z\in\mathbb{A}(\rho_0,\rho_0^{-1}),$$

and further away,

$$\inf_{z\in\mathbb{D}_{\mathrm{e}}(0,\rho_0^{-1})}\frac{R(z)}{\log|z|} = \theta(R) > 0.$$

We say that a subset $\mathfrak{S}\subset\mathcal{W}(\rho_0,\sigma_0)$ is a *uniform family*, provided that for each $R\in\mathfrak{S}$, the corresponding $R_0(z,w)$ is uniformly bounded on $\hat{\mathbb{A}}(\rho_0,\sigma_0)$ while the controlling constants such as $\alpha(R)$ and $\theta(R)$ are uniformly bounded away from 0.

If a function $f(z,w)$ is holomorphic in $(z,\bar w)$, we may consider the associated function

$$f_{\mathbb{T}}(z)=f\Big(z,\frac{1}{\bar z}\Big) \tag{4.2.1}$$

which is then holomorphic in z, wherever it is well-defined. We note that $f_{\mathbb{T}}(z)=f(z,z)$ on the circle $\mathbb{T}$. We recall the notation of Definition 4.2.1.

Proposition 4.2.2. *Suppose that $f(z,w)$ is holomorphic in $(z,\bar w)$ on the domain $\hat{\mathbb{A}}(\rho,\sigma)$, where $0<\rho<1$ and $\sigma>0$. Then the function $f_{\mathbb{T}}(z)$, which extends the restriction of the diagonal function $f(z,z)$, where $z\in\mathbb{T}$, is holomorphic on the annulus*

$$\rho'<|z|<(\rho')^{-1},$$

where

$$\rho'=\max\Big\{\rho,\sqrt{1+\sigma^2}-\sigma\Big\}.$$

Proof. The function $f_{\mathbb{T}}(z)=f(z,\bar z^{-1})$ is automatically holomorphic in the variable z in the domain

$$\Big|z-\frac{1}{\bar z}\Big|\le 2\sigma,$$

provided that $z\in\mathbb{A}(\rho,\rho^{-1})$. By checking these requirements carefully, the assertion follows. □

The setting which will prove useful to us is when we may control certain related quantities and their polarizations, which is possible on smaller thickened annuli. The polarization of $\log\varDelta R$ appears later in the induction algorithm, while $\log(z\partial\hat R)$ is important for the control associated with the implicit function theorem.

Proposition 4.2.3. *If R belongs to a uniform family $\mathfrak{S}\subset\mathcal{W}(\rho_0,\sigma_0)$ for some positive reals ρ_0,σ_0 with $\rho_0<1$, and if $\hat R=\sqrt{R}$ is chosen so that $\hat R(z)$ is positive for $|z|>1$ and negative for $|z|<1$, then there exist positive ρ_1,σ_1 with $\rho_0\le\rho_1<1$ and $\sigma_1\le\sigma_0$, such that the polarizations of the functions $\log\varDelta R$, $\hat R$, $\log(z\partial\hat R)$ are all holomorphic in $(z,\bar w)$ and uniformly bounded on the $2\sigma_1$-fattened diagonal annulus $\hat{\mathbb{A}}(\rho_1,\sigma_1)$.*

Proof. This follows from the assumptions on the uniform family, if we use the standard Cauchy estimates plus the fact that $\log\varDelta R=\log(2R_0)$ and $\log(z\partial\hat R)=\frac12\log R_0$ hold on the unit circle $\mathbb{T}$. □

Remark 4.2.4. Suppose a real-analytic function $F(z)$ admits a polarization $F(z,w)$ which is holomorphic in $(z,\bar w)$ for $(z,w)\in\hat{\mathbb{A}}(\rho,\sigma)$, and let f be given in terms of the Herglotz kernel by $f=\mathbf{H}_{\mathbb{D}_{\mathrm{e}}}[F|_{\mathbb{T}}]$. We note that by the properties of the Herglotz kernel, f may be obtained by the formula $f=2P_{H^2_{-,0}}[F|_{\mathbb{T}}]+\langle F\rangle_{\mathbb{T}}$, where $\langle F\rangle_{\mathbb{T}}$ denotes the average of F on the unit circle. Let $F_{\mathbb{T}}$ be as in (4.2.1), and express it in terms of its Laurent series, which by Proposition 4.2.2 converges in the annulus $\mathbb{A}(\rho',(\rho')^{-1})$:

$$F_{\mathbb{T}}(z)=\sum_{n\in\mathbb{Z}}a_nz^n.$$

In terms of the Laurent series, $P_{H^2_{-,0}}[F|_{\mathbb{T}}]$ equals $\sum_{n<0}a_nz^n$ and $\langle F\rangle_{\mathbb{T}}=a_0$. As a consequence, $\mathbf{P}_{H^2_{-,0}}[F|_{\mathbb{T}}]$ defines a holomorphic function on the exterior disk $\mathbb{D}_{\mathrm{e}}(0,\rho')$ and hence, f is holomorphic on $\mathbb{D}_{\mathrm{e}}(0,\rho')$ as well.

For an integer n, we denote by $\mathcal{I}_n$ the triangular index set

$$\mathcal{I}_n=\{(j,l)\in\mathbb{N}^2:\ 2j+l\le n\},$$

and supply it with the inherited lexicographic ordering $\prec$:

$$(i,k)\prec(j,l)\ \text{ if }\ i<j\ \text{ or }\ i=j\ \text{ and }\ k<l.$$

Recall the number ρ' from Proposition 4.2.2, which is defined in terms of the pair ρ_1,σ_1 supplied in Proposition 4.2.3.

The following is an analogue of Lemma 4.1.2.

Proposition 4.2.5. *Let κ be a given positive integer and let $R \in \mathcal{W}(\rho_0, \sigma_0)$, for some ρ_0, σ_0 with $0 < \rho_0 < 1$ and $\sigma_0 > 0$. Then there exist a radius ρ'' with $\rho' < \rho'' < 1$, bounded holomorphic functions $f_s = \sum_{j=0}^{\kappa} s^j B_j$ on $\mathbb{D}_{\mathrm{e}}(0, \rho')$ and normalized conformal mappings*

$$\psi_{s,t} = \psi_{0,t} + \sum_{\substack{(j,l)\in\mathcal{I}_{2\kappa+1} \\ j\geq 1}} s^j t^l \hat{\psi}_{j,l}$$

defined on $\mathbb{D}_{\mathrm{e}}(0, \rho'')$, such that $\psi_{s,t}(\mathbb{D}_{\mathrm{e}}(0, \rho'')) \subset \mathbb{D}_{\mathrm{e}}(0, \rho')$ holds for s and t close to 0, while the domains $\psi_{s,t}(\mathbb{D}_{\mathrm{e}})$ increase with t. Moreover, on the circle $\mathbb{T}$ the functions f_s and $\psi_{s,t}$ have the property that

$$\begin{aligned}(4.2.2)\quad |f_s \circ \psi_{s,t}(\zeta)|^2 \mathrm{e}^{-2s^{-1} R\circ\psi_{s,t}(\zeta)} \operatorname{Re}\big(- \bar{\zeta}\partial_t \psi_{s,t}(\zeta)\overline{\psi'_{s,t}(\zeta)}\big) \\ = \mathrm{e}^{-t^2/s}\Big\{(4\pi)^{-\frac{1}{2}} + \mathrm{O}\big(|s|^{\kappa+\frac{1}{2}} + |t|^{2\kappa+1}\big)\Big\}.\end{aligned}$$

Here, the implicit constant remains uniformly bounded as long as R is confined to a uniform family in $\mathcal{W}(\rho_0, \sigma_0)$, for fixed ρ_0, σ_0.

The first step towards finding the conformal mappings $\psi_{s,t}$ is to note the following: if we set $h_s = \log|f_s|^2$, we find by taking logarithms that

$$(4.2.3)\qquad h_s \circ \psi_{s,t}(\zeta) - 2s^{-1}(R \circ \psi_{s,t})(\zeta) + \log \operatorname{Re}\Big(- \bar{\zeta}\partial_t \psi_{s,t}\overline{\psi'_{s,t}(\zeta)}\Big) = -s^{-1}t^2 + \mathrm{O}(1),$$

as $s, t \to 0$. Next, we multiply both sides by s, to obtain

$$(4.2.4)\qquad s h_s \circ \psi_{s,t}(\zeta) - 2\, R \circ \psi_{s,t}(\zeta) + s \log \operatorname{Re}\Big(- \bar{\zeta}\partial_t \psi_{s,t}\overline{\psi'_{s,t}(\zeta)}\Big) = -t^2 + \mathrm{O}(s).$$

Finally, we take the limit as $s \to 0$ expecting that $h_s \circ \psi_{s,t}$ and $\log \operatorname{Re}(-\bar{\zeta}\partial_t \psi_{s,t}\bar{\psi}'_{s,t})$ remain bounded, and arrive at the equation

$$R \circ \psi_{0,t}(\zeta) = \frac{t^2}{2}.$$

As a consequence, $\psi_{0,t}$ should be a conformal mapping onto the exterior of the appropriate level curve of the weight R.

Proposition 4.2.6. *There exists a positive number t_0, and a real-analytically smooth family $\{\psi_{0,t}\}_{t\in(-t_0,t_0)}$ of normalized conformal mappings $\mathbb{D}_{\mathrm{e}} \to \Omega_t$, where Ω_t is the unbounded component of $\mathbb{C} \setminus \Gamma_t$, and where Γ_t are real-analytically smooth, simple closed level curves of R:*

$$R|_{\Gamma_t} = \frac{t^2}{2}.$$

Moreover, $\Omega_0 = \mathbb{D}_{\mathrm{e}}$ and Ω_t increases with t.

Proof. The assumed strict subharmonicity of R gives that there exists a neighbourhood U of $\mathbb{T}$ such that $\nabla R\big|_{U\setminus\mathbb{T}} \neq 0$. This shows that the level sets must be simple and closed curves, for $|t|$ sufficiently small. Indeed, if a curve would possess a loop, then R would have to have a local extremal point inside the loop, which is impossible. Since ∇R vanishes on $\mathbb{T}$, we cannot apply the implicit function theorem directly to R to obtain the result. However, the function

$$\tilde{R}(r\mathrm{e}^{\mathrm{i}\theta}) := \frac{R(r\mathrm{e}^{\mathrm{i}\theta})}{(r-1)^2}$$

is, in view of Proposition 3.1.3, strictly positive and real-analytic in a neighbourhood of the unit crcle $\mathbb{T}$. We form the square root $\hat{R} = \sqrt{R}$ by

$$\hat{R}(r\mathrm{e}^{\mathrm{i}\theta}) = (r-1)\sqrt{\tilde{R}(r\mathrm{e}^{\mathrm{i}\theta})},$$

where the square root on the right-hand side is the standard square root of a positive number. We may now apply the implicit function theorem to the function $\hat{R}$. The result follows immediately by applying the Riemann mapping theorem to the exterior of the resulting analytic level curves of $\hat{R}$. □

Proposition 4.2.6 tells us that the conformal mappings $\psi_{0,t}$ extend to some domain containing $\bar{\mathbb{D}}_{\mathrm{e}}$, but supplies little information on how much bigger such a domain is allowed to be. We will discuss this issue in Subsection 4.3 below. Along the way, we also obtain an alternative proof of Proposition 4.2.6, which may be viewed as a quantitative version of the implicit function theorem in the given context.

The Taylor coefficients $\hat{\psi}_{0,l}$ (in the flow variable t) of the conformal mappings $\psi_{0,t}$ can be explicitly computed in terms of R, using a higher order version of Nehari's formula for conformal mappings to nearly circular domains. We will return to this in Section 4.6. Before we continue, we recall the following elementary lemma, which allows us to draw the conclusion that the mappings $\psi_{s,t}$ are actually conformal.

Lemma 4.2.7. *Let $\mathcal{D}$ denote a simply connected domain in $\widehat{\mathbb{C}}$ containing the point at infinity, whose boundary $\partial\mathcal{D}$ is a simple closed curve. Denote by f a holomorphic function $f : \mathbb{D}_{\mathrm{e}} \to \mathcal{D}$ such that $f'(z) \neq 0$ in $\mathbb{D}_{\mathrm{e}}$ and $f(z) = cz + \mathrm{O}(1)$ as $|z| \to \infty$, which maps $\mathbb{T}$ bijectively onto $\partial\mathcal{D}$. Then f is a conformal mapping.*

This result is well-known.

From this it immediately follows that the mappings $\psi_{s,t}$ are conformal when s is small enough. That the derivatives are non-vanishing is immediate, and a small computation allows one to show that for small enough $|s|$ and $|t|$, $\psi_{s,t}$ is injective on $\mathbb{T}$. Let us expand on the latter claim: we write

$$\psi_{s,t}(\zeta) = \psi_{0,t}(\zeta) + s\psi_{s,t}(\zeta),$$

where $\psi_{s,t}$ is some holomorphic function, which is uniformly bounded in the relevant parameter range. But then, if $\psi_{s,t}$ maps two distinct points $\zeta_1, \zeta_2 \in \mathbb{T}$ to the same point, it follows that

$$s^{-1} = \frac{\psi_{s,t}(\zeta_1) - \psi_{s,t}(\zeta_2)}{\psi_{0,t}(\zeta_1) - \psi_{0,t}(\zeta_2)}. \tag{4.2.5}$$

Since the functions $\psi_{0,t}$ and $\psi_{s,t}$ are smooth up to the boundary, and since $\psi_{0,t}' \neq 0$, the quotients

$$D_{s,t}(\zeta, \eta) := \frac{\psi_{s,t}(\zeta) - \psi_{s,t}(\eta)}{\psi_{0,t}(\zeta) - \psi_{0,t}(\eta)}$$

are uniformly bounded. It follows that if

$$s^{-1} > D := \sup_{s,t} D_{s,t}(\zeta, \eta),$$

where the supremum is taken over all s and t sufficiently close to zero, we arrive at a contradiction in (4.2.5), so $\psi_{s,t}$ must be univalent, and thus conformal.

4.3. **The smoothness of level curves, the implicit function theorem, and Toeplitz kernel equations.** We consider a function R, which is assumed to belong to the class $\mathcal{W}(\rho_0, \sigma_0)$ of Definition 4.2.1, which is a quantitative way to say that R is real-analytic near the unit circle $\mathbb{T}$, and vanishes along with its normal derivative on $\mathbb{T}$, while ΔR is positive on $\mathbb{T}$. Next, we recall the definition of the square root $\hat{R}$ of R from the proof of Proposition 4.2.6. This function is also real-analytic near the circle, vanishes on $\mathbb{T}$ but its gradient there is non-zero and points in the direction of the outward normal. As such, there exists a quantitative way to say this, which we now state. To this end, in analogy with Definition 4.2.1, we let $\hat{\rho}$ and $\hat{\sigma}$ denote numbers with $0 < \hat{\rho} < 1$ and $\hat{\sigma} > 0$ such that $\hat{R}$ has a polarization $\hat{R}(z, w)$ which is holomorphic in $(z, \bar{w})$ for (z, w) in the $2\hat{\sigma}$-fattened diagonal annulus $\hat{A}(\hat{\rho}, \hat{\sigma})$, such that

$$\inf_{(z,w)\in\hat{A}(\hat{\rho},\hat{\sigma})} |\partial_z \hat{R}(z, w)| > 0.$$

In this subsection, we will focus on the conformal mappings $\psi_{0,t}$, and specifically the region to which they extend holomorphically. We recall that these mappings satisfy

$$\hat{R} \circ \psi_{0,t}(\zeta) = -\frac{t}{\sqrt{2}}, \qquad \zeta \in \mathbb{T}.$$

Upon differentiating in t, we obtain

$$\partial_r \hat{R} \circ \psi_{0,t}\ \partial_t |\psi_{0,t}| + \partial_\theta \hat{R} \circ \psi_{0,t}\ \partial_t \arg \psi_{0,t} = -\frac{1}{\sqrt{2}},$$

which we may rewrite as

$$r\partial_r \hat{R} \circ \psi_{0,t}\ \partial \log|\psi_{0,t}| + \partial_\theta \hat{R} \circ \psi_{0,t}\ \partial_t \arg \psi_{0,t} \\ = \mathrm{Re}\left\{ \left(r\partial_r \hat{R} - \mathrm{i}\partial_\theta \hat{R} \right) \circ \psi_{0,t}\ \partial_t \log \frac{\psi_{0,t}}{\zeta} \right\} = -\frac{1}{\sqrt{2}},$$

where we have divided by the coordinate function ζ in order to avoid issues with branch cuts of the logarithm. The differential operator acting on $\hat{R}$ may be rewritten as $2z\partial_z$, so we may once again simplify

$$\mathrm{Re}\left\{ \left(2z\partial_z \hat{R} \right) \circ \psi_{0,t}\ \partial_t \log \frac{\psi_{0,t}}{\zeta} \right\} = -\frac{1}{\sqrt{2}}.$$

If we introduce the notation $\nu_t = \log\left(2z\partial_z \hat{R}\right) \circ \psi_{0,t}$ and $f_t = \partial_t \log \frac{\psi_{0,t}}{\zeta}$, this may be rewritten

$$\mathrm{e}^{\nu_t} f_t + \mathrm{e}^{\bar{\nu}_t} \bar{f}_t = -\sqrt{2}.$$

Here, the function $\left(2z\partial_z \hat{R}\right) \circ \psi_{0,t}$ evaluated at $t = 0$ is just $\sqrt{2\Delta R}$ on the circle $\mathbb{T}$, so there are no problems with taking the logarithm in the definition of ν_t for small t. Next, we make the decomposition $\nu_t = \nu_t^+ + \nu_t^-$, where $\nu_t^+ \in H^2$ and $\nu_t^- \in H^2_{-,0}$, and write $\tilde{f}_t = \mathrm{e}^{\nu_t^-} f_t$. It is clear that $\tilde{f}_t \in H^2_-$. If we multiply the above equation by $\mathrm{e}^{-2\,\mathrm{Re}\,\nu_t^+}$, we arrive at

$$\mathrm{e}^{-\bar{\nu}_t{}^+} \tilde{f}_t + \mathrm{e}^{-\nu_t^+} \bar{\tilde{f}}_t = 2\,\mathrm{Re}\left\{ \mathrm{e}^{-\bar{\nu}_t^+} \tilde{f}_t \right\} = -\sqrt{2}\,\mathrm{e}^{-2\,\mathrm{Re}\,\nu_t^+},$$

where we point out that $\mathrm{e}^{-\bar{\nu}_t^+} \tilde{f}_t \in H^2_-$. This may be recognized as a Toeplitz kernel condition, and it has the solution

$$\tilde{f}_t = -\frac{1}{\sqrt{2}}\,\mathrm{e}^{\bar{\nu}_t^+} \mathbf{H}_{\mathbb{D}_{\mathrm{e}}}\left[\mathrm{e}^{-2\,\mathrm{Re}\,\nu_t^+}\right],$$

that is,

$$f_t = -\frac{1}{\sqrt{2}}\,\mathrm{e}^{\bar{\nu}_t^+ - \nu_t^-} \mathbf{H}_{\mathbb{D}_{\mathrm{e}}}\left[\mathrm{e}^{-2\,\mathrm{Re}\,\nu_t^+}\right]. \tag{4.3.1}$$

Let us recall that in the equation (4.3.1), the functions f_t and ν_t may be expressed in terms of $\hat{R}$ and $\psi_{0,t}$. Let us write

$$g_t(\zeta) = \log \frac{\psi_{0,t}(\zeta)}{\zeta} \quad \text{and} \quad \mu(z) = \log(2z\partial_z \hat{R}(z)), \tag{4.3.2}$$

where both logarithms may be understood in terms of the principal branch of the logarithm. In terms of these functions, the equation (4.3.1) becomes the following non-linear differential equation in t:

$$\partial_t g_t = -\frac{1}{\sqrt{2}} \exp\left\{ \overline{\mathbf{P}_{H^2}[\mu \circ \psi_{0,t}]} - \mathbf{P}_{H^2_{-,0}}[\mu \circ \psi_{0,t}] \right\} \mathbf{H}_{\mathbb{D}_{\mathrm{e}}}\left[\exp\left\{ -2\,\mathrm{Re}\,\mathbf{P}_{H^2}[\mu \circ \psi_{0,t}] \right\} \right]. \tag{4.3.3}$$

It is not difficult to see that the equation (4.3.3) may be solved by an iterative procedure, if we rewrite it in integral form

$$g_t = -\int_0^t \frac{1}{\sqrt{2}} \exp\left\{ \overline{\mathbf{P}_{H^2}[\mu \circ \psi_{0,\theta}]} - \mathbf{P}_{H^2_{-,0}}[\mu \circ \psi_{0,\theta}] \right\} \mathbf{H}_{\mathbb{D}_{\mathrm{e}}}\left[\exp\left\{ -2\,\mathrm{Re}\,\mathbf{P}_{H^2}[\mu \circ \psi_{0,\theta}] \right\} \right] \mathrm{d}\theta. \tag{4.3.4}$$

As a first order approximation, we start with $\psi_{0,t}^{[0]}(\zeta) = \zeta$, and use the formula (4.3.4) to define $g_t^{[j+1]}$ in terms of $\psi_{0,t}^{[j]}$, for $j = 0, 1, 2, \ldots$ by integration. The process is interlaced with computing $\psi_{0,t}^{[j+1]} := \zeta \exp(g_t^{[j+1]})$, and results in convergent sequences $g_t^{[j]}$ and $\psi_{0,t}^{[j]}$.

Next, we are interested in analyzing where the function $\psi_{0,t}$ extends to as a holomorphic mapping. To this end, we recall that the function μ given by (4.3.2) has a well-defined polarization to $\hat{\mathbb{A}}(\hat\rho,\hat\sigma)$. It is clear that if $\psi_{0,t}$ maps $\mathbb{A}(\rho_t,\rho_t^{-1})$ into $\mathbb{A}(\hat\rho',(\hat\rho')^{-1})$, we obtain the estimate

$$\|\partial_t g_t\|_{H^\infty(\mathbb{A}(\rho_t,\rho_t^{-1}))}\le\frac{\sqrt{2}}{1-\rho_t^2}\exp\Big\{5\frac{\|\mu\|_{H^\infty(\mathbb{A}(\hat\rho',(\hat\rho')^{-1})}}{1-\rho_t^2}\Big\},$$

where we use the estimate

$$\|\mathbf{P}_{H^2}[f]\|_{H^\infty(\mathbb{D}(0,\rho_t^{-1}))}\le\frac{\|f\|_{H^\infty(\mathbb{A}(\rho_t,\rho_t^{-1}))}}{1-\rho_t^2},$$

and the analogous estimate for $\mathbf{P}_{H^2_{-,0}}[f]$. Assume for the moment that $\rho_t<1$ is monotonically increasing in $|t|$, and recall that $\psi_{0,t}(\zeta)=\zeta\exp(g_t)$. In light of the above estimate of $\partial_t g_t$, we obtain

$$\|g_t\|_{H^\infty(\mathbb{A}(\rho_t,\rho_t^{-1}))}\le\frac{\sqrt{2}|t|}{1-\rho_t^2}\exp\Big\{5\frac{\|\mu\|_{H^\infty(\mathbb{A}(\hat\rho',(\hat\rho')^{-1})}}{1-\rho_t^2}\Big\}=:C_t|t|,$$

where C_t is defined implicitly by the last relation. This leads to the control

$$\mathrm{e}^{-C_t|t|}\rho_t\le|\psi_{0,t}(\zeta)|\le\mathrm{e}^{C_t|t|}(\rho_t)^{-1},\qquad\zeta\in\mathbb{A}(\rho_t,\rho_t^{-1}),$$

which means that $\psi_{0,t}$ maps the annulus $\mathbb{A}(\rho_t,\rho_t^{-1})$ into $\mathbb{A}(\hat\rho',(\hat\rho')^{-1})$, provided that

$$\mathrm{e}^{-C_t|t|}\rho_t\ge\hat\rho'.$$

Let us make the ansatz $\rho_t=\hat\rho'\mathrm{e}^{A|t|}$, for some constant A. The above requirement is then satisfied provided that $A\ge C_t$. If we restrict t to have

$$|t|\le\frac{\log\frac{1}{\hat\rho'}}{2A},\tag{4.3.5}$$

it is immediate that

$$\frac{1}{1-\rho_t^2}\le\frac{1}{1-\hat\rho'}.$$

This then gives the estimate for C_t

$$C_t\le\frac{\sqrt{2}}{1-\hat\rho'}\exp\Big\{5\frac{\|\mu\|_{H^\infty(\mathbb{A}(\hat\rho',(\hat\rho')^{-1})}}{1-\hat\rho'}\Big\},$$

where the right-hand side does not depend on t. We may finally choose A to be this constant,

$$A=\frac{\sqrt{2}}{1-\hat\rho'}\exp\Big\{5\frac{\|\mu\|_{H^\infty(\mathbb{A}(\hat\rho',(\hat\rho')^{-1})}}{1-\hat\rho'}\Big\}$$

and obtain that $\psi_{0,t}$ is holomorphic in the exterior disk $\mathbb{D}_{\mathrm{e}}(0,\rho_t)$, where

$$\rho_t=\hat\rho'\mathrm{e}^{A|t|},$$

provided that t satisfies (4.3.5).

4.4. **The orthogonal foliaton flow II. Overview of the algorithm.** We now proceed to describe the outlines of the algorithm. With the notation

$$\omega_{s,t}(\zeta)=|(f_s\circ\psi_{s,t})(\zeta)|^2\mathrm{e}^{-2s^{-1}\{(R\circ\psi_{s,t})(\zeta)-\frac{t^2}{2}\}}\operatorname{Re}\Big(-\bar\zeta\partial_t\psi_{s,t}(\zeta)\overline{\psi'_{s,t}(\zeta)}\Big)\tag{4.4.1}$$

we may rewrite the flow condition (4.2.2) as

$$\partial_s^j\partial_k^l\omega_{s,t}(\zeta)\big|_{s=t=0}=\begin{cases}(4\pi)^{-\frac12}&\text{for}\quad\zeta\in\mathbb{T}\ \text{ and }\ (j,l)=(0,0),\\0&\text{for}\quad\zeta\in\mathbb{T}\ \text{ and }\ (j,l)\in\mathcal{I}_{2\kappa}\setminus\{(0,0)\}.\end{cases}\tag{4.4.2}$$

provided that the functions f_s and $\psi_{s,t}$ obtained by solving these equations do not degenerate, as long as R remains in a bounded set of $\mathcal{W}(\rho_0,\sigma_0)$ for some ρ_0 which is bounded away from 1 and $\sigma_0>0$. It turns out (see Proposition 4.6.4) that for $j,l\ge1$, it holds that

(4.4.3)

$$\frac{1}{(j-1)!l!}\partial_s^{j-1}\partial_t^l\omega_{s,t}(\zeta)\big|_{s=t=0}=4(4\pi)^{-\frac12}\Delta R(\zeta)\ \operatorname{Re}(\bar\zeta\hat\psi_{j,l-1}(\zeta))\operatorname{Re}(\bar\zeta\hat\psi_{0,1}(\zeta))+\mathfrak{F}_{j-1,l}(\zeta),$$

where $\mathfrak{F}_{j-1,l}$ is real-valued and real-analytic, and depends only on $B_0, \dots, B_{j-1}$ and $\hat{\psi}_{p,q}$ where $(p,q) \prec (j, l-1)$. Moreover, when $l = 0$ we have

$$\frac{1}{j!}\partial_s^j \omega_{s,t}(\zeta)\big|_{s=t=0} = 2\,\mathrm{Re}\,\big(\bar{B}_0(\zeta)B_j(\zeta)\big) + \mathfrak{F}_{j,0}(\zeta) \tag{4.4.4}$$

where $\mathfrak{F}_{j,0}$ depends only on $B_0, \dots, B_{j-1}$ and $\hat{\psi}_{p,q}$ for $(p,q) \prec (j+1, 0)$.

Step 1. Let $\psi_{0,t}$ be the normalized conformal mappings to the exterior of level curves of R, as given by Proposition 4.2.6. In particular, this determines uniquely the coefficient functions $\hat{\psi}_{0,l}$, for $l = 0, \dots, 2\kappa + 1$ (see Proposition 4.6.1 below).

Step 2. By evaluating $\omega_{s,t}\big|_{s=t=0}$, we obtain from (4.4.2) that

$$|B_0(\zeta)|^2\,\mathrm{Re}(-\bar{\zeta}\hat{\psi}_{0,1}(\zeta)) = (4\pi)^{-\frac{1}{2}}.$$

As $\hat{\psi}_{0,1}$ is already known and the above real part is strictly positive on $\mathbb{T}$ (see Proposition 4.6.1 below), this gives $|B_0|^2$ on the unit circle $\mathbb{T}$. We choose B_0 to be the outer function in $\mathbb{D}_\mathrm{e}$ with these boundary values, with the additional normalization $B_0(\infty) > 0$.

We proceed from Step 2 to Step 3 with $j = 1$.

Step 3. We have determined $B_0, \dots, B_{j-1}$ and $\hat{\psi}_{a,b}$ for all $(a,b) \prec (j, 0)$. For $l = 1$, we may then compute $\mathfrak{F}_{j-1,1}$ (see Proposition 4.6.4 below), which by the equations (4.4.2) and (4.4.3) gives an equation for $\hat{\psi}_{j,0}$. The equation takes the form $\mathrm{Re}(\bar{\zeta}\hat{\psi}_{j,0}) = g_{j,0}$ on $\mathbb{T}$, for some known real-valued real-analytic expression $g_{j,0}$, and we solve it by the formula

$$\hat{\psi}_{j,0}(\zeta) = \zeta \mathbf{H}_{\mathbb{D}_\mathrm{e}}[g_{j,0}](\zeta).$$

This means that we extend the background data to all $\hat{\psi}_{p,q}$ with $(p,q) \prec (j,1)$, and we may apply the same procedure with $l = 2$ to determine $\mathfrak{F}_{j-1,2}$ followed by $\hat{\psi}_{j,1}$. We continue iterating the same procedure to cover $l = 3, 4, \dots, 2(\kappa - j) + 2$. Ultimately, in this step we obtain the functions $\hat{\psi}_{j,l}$ for $l = 0, \dots, 2(\kappa - j) + 1$.

Step 4. At this stage, using Step 3, we have at our disposal the functions $B_0, \dots, B_{j-1}$, and $\hat{\psi}_{p,q}$ for all $(p,q) \prec (j+1, 0)$. This allows us to compute $\mathfrak{F}_{j,0}$ (see Proposition 4.6.4 below), and from (4.4.2) and (4.4.4), we derive an equation of the form $\mathrm{Re}(\bar{B}_0 B_j) = h_j$, for some known real-analytic real-valued expression h_j. We solve this explicitly by

$$B_j(\zeta) = B_0(\zeta)\mathbf{H}_{\mathbb{D}_\mathrm{e}}\Big[\frac{h_j}{|B_0|^2}\Big](\zeta), \qquad \zeta \in \mathbb{D}_\mathrm{e}.$$

Due to the smoothness of the function $h_j/|B_0|^2$, the function B_j extends holomorphically across the boundary $\mathbb{T}$.

Step 5. Finally, we iterate Steps 3 and 4 with j replaced by $j + 1$, until all coefficients B_j for $j = 0, \dots, \kappa$ and $\hat{\psi}_{j,l}$ for all $(j,l) \in \mathcal{I}_{2\kappa+1}$ have been determined.

Remark 4.4.1. (a) If we apply the above algorithm to the function $R = R_\tau$, the functions B_j obtained here are (up to a constant multiple) the same as those appearing in Theorem 1.3.7. This algorithm is in principle an alternative route towards finding them explicitly. However, since this algorithm involves the additional functions $\hat{\psi}_{j,l}$, this is not feasible even for $\kappa = 2$.

(b) The functions B_j and $\hat{\psi}_{j,l}$ determined iteratively by the above algorithmic procedure all have the required properties: each function B_j is bounded and holomorphic in the exterior disk, and each function $\hat{\psi}_{j,l}$ is holomorphic in the exterior disk and meets the normalization

$\hat{\psi}'_{j,l}(\infty) > 0$. Moreover, all the above-mentioned functions extend holomorphically across the boundary $\mathbb{T}$, and we tacitly extend them to the larger region $\mathbb{D}_{\mathrm{e}}(0,\rho)$ for some positive $\rho < 1$.

4.5. **The multivariate Faà di Bruno formula.** We recall Faà di Bruno's formula in several variables, and study some of the properties. Using standard multi-index notation, we may introduce the *lexicographic ordering*: if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two multi-indices

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \quad \text{and} \quad \boldsymbol{\beta} = (\beta_1, \ldots, \beta_n),$$

we say that $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$, if either $\alpha_1 < \beta_1$ or $\alpha_1 = \beta_1, \ldots, \alpha_k = \beta_k$ while $\alpha_{k+1} < \beta_{k+1}$ holds for some $1 \le k \le n$. If $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$ or $\boldsymbol{\alpha} = \boldsymbol{\beta}$, we agree that $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$. For multi-indices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in $\mathbb{N}^n$ we write

$$\begin{aligned}
|\boldsymbol{\alpha}| &= \sum_i |\alpha_i| \\
\boldsymbol{\alpha}! &= \prod_i (\alpha_i!) \\
\boldsymbol{\xi}^{\boldsymbol{\beta}} &= \prod_i \xi_i^{\beta_i}, \qquad \boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n, \\
\partial^{\boldsymbol{\alpha}} f(\boldsymbol{x}) &= \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f(\boldsymbol{x}), \qquad \boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n.
\end{aligned}$$

We will need the index set

$$\begin{aligned}
\mathcal{T}_{m;d,n} = \big\{ &(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m; \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_m) \in (\mathbb{N}^d)^m \times (\mathbb{N}^n)^m \ : \\
&0 \prec \boldsymbol{\alpha}_1 \prec \boldsymbol{\alpha}_2 \prec \ldots \prec \boldsymbol{\alpha}_m \ \text{ and } \ \forall i = 1, \ldots, m: \ |\boldsymbol{\beta}_i| > 0 \big\}.
\end{aligned}$$

Proposition 4.5.1 (Faà di Bruno's formula, [11])**.** *Let f be a real-valued function defined in a domain $\mathcal{D} \subset \mathbb{R}^n$, of class C^k, and let g be defined and C^k in a domain $\mathcal{D}' \subset \mathbb{R}^d$ such that g takes values in $\mathcal{D}$. Then, for any multi-index $\boldsymbol{\nu}$ with $|\boldsymbol{\nu}| = k$, we have on $\mathcal{D}'$*

$$\partial^{\boldsymbol{\nu}}(f \circ g) = \sum_{1 \le |\boldsymbol{\mu}| \le k} (\partial^{\boldsymbol{\mu}} f) \circ g \, \mathcal{G}_{\boldsymbol{\mu},\boldsymbol{\nu}}(g),$$

where the function $\mathcal{G}_{\boldsymbol{\mu},\boldsymbol{\nu}}(g)$ is given by

$$\mathcal{G}_{\boldsymbol{\mu},\boldsymbol{\nu}}(g) = \boldsymbol{\nu}! \sum_{m=1}^{k} \sum_{(\boldsymbol{\alpha};\boldsymbol{\beta}) \in S_m(\boldsymbol{\mu},\boldsymbol{\nu})} \prod_{j=1}^{m} \frac{[\partial^{\boldsymbol{\alpha}_j} g]^{\boldsymbol{\beta}_j}}{\boldsymbol{\beta}_j! [\boldsymbol{\alpha}_j!]^{|\boldsymbol{\beta}_j|}},$$

and where the index set is defined as

$$S_m(\boldsymbol{\mu},\boldsymbol{\nu}) = \Big\{ (\boldsymbol{\alpha};\boldsymbol{\beta}) = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m; \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_m) \in \mathcal{T}_{m;d,n} \ : \ \sum \boldsymbol{\beta}_i = \boldsymbol{\mu}, \ \sum |\boldsymbol{\beta}_i| \boldsymbol{\alpha}_i = \boldsymbol{\nu} \Big\}.$$

Here, since g is assumed vector-valued, the derivative $\partial^{\boldsymbol{\alpha}_j} g$ is also vector-valued, and the power $[\partial^{\boldsymbol{\alpha}_j} g]^{\boldsymbol{\beta}_j}$ is taken with respect to the multi-index notation and produces a real-valued function.

Let us now specialize Proposition 4.5.1 to our situation. We will be interested in the case of $n = d = 2$. By setting $F(r,\theta) = R(r\mathrm{e}^{\mathrm{i}\theta})$, we may write

$$R \circ \psi_{s,t} = F \circ \boldsymbol{\Phi}_{s,t}, \qquad \boldsymbol{\Phi}_{s,t} = (|\psi_{s,t}|, \arg \psi_{s,t}).$$

If we denote by $D^{\boldsymbol{\mu}}_{r,\theta}$ the differential operator

$$D^{\boldsymbol{\mu}}_{r,\theta} = \partial_r^{\mu_1} \partial_\theta^{\mu_2}, \quad \boldsymbol{\mu} = (\mu_1, \mu_2),$$

we obtain by applying Proposition 4.5.1 to $F \circ \Phi_{s,t}$ with $\boldsymbol{\nu} = (j,l)$ that on the circle $\mathbb{T}$,

$$\partial_s^j \partial_t^l (R \circ \psi_{s,t}) \big|_{s=t=0} = \sum_{2 \le |\boldsymbol{\mu}| \le j+l} D^{\boldsymbol{\mu}}_{r,\theta} R(\psi_{s,t}) \, \mathcal{G}_{\boldsymbol{\mu},(j,l)}(\boldsymbol{\Phi}_{s,t}) \big|_{s=t=0}, \tag{4.5.1}$$

where the terms corresponding to indices $\boldsymbol{\mu}$ with $|\boldsymbol{\mu}| = 1$ are dropped, and the reason is the following. First, we have that $\psi_{0,0}(\zeta) = \zeta$, and second, the function R together with its gradient vanish along the unit circle $\mathbb{T}$. In the context of (4.5.1), we should point out that the multi-index derivatives that appear in the expression $\mathcal{G}_{\boldsymbol{\mu},(j,l)}(\boldsymbol{\Phi}_{s,t})$, as defined in Proposition 4.5.1, are taken with respect to the variables (s,t). As for the (suppressed) variable $\zeta \in \mathbb{T}$, it is considered fixed.

We will be interested in identifying the *maximal* index (a,b) with respect to the lexicographical ordering, such that the partial derivative $\partial_s^a \partial_t^b \boldsymbol{\Phi}_{s,t}$ appears non-trivially in the right-hand side expression of (4.5.1).

Proposition 4.5.2. *Let $\boldsymbol{\nu}$ and $\boldsymbol{\mu}$ be double-indices with $2 \le |\boldsymbol{\mu}| \le |\boldsymbol{\nu}|$ and $\boldsymbol{\mu} \notin \{(1,1),(0,2)\}$. Let $(\boldsymbol{\alpha};\boldsymbol{\beta}) \in S_m(\boldsymbol{\mu},\boldsymbol{\nu})$. Then*

(i) If $\boldsymbol{\nu} = (j,l)$, where $j,l \ge 1$, then for all $i = 1,\ldots,m$, we have that $\boldsymbol{\alpha}_i \preceq (j,l-1)$, where equality holds if and only if $i = m = 2$, $\boldsymbol{\mu} = (2,0)$, and

$$(\boldsymbol{\alpha};\boldsymbol{\beta}) = ((0,1),(j,l-1);(1,0),(1,0)).$$

(ii) If $\boldsymbol{\nu} = (j,0)$ with $j \ge 3$, then $\boldsymbol{\alpha}_i$ is of the form $(a,0)$ with $a \le j-1$. Moreover, equality holds if and only if $i = m = 2$, $\boldsymbol{\mu} = (2,0)$, and

$$(\boldsymbol{\alpha};\boldsymbol{\beta}) = ((1,0),(j-1,0);(1,0),(1,0)).$$

(iii) If $\boldsymbol{\nu} = (0,l)$ with $l \ge 3$, then $\boldsymbol{\alpha}_i$ is of the form $(0,b)$ with $b \le l-1$. Moreover, equality holds if and only if $\boldsymbol{\mu} = (2,0)$ and

$$(\boldsymbol{\alpha};\boldsymbol{\beta}) = ((0,1),(0,l-1);(1,0),(1,0)).$$

(iv) If $\boldsymbol{\nu} = (2,0)$, then necessarily $\boldsymbol{\mu} = (2,0)$ and the only non-trivial index $(\boldsymbol{\alpha};\boldsymbol{\beta})$ is

$$(\boldsymbol{\alpha};\boldsymbol{\beta}) = ((1,0);(0,2)).$$

(v) If $\boldsymbol{\nu} = (0,2)$, then necessarily $\boldsymbol{\mu} = (2,0)$ and the only non-trivial index $(\boldsymbol{\alpha};\boldsymbol{\beta})$ is

$$(\boldsymbol{\alpha};\boldsymbol{\beta}) = ((0,1);(0,2)).$$

Note that since $|\boldsymbol{\nu}| \ge 2$, the above list covers all the possibilities.

Proof of Proposition 4.5.2. We will show how to obtain (i), (ii) and (iv). The remaining cases (iii) and (v) are analogous and omitted. We recall the compatibility conditions of the index set $S_m(\boldsymbol{\mu},\boldsymbol{\nu})$: the assertion $(\boldsymbol{\alpha};\boldsymbol{\beta}) \in S_m(\boldsymbol{\mu},\boldsymbol{\nu})$ means that

$$\sum_{i=1}^{m} |\boldsymbol{\beta}_i| \boldsymbol{\alpha}_i = \boldsymbol{\nu}, \qquad \sum_{i=1}^{m} \boldsymbol{\beta}_i = \boldsymbol{\mu}, \tag{4.5.2}$$

where each $\boldsymbol{\beta}_i$ meets $|\boldsymbol{\beta}_i| \ge 1$, and the multi-indices α_{i} are strictly increasing with i in the lexicographical ordering. From these assumptions it follows that each $\boldsymbol{\alpha}_i$ satisfies $\boldsymbol{\alpha} \preceq \boldsymbol{\nu}$.

Turning to (i), we see that equality $\boldsymbol{\alpha}_i = (j,l)$ could hold only if $m = 1$, $\boldsymbol{\alpha}_1 = (j,l)$ and $\boldsymbol{\beta}_1 = (1,0)$. But then $|\boldsymbol{\mu}| = 1$, which contradicts the assumption that $|\boldsymbol{\mu}| \ge 2$. Hence, for any index i, we have $\boldsymbol{\alpha}_i \preceq (j,l-1)$. However, if equality holds here, that is, if for some i_0 we have $\boldsymbol{\alpha}_{i_0} = (j,l-1)$, we find from (4.5.2) that $|\boldsymbol{\beta}_{i_0}| = 1$, the sum on the left-hand side, taken over all other indices $i \ne i_0$, equals $(0,1)$. As a consequence, we find that $m = 2$ and that $\boldsymbol{\alpha} = ((0,1),(j,l-1))$. Since the only admissible index $\boldsymbol{\mu}$ with $|\boldsymbol{\mu}| = 2$ is $\boldsymbol{\mu} = (2,0)$, the rest of the claim is immediate.

We next consider (ii). In a similar manner as above, since the weighted sum of the multi-indices $\boldsymbol{\alpha}_i$ equals $(j,0)$, we see that $\boldsymbol{\alpha}_i$ is of the form $\boldsymbol{\alpha}_i = (a_i,0)$ for any i. It is moreover clear that $a_{i_0} = j$ could only occur for some i_0 only if $i_0 = m = 1$, $|\boldsymbol{\beta}_1| = 1$ and $|\boldsymbol{\mu}| = 1$, which again is contrary to our assumption that $|\boldsymbol{\mu}| \ge 2$. Next, the only way we could have $\boldsymbol{\alpha}_{i_0} = (j-1,0)$ for some i_0 is if $i_0 = m = 2$ and correspondingly $\boldsymbol{\alpha} = ((1,0),(j-1,0))$. The remaining properties are immediate.

Finally, to see why (iv) holds, we again find that each $\boldsymbol{\alpha}_i$ is of the form $\boldsymbol{\alpha}_i = (a_i,0)$. Since the multi-indices $\boldsymbol{\alpha}_i$ are assumed strictly increasing with i, the combined index $\boldsymbol{\alpha} = ((1,0),(1,0))$ is not admissible. Recall that $\{|\boldsymbol{\beta}_i|\boldsymbol{\alpha}_i : i = 1,\ldots,m\}$ sums up to $\boldsymbol{\nu}$. The only remaining way to obtain $\boldsymbol{\nu} = (2,0)$ is if $m = 1$ so that the sum has only one element, which is automatically $(2,0)$ itself. If $\boldsymbol{\alpha}_1 = (2,0)$, then $|\boldsymbol{\beta}_1| = 1$, so $|\boldsymbol{\mu}| = 1$, which is contrary to our assumption that $|\boldsymbol{\mu}|\ge 2$. The only remaining alternative is that $\boldsymbol{\alpha}_1=(1,0)$, and $|\boldsymbol{\beta}_1|=2$. Since $\boldsymbol{\beta}_1=\boldsymbol{\mu}$, and the only admissible $\boldsymbol{\mu}$ of length 2 is $\boldsymbol{\mu}=(2,0)$, it follows that $\boldsymbol{\beta}_1=(2,0)$, and the claim follows. $\square$

We note that in each of the cases (i)-(v), the maximal $\boldsymbol{\alpha}_i$ occurs as the index $\boldsymbol{\alpha}_m$, where $(\boldsymbol{\alpha};\boldsymbol{\beta})\in S_m(\boldsymbol{\mu},\boldsymbol{\nu})$ and $\boldsymbol{\mu}=(2,0)$. We will find the notation

$$\mathcal{G}^{\star}_{(2,0),\boldsymbol{\nu}}(\boldsymbol{\Phi}_{s,t})=\boldsymbol{\nu}!\sum_{m=1}^{|\boldsymbol{\nu}|}\sum_{(\boldsymbol{\alpha};\boldsymbol{\beta})\in S^{\star}_m(\boldsymbol{\nu})}\prod_{j=1}^{m}\frac{[\partial^{\boldsymbol{\alpha}_j}\boldsymbol{\Phi}_{s,t}]^{\boldsymbol{\beta}_j}}{\boldsymbol{\beta}_j![\boldsymbol{\alpha}_j!]^{|\boldsymbol{\beta}_j|}} \tag{4.5.3}$$

useful. Here, the sum is taken over the index set $S^{\star}_m(\boldsymbol{\nu})$, defined as follows. Let

$$A(\boldsymbol{\nu})=\max_{m}\max_{(\boldsymbol{\alpha};\boldsymbol{\beta})\in S_m((2,0),\boldsymbol{\nu})}\boldsymbol{\alpha}_m$$

where the maximum is taken lexicographically over the range $m=1,\ldots,|\boldsymbol{\nu}|$. Moreover, denote by $(\boldsymbol{\alpha}^{\star};\boldsymbol{\beta}^{\star})$ the unique pair such that $\boldsymbol{\alpha}^{\star}_i=A(\boldsymbol{\nu})$ for some i (see Proposition 4.5.2 for details). We then put

$$S^{\star}_m(\boldsymbol{\nu})=\begin{cases}S_m((2,0),\boldsymbol{\nu}), & \text{if } (\boldsymbol{\alpha}^{\star};\boldsymbol{\beta}^{\star})\notin S_m((2,0),\boldsymbol{\nu}),\\ S_m((2,0),\boldsymbol{\nu})\setminus\{(\boldsymbol{\alpha}^{\star};\boldsymbol{\beta}^{\star})\}, & \text{if } (\boldsymbol{\alpha}^{\star};\boldsymbol{\beta}^{\star})\in S_m((2,0),\boldsymbol{\nu}).\end{cases}$$

4.6. **The orthogonal foliation flow III. The construction.** In this section we obtain Proposition 4.2.5. To this end, we need to fill in the blanks of the algorithmic procedure outlined in Subsection 4.4.

We now explore STEP 1 of the algorithmic procedure outlined in Subsection 4.4. We recall the notation $\boldsymbol{\Phi}_{0,t}=(|\psi_{0,t}|,\arg\psi_{0,t})$.

Proposition 4.6.1. *The coefficients $\hat\psi_{0,l}$ of the conformal mapping $\psi_{0,t}$ with Taylor expansion near the origin*

$$\psi_{0,t}(\zeta)=\sum_{l=0}^{2\kappa+1}t^l\hat\psi_{0,l}(\zeta)+\mathrm{O}(t^{2\kappa+2}),$$

are uniquely determined by the level-curve requirement

$$R\circ\psi_{0,t}(\zeta)=\frac{t^2}{2},\qquad \zeta\in\mathbb{T},$$

the monotonicity condition that the images $\psi_{0,t}(\mathbb{D}_{\mathrm{e}})$ grow with t, and the normalization $\psi'_{0,t}(\infty)>0$. Moreover, as such they are given by

$$\hat\psi_{0,0}(\zeta)=\zeta,$$
$$\hat\psi_{0,1}(\zeta)=-\zeta\mathbf{H}_{\mathbb{D}_{\mathrm{e}}}\big[(4\Delta R)^{-\frac12}\big](\zeta),$$

and, more generally, by

$$\hat\psi_{0,l}(\zeta)=\zeta\mathbf{H}_{\mathbb{D}_{\mathrm{e}}}\big[(4\Delta R)^{-\frac12}\mathfrak{G}_l\big](\zeta),\qquad l=2,\ldots,2\kappa+3,$$

where $\mathfrak{G}_l(\zeta)$ is a real-analytic function on the circle $\mathbb{T}$, which may be expressed in terms of the functions $\hat\psi_{0,0},\ldots,\hat\psi_{0,l-1}$ by the formula

$$\mathfrak{G}_l(\zeta):=\frac{1}{(l+1)!}\Bigg\{4\Delta R(\zeta)\mathcal{G}^{\star}_{(2,0),(0,l+1)}(\boldsymbol{\Phi}_{0,t})\big|_{t=0}+\mathfrak{H}_l + \sum_{3\le|\boldsymbol{\mu}|\le l+1}\partial_r^{\mu_1}\partial_\theta^{\mu_2}R(\zeta)\mathcal{G}_{\mu,(0,l+1)}(\boldsymbol{\Phi}_{0,t})\big|_{t=0}\Bigg\},$$

where

$$\mathfrak{H}_l:=\partial_t^l|\psi_{0,t}|\big|_{t=0}-l!\,\mathrm{Re}(\bar\zeta\hat\psi_{0,l}).$$

The coefficient functions $\hat\psi_{j,l}$ extend holomorphically to the domain $\mathbb{D}_{\mathrm{e}}(0,\rho')$.

Proof. That $\hat\psi_{0,0}(\zeta)=\zeta$ follows since R vanishes only on $\mathbb{T}$. By Taylor's formula, we have that

$$(R\circ\psi_{0,t})(\zeta)=\sum_{j=0}^{2\kappa+1}\frac{t^l}{l!}\partial_t^l(R\circ\psi_{0,t}(\zeta))\big|_{t=0}+\mathrm{O}(|t|^{2\kappa+2}). \tag{4.6.1}$$

Since by assumption $R\circ\psi_{0,t}(\zeta)=\frac{t^2}{2}$ holds on $\mathbb{T}$, we find that

$$\partial_t^l(R\circ\psi_{0,t})(\zeta)\big|_{t=0}=\begin{cases}1, & \text{for } l=2\\ 0, & \text{otherwise.}\end{cases} \tag{4.6.2}$$

The above formula for $\hat\psi_{0,0}$ is readily verified. Indeed, the level curve $R=0$ corresponding to $t=0$ is the unit circle, and the normalization at infinity forces $\hat\psi_{0,0}(\zeta)=\zeta$. Moreover, expression

$$\partial_t(R\circ\psi_{0,t})\big|_{t=0}=0$$

as the function R and its gradient vanish on the circle $\mathbb{T}$. Thus the equation (4.6.2) for $l=1$ gives us no additional information. We now consider the second derivative separately. A short computation using the chain rule and the flatness of R near the unit circle $\mathbb{T}$ shows that

$$\partial_t^2(R\circ\psi_{0,t})\big|_{t=0}=4\Delta R\,[\mathrm{Re}(\bar\zeta\hat\psi_{0,1})]^2.$$

If we solve for $\mathrm{Re}(\bar\zeta\hat\psi_{0,1})$ in (4.6.2) for $l=2$ using the negative square root, we find that

$$\mathrm{Re}(\bar\zeta\hat\psi_{0,1})=-(4\Delta R)^{-\frac12},\qquad \text{on }\ \mathbb{T},$$

which is compatible with the growth of the domains $\psi_{0,t}(\mathbb{D}_\mathrm{e})$ as t increases. We finally solve this equation by the formula

$$\hat\psi_{0,1}(\zeta)=-\zeta\mathbf{H}_{\mathbb{D}_\mathrm{e}}\big[(4\Delta R)^{-\frac12}\big](\zeta), \tag{4.6.3}$$

as in Step 3 of the algorithmic procedure in Subsection 4.4. As $(4\Delta R)^{-\frac12}$ has a polarization which is holomorphic in $(z,\bar w)$ for $(z,w)\in\hat{\mathbb{A}}(\rho,\sigma)$, the function $\hat\psi_{0,1}$ extends holomorphically to $\mathbb{D}_\mathrm{e}(0,\rho')$, by Proposition 4.2.2 and Remark 4.2.4.

We find the higher order Taylor coefficients by applying Faà di Bruno's formula to the composition $F\circ\mathbf{\Phi}_t$, where $F(r,\theta):=R(r\mathrm{e}^{\mathrm{i}\theta})$. First, recall that by the properties of R, we have that

$$D^{\boldsymbol{\mu}}_{r,\theta}R|_{\mathbb{T}}=0,\qquad \boldsymbol{\mu}\in\big\{(0,0),(1,0),(0,1),(1,1),(0,2)\big\}.$$

Consequently, we cannot have non-zero contributions from $\boldsymbol{\mu}$ with $|\boldsymbol{\mu}|\le 1$. As a result, on the circle $\zeta\in\mathbb{T}$ we have for $l=2,3,4,\dots$

$$\begin{aligned}\partial_t^{l+1}(R\circ\psi_{0,t})\big|_{t=0}&=\sum_{2\le|\boldsymbol{\mu}|\le l+1}(\partial_r^{\mu_1}\partial_\theta^{\mu_2}R)\,\mathcal{G}_{\boldsymbol{\mu},(0,l+1)}(\mathbf{\Phi}_{0,t})\big|_{t=0}\\ &=4(l+1)\Delta R\ \partial_t^l|\psi_{0,t}|\,\partial_t|\psi_{0,t}|\big|_{t=0}+\mathcal{G}^\star_{(2,0),l+1}(\mathbf{\Phi}_{0,t})\big|_{t=0}\\ &\quad+\sum_{3\le|\boldsymbol{\mu}|\le l+1}(\partial_r^{\mu_1}\partial_\theta^{\mu_2}R)\,\mathcal{G}_{\boldsymbol{\mu},(0,l+1)}(\mathbf{\Phi}_{0,t})\big|_{t=0}\\ &=4(l+1)!\Delta R\ \ \mathrm{Re}(\bar\zeta\hat\psi_{0,l})\,\mathrm{Re}(\bar\zeta\hat\psi_{0,1})+\mathfrak{H}_l+\mathcal{G}^\star_{(2,0),(0,l+1)}(\mathbf{\Phi}_{0,t})\big|_{t=0}\\ &\quad+\sum_{3\le|\boldsymbol{\mu}|\le l+1}(\partial_r^{\mu_1}\partial_\theta^{\mu_2}R)\,\mathcal{G}_{\boldsymbol{\mu},(0,l+1)}(\mathbf{\Phi}_{0,t})\big|_{t=0},\end{aligned} \tag{4.6.4}$$

where we recall that

$$\mathfrak{H}_l=\partial_t^l|\psi_{0,t}|\big|_{t=0}-l!\,\mathrm{Re}(\bar\zeta\hat\psi_{0,l}).$$

A short computation shows that the highest order derivatives cancel out, and that $\mathfrak{H}_l$ may be expressed in terms of the lower order Taylor coefficients $\hat\psi_{0,b}$ for $b\le l-1$. We recall that the expression $\mathcal{G}^\star_{(2,0),(0,l+1)}(\mathbf{\Phi}_{0,t})$ appearing in the above formula is defined in (4.5.3). If we write

$$\mathfrak{G}_l(\zeta)=\frac{1}{(l+1)!}\Big\{\mathfrak{H}_l(\zeta)+\mathcal{G}^\star_{(2,0),(0,l+1)}(\mathbf{\Phi}_{0,t})\big|_{t=0}+\sum_{3\le|\boldsymbol{\mu}|\le l+1}(\partial_r^{\mu_1}\partial_\theta^{\mu_2}R)\,\mathcal{G}_{\boldsymbol{\mu},(0,l+1)}(\mathbf{\Phi}_t)\big|_{t=0}\Big\},$$

we claim that $\mathfrak{G}_l$ may be expressed in terms of the functions $\hat{\psi}_{0,b}$ for $b \le l-1$. Indeed, we saw previously that $\mathfrak{H}_l$ has this property. That the same holds for the remaining two terms of the above formula is a consequence of Proposition 4.5.2.

By (4.6.3), we have on the unit circle $\mathbb{T}$ that

$$4\Delta R\ \mathrm{Re}(\bar{\zeta}\hat{\psi}_{0,1}) = (4\Delta R)^{\frac12}(4\Delta R)^{\frac12}\,\mathrm{Re}(\bar{\zeta}\hat{\psi}_{0,1}) = -(4\Delta R)^{\frac12}.$$

In view of this and the calculation (4.6.4), we may express the condition (4.6.2) in the form

$$-(4\Delta R)^{\frac12}\,\mathrm{Re}(\bar{\zeta}\hat{\psi}_{0,l}) + \mathfrak{G}_l = 0.$$

This type of equation we have encountered previously, and we know that a solution $\hat{\psi}_{0,l}$ is given by the formula

$$\hat{\psi}_{0,l}(\zeta) = \zeta\mathbf{H}_{\mathbb{D}_\mathrm{e}}\Big[\frac{\mathfrak{G}_l}{(4\Delta R)^{\frac12}}\Big](\zeta).$$

In this manner, we iteratively determine the functions $\hat{\psi}_{0,l}$ for $l = 2, \ldots, 2\kappa+1$. As the function $\mathfrak{G}_l(4\Delta R)^{-\frac12}$ has a polarization which is holomorphic in $(z, \bar{w})$ for $(z, w) \in \hat{\mathbb{A}}(\rho, \sigma)$, it follows that $\hat{\psi}_{0,l}$ is holomorphic in $\mathbb{D}_\mathrm{e}(0, \rho')$. □

We need three further propositions of a rather technical character. The following proposition gives us structural information regarding the expansion of the composition $R \circ \psi_{s,t}$ in terms of powers of s and t. As for the formulation, we retain the notation introduced in connection with Faà di Bruno's formula in Subsection 4.5.

Proposition 4.6.2. *On the unit circle $\mathbb{T}$, the function $R \circ \psi_{s,t}$ enjoys the expansion*

$$R \circ \psi_{s,t} = R \circ \psi_{0,t} + \sum_{(j,l)\in\mathcal{I}_{2\kappa}} s^{j+1}t^l\mathfrak{A}_{j,l} + \mathrm{O}\big(|s|\big(|s|^{\kappa+\frac12} + |t|^{2\kappa+1}\big)\big), \tag{4.6.5}$$

where $\mathfrak{A}_{0,0} = 0$, while for the remaining indices $(j, l) \ne (0, 0)$, we have

$$\mathfrak{A}_{j,l} = \begin{cases} \frac{1}{(j+1)!(l-1)!}4\Delta R\ (\partial_s^{j+1}\partial_t^{l-1}|\psi_{s,t}|)(\partial_t|\psi_{0,t}|)\Big|_{s=t=0} + \frac{1}{(j+1)!l!}\mathfrak{R}_{j,l}, & \text{for } j \ge 0 \text{ and } l \ge 1, \\ \frac{1}{(j+1)!}\sum_{2\le|\boldsymbol{\mu}|\le j+1} D^{\boldsymbol{\mu}}_{r,\theta}R\ \mathcal{G}_{\boldsymbol{\mu},(j+1,0)}(\boldsymbol{\Phi}_{s,t})\big|_{s=t=0}, & \text{for } j \ge 2 \text{ and } l = 0, \\ 4\Delta R\ [\mathrm{Re}(\bar{\zeta}\hat{\psi}_{1,0})]^2, & \text{for } j = 1 \text{ and } l = 0. \end{cases}$$

Here, the functions $\mathfrak{R}_{j,l}$ are given by

$$\mathfrak{R}_{j,l} = 4\Delta R\ \mathcal{G}^\star_{(2,0),(j+1,l)}(\boldsymbol{\Phi}_{s,t})\Big|_{s=t=0} + \sum_{3\le|\boldsymbol{\mu}|\le j+l+1} D^{\boldsymbol{\mu}}_{r,\theta}R\ \mathcal{G}_{\boldsymbol{\mu},(j+1,l)}(\boldsymbol{\Phi}_{s,t})\Big|_{s=t=0}.$$

In particular, the terms $\mathfrak{R}_{j,l}$ may be expressed in terms of the partial derivatives $\partial_s^a\partial_t^b\psi_{s,t}$ with $(a, b) \prec (j + 1, l - 1)$, while the coefficients $\mathfrak{A}_{j,0}$ may be expressed in terms of the partial derivatives $\partial_s^a\partial_t^b\psi_{s,t}$ with $b = 0$ and $a \le j$. Moreover, the implied constant in (4.6.5) remains bounded if the weight R is confined to a uniform family in $\mathcal{W}(\rho_0, \sigma_0)$ for some fixed $\rho_0 < 1$ and $\sigma_0 > 0$, while the functions $\psi_{s,t}$ are assumed smooth with bounded norms in $C^{2\kappa+4}$ with respect to (s, t) in a neighbourhood of $(0, 0)$, uniformly on the circle $\mathbb{T}$.

Proof. The fact that $R \circ \psi_{s,t}$ enjoys an expansions of the form (4.6.5) for some coefficients $\mathfrak{A}_{j,l}$ with the given error term is an immediate consequence of the multivariate Taylor's formula. The coefficients $\mathfrak{A}_{j,l}$ are then obtained from an of application Faà di Bruno's formula (4.5.1). It remains to identify the coefficients $\mathfrak{A}_{j,l}$. By the flatness of R near the circle $\mathbb{T}$, we have

$$D^{\boldsymbol{\mu}}_{r,\theta}R|_{\mathbb{T}} = 0, \qquad \boldsymbol{\mu} \in \big\{(0, 0), (1, 0), (0, 1), (1, 1), (0, 2)\big\},$$

so no index $\boldsymbol{\mu}$ with $|\boldsymbol{\mu}| \le 1$ gives non-trivial contributions.

We first consider the case $l \ge 1$, and study the derivative $\partial^{\boldsymbol{\nu}}(R \circ \psi_{s,t})\big|_{s=t=0}$, where $\boldsymbol{\nu} = (j + 1, l)$. If $|\boldsymbol{\mu}| = 2$, and $(\boldsymbol{\alpha}; \boldsymbol{\beta}) \in S_m(\boldsymbol{\mu}, \boldsymbol{\nu})$, then Proposition 4.5.2 gives that $\boldsymbol{\alpha}_m$, which is maximal among the indices $\boldsymbol{\alpha}_i$ in our lexicographical ordering, is bounded above by $(j+1, l-1)$. Moreover, the index $(j + 1, l - 1)$ appears only once across all summation indices m and $\boldsymbol{\mu}$ with

$|\boldsymbol{\mu}|=2$, and the only way that we can achieve $\boldsymbol{\alpha}_m=(j+1,l-1)$ is if $m=2$, $\boldsymbol{\alpha}_1=(0,1)$, and $\boldsymbol{\beta}_1=\boldsymbol{\beta}_2=(1,0)$.

In a similar fashion, Proposition 4.5.2 shows that if $(\boldsymbol{\alpha},\boldsymbol{\beta})\in S_m(\boldsymbol{\mu},\boldsymbol{\nu})$ with $|\boldsymbol{\mu}|\geq 3$, then $\boldsymbol{\alpha}_m\prec(j+1,l-1)$.

A short computation using the fact that $\partial_r^2 R=4\Delta R$ on $\mathbb{T}$ (see Proposition 3.1.3) shows that the coefficient $\mathfrak{A}_{j,l}$ has the required form, and since the remainder term $\mathfrak{R}_{j,l}$ is not summed over the index $(\boldsymbol{\alpha}^\star;\boldsymbol{\beta}^\star)$, which is strictly maximal according to the above discussion, it follows that $\mathfrak{R}_{j,l}$ has the asserted properties.

The assertion $\mathfrak{A}_{0,0}=0$ is a consequence of the fact that both the function R and its gradient ∇R vanish along $\mathbb{T}$. The formula supplied for $\mathfrak{A}_{j,0}$ with $j\geq 1$ is a direct consequence of the Faà di Bruno formula (4.5.1). With regards to the claimed property of $\mathfrak{A}_{j,0}$, it suffices to note that any index $\boldsymbol{\alpha}_i$, where $(\boldsymbol{\alpha};\boldsymbol{\beta})\in S_m(\boldsymbol{\mu},\boldsymbol{\nu})$ with $\boldsymbol{\nu}=(j+1,0)$ is of the form $\boldsymbol{\alpha}_i=(a,0)$ for some a with $1\leq a\leq j$, by Proposition 4.5.2. This finishes the proof. $\square$

For a positive integer $r\in\mathbb{Z}_+$ and $(j,l)\in\mathcal{I}_{2\kappa}$, we use the notation $X_r(j,l)$ for the set

$$X_r(j,l)=\Big\{(j_i,l_i)_{i=1}^r:\sum_{i=1}^r j_i=j,\ \sum_{i=1}^r l_i=l\Big\}.$$

In preparation for the next proposition, we observe that by Proposition 4.6.2

$$\frac{1}{s}\Big\{R\circ\psi_{s,t}-\frac{t^2}{2}\Big\}=\sum_{(i,l)\in\mathcal{I}_{2\kappa}}s^jt^l\mathfrak{A}_{j,l}+\mathrm{O}\big(|s|^{\kappa+\frac12}+|t|^{2\kappa+1}\big).\tag{4.6.6}$$

holds on the unit circle $\mathbb{T}$, since there, $R\circ\psi_{0,t}=\frac{t^2}{2}$ holds as a matter of definition. We recall that f_s and $\psi_{s,t}$ denote functions of the form

$$f_s(\zeta)=\sum_{j=0}^{\kappa}s^jB_j(\zeta)\quad\text{and}\quad\psi_{s,t}(\zeta)=\psi_{0,t}(\zeta)+\sum_{\substack{(j,l)\in\mathcal{I}_{\kappa+1}\\ j\geq1}}s^jt^l\hat\psi_{j,l}(\zeta),$$

where the B_j are some bounded holomorphic functions in a neighbourhood of the exterior disk $\bar{\mathbb{D}}_{\mathrm{e}}$, and where $\psi_{0,t}$ is a conformal mapping onto the exterior of the level curves Γ_t of R as above, and where $\hat\psi_{j,l}$ are holomorphic functions on $\mathbb{D}_{\mathrm{e}}(0,\rho')$ with bounded derivative normalized at infinity by $\hat\psi_{j,l}'(\infty)\in\mathbb{R}$. Let us denote by $\mathfrak{U}_{j,l}$, $\mathfrak{V}_{j,l}$ and $\mathfrak{W}_{j,l}$ the coefficients from the following three expansions (for $\zeta\in\mathbb{T}$):

$$|f_s\circ\psi_{s,t}(\zeta)|^2=\sum_{(j,l)\in\mathcal{I}_{2\kappa}}s^jt^l\mathfrak{U}_{j,l}(\zeta)+\mathrm{O}\big(|s|^{\kappa+\frac12}+|t|^{2\kappa+1}\big),\tag{4.6.7}$$

$$\mathrm{e}^{-2s^{-1}\{R\circ\psi_{s,t}(\zeta)-\frac{t^2}{2}\}}=\sum_{(j,l)\in\mathcal{I}_{2\kappa}}s^jt^l\mathfrak{V}_{j,l}(\zeta)+\mathrm{O}\big(|s|^{\kappa+\frac12}+|t|^{2\kappa+1}\big),\tag{4.6.8}$$

$$\mathrm{Re}\big(-\bar\zeta\partial_t\psi_{s,t}(\zeta)\overline{\psi_{s,t}'(\zeta)}\big)=\sum_{(j,l)\in\mathcal{I}_{2\kappa}}s^jt^l\mathfrak{W}_{j,l}(\zeta)+\mathrm{O}\big(|s|^{\kappa+\frac12}+|t|^{2\kappa+1}\big).\tag{4.6.9}$$

The following proposition tells us how to obtain these coefficients in terms of the successive partial derivatives of the functions B_j and $\psi_{s,t}$.

Proposition 4.6.3. *In the above context, the coefficients $\mathfrak{U}_{j,l}$, $\mathfrak{V}_{j,l}$ and $\mathfrak{W}_{j,l}$ are obtained on the unit circle $\mathbb{T}$ as follows. We have that $\mathfrak{V}_{0,0}=1$, while for $(j,l)\in\mathcal{I}_{2\kappa}\setminus\{(0,0)\}$*

$$\mathfrak{V}_{j,l}=\sum_{r=1}^{j+l}\sum_{X_r(j,l)}\frac{r!}{(j_1')!\cdots(j_r')!(l_1')!\cdots(l_r')!}\mathfrak{A}_{j_1',l_1'}\cdots\mathfrak{A}_{j_r',l_r'},$$

where $\mathfrak{A}_{j,l}$ denote the coefficients obtained in Proposition 4.6.2. Moreover, the remaining coefficients are obtained by

$$\mathfrak{U}_{j,l}=\sum_{\substack{p_1+p_2+i+k=j\\ q_1+q_2=l}}\frac{\partial_s^{p_1}\partial_t^{q_1}(B_i\circ\psi_{s,t})\,\partial_s^{p_2}\partial_t^{q_2}(\bar{B}_k\circ\psi_{s,t})}{p_1!p_2!q_1!q_2!}\Big|_{s=t=0},\qquad (j,l)\in\mathcal{I}_{2\kappa},$$

$$\mathfrak{W}_{j,l}=\sum_{\substack{p+i=j\\ q+k=l}}(k+1)\operatorname{Re}\Big(-\bar{\zeta}\hat{\psi}_{i,k+1}\overline{\hat{\psi}'_{p,q}}\Big),\qquad (j,l)\in\mathcal{I}_{2\kappa}.$$

Here, the summation takes place over all the relevant tuples (p_1,p_2,q_1,q_2,i,k) and (p,q,i,k), respectively, where the entries are all non-negative integers.

Proof. This follows from an application of the multivariate Taylor's formula, together with Faà di Bruno's formula (Proposition 4.5.1), and the equation (4.6.6) above. □

Next, let $\omega_{s,t}$ denote the function

$$\omega_{s,t}=|f_s\circ\psi_{s,t}|^2\mathrm{e}^{-2s^{-1}\{R\circ\psi_{s,t}-\frac{t^2}{2}\}}\operatorname{Re}\Big(-\bar{\zeta}\partial_t\psi_{s,t}\overline{\psi'_{s,t}}\Big),$$

and define implicitly the coefficients $\mathfrak{C}_{j,l}$ of the expansion of $\omega_{s,t}$ on the circle $\mathbb{T}$:

$$\omega_{s,t}=\sum_{(j,l)\in\mathcal{I}_{2\kappa}}s^jt^l\mathfrak{C}_{j,l}+\mathrm{O}\big(|s|^{\kappa+\frac12}+|t|^{2\kappa+1}\big).$$

Proposition 4.6.4. *The coeffients $\mathfrak{C}_{j,l}$ in the above expansion are given by*

$$\mathfrak{C}_{0,0}=|B_0|^2\operatorname{Re}\big(-\bar{\zeta}\hat{\psi}_{0,1}\big)=|B_0|^2(4\Delta R)^{-\frac12},$$

and for $l=0$ and $j=1,2,3,\ldots$ by

$$\mathfrak{C}_{j,0}=2\operatorname{Re}\big(\bar{B}_0B_j\big)(4\Delta R)^{-\frac12}+\mathfrak{F}_{j,0},$$

where

$$\mathfrak{F}_{j,0}=(4\Delta R)^{-\frac12}\sum_{\substack{0\le p,q\le j-1\\ p+q+r=j}}2\operatorname{Re}\big(\bar{B}_pB_q\big)\mathfrak{A}_{r,0},$$

while for $j=0,1,2,\ldots$ and $l=1,2,3,\ldots$ the coefficient function $\mathfrak{C}_{j,l}$ meets

$$\mathfrak{C}_{j,l}=(4\pi)^{\frac12}\operatorname{Re}(\bar{\zeta}\hat{\psi}_{j+1,l-1})(4\Delta R)^{\frac12}+\mathfrak{F}_{j,l},$$

where

$$\mathfrak{F}_{j,l}=(4\pi)^{-\frac12}(4\Delta R)^{\frac12}\big\{\mathfrak{S}_{j,l}+\mathfrak{R}_{j,l}\big\}+\sum_{\substack{(j_1,l_1,j_2,l_2,j_3,l_3)\in X_3(j,l)\\ (j_2,l_2)\prec(j,l)}}\mathfrak{U}_{j_1,l_1}\mathfrak{V}_{j_2,l_2}\mathfrak{W}_{j_3,l_3}.$$

Here, we recall that the functions $\mathfrak{R}_{j,l}$ were defined in Proposition 4.6.2, and on the unit circle $\mathbb{T}$, the function $\mathfrak{S}_{j,l}$ is given by

$$\begin{aligned}\mathfrak{S}_{j,l}&=\partial_s^{j+1}\partial_t^{l-1}|\psi_{s,t}|\big|_{s=t=0}-(j+1)!(l-1)!\operatorname{Re}\big(\bar{\zeta}\hat{\psi}_{j+1,l-1}\big)=\sum_{k=2}^{j+l}(-1)^k\big(-\tfrac12\big)_k(j+1)!(l-1)!\\ &\times\sum_{m=1}^{j+l}\sum_{S_m(k,(j+1,l-1))}\prod_{i=1}^m\frac{1}{\beta_i![\boldsymbol{\alpha}_i!]^{\beta_i}}\Big(\sum_{\mathbf{0}\preceq\boldsymbol{\gamma}\preceq\boldsymbol{\alpha}_i}\frac{\boldsymbol{\alpha}_i!}{\boldsymbol{\gamma}!(\boldsymbol{\alpha}_i-\boldsymbol{\gamma})!}\partial_{s,t}^{\boldsymbol{\gamma}}\psi_{s,t}\partial_{s,t}^{\boldsymbol{\alpha}_i-\boldsymbol{\gamma}}\bar{\psi}_{s,t}\Big)^{\beta_i}\Big|_{s=t=0}\\ &\qquad+\frac12\sum_{\mathbf{0}\prec\boldsymbol{\gamma}\prec(j+1,l-1)}\frac{(j+1)!(l-1)!}{\boldsymbol{\gamma}!((j+1,l-1)-\boldsymbol{\gamma})!}\partial_{s,t}^{\boldsymbol{\gamma}}\psi_{s,t}\partial_{s,t}^{(j+1,l-1)-\boldsymbol{\gamma}}\bar{\psi}_{s,t}\Big|_{s=t=0},\end{aligned}$$

where for a multi-index $\boldsymbol{\gamma}\in\mathbb{N}^2$, $\partial_{s,t}^{\boldsymbol{\gamma}}=\partial_s^{\gamma_1}\partial_t^{\gamma_2}$. In particular, the expression for the function $\mathfrak{F}_{j,0}$ involves only the coefficients $B_0,\ldots,B_{j-1}$ and $\hat{\psi}_{a,b}$ for $(a,b)\prec(j+1,0)$, whereas for $l\ge1$, the expression for the function $\mathfrak{F}_{j,l}$ involves only $B_0,\ldots,B_j$ and $\hat{\psi}_{a,b}$ for $(a,b)\prec(j+1,l-1)$.

Proof. Again, this follows from an application of Taylor's formula, together with Faà di Bruno's formula. Note that the endpoints of the ordering in the final sum defining $\mathfrak{S}_{j,l}$ are not included in the summation, as a consequence of cancellation in the defining expression

$$\mathfrak{S}_{j,l} = \partial_s^{j+1}\partial_t^{l-1}|\psi_{s,t}|\big|_{s=t=0} - (j+1)!(l-1)!\,\mathrm{Re}\,\big(\bar{\zeta}\hat{\psi}_{j+1,l-1}\big).$$

The fact that the expression for $\mathfrak{S}_{j,l}$ does not include the maximal partial derivative of $\psi_{s,t}$, corresponding to the index $(j+1, l-1)$, is a consequence of an analogue of Proposition 4.5.2 (adapted to include the simpler instance of compositions $f \circ g$ when g is a scalar-valued function, and the gradient of f does not vanish). The omitted details are left to the reader. □

4.7. **The orthogonal foliation flow IV. Putting the pieces together.** We are now ready to formalize the algorithm of Subsection 4.4. We restate the flow equation (4.4.2) in terms of the coefficient functions $\mathfrak{C}_{j,l}$ for the expansion of $\omega_{s,t}$ on the circle:

$$\mathfrak{C}_{j,l} = \begin{cases} (4\pi)^{-\frac{1}{2}} & \text{for} \quad \zeta \in \mathbb{T} \;\text{ and }\; (j,l) = (0,0), \\ 0 & \text{for} \quad \zeta \in \mathbb{T} \;\text{ and }\; (j,l) \in \mathcal{I}_{2\kappa} \setminus \{(0,0)\}. \end{cases} \tag{4.7.1}$$

This property is key to our completing the proof of Proposition 4.2.5. We solve for the coefficients of f_s and $\psi_{s,t}$ iteratively, according to the algorithm outlined in Subsection 4.4.

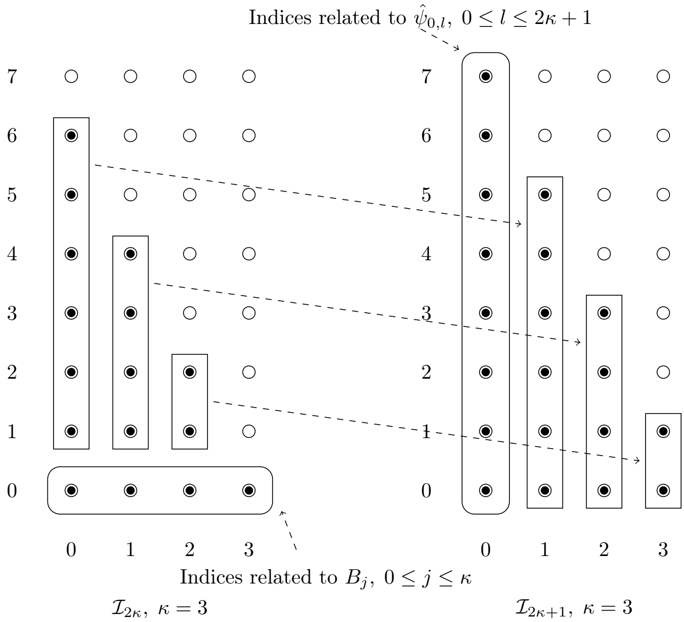

FIGURE 4.1. Illustrations of the index sets $\mathcal{I}_{2\kappa}$ and $\mathcal{I}_{2\kappa+1}$ (marked in black), related to the coefficients $\mathfrak{C}_{j,l}$ (left) and the conformal maps $\hat{\psi}_{j+1,l-1}$ (right), respectively. Indices in sharp boxes are involved in STEP 3, in the determination of the coefficients $\hat{\psi}_{j,l}$ with $j \ge 1$.

Proof of Proposition 4.2.5. In view of Propositions 4.2.6 and 4.6.1, the conformal mapping $\psi_{0,t}$ and its Taylor coefficients $\hat{\psi}_{0,l}$ for $l = 0, 1, \ldots, 2\kappa+1$ with respect to the time parameter t of the flow are well-defined, and they satisfy the required smoothness properties: for t near the origin, $\psi_{0,t}$ extends conformally across the boundary $\mathbb{T}$ to an exterior disk $\mathbb{D}_{\mathrm{e}}(0, \rho'')$ for some

$0<\rho''<1$. Moreover, the derivative $\psi'_{0,t}$ remains uniformly bounded as long as the weight R is confined to a uniform family in $\mathcal{W}(\rho_0,\sigma_0)$ for fixed ρ_0 and σ_0. In addition, the coefficient functions extend holomorphically to $\mathbb{D}_{\mathrm{e}}(0,\rho')$, by the token of Remark 4.2.4. This completes STEP 1 of the algorithmic procedure.

Turning our attention to STEP 2, we recall from Proposition 4.6.1 that on the circle $\mathbb{T}$, we have $\mathrm{Re}(-\bar\zeta\hat\psi_{0,1})=(4\Delta R(\zeta))^{-\frac12}$. Hence, the equation

$$\mathfrak{C}_{0,0}=|B_0(\zeta)|^2\,\mathrm{Re}(-\bar\zeta\hat\psi_{0,1}(\zeta))=(4\pi)^{-\frac12},$$

which is (4.7.1) for $(j,l)=(0,0)$, together with our requirements that the function B_0 be outer on $\mathbb{D}_{\mathrm{e}}$ with $B_0(\infty)>0$, tell us that

$$B_0(\zeta)=(4\pi)^{-\frac14}\exp\big\{\tfrac12\mathbf{H}_{\mathbb{D}_{\mathrm{e}}}[\log(4\Delta R)^{\frac12}](\zeta)\big\}=(4\pi)^{-\frac14}\exp\big\{\tfrac14\mathbf{H}_{\mathbb{D}_{\mathrm{e}}}[\log 4\Delta R](\zeta)\big\}.$$

This formula initially defines B_0 in the exterior disk $\mathbb{D}_{\mathrm{e}}$. In view of the given smoothness of the weight R, and the fact that the polarization of ΔR remains zero-free on $\hat{\mathbb{A}}(\rho,\sigma)$, the function $\log(4\Delta R)$ has a holomorphic polarization on $\hat{\mathbb{A}}(\rho,\sigma)$. It follows from Remark 4.2.4 that B_0 extends holomorphically across the unit circle $\mathbb{T}$ to the domain $\mathbb{D}_{\mathrm{e}}(0,\rho')$ Moreover, B_0 remains uniformly bounded provided that R is confined to a uniform family in $\mathcal{W}(\rho_0,\sigma_0)$. This completes STEP 2.

We proceed to STEP 3 of the algorithmic procedure. We are now in the following situation. For some $j_0\ge1$, the functions $B_0,\ldots,B_{j_0-1}$ and $\hat\psi_{j,l}$ for all $(j,l)\in\mathcal{I}_{2\kappa+1}$ with $(j,l)\prec(j_0,0)$ are already known, and in addition, the relations (4.7.1) are met for all $(j,l)\in\mathcal{I}_{2\kappa}$ with $(j,l)\preceq(j_0-1,0)$. Moreover, all the above-mentioned functions are holomorphic on $\mathbb{D}_{\mathrm{e}}(0,\rho')$. We will now show how this allows us to obtain the relations (4.7.1) for all indices $(j,l)\in\mathcal{I}_{2\kappa}$ with $(j,l)\prec(j_0,0)$, by making appropriate choices of the functions $\hat\psi_{j_0,l-1}$ for $l\ge1$ with $(j_0,l-1)\in\mathcal{I}_{2\kappa+1}$. The only additional tuples in (4.7.1) are those $(j,l)\in\mathcal{I}_{2\kappa}$ of the form $(j,l)=(j_0-1,l)$, where $l\ge1$.

To achieve the above, we assume that we have completed this procedure up to some $l=l_0\ge0$, and turn to the next equation, which reads $\mathfrak{C}_{j_0-1,l_0+1}=0$, as long as $(j_0-1,l_0+1)\in\mathcal{I}_{2\kappa}$. If l_0 is too large for this to happen, we are in fact done. In view of Proposition 4.6.4, the equation $\mathfrak{C}_{j_0-1,l_0+1}=0$ may be written in the form

$$(4\pi)^{-\frac14}(4\Delta R)^{\frac12}\,\mathrm{Re}(\bar\zeta\hat\psi_{j_0,l_0})+\mathfrak{F}_{j_0-1,l_0+1}=0,$$

where $\mathfrak{F}_{j_0-1,l_0+1}$ are real-analytic functions on the circle $\mathbb{T}$ that may be expressed in terms of the known functions $B_0,\ldots,B_{j_0-1}$ and $\hat\psi_{j,l}$ for $(j,l)\in\mathcal{I}_{2\kappa+1}$ with $(j,l)\preceq(j_0,l_0-1)$. We provide a solution to this equation by the formula

$$\hat\psi_{j_0,l_0}=-(4\pi)^{\frac14}\zeta\mathbf{H}_{\mathbb{D}_{\mathrm{e}}}\Big[\frac{\mathfrak{F}_{j_0-1,l_0+1}}{(4\Delta R)^{\frac12}}\Big].$$

The function $\mathfrak{F}_{j_0-1,l_0+1}$ has a polarization which is holomorphic in $(z,\bar w)$ for $(z,w)\in\hat{\mathbb{A}}(\rho,\sigma)$, and the same holds for the weight R. As a consequence, it follows that $\hat\psi_{j_0,l_0}$ extends holomorphically to exterior the disk $\mathbb{D}_{\mathrm{e}}(0,\rho')$, and that $\hat\psi_{j_0,l_0}(\zeta)=\mathrm{O}(|\zeta|)$ with an implicit constant which is uniformly bounded, provided that R is confined to a uniform family in $\mathcal{W}(\rho_0,\sigma_0)$.

We now turn to STEP 4. After the completion of STEP 3, the situation is as follows: The functions $B_0,\ldots,B_{j_0-1}$ and $\hat\psi_{j,l}$ for $(j,l)\in\mathcal{I}_{2\kappa+1}$ with $(j,l)\prec(j_0+1,0)$ are known, and the relations (4.7.1) are met for all $(j,l)\in\mathcal{I}_{2\kappa}$ with $(j,l)\prec(j_0,0)$. In this step, we need to find the function B_{j_0}, and verify that the relation (4.7.1) is then met with $(j,l)=(j_0,0)$. To this end, we apply Proposition 4.6.4, and observe that the equation (4.7.1) with $(j,l)=(j_0,0)$ is equivalent to having

$$\mathfrak{C}_{j_0,0}=2\,\mathrm{Re}\,\big(\bar B_0B_{j_0}\big)(4\Delta R)^{-\frac12}+\mathfrak{F}_{j_0,0}=0,$$

where $\mathfrak{F}_{j_0,0}$ is a real-valued real-analytic function expressed in terms of the known functions $B_0,\ldots,B_{j_0-1}$ and $\hat\psi_{j,l}$ for $(j,l)\in\mathcal{I}_{2\kappa+1}$ with $(j,l)\prec(j_0+1,0)$. The above equation $\mathfrak{C}_{j_0,0}=0$

may be interpreted as an equation for the unknown function B_{j_0}, with solution

$$B_{j_0} = -\frac{1}{2} B_0 \mathbf{H}_{\mathbb{D}_\mathrm{e}}\Big[\frac{(4\Delta R)^{\frac{1}{2}}\mathfrak{F}_{j_0,0}}{|B_0|^2}\Big] = -\pi^{\frac{1}{2}} B_0 \mathbf{H}_{\mathbb{D}_\mathrm{e}}\big[\mathfrak{F}_{j_0,0}\big].$$

This function B_{j_0} extends holomorphically to the exterior disk $\mathbb{D}_\mathrm{e}(0,\rho')$, and remains uniformly bounded if the weight R is confined to a uniform family in $\mathcal{W}(\rho_0,\sigma_0)$. Moreover, we observe that B_{j_0} has the required normalization at infinity: $\operatorname{Im} B_{j_0}(\infty) = 0$.

We finally turn to Step 5. The key observation is that we are now in a position to return to Step 3 followed by Step 4 with j_0 replaced by j_0+1. Since Step 1 and Step 2 combine to form the initial data for Steps 3 and 4 with $j_0 = 1$, the algorithm produces iteratively the entire set of coefficient functions, and solves in the process all the equations (4.7.1) for $(j,l) \in \mathcal{I}_{2\kappa}$.

Equipped with the functions B_j for $j = 0,\ldots,\kappa$, the conformal mappings $\psi_{0,t}$ and the coefficients $\hat{\psi}_{j,l}$ for $(j,l) \in \mathcal{I}_{2\kappa+1} \cap \{(j,l) : j \geq 1\}$, we observe that the functions f_s and $\psi_{s,t}$ given by

$$f_s(\zeta) = \sum_{j=0}^{\kappa} s^j B_j(\zeta) \qquad \psi_{s,t}(\zeta) = \psi_{0,t} + \sum_{\substack{(j,l)\in\mathcal{I}_{2\kappa+1}\\ j\geq 1}} s^j t^l \hat{\psi}_{j,l}(\zeta)$$

are well-defined, and have the desired smoothness and mapping properties. As $\psi_{s,t}$ is a perturbation of the identity, and as the finite collection of coefficient functions all extend holomorphically past the boundary $\mathbb{T}$ of the exterior disk, it follows that the radii ρ'' may be chosen with the stated properties. The conclusion of Proposition 4.2.5 is now an immediate consequence of the relations (4.7.1) for the Taylor coefficients of the function

$$\omega_{s,t}(\zeta) = |f_s \circ \psi_{s,t}(\zeta)|^2 \mathrm{e}^{-2s^{-1}\{(R\circ\psi_{s,t})(\zeta) - \frac{t^2}{2}\}} \operatorname{Re}\big(-\bar{\zeta}\partial_t \psi_{s,t}(\zeta)\overline{\psi'_{s,t}(\zeta)}\big), \qquad \zeta \in \mathbb{T}$$

in the variables (s,t) near $(0,0)$, verified in the above algorithm. These Taylor coefficients were calculated in (4.6.7), (4.6.8) and (4.6.9) and later combined in Proposition 4.6.4. □

4.8. **The orthogonal foliation flow IV. The implementation scheme.** The hard work was completed in the previous subsection. The existence of the orthogonal foliation flow now follows succintly, if we put $s = m^{-1}$. In terms of notation, we change to more convenient subscripts. So, we write $\psi_{n,m,t}$ and $f^{\langle\kappa\rangle}_{n,m}$ when we mean the mapping $\psi_{s,t}$ and the function f_s, respectively, associated with the choices $s = m^{-1}$ and $R = R_\tau$ with $\tau = n/m$. In the previous section the parameter κ was needed but it was suppressed in the notation, here we sometimes prefer to express the dependence explicitly.

Proof of Lemma 4.1.2. In view of Proposition 3.1.3, the collection R_τ of weights with $\tau \in \mathcal{I}_{\epsilon_0}$, is a uniform family in $\mathcal{W}(\rho_0,\sigma_0)$ for some numbers ρ_0,σ_0 with $0<\rho_0<1$ and $\sigma_0>0$. The flow equation (4.1.3) of Lemma 4.1.2 now follows immediately from the assertion of Proposition 4.2.5, together with the observation that

$$\begin{aligned} m^{\frac{1}{2}} &\int_{\mathcal{D}_{n,m}} |f^{\langle\kappa\rangle}_{n,m}|^2 \mathrm{e}^{-2mR_\tau}\mathrm{dA} \\ &= 2m^{\frac{1}{2}} \int_{-\delta_m}^{\delta_m}\int_{\mathbb{T}} \big|f^{\langle\kappa\rangle}_{n,m}\circ\psi_{n,m,t}(\zeta)\big|^2 \mathrm{e}^{-2m\,R_\tau\circ\psi_{n,m,t}(\zeta)} \operatorname{Re}\big(-\bar{\zeta}\partial_t\psi_{n,m,t}(\zeta)\overline{\psi'_{n,m,t}(\zeta)}\big)\mathrm{ds}(\zeta)\mathrm{d}t \\ &\qquad = 2m^{\frac{1}{2}}\int_{-\delta_m}^{\delta_m}\big((4\pi)^{-\frac{1}{2}} + \mathrm{O}(\delta_m^{2\kappa+1})\big)\mathrm{e}^{-mt^2}\mathrm{d}t = 1 + \mathrm{O}(\delta_m^{2\kappa+1}) = 1 + \mathrm{O}(m^{-\kappa-\frac{1}{3}}), \end{aligned}$$

where we use (4.1.2) to integrate over the flow in the coordinates $(t,\zeta) \in [-\delta_m,\delta_m]\times\mathbb{T}$. Moreover, we observe that $f^{\langle\kappa\rangle}_{n,m}$ is zero-free in $\mathbb{D}_\mathrm{e}(0,\rho')$ for large enough m, as the leading term $B_{0,\tau}$ has this property. Here, we recall that ρ' is defined in terms of ρ and σ' in Proposition 4.2.2. □

We next turn to the result on the structure of the normalized approximately orthogonal quasipolynomials, Proposition 4.1.1.

Proof of Proposition 4.1.1. We recall the definition of the canonical positioning operator $\mathbf{\Lambda}_{n,m}$ from Subsection 3.1, and write

$$F_{n,m}^{\langle\kappa\rangle}(z) = m^{\frac{1}{4}}\mathbf{\Lambda}_{n,m}[f_{n,m}^{\langle\kappa\rangle}](z), \tag{4.8.1}$$

where $f_{n,m}^{\langle\kappa\rangle}$ is the function from Lemma 4.1.2, which is defined and holomorphic on $\mathcal{K}_\tau^c$, for some compact subset $\mathcal{K}_\tau$ of the interior $\mathcal{S}_\tau^\circ$ of $\mathcal{S}_\tau$, which stays away from the boundary $\partial\mathcal{S}_\tau$ in the sense that

$$\inf_{\tau\in\mathcal{I}_{\epsilon_0}} \mathrm{dist}_{\mathbb{C}}(\mathcal{K}_\tau, \partial\mathcal{S}_\tau) > 0.$$

We recall that $\chi_{0,\tau}$ denotes a smooth cut-off function which vanishes on $\mathcal{K}_\tau$ and equals 1 on $\mathcal{X}^\tau$, where $\mathcal{X}_\tau$ is an intermediate set between $\mathcal{K}_\tau$ and $\mathcal{S}_\tau^c$. In line with Remark 1.3.4(a), we may insert a further intermediate set $\mathcal{X}_\tau'$ between $\mathcal{K}_\tau$ and $\mathcal{X}_\tau$, such that $\chi_{0,\tau}$ vanishes on $\mathcal{X}_\tau'$ as well (and not just on $\mathcal{K}_\tau$).

The functions $f_{n,m}^{\langle\kappa\rangle}$ are bounded and holomorphic on $\mathcal{K}_\tau^c$. As the leading term $B_{0,\tau}$ in the expansion of $f_{n,m}^{\langle\kappa\rangle}$ does not vanish at infinity, it follows that for large enough m, the same can be said for $f_{n,m}^{\langle\kappa\rangle}$. In view of this, the functions $F_{n,m}^{\langle\kappa\rangle}$ are quasipolynomials of order n on $\mathcal{K}_\tau^c$ in the sense of Definition 3.1.1. Moreover, the definition (4.8.1) of the functions $F_{n,m}^{\langle\kappa\rangle}$ together with the definition (3.3.1) of the functions $f_{n,m}^{\langle\kappa\rangle}$ shows that $F_{n,m}^{\langle\kappa\rangle}$ has the indicated form. What remains for us to do is to verify the properties (i), (ii), and (iii) of Definition 3.1.2.

To this end, we recall the definition of the domain $\mathcal{D}_{n,m}$ from Lemma 4.1.2, which is a certain closed neighbourhood of the unit circle which arises from our orthogonal foliation flow. We recall that

$$\mathrm{dist}_{\mathbb{C}}(\mathcal{D}_{n,m}^c, \mathbb{T})) \geq c_0\delta_m$$

holds for some fixed constant $c_0 > 0$. We first check property (ii) of Definition 3.1.2. As a step in this direction, we claim that most of the weighted L^2-mass of the function $\chi_{1,\tau}f_{n,m}^{\langle\kappa\rangle}$ lies in the domain $\mathcal{D}_{n,m}$. We know that the functions $f_{n,m}^{\langle\kappa\rangle}$ are bounded uniformly in $\mathcal{K}_\tau^c$ independently of m and n while $\tau \in I_{\epsilon_0}$, so that

$$\chi_{1,\tau}|f_{n,m}^{\langle\kappa\rangle}| \leq C_0 \tag{4.8.2}$$

holds in the whole plane $\mathbb{C}$, for some constant C_0. Let $\mathcal{D}_\tau$ denote a bounded domain which only depends only on τ and contains $\mathbb{D}\cup\mathcal{D}_{n,m}$, such that the bound from below $R_\tau(z) \geq \theta_0\log|z|$ holds outside $\mathcal{D}_\tau$, for some $\theta_0 > 0$. That such a domain exists for sufficiently large m is shown in Proposition 3.1.3. On the other hand, as $R_\tau \in \mathcal{W}(\rho_0, \sigma_0)$ in the sense of Definition 4.2.1, on the set $\mathcal{D}_\tau \cap \mathbb{D}_{\mathrm{e}}(0,\rho_0)\setminus\mathcal{D}_{n,m}$ we have the estimate

$$\mathrm{e}^{-2mR_\tau} \leq \mathrm{e}^{-\alpha_0(\log m)^2}$$

for some constant $\alpha_0 > 0$, at least if $\mathcal{D}_\tau$ is chosen small enough. Here, we recall that ρ_0 is a radius with $0 < \rho_0 < 1$. It now follows that we have

$$\begin{aligned} m^{\frac{1}{2}}\int_{\mathbb{C}\setminus\mathcal{D}_{n,m}} \chi_{1,\tau}^2|f_{n,m}^{\langle\kappa\rangle}|^2\mathrm{e}^{-2mR_\tau} &\leq C_0^2 m^{\frac{1}{2}}\int_{\mathbb{C}\setminus\mathcal{D}_\tau} \mathrm{e}^{-2m\theta_0\log|z|}\mathrm{dA} \\ &+ C_0^2 m^{\frac{1}{2}}\int_{\mathcal{D}_\tau\cap\mathbb{D}(0,\rho_0)\setminus\mathcal{D}_{n,m}} \mathrm{e}^{-\alpha_0(\log m)^2}\mathrm{dA} = \mathrm{O}(m^{\frac{1}{2}}\mathrm{e}^{-\alpha_0(\log m)^2}) = \mathrm{O}(m^{-\alpha_0\log m+\frac{1}{2}}). \end{aligned} \tag{4.8.3}$$

In the above calculation, we observe that the integral over $\mathbb{C}\setminus\mathcal{D}_\tau$ gives an exponentially small contribution, dominated by the indicated error term. It follows that

$$m^{\frac12}\int_{\mathbb{C}}\chi_{1,\tau}^2|f_{n,m}^{\langle\kappa\rangle}|^2\mathrm{e}^{-2mR_\tau}\mathrm{dA}=m^{\frac12}\int_{\mathcal{D}_{n,m}}|f_{n,m}^{\langle\kappa\rangle}|^2\mathrm{e}^{-2mR_\tau}\mathrm{dA}\\ +m^{\frac12}\int_{\mathbb{C}\setminus\mathcal{D}_{n,m}}\chi_{1,\tau}^2|f_{n,m}^{\langle\kappa\rangle}|^2\mathrm{e}^{-2mR_\tau}\mathrm{dA}=1+\mathrm{O}(m^{-\kappa-\frac13}),$$

where we use that $\chi_{1,\tau}=1$ holds on $\mathcal{D}_{n,m}$ together with our foliation flow Lemma 4.1.2 and the estimate (4.8.3). Hence, by the isometric property of $\mathbf{\Lambda}_{n,m}$ from Proposition 3.1.5, it follows that

$$\int_{\mathbb{C}}\chi_{0,\tau}^2|F_{n,m}^{\langle\kappa\rangle}|^2\mathrm{e}^{-2mQ}\mathrm{d}A=1+\mathrm{O}(m^{-\kappa-\frac13}),$$

as required by property (i) of Definition 3.1.2.

We turn to property (i) of Definition 3.1.2, the approximate orthogonality property. For a polynomial $p\in\mathrm{Pol}_n$ of degree at most $n-1$, we put $g=\mathbf{\Lambda}_{n,m}^{-1}[p]$. The function $f_{n,m}^{\langle\kappa\rangle}$ is zero-free in a neighbourhood of the exterior disk $\bar{\mathbb{D}}_\mathrm{e}$, which we may assume to be a fixed exterior disk $\mathbb{D}_\mathrm{e}(0,\rho_0)$ for some fixed $\rho_0<1$ for all large enough n and m with $\tau=\frac{n}{m}\in I_{\epsilon_0}$, and by the isometric property of $\mathbf{\Lambda}_{n,m}$, we find that

$$\int_{\mathbb{C}}\chi_{0,\tau}p\,\overline{F_{n,m}^{\langle\kappa\rangle}}\mathrm{e}^{-2mQ}\mathrm{dA}=m^{\frac14}\int_{\mathbb{C}}\chi_{1,\tau}g\,\overline{f_{n,m}^{\langle\kappa\rangle}}\mathrm{e}^{-2mR_\tau}\mathrm{dA}(z)\\ =m^{\frac14}\int_{\mathcal{D}_{n,m}}\frac{g}{f_{n,m}^{\langle\kappa\rangle}}|f_{n,m}^{\langle\kappa\rangle}|^2\mathrm{e}^{-2mR_\tau}\mathrm{dA}+\mathrm{O}(m^{-\frac{\alpha_0}{2}\log m+\frac34}\|p\|_{2mQ}), \tag{4.8.4}$$

where we are required to justify the indicated error term estimate. To do this, we need Proposition 2.2.2, or more accurately, Lemma 3.5 in [1], which gives the estimate for $p\in\mathrm{Pol}_n$

$$|p|\le C_1m^{\frac12}\|p\|_{2mQ}\mathrm{e}^{m\hat{Q}_\tau} \tag{4.8.5}$$

in the whole plane $\mathbb{C}$ for some constant C_1, independent of $\tau=\frac{n}{m}\in I_{\epsilon_0}$. The missing term on the right-hand side of (4.8.4) equals

$$m^{\frac14}\int_{\mathbb{C}\setminus\mathcal{D}_{n,m}}\chi_{1,\tau}g\,\overline{f_{n,m}^{\langle\kappa\rangle}}\mathrm{e}^{-2mR_\tau}\mathrm{dA}=\int_{\mathbb{C}\setminus\phi_\tau^{-1}(\mathcal{D}_{n,m})}\chi_{0,\tau}p\,\overline{F_{n,m}^{\langle\kappa\rangle}}\mathrm{e}^{-2mQ}\mathrm{dA},$$

and if we apply the pointwise estimate (4.8.5), we obtain

$$\int_{\mathbb{C}\setminus\phi_\tau^{-1}(\mathcal{D}_{n,m})}\chi_{0,\tau}|p\,F_{n,m}^{\langle\kappa\rangle}|\,\mathrm{e}^{-2mQ}\mathrm{dA}\le C_1m^{\frac12}\int_{\mathbb{C}\setminus\phi_\tau^{-1}(\mathcal{D}_{n,m})}\chi_{0,\tau}|F_{n,m}^{\langle\kappa\rangle}|\,\mathrm{e}^{-2mQ+m\hat{Q}_\tau}\mathrm{dA}\\ =C_1m^{\frac34}\int_{\mathbb{C}\setminus\mathcal{D}_{n,m}}\chi_{1,\tau}|f_{n,m}^{\langle\kappa\rangle}|\,\mathrm{e}^{m(\hat{Q}_\tau-Q)\circ\phi_\tau^{-1}-mR_\tau}\mathrm{dA}\le C_0C_1m^{\frac34}\int_{\mathbb{D}_\mathrm{e}(0,\rho_0)\setminus\mathcal{D}_{n,m}}\mathrm{e}^{-mR_\tau}\mathrm{dA},$$

where in the last step, we applied the estimate (4.8.2) and the fact that $\hat{Q}_\tau\le Q$. The rest of the argument that gives (4.8.4) involves splitting the domain of integration using the set $\mathcal{D}_\tau$, and proceeds as in (4.8.3). This establishes (4.8.4), although we still need to control the main term on the right-hand side. To this end, we denote by h the ratio $h=g/f_{n,m}^{\langle\kappa\rangle}$. Then h is holomorphic outside $\mathcal{K}_\tau$, and vanishes at infinity, since $f_{n,m}$ is zero-free there. Using the foliation flow as coordinates on $\mathcal{D}_{n,m}$ in terms of $(t,\zeta)\in[-\delta_m,\delta_m]\times\mathbb{T}$, we find as in the above proof of Lemma 4.1.2 that

$$
\begin{aligned}
(4.8.6)\quad m^{\frac14}\int_{\mathcal{D}_{n,m}} & h(z)|f_{n,m}^{\langle\kappa\rangle}(z)|^2\mathrm{e}^{-2mR_\tau(z)}\mathrm{dA}(z) \\
&= 2m^{\frac14}\int_{-\delta_m}^{\delta_m}\int_{\mathbb{T}} h\circ\psi_{n,m,t}(\zeta)\big|f_{n,m}^{\langle\kappa\rangle}\circ\psi_{n,m,t}(\zeta)\big|^2\mathrm{e}^{-2m\,R_\tau\circ\psi_{n,m,t}(\zeta)} \\
&\qquad\times\operatorname{Re}\big\{-\bar\zeta\partial_t\psi_{n,m,t}(\zeta)\overline{\psi'_{n,m,t}(\zeta)}\big\}\mathrm{ds}(\zeta)\mathrm{d}t \\
&= 2m^{\frac14}\int_{-\delta_m}^{\delta_m}\int_{\mathbb{T}} h\circ\psi_{n,m,t}(\zeta)\Big\{(4\pi)^{-\frac12}\mathrm{e}^{-mt^2}+\mathrm{O}\big(m^{-\kappa-\frac13}\mathrm{e}^{-mt^2}\big)\Big\}\mathrm{ds}(\zeta)\mathrm{d}t \\
&= \mathrm{O}\bigg(m^{-\kappa-\frac{1}{12}}\int_{-\delta_m}^{\delta_m}\int_{\mathbb{T}}|h\circ\psi_{n,m,t}(\zeta)|\mathrm{ds}(\zeta)\,\mathrm{e}^{-mt^2}\mathrm{d}t\bigg).
\end{aligned}
$$

Here, the crucial reduction in the last step of (4.8.6) is based on the fact that the function $h\circ\psi_{n,m,t}$ is holomorphic in $\bar{\mathbb{D}}_{\mathrm{e}}$ and vanishes at infinity, so that by the mean value property

$$\int_{\mathbb{T}} h\circ\psi_{n,m,t}\,\mathrm{ds}=0.$$

Now that (4.8.6) is established, we need to simplify the error term further. We will use the observation that all the steps before the last in (4.8.6) apply to a fairly general sufficiently integrable function in place of h, for instance $|h|$ will work. It then follows from (4.8.6) with $|h|$ instead that large enough m, we have

$$
\begin{aligned}
\int_{-\delta_m}^{\delta_m}\int_{\mathbb{T}}|h\circ\psi_{n,m,t}(\zeta)|\,\mathrm{e}^{-mt^2}\mathrm{ds}(\zeta)\mathrm{d}t &\le 2\int_{\mathcal{D}_{n,m}}|h(z)|\,|f_{n,m}^{\langle\kappa\rangle}(z)|^2\mathrm{e}^{-2mR_\tau(z)}\mathrm{dA}(z) \\
&= 2\int_{\mathcal{D}_{n,m}}|g(z)\,f_{n,m}^{\langle\kappa\rangle}(z)|\mathrm{e}^{-2mR_\tau(z)}\mathrm{dA}(z)\le 2C_0\int_{\mathcal{D}_{n,m}}|g(z)|\,\mathrm{e}^{-2mR_\tau(z)}\mathrm{dA}(z),
\end{aligned}
$$

where in the last step we applied the bound (4.8.2). Finally, we apply the Cauchy-Schwarz inequality, and recall that recall that $g=\mathbf{\Lambda}_{n,m}^{-1}[p]$ where $\mathbf{\Lambda}_{n,m}$ has the isometry property of Proposition 3.1.5:

$$
\begin{aligned}
(4.8.7)\quad \int_{-\delta_m}^{\delta_m}\int_{\mathbb{T}}|h\circ\psi_{n,m,t}(\zeta)|\,\mathrm{e}^{-mt^2}\mathrm{ds}(\zeta)\mathrm{d}t &\le 2C_0\int_{\mathcal{D}_{n,m}}|g(z)|\,\mathrm{e}^{-2mR_\tau(z)}\mathrm{dA}(z) \\
&\le 2C_0\|g\|_{L^2(\mathcal{D}_{n,m},\mathrm{e}^{-2mR_\tau})}\bigg\{\int_{\mathcal{D}_{n,m}}\mathrm{e}^{-2mR_\tau}\mathrm{dA}\bigg\}^{1/2}=\mathrm{O}\big(m^{-\frac14}\|p\|_{2mQ}\big).
\end{aligned}
$$

Here, we used a simple decay estimate of the integral of the "Gaussian ridge" e^{-2mR_τ}. Next, we write $g/f_{n,m}^{\langle\kappa\rangle}$ in place of h, and combine the estimates (4.8.6) and (4.8.7), and arrive at

$$
\begin{aligned}
(4.8.8)\quad m^{\frac14}\int_{\mathcal{D}_{n,m}} g\,\overline{f_{n,m}^{\langle\kappa\rangle}}\mathrm{e}^{-2mR_\tau(z)}\mathrm{dA}(z) &= m^{\frac14}\int_{\mathcal{D}_{n,m}} h(z)|f_{n,m}^{\langle\kappa\rangle}(z)|^2\mathrm{e}^{-2mR_\tau(z)}\mathrm{dA}(z) \\
&= \mathrm{O}\big(m^{-\kappa-\frac13}\|p\|_{2mQ}\big).
\end{aligned}
$$

In view of (4.8.4) and (4.8.8), we find that for all polynomials $p\in\mathrm{Pol}_n$,

$$(4.8.9)\qquad \int_{\mathbb{C}}\chi_{0,\tau}p\,\overline{F_{n,m}^{\langle\kappa\rangle}}\mathrm{e}^{-2mQ}\mathrm{dA}=\mathrm{O}\big(m^{-\kappa-\frac13}\|p\|_{2mQ}\big),$$

as required. Since in addition, $f_{n,m}^{\langle\kappa\rangle}(\infty)>0$ by construction, and we have made sure that $\mathcal{Q}_\tau(\infty)\in\mathbb{R}$ as well as $\phi'_\tau(\infty)>0$, the leading coefficient of the quasipolynomial $F_{n,m}^{\langle\kappa\rangle}$ is now positive, which settles property (iii) of Definition 3.1.2 as well. This completes the proof. □

4.9. **Polynomialization of the quasipolynomials.** We have by now constructed our quasipolynomials $F_{n,m}^{\langle\kappa\rangle}$, of degree n and accuracy κ, and shown that they are approximately orthogonal and normalized. It remains to show that they are indeed good approximations of the true normalized orthogonal polynomials $P_{n,m}$.

Proof of Theorem 1.3.3. We retain the above notation, and consider the $\bar\partial$-problem

$$\bar\partial_z u(z) = F_{n,m}^{\langle\kappa\rangle}(z)\bar\partial_z\chi_{0,\tau}(z).$$

In view of Corollary 2.4.2, the $L^2_{2mQ,n}$-norm minimal solution u_0, which then has the growth $u_0(z) = \mathrm{O}(|z|^{n-1})$ near infinity, enjoys the norm bound

$$\int_{\mathbb{C}} |u_0|^2 \mathrm{e}^{-2mQ}\mathrm{dA} \le \frac{1}{\alpha_1 m}\int_{\mathcal{S}_\tau} |F_{n,m}^{\langle\kappa\rangle}|^2|\bar\partial\chi_{0,\tau}|^2\mathrm{e}^{-2mQ}\mathrm{dA}, \tag{4.9.1}$$

where $\alpha_1 > 0$ stands for the minimum of ΔQ on the biggest droplet $\mathcal{S}_\tau$ with $\tau\in I_{\epsilon_0}$ (which is attained for the rightmost endpoint $\tau = 1+\epsilon_0$). Next, given that the quasipolynomials of degree n are of the form $F_{n,m}^{\langle\kappa\rangle} = m^{\frac14}\mathbf{\Lambda}_{n,m}[f_{n,m}^{\langle\kappa\rangle}]$, where the functions $f_{n,m}^{\langle\kappa\rangle}$ are uniformly bounded in $\mathbb{D}_{\mathrm{e}}(0,\rho_0)$ for some radius $\rho_0<1$, we find that

$$\begin{aligned}\int_{\mathcal{S}_\tau} |F_{n,m}^{\langle\kappa\rangle}|^2|\bar\partial\chi_{0,\tau}|^2\mathrm{e}^{-2mQ}\mathrm{dA} &= m^{\frac12}\int_{\mathbb{D}} |f_{n,m}^{\langle\kappa\rangle}|^2|\bar\partial\chi_{1,\tau}|^2|\phi_\tau'\circ\phi_\tau^{-1}|^2\mathrm{e}^{-2mR_\tau}\mathrm{dA}\\ &= \mathrm{O}(m^{\frac12}\mathrm{e}^{-\alpha_2 m})\end{aligned} \tag{4.9.2}$$

for some $\alpha_2>0$ such that $2R_\tau\ge\alpha_2$ on the support of $\bar\partial\chi_{1,\tau}$. This exponential decay estimate is possible since the support of $\bar\partial\chi_{1,\tau}$ is located inside $\mathbb{D}$ away from the boundary. Note that in the context of the estimate (4.9.2) it is important as well that the expression $|\phi_\tau'\circ\phi_\tau^{-1}|^2$ is uniformly bounded on the support of $\bar\partial\chi_{1,\tau}$ as well. If we combine the above estimates (4.9.1) and (4.9.2), we find that

$$\int_{\mathbb{C}} |u_0|^2\mathrm{e}^{-2mQ}\mathrm{dA} = \mathrm{O}(m^{-\frac12}\mathrm{e}^{-\alpha_2 m}), \tag{4.9.3}$$

as $m\to\infty$ while $\tau=\frac{n}{m}\in I_{\epsilon_0}$, with a uniform implicit constant. Next, we put

$$P_{n,m}^\star := F_{n,m}^{\langle\kappa\rangle}\chi_{0,\tau} - u_0(z)$$

which is then automatically a polynomial of degree n, since the function is entire and grows $\asymp |z|^n$ near infinity. Moreover, in view of (4.9.3), this polynomial is very close to the function $F_{n,m}^{\langle\kappa\rangle}\chi_{0,\tau}$ in the norm of $L^2(\mathbb{C},\mathrm{e}^{-2mQ})$:

$$\int_{\mathbb{C}} |P_{n,m}^\star - F_{n,m}^{\langle\kappa\rangle}\chi_{0,\tau}|^2\mathrm{e}^{-2mQ}\mathrm{dA} = \int_{\mathbb{C}}|u_0|^2\mathrm{e}^{-2mQ}\mathrm{dA} = \mathrm{O}(m^{-\frac12}\mathrm{e}^{-\alpha_2 m}). \tag{4.9.4}$$

It now follows from (4.8.9) and (4.9.4) that of the function for all polynomials $p\in\mathrm{Pol}_n$ of degree $\le n-1$,

$$\int_{\mathbb{C}} P_{n,m}^\star\bar p\,\mathrm{e}^{-2mQ}\mathrm{dA} = \mathrm{O}(m^{-\kappa-\frac13}\|p\|_{2mQ}), \tag{4.9.5}$$

while

$$\int_{\mathbb{C}} |P_{n,m}^\star|^2\mathrm{e}^{-2mQ}\mathrm{dA} = 1+\mathrm{O}(m^{-\kappa-\frac13}). \tag{4.9.6}$$

We observe also that by duality, (4.9.5) asserts that

$$\|\mathbf{P}_{n,m}P_{n,m}^\star\|_{2mQ} = \mathrm{O}(m^{-\kappa-\frac13}), \tag{4.9.7}$$

where $\mathbf{P}_{n,m}$ denotes the orthogonal projection in $L^2(\mathbb{C},\mathrm{e}^{-2mQ})$ onto the subspace Pol_n of polynomials of degree $\le n-1$. If we use this to correct the polynomial $P_{n,m}^\star$, and put $\tilde P_{n,m} := \mathbf{P}_{n,m}^\perp P_{n,m}^\star = P_{n,m}^\star - \mathbf{P}_{n,m}P_{n,m}^\star$, then automatically $\tilde P_{n,m}$ has degree n and it is also orthogonal to all the lower degree polynomials. As a consequence, $\tilde P_{n,m}$ must be a scalar multiple of $P_{n,m}$, the orthogonal polynomial we are looking for, which we write as $\tilde{P}_{n,m} = cP_{n,m}$ for a constant c. Putting things together so far, we have obtained that

$$\big\|\tilde{P}_{n,m} - F_{n,m}^{\langle\kappa\rangle}\chi_{0,\tau}\big\|_{2mQ} = \mathrm{O}(m^{-\kappa-\frac{1}{3}}) \tag{4.9.8}$$

with a uniform implied constant. Moreover, by (4.9.6) and (4.9.7), the norm of $\tilde{P}_{n,m}$ equals

$$|c| = \|cP_{n,m}\|_{2mQ} = \big\|\tilde{P}_{n,m}\big\|_{2mQ} = 1 + \mathrm{O}(m^{-\kappa-\frac{1}{3}}), \tag{4.9.9}$$

Next, by Proposition 2.2.2, it follows from (4.9.8) that

$$\big|cP_{n,m} - F_{n,m}^{\langle\kappa\rangle}\big| = \big|\tilde{P}_{n,m} - F_{n,m}^{\langle\kappa\rangle}\big| = \mathrm{O}(m^{-\kappa+\frac{1}{6}}\mathrm{e}^{m\hat{Q}_\tau}) \tag{4.9.10}$$

holds in $\mathbb{C} \setminus \mathcal{S}_\tau$. Since by construction

$$\lim_{|z|\to\infty} z^{-n}F_{n,m}^{\langle\kappa\rangle}(z) > 0,$$

and the function $\mathrm{e}^{m\hat{Q}_\tau}$ grows like $|z|^{-n}$ as $|z| \to \infty$, it follows from (4.9.10) that the above constant $c = c_{n,m}$ must have

$$\frac{\operatorname{Im} c}{\operatorname{Re} c} = \mathrm{O}\big(m^{-\kappa-\frac{1}{12}}\big),$$

as $n, m \to \infty$ while $\tau = \frac{n}{m} \in I_{\epsilon_0}$. Hence we may write $c = \gamma c'$, where c' is real and positive and $\gamma \in \mathbb{C}$ with $\gamma = 1 + \mathrm{O}(m^{-\kappa-\frac{1}{12}})$. Taking into account the relation (4.9.9) as well, we find that $c' = 1 + \mathrm{O}(m^{-\kappa-\frac{1}{12}})$. It now follows from this observation combined with (4.9.8) that

$$\|P_{n,m} - \chi_{0,\tau}F_{n,m}^{\langle\kappa\rangle}\|_{2mQ} = \mathrm{O}(m^{-\kappa-\frac{1}{12}}).$$

This falls slightly short of allowing us to obtain Theorem 1.3.3 right away. The problem is that our error term is larger than what is claimed. However, since the precision κ is arbitrary, we might as well replace κ by $\kappa + 1$ and see what we get. This would give that

$$\|P_{n,m} - \chi_{0,\tau}F_{n,m}^{\langle\kappa+1\rangle}\|_{2mQ} = \mathrm{O}(m^{-\kappa-1-\frac{1}{12}}). \tag{4.9.11}$$

By analyzing a single term in the asymptotic expansion, it is easy to verify that

$$\|\chi_{0,\tau}F_{n,m}^{\langle\kappa+1\rangle} - \chi_{0,\tau}F_{n,m}^{\langle\kappa\rangle}\|_{2mQ} = \mathrm{O}(m^{-\kappa-1}),$$

and hence the assertion of the theorem immediate from this estimate and (4.9.11). □

Proof of Theorem 1.3.5. The quasipolynomials $F_{n,m}^{\langle\kappa\rangle}$ that we obtained above can be written in the form

$$F_{n,m}^{\langle\kappa\rangle} = \Big(\frac{m}{2\pi}\Big)^{\frac{1}{4}}\sqrt{\phi_\tau'}[\phi_\tau]^n\mathrm{e}^{m\mathcal{Q}_\tau}\sum_{j=0}^{\kappa} m^{-j}\mathcal{B}_{j,\tau},$$

where $\mathcal{B}_{j,\tau}$ are uniformly bounded, and holomorphic in a fixed neighbourhood of the exterior disk $\bar{\mathbb{D}}_\mathrm{e}$. To obtain the theorem, we need to show that $F_{n,m}^{\langle\kappa\rangle}$ is close to $P_{n,m}$ pointwise in some domains $\mathcal{K}_{m,\tau}^c$, where $\mathcal{K}_\tau$ are compact subsets of $\mathcal{S}_\tau$ that remain far enough away from the boundary $\partial\mathcal{S}_\tau$. To this end, we consider the level sets $\gamma_{t,\tau}$

$$\gamma_{t,\tau} := \{z \in \mathcal{S}_\tau \,:\, \operatorname{dist}_\mathbb{C}(z, \partial\mathcal{S}_\tau) = t\},$$

and observe that by the implicit function theorem, these sets are closed simple curves, provided that t is small enough. Next, we define the compact sets $\mathcal{K}_{m,\tau}$ as the closure of the bounded component of $\mathbb{C}\setminus\gamma_{t_m,\tau}$, where $t_m = (m^{-1}\log\log m)^{\frac{1}{2}}$. On the set $\mathbb{C}\setminus\mathcal{K}_{m,\tau}$, we have the estimate

$$0 \le m(\hat{Q}_\tau - \breve{Q})(z) \le D\log\log m,$$

where D is some positive constant, which is uniformly bounded while $\tau \in I_{\epsilon_0}$. Thus

$$\mathrm{e}^{m(\hat{Q}_\tau - \breve{Q}_\tau)} \le \mathrm{e}^{D\log\log m} = (\log m)^D,$$

which grows slower than m^ε for arbitrarily small ε, and in particular slower than $m^{\frac{1}{6}}$. In view of Theorem 1.3.3, and the pointwise estimate of Proposition 2.2.2 applied to the intermediate set $\mathcal{X}_\tau$ between $\mathcal{K}_\tau$ and $\mathcal{S}_\tau^c$ where the cut-off function $\chi_{0,\tau}$ assumes the value 1, we find that

$$|P_{n,m}(z) - F_{n,m}^{\langle\kappa\rangle}(z)| = \mathrm{O}\big(m^{-\kappa-\frac{1}{2}}\mathrm{e}^{m\hat{Q}_\tau(z)}\big) = \mathrm{O}\big(m^{-\kappa-\frac{1}{3}}\mathrm{e}^{m\check{Q}_\tau(z)}\big), \qquad z \in \mathcal{K}_{m,\tau}^c,$$

where the implicit constant again is uniform in the relevant parameter range. We may rephrase this as saying that

$$P_{n,m}(z) = F_{n,m}^{\langle\kappa\rangle}(z) + \mathrm{O}\big(m^{-\kappa-\frac{1}{3}}\mathrm{e}^{m\check{Q}_\tau(z)}\big) = \Big(\frac{m}{2\pi}\Big)^{\frac{1}{4}}\sqrt{\phi_\tau'}[\phi_\tau]^n\mathrm{e}^{m\mathcal{Q}_\tau}\Big(\sum_{j=0}^{\kappa}\mathcal{B}_{j,\tau} + \mathrm{O}\big(m^{-\kappa-\frac{7}{12}}\big)\Big),$$

for $z \in \mathcal{K}_{m,\tau}^c$. This essentially proves the theorem, except that the error term is now slightly worse than claimed. However, we may fix this by replacing κ by $\kappa+1$ in the above argument, to obtain

$$\begin{aligned}P_{n,m}(z) = \Big(\frac{m}{2\pi}\Big)^{\frac{1}{4}}\sqrt{\phi_\tau'}[\phi_\tau]^n\mathrm{e}^{m\mathcal{Q}_\tau}\Big(\sum_{j=0}^{\kappa+1}m^{-j}\mathcal{B}_{j,\tau} + \mathrm{O}(m^{-\kappa-\frac{19}{12}})\Big)\\ = \Big(\frac{m}{2\pi}\Big)^{\frac{1}{4}}\sqrt{\phi_\tau'}[\phi_\tau]^n\mathrm{e}^{m\mathcal{Q}_\tau}\Big(\sum_{j=0}^{\kappa}m^{-j}\mathcal{B}_{j,\tau} + \mathrm{O}(m^{-\kappa-1})\Big),\end{aligned}$$

where the last step follows since the function $\mathcal{B}_{\kappa+1,\tau}$ is bounded in the relevant region. The proof is complete. □

5. Boundary Universality in the Random normal matrix model

5.1. **Uniform asymptotics near $\tau = 1$.** We take as our starting point the first term of the asymptotic expansion of Theorem 1.3.5. Recall from Theorem 1.3.3 that $\mathcal{K}_{m,\tau}$ is a compact subset of $\mathcal{S}_\tau$ with $\mathrm{dist}_{\mathbb{C}}(\mathcal{K}_{m,\tau}, \mathcal{S}_\tau^c) \geq (m^{-1}\log\log m)^{1/2}$.

Corollary 5.1.1. *Let $\mathcal{H}_{Q,\tau}$ be the bounded holomorphic function in the set $\mathcal{K}_\tau^c$ with real part $\mathrm{Re}\,\mathcal{H}_{Q,\tau} = \frac{1}{4}\log(2\Delta Q)$ on the boundary $\partial\mathcal{S}_\tau$, which is real-valued at infinity. Then, in the limit as $m, n \to \infty$ while $\tau = \frac{n}{m} \in I_{\epsilon_0}$, we have the asymptotics*

$$|P_{n,m}(z)|^2\mathrm{e}^{-2mQ(z)} = \sqrt{\frac{m}{\pi}}|\phi_\tau'(z)|\,\mathrm{e}^{-2m(Q-\check{Q}_\tau)(z)}\Big(\mathrm{e}^{2\,\mathrm{Re}\,\mathcal{H}_{Q,\tau}(z)} + \mathrm{O}\big(m^{-1}\big)\Big),$$

where the implied constant is uniform for $z \in \mathcal{K}_{m,\tau}^c$.

Proof. We recall that

$$\check{Q}_\tau = \mathrm{Re}\,\mathcal{Q}_\tau + \tau\log|\phi_\tau| = \mathrm{Re}\,\mathcal{Q}_\tau + \tfrac{n}{m}\log|\phi_\tau|,$$

and in view of Theorems 1.3.5 and 1.3.7, we may write

$$\begin{aligned}|P_{n,m}|^2 = \sqrt{\frac{m}{\pi}}|\phi_\tau'(z)||\phi_\tau|^{2n}\mathrm{e}^{2m\,\mathrm{Re}\,\mathcal{Q}_\tau}\big|\mathcal{B}_{0,\tau} + \mathrm{O}(m^{-1})\big|^2\\ = \sqrt{\frac{m}{\pi}}|\phi_\tau'(z)|\mathrm{e}^{2m\check{Q}_\tau}\Big(\mathrm{e}^{2\,\mathrm{Re}\,\mathcal{H}_{Q,\tau}(z)} + \mathrm{O}(m^{-1})\Big),\end{aligned}$$

and the assertion follows. □

We now obtain Theorem 1.4.1.

5.2. **Error function asymptotics.** In view of Corollary 5.1.1, we observe that the probability density $|P_{n,m}|^2\mathrm{e}^{-2mQ}$ resembles a Gaussian wave which crests around the boundary $\partial\mathcal{S}_\tau$ of the droplet, where $\tau = \frac{n}{m}$. As a consequence, we expect the density to be obtained as the sum of such Gaussians. Near the droplet boundary, this effect is the strongest, and adding a large but finite number of such Gaussian waves crested along boundary curves $\partial\mathcal{S}_\tau$ which move with the degree parameter n results in error function asymptotics.

Proof of Theorem 1.4.1. We recall the rescaled variable from the introduction

$$z_m(\xi) = z_0 + \mathrm{n}\frac{\xi}{\sqrt{2m\Delta Q(z_0)}},$$

where $z_0 \in \partial\mathcal{S}_\tau$ and n is the outward unit normal to $\mathcal{S}_\tau$ at z_0, and the rescaled density $\rho_m(\xi)$ given by (1.2.2). In terms of orthogonal polynomials, the object of study is the function

$$\rho_m(\xi) = \frac{1}{2m\Delta Q(z_0)}\sum_{n=0}^{m-1}|P_{n,m}(z_m(\xi))|^2\mathrm{e}^{-2mQ(z_m(\xi))}.$$

We begin by noting that $z_m(\xi)$ is in the set $\mathcal{K}^c_{m,1}$ (see Theorem 1.3.5), provided that ξ is confined to the disk $\mathbb{D}(0,r_m)$, where $r_m = \sqrt{2\Delta Q(z_0)\log\log m}$. We shall assume throughout that $\xi \in \mathbb{D}(0,r_m)$.

Next, we write

$$\rho_{m_1,m}(\xi) = \frac{1}{2m\Delta Q(z_0)}\sum_{n=0}^{m_1-1}|P_{n,m}(z_m(\xi))|^2\mathrm{e}^{-2mQ(z_m(\xi))}$$

and split accordingly for $m_1 < m$

$$\rho_m(\xi) = \frac{1}{2m\Delta Q(z_0)}\sum_{n=m_1}^{m-1}|P_{n,m}(z_m(\xi))|^2\mathrm{e}^{-2mQ(z_m(\xi))} + \rho_{m_1,m}(\xi). \tag{5.2.1}$$

We choose m_1 to be the integer part of $m - m^{\frac12}\log m$.

By Proposition 2.2.2 that for $n \le m_1$,

$$|P_{n,m}(z)|^2\mathrm{e}^{-2mQ(z)} \le Cm\mathrm{e}^{-2m(Q-\hat{Q}_{\tau_1})(z)}, \qquad z \in \mathbb{D}(z_0,\alpha_0\delta_m), \tag{5.2.2}$$

where $\tau_1 = m_1/m$. By Taylor's formula applied to $Q-\check{Q}_{\tau_1} = R_{\tau_1}\circ\phi_{\tau_1}$ in $\mathcal{S}^c_{\tau_1}$ (Proposition 3.1.3), it follows that

$$(Q-\hat{Q}_{\tau_1})(z) \ge \beta_0\mathrm{dist}_{\mathbb{C}}(z,\partial\mathcal{S}_{\tau_1})^2 \tag{5.2.3}$$

for some constant $\beta_0 > 0$, provided that $z \in \mathcal{S}^c_{\tau_1}$ is close enough to $\partial\mathcal{S}_{\tau_1}$. For instance, this estimate holds for $z \in \mathcal{S}_1 \setminus \mathcal{S}_{\tau_1}$. Moreover, as $\tau_1 = \frac{m_1}{m}$ eventually is in I_{ϵ_0}, the function $Q - \hat{Q}_{\tau_1}$ does not vanish on $\mathcal{S}^c_{\tau_1}$, and tends to infinity at infinity. The latter observation shows that further away from the boundary $\partial\mathcal{S}_{\tau_1}$, the right-hand side of (5.2.2) decays exponentially.

If $n \le m_1$ and $\tau = \frac{n}{m}$, then $1-\tau \ge m^{-\frac12}\log m = \delta_m$. As a consequence of Lemma 2.3.1 we obtain that the boundary $\partial\mathcal{S}_\tau$ moves at a positive speed in τ. In particular, for $\tau = \frac{n}{m}$ where $n \le m_1$ we have that the distance $\mathrm{dist}_{\mathbb{C}}(\partial\mathcal{S}_\tau,\partial\mathcal{S}_1)$ is at least $2\alpha_0\delta_m$, for some fixed positive α_0. Since $\mathrm{dist}_{\mathbb{C}}(\partial\mathcal{S}_{\tau_1},\partial\mathcal{S}_1)$ is at least $2\alpha_0\delta_m$, we have that

$$\mathrm{dist}_{\mathbb{C}}(z,\partial\mathcal{S}_{\tau_1}) \ge \alpha_0\delta_m, \qquad z \in \mathbb{D}(z_0,\alpha_0\delta_m). \tag{5.2.4}$$

Next, we note that if $\zeta \in \mathbb{D}(0,r_m)$, then for large enough m we have $z_m(\zeta) \in \mathbb{D}(z_0,\alpha_0\delta_m)$. This follows from the obvious fact that $\log\log m = \mathrm{o}(\log m)$. By a combination of (5.2.3) and (5.2.4) it follows that

$$(Q-\hat{Q}_{\tau_1})(z_m(\zeta)) \ge \beta_0\alpha_0^2\delta_m^2.$$

Now, it follows from the above estimates (5.2.2) and (5.2.3) that for $n \le m_1$

$$|P_{n,m}(z_m(\xi))|^2\mathrm{e}^{-2mQ(z_m(\xi))} = \mathrm{O}(m\,\mathrm{e}^{-2\beta_0\alpha_0^2(\log m)^2}),$$

where the constant $c_0 > 0$ can be taken to be independent of $\xi \in \mathbb{D}(0, r_m)$. It follows that

$$\rho_{m_1,m}(\xi) = \mathrm{O}\big(m^2 \mathrm{e}^{-\beta_0\alpha_0^2(\log m)^2}\big), \qquad \xi \in \mathbb{D}(0, r_m)$$

which shows in particular $\rho_{m_1,m}(\xi) = \mathrm{O}(m^{-M})$ for arbitrarily large M.

As a result of the above considerations, it follows that we may focus on the remaining sum in (5.2.1) over the degrees n with $m_1 \le n \le m-1$, that is, $\tau = \frac{n}{m}$ with $\frac{m_1}{m} \le \tau \le 1$. In particular, the asymptotics of Corollary 5.1.1 applies for the whole range. Set $\tau(j) = \tau_m(j) = 1 - \frac{j}{m}$, where j ranges from 1 to $m - m_1$, which is approximately $m^{\frac{1}{2}} \log m$. We obtain

(5.2.5)
$$\rho_m(\xi) = \frac{(\pi m)^{-\frac{1}{2}}}{2\Delta Q(z_0)} \sum_{j=1}^{m-m_1} \big|\phi'_{\tau(j)}(z_m(\xi))\big|\, \mathrm{e}^{-2m(Q-\check{Q}_{\tau(j)})(z_m(\xi)) + 2\operatorname{Re}\mathcal{H}_{Q,\tau(j)}(z_m(\xi))} + \mathrm{O}(m^{-M}).$$

By Taylor expansion, it follows that

$$|\phi'_{\tau(j)}(z_m(\xi))| = |\phi'_1(z_0)| + \mathrm{O}(m^{-1/2}\log\log m),$$

and by the same token that

$$2\operatorname{Re}\mathcal{H}_{Q,\tau(j)}(z_m(\xi)) = \frac{1}{2}\log\Delta Q(z_0) + \mathrm{O}(m^{-1/2}\log\log m)$$

as $m \to \infty$ for all $j \le m - m_1$. The next thing to consider is the movement of $\partial\mathcal{S}_\tau$, for $\tau = \tau(j)$ as j increases. Recalling that n denotes the outward pointing unit normal to $\partial\mathcal{S}_1$ at the point z_0, Lemma 2.3.1 tells us that the line $z_0 + \mathrm{n}\mathbb{R}$ intersects $\partial\mathcal{S}_{\tau(j)}$ at the nearest point

$$z_j = z_0 - \mathrm{n}\frac{j}{m}\frac{|\phi'_1(z_0)|}{4\Delta Q(z_0)} + \mathrm{O}\Big(\Big(\frac{j}{m}\Big)^2\Big),$$

and the outer unit normal n_j to $\partial\mathcal{S}_{\tau(j)}$ at the point z_j will satisfy

$$\mathrm{n}_j = \mathrm{n} + \mathrm{O}\Big(\frac{j}{m}\Big) = \mathrm{n} + \mathrm{O}(m^{-\frac{1}{2}}\log m).$$

We may hence write

$$(Q - \check{Q}_{\tau(j)})(z_m(\xi)) = (Q - \check{Q}_{\tau(j)})\Bigg(z_j + \mathrm{n}_j \frac{\xi + \frac{j}{2}\frac{|\phi'_1(z_0)|}{\sqrt{2m\Delta Q(z_0)}} + \mathrm{O}\big(m^{-\frac{1}{2}}(\log m)^2\big)}{\sqrt{2m\Delta Q(z_0)}}\Bigg).$$

A simple Taylor series expansion in normal and tangential coordinates at the point z_j gives that

$$(Q - \check{Q}_{\tau_j})(z_j + \mathrm{n}_j\eta) = 2\Delta Q(z_j)\,(\operatorname{Re}\eta)^2 + \mathrm{O}(|\eta|^3) = 2\Delta Q(z_0)\operatorname{Re}(\eta)^2 + \mathrm{O}\Big(|\eta|^2\frac{j}{m} + |\eta|^3\Big),$$

for η close to 0. From this we deduce that for η with $|\eta| = \mathrm{O}(\log m)$ we have

$$2m(Q - \check{Q}_{\tau(j)})\Big(z_{\tau_j} + \mathrm{n}_j\frac{\eta}{\sqrt{2m\Delta Q(z_0)}}\Big) = \frac{1}{2}(2\operatorname{Re}\eta)^2 + \mathrm{O}(m^{-1/2}(\log m)^3), \quad m \to \infty.$$

We apply this with η given by

$$\eta = \xi + \frac{j}{2}\frac{|\phi'_1(z_0)|}{\sqrt{2m\Delta Q(z_0)}} + \mathrm{O}\big(m^{-\frac{1}{2}}(\log m)^2\big),$$

which then gives that

$$(2\operatorname{Re}\eta)^2 = \Big(2\operatorname{Re}\xi + j\frac{|\phi'_1(z_0)|}{\sqrt{2m\Delta Q(z_0)}}\Big)^2 + \mathrm{O}\big(m^{-\frac{1}{2}}(\log m)^3\big).$$

Putting these asymptotic relations together, we find that

(5.2.6)
$$\rho_m(\xi) = \frac{1}{\sqrt{2\pi}}\Big(1 + \mathrm{O}\big(m^{-\frac{1}{2}}(\log m)^3\big)\Big) \times \sum_{j=1}^{m-m_1} \frac{|\phi'_1(z_0)|}{\sqrt{2m\Delta Q(z_0)}}\, \exp\Big\{-\frac{1}{2}\Big(2\operatorname{Re}\xi + j\frac{|\phi'(z_0)|}{\sqrt{2m\Delta Q(z_0)}}\Big)^2\Big\} + \mathrm{O}(m^{-M}).$$

We recognize immediately (5.2.6) as an approximate Riemann sum for

$$\operatorname{erf}(2\operatorname{Re}\xi) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \mathrm{e}^{-\frac{1}{2}(2\operatorname{Re}\xi+t)^2}\mathrm{d}t$$

with respect to a partition of the interval $[0, \gamma_0 \log m]$, with step length $m^{-\frac{1}{2}}\gamma_0$, where

$$\gamma_0 = \frac{|\phi'(z_0)|}{\sqrt{2\Delta Q(z_0)}}.$$

Since such Riemann sums converge to the corresponding integral with small error, this implies that

$$\lim_{m\to\infty} \rho_m(\xi) = \operatorname{erf}\big(2\operatorname{Re}\xi\big),$$

which completes the proof. □

5.3. **Convergence of correlation kernels.** Finally, we turn to the convergence of the rescaled kernels $\mathrm{k}_m(z_m(\xi), z_m(\eta))$ as $m \to \infty$. In principle, this should follow from our expansion of the orthogonal polynomials, but to do this directly seems a bit tricky. However, given the work of Ameur, Kang, and Makarov [4], it turns out to be enough to obtain the more straightforward diagonal convergence of the correlation kernel.

Proof of Corollary 1.4.2. As in the introduction, we denote by $G(z, w)$ the Ginibre-∞ kernel

$$G(\xi, \eta) = \mathrm{e}^{\xi\bar{\eta}-\frac{1}{2}(|\xi|^2+|\eta|^2)},$$

which is the correlation kernel of a translation invariant planar point process. We now present some material from [4]. An important concept is that of cocycles. We recall Theorem 2.6.1, which tells us that there exists a sequence of continuous functions $c_m : \mathbb{C} \to \mathbb{T}$ such that, for any subsequence $\mathcal{N}$ of the natural numbers $\mathbb{N}$, there exists a Hermitian entire function $F(\xi, \eta)$ and a further subsequence $\mathcal{N}^\star \subset \mathcal{N}$ such that

$$c_m(\xi)\bar{c}_m(\eta)\,\mathrm{k}_m(z_m(\xi), z_m(\eta)) \to G(\xi, \eta)F(\xi, \eta), \quad m \in \mathcal{N}^\star, m \to \infty, \tag{5.3.1}$$

where the convergence is uniform on compact subsets of $\mathbb{C}^2$. Here, we recall the familiar notion that a function $F(\xi, \eta)$ is Hermitian entire if it is an entire function of the two variables $(\xi, \bar{\eta})$ with the symmetry property $F(z, w) = \bar{F}(w, z)$. For such functions, the diagonal restriction $F(\xi, \xi)$ determines the function uniquely. Indeed, the polarization of the diagonal restriction gives back our function $F(\xi, \eta)$. We denote by $\rho(\xi)$ the limiting density

$$\rho(\xi) = \lim_{m\to\infty, m\in\mathcal{N}^\star} \mathrm{k}_m(z_m(\xi), z_m(\xi)) = G(\xi, \xi)F(\xi, \xi),$$

and since $G(\xi, \xi) \equiv 1$, it follows that $F(\xi, \xi) = \rho(\xi)$. By Theorem 1.4.1, it follows that

$$\rho(\xi) = \operatorname{erf}(2\operatorname{Re}\xi).$$

By the uniqueness property of diagonal restriction, the only possibility for the entire Hermitian kernel is

$$F(\xi, \eta) = \operatorname{erf}(\xi + \bar{\eta}).$$

This shows that the limit along some subsequence of any given sequence of positive integers is always the same. We claim that this means that the whole sequence converges. Before we turn to this, we need to observe that the method of [4] gives a locally uniform bound on the blow-up correlation kernels, which in our setting means that the entire Hermitian kernels that converge to $F(\xi, \xi)$ on the diagonal form a normal family on $\mathbb{C}^2$. In case the convergence (5.3.1) were to fail along the positive integers, by normal families, we could distill a sequence $\mathcal{N}_0$ such that the left-hand side of (5.3.1) would converge to something else along the subsequence $\mathcal{N}_0$. This would contradict what we have already established, which is that the we have convergence along a subsequence $\mathcal{N}_0^\star$ of $\mathcal{N}_0$. The assertion of the corollary follows. □

6. Connection with the matrix $\bar\partial$-problem of Its and Takhtajan

6.1. **Matrix $\bar\partial$-problems and orthogonal polynomials.** Given the successful application of Riemann-Hilbert problem methods to the study of orthogonal polynomials in the context of the real line and the unit circle, it has been proposed that the planar orthogonal polynomials should be approached in a similar fashion. Following Its and Takhtajan [30], we consider a matrix $\bar\partial$-problem (or a *thick Riemann-Hilbert problem*) and see how it fits in with our orthogonal foliation flow.

Denote by Ω a bounded domain in $\mathbb{C}$ (chosen to be sufficiently large), and let $\pi_{n,m}$ denote the monic orthogonal polynomial of degree n with respect to the measure $1_\Omega\,\mathrm{e}^{-2mQ}\mathrm{dA}$. That is, $\pi_{n,m}$ is given by

$$\pi_{n,m}(z)=\kappa_{n,m}^{-1}P_{n,m,\Omega}(z),$$

where $\kappa_{n,m}$ is the leading coefficient of the normalized orthogonal polynomial $P_{n,m,\Omega}$, taken with respect to $1_\Omega\,\mathrm{e}^{-2mQ}\mathrm{dA}$. The requirement on the bounded domain Ω is that it should contain the spectral droplet $\mathcal{S}_1$. Under this condition, the polynomials $P_{n,m,\Omega}$ can be shown to be close to the orthogonal polynomials $P_{n,m}$ considered previously in this work, with an exponentially decaying error term.

For a compactly supported function f, let $\mathbf{C}[f]$ be its Cauchy transform, given by

$$\mathbf{C}[f](z)=\int_{\mathbb{C}}\frac{f(w)}{z-w}\mathrm{dA}(w).$$

The Cauchy transform may immediately be extended past compactly supported functions, e.g. to $L^p(\mathbb{C})$ for $1\le p<2$. This is not optimal, but for our purposes there will be no need to push the matter further. The importance of the Cauchy transform comes from the fact that in the sense of distribution theory, $\bar\partial\mathbf{C}[f]=f$.

In [30], Its and Takhtajan propose to study the asymptotics of $\pi_{n,m}$ starting from the observation that the matrix-valued function

$$Y_{n,m}(z)=\begin{pmatrix}\pi_{n,m}(z) & -\mathbf{C}\big[1_\Omega\,\bar\pi_{n,m}\mathrm{e}^{-2mQ}\big](z)\\ -\kappa_{n-1,m}^2\pi_{n-1,m}(z) & \kappa_{n-1,m}^2\mathbf{C}\big[1_\Omega\,\bar\pi_{n-1,m}\mathrm{e}^{-2mQ}\big](z),\end{pmatrix}\tag{6.1.1}$$

solves the $\bar\partial$-problem

$$\begin{cases}\bar\partial Y(z)=-\bar Y(z)W(z), & \text{for } z\in\mathbb{C},\\ Y(z)=\big(I+\mathrm{O}(z^{-1})\big)\begin{pmatrix}z^n & 0\\ 0 & z^{-n}\end{pmatrix}, & \text{as } |z|\to+\infty,\end{cases}\tag{6.1.2}$$

where $W(z)=W_m(z)$ is the matrix-valued function

$$W(z)=\begin{pmatrix}0 & 1_\Omega(z)\,\mathrm{e}^{-2mQ(z)}\\ 0 & 0\end{pmatrix}.$$

Moreover, the solution is unique, as shown in [30]. We remark that classical Riemann-Hilbert problems where a jump occurs on a curve may be phrased as $\bar\partial$-problems where $\bar\partial Y(z)$ is understood as a measure supported on a curve Γ, and the above problem is a natural generalization to a more genuinely two-dimensional situation.

The idea that underlies the Its-Takhtajan approach, as well as the classical Riemann-Hilbert approach to orthogonal polynomials, is the expectation that one may constructively obtain an approximate solution $\tilde Y=\tilde Y_{n,m}(z)$ to the problem (6.1.1) (or the corresponding RHP), which should then produce a component $(\tilde Y_{n,m})_{1,1}$ approximately equal to $\pi_{n,m}(z)$.

6.2. **Integration of Riemann-Hilbert problems along curve families.** Unfortunately, it has proven difficult to solve the problem (6.1.1) constructively. The following simple observation shows how our orthogonal foliation flow reduces the $\bar\partial$-problem to a family of classical Riemann-Hilbert problems along closed curves.

In order to describe this problem, we denote by $J(z)$ a 2×2 jump matrix and let Γ be an oriented smooth simple closed curve in $\mathbb{C}$. We denote by Ω^+ and Ω^- the interior and exterior components of the complement $\mathbb{C}\setminus\Gamma$, respectively. If f is a function defined on $\mathbb{C}\setminus\Gamma$, which is continuous up to the boundary Γ as seen from each component we define the two boundary value functions f^+ and f^- on Γ by

$$f^{\pm}(\zeta)=\lim_{\substack{z\to\zeta\\ z\in\Omega^{\pm}}} f(z),\qquad \zeta\in\Gamma.$$

We consider the Riemann-Hilbert problem of finding a 2×2 matrix-valued function $Y(z)$ which meets

$$\begin{cases} Y \text{ is holomorphic on } \mathbb{C}\setminus\Gamma, & \\ Y^+(z)=Y^-(z)+\bar{Y}^-(z)J(z), & \text{for } z\in\Gamma,\\ Y(z)=(I+\mathrm{O}(z^{-1}))\begin{pmatrix} z^n & 0\\ 0 & z^{-n}\end{pmatrix}, & \text{as } |z|\to+\infty. \end{cases}\tag{6.2.1}$$

In order to analyze this problem, we need a variant of the Cauchy transform, which applies to functions defined on Γ. For smooth Γ, we write

$$\mathbf{C}_\Gamma[f](z)=\frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{f(w)}{w-z}\mathrm{d}w,\qquad z\in\mathbb{C}\setminus\Gamma.$$

As is well-known, the classical Plemelj formula is a useful tool in the study of Riemann-Hilbert problems:

$$(\mathbf{C}_\Gamma[f])^+(z)=(\mathbf{C}_\Gamma[f])^-(z)+f(z).\tag{6.2.2}$$

We now connect the classical Riemann-Hilbert problem (6.2.1) with the matrix $\bar\partial$-problem (6.1.2).

Proposition 6.2.1. *Let $\{\Gamma_t\}_{t\in I}$ be a smooth strictly expanding flow of positively oriented simple closed curves, and denote by $\mathcal{D}$ the union $\mathcal{D}=\bigcup_{t\in I}\Gamma_t$. Let $\omega(z)$ denote a smooth positive function on $\mathcal{D}$, and denote by $\xi:\mathcal{D}\to\mathbb{C}$ the vector field $\nu\bar\eta$, where $\eta(z)$ denotes the outward unit normal field to the curve family and ν denotes the scalar normal velocity of the flow. Then, for each $t\in I$, there is a unique solution $Y_t(z)$ to the Riemann-Hilbert problem* (6.2.1) *with jump matrix*

$$J=\begin{pmatrix}0 & 2\omega\xi\\ 0 & 0\end{pmatrix}.$$

Moreover, if there exists a continuous positive function $\lambda(t)$ such that $(Y_t)_{1,1}$ and $\lambda(t)(Y_t)_{2,1}$ are independent of t, then the matrix-valued function

$$Y(z)=\Lambda_1^{-1}\int_I \Lambda(t)Y_t(z)\mathrm{d}t\,\Lambda_2^{-1}$$

is the unique solution to (6.1.2), *with $W=\begin{pmatrix}0 & 1_{\mathcal{D}}\,\omega\\ 0 & 0\end{pmatrix}$, provided that*

$$\Lambda(t)=\begin{pmatrix}1 & 0\\ 0 & \lambda(t)\end{pmatrix},\quad \Lambda_1=\begin{pmatrix}1 & 0\\ 0 & \int_I\lambda(t)\mathrm{d}t\end{pmatrix},\quad \Lambda_2=\begin{pmatrix}|I| & 0\\ 0 & 1\end{pmatrix}.$$

Proof. We first establish the existence of solutions to the problem (6.2.1) of Γ_t, which may be expressed in terms of a family of t-dependent orthogonal polynomials. We recall that ξ factors as $\nu\bar\eta$, where ν denotes the speed of the boundary in the normal direction while η denotes the outward pointing unit normal field. Since arc-length measure $|\mathrm{d}z|$ on Γ_t relates to the complex line element $\mathrm{d}z$ by $\mathrm{d}z=\tau|\mathrm{d}z|$ where τ denotes the unit tangent vector field along Γ_t, it follows that

$$\frac{1}{2\pi\mathrm{i}}\mathrm{d}z=\frac{1}{2\pi}(-\mathrm{i}\tau)|\mathrm{d}z|=\eta\,\mathrm{d}s\tag{6.2.3}$$

where we recall the convention $\mathrm{ds} = \frac{|\mathrm{d}z|}{2\pi}$. From this it follows that $(2\pi\mathrm{i})^{-1}\xi\,\mathrm{d}z = \nu\,\mathrm{ds}$, and we may consequently define an inner product by

$$\langle f, g\rangle_t := \int_{\Gamma_t} f(z)\bar{g}(z)\nu(z)\mathrm{ds}(z) = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_t} f(z)\bar{g}(z)\xi(z)\mathrm{d}z.$$

Let $\{\pi^\star_{n,t}\}_n$ denote the sequence of monic orthogonal polynomials with respect to this inner product, such that $\pi^\star_{n,t}$ has degree n, and denote by $\kappa^\star_{n,t}$ the sequence of leading coefficient of the corresponding normalized orthogonal polynomials $P^\star_{n,t} = \kappa^\star_{n,t}\pi^\star_{n,t}$. It is straightforward to check that the function

$$\begin{pmatrix} \pi^\star_{n,t} & 2\mathbf{C}_{\Gamma_t}[\bar{\pi}^\star_{n,t}\omega\xi] \\ -\frac{1}{2}(\kappa^\star_{n-1,t})^2\pi^\star_{n-1,t} & -(\kappa^\star_{n-1,t})^2\mathbf{C}_{\Gamma_t}[\bar{\pi}^\star_{n-1,t}\omega\xi](z) \end{pmatrix}$$

supplies a solution to the Riemann-Hilbert problem (6.2.1).

Turning to unicity, it is clear from Plemelj's formula (6.2.2) and the jump condition that any solution $Y_t(z)$ must take the form

$$Y_t(z) = \begin{pmatrix} a_t(z) & u_t(z) + 2\mathbf{C}_{\Gamma_t}[\bar{a}_t\omega\xi](z) \\ b_t(z) & v_t(z) + 2\mathbf{C}_{\Gamma_t}[\bar{b}_t\omega\xi](z) \end{pmatrix},$$

where a_t, b_t, u_t, v_t are entire functions. From the growth constraint at infinity, we see that these four functions are all polynomials. Moreover, $u_t = v_t = 0$ for the same reason. A standard expansion of the Cauchy kernel at infinity shows that a_t is a monic polynomial of degree n which is orthogonal to the lower degree polynomials Pol_n with respect to $\omega\xi\mathrm{d}z$ on Γ_t. It follows that $a_t = \pi^\star_{n,t}$. Analogously, b_t is given by $b_t = -\frac{1}{2}(\kappa^\star_{n-1,t})^2\pi^\star_{n-1,t}$. We have established the unique solvability of the Riemann-Hilbert problem (6.2.1) with the given jump matrix.

Next, we turn to the connection with the $\bar\partial$-problem (6.1.2). Under the assumption that $(Y_t)_{1,1} = a_t = A$ and $\lambda(t)(Y_t)_{2,1} = \lambda(t)b_t = B$ are independent of t, we may consequently write

$$\Lambda(t)\,Y_t(z) = \begin{pmatrix} A(z) & 2\mathbf{C}_{\Gamma_t}[\bar{A}\omega\xi](z) \\ B(z) & 2\mathbf{C}_{\Gamma_t}[\bar{B}\omega\xi](z) \end{pmatrix}.$$

Recall that we may integrate over the flow $\{\Gamma_t\}_t$ using the disintegration

$$\int_{t\in I}\left\{2\int_{\Gamma_t} u(z)\nu(z)\mathrm{ds}\right\}\mathrm{d}t = \int_{\mathcal{D}} u(z)\mathrm{dA}(z),$$

for functions u such that the indicated integrals have a well-defined meaning. It now follows that if $\langle\lambda\rangle_I = \int_I \lambda(t)\mathrm{d}t$, the matrix-valued function

$$\begin{aligned}\hat{Y}(z) := \Lambda_1^{-1}\int_I \Lambda(t)Y_t(z)\mathrm{d}t\,\Lambda_2^{-1} &= \Lambda_1^{-1}\begin{pmatrix} |I|\,A(z) & -\mathbf{C}[\bar{A}\omega\,1_{\mathcal{D}}](z) \\ |I|\,B(z) & -\mathbf{C}[\bar{B}\omega\,1_{\mathcal{D}}](z) \end{pmatrix}\Lambda_2^{-1} \\ &= \begin{pmatrix} A(z) & -\mathbf{C}[\bar{A}\omega\,1_{\mathcal{D}}](z) \\ (\langle\lambda\rangle_I)^{-1}\,B(z) & -(\langle\lambda\rangle_I)^{-1}\,\mathbf{C}[\bar{B}\omega\,1_{\mathcal{D}}](z) \end{pmatrix}\end{aligned}$$

solves

$$\bar\partial\hat{Y}(z) = \begin{pmatrix} 0 & -\bar{A}\omega\,1_{\mathcal{D}} \\ 0 & -(\langle\lambda\rangle_I)^{-1}\bar{B}\omega\,1_{\mathcal{D}} \end{pmatrix} = -\overline{\hat{Y}(z)}\begin{pmatrix} 0 & \omega\,1_{\mathcal{D}} \\ 0 & 0 \end{pmatrix}$$

with asymptotics

$$\hat{Y}(z) = \big(I + \mathrm{O}(z^{-1})\big)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \text{ as } |z| \to +\infty,$$

as a consequence of the corresponding asymptotics of Y_t for each $t \in I$. □

Remark 6.2.2. For the orthogonal foliation flow, in the context of a neighborhood of the boundary curve of the droplet $\mathcal{S}_\tau$ with $\tau = n/m$, the (approximate) orthogonal polynomial of degree n is also approximately orthogonal to the lower degree polynomials along the individual flow loops corresponding to $\omega = \mathrm{e}^{-2mQ}$. So, in view of Proposition 6.2.1, the conditions

$$\partial_t(Y_t)_{1,1} = 0, \quad \partial_t\big(\lambda(t)(Y_t)_{2,1}\big) = 0 \tag{6.2.4}$$

should be met at least approximately for some appropriate scalar-valued function $\lambda(t)$. Alternatively, we could use the (6.2.4) as a criterion to define a flow of curves. In the given setting, this should give us back our orthogonal foliation flow. In other words, (6.2.4) should be analogous to the condition (4.4.2), once the Riemann-Hilbert problems of Proposition 6.2.1 are approximately soved in a constructive fashion, and we would expect that in an approximate sense,

$$\Gamma_t \sim \phi_\tau^{-1}(\psi_{n,m,-t}(\mathbb{T})).$$

It is entirely possible that the conditions (6.2.4) would be more stable close to the zeros of the given orthogonal polynomial of degree n. For instance, this might be the case with a highly eccentric ellipse.

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Paper B

Off-spectral analysis of Bergman kernels
(joint with H. Hedenmalm)
Preprint: arXiv (2018), 28 pages.

OFF-SPECTRAL ANALYSIS OF BERGMAN KERNELS

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Abstract. The asymptotic analysis of Bergman kernels with respect to exponentially varying measures near emergent interfaces has attracted recent attention. Such interfaces typically occur when the associated limiting Bergman density function vanishes on a portion of the plane, *the off-spectral region*. This type of behaviour is observed when the metric is negatively curved somewhere, or when we study partial Bergman kernels in the context of positively curved metrics. In this work, we cover these two situations in a unified way, for exponentially varying planar measures on the complex plane. We obtain uniform asymptotic expansions of *root functions*, which are essentially normalized partial Bergman kernels at an off-spectral point, valid in the entire off-spectral component and protruding into the spectrum as well, which allows us to show error function transition behaviour of the original kernel along the interface. In contrast, previous work on asymptotic expansions of Bergman kernels is typically local, and valid only in the bulk region of the spectrum.

1. Introduction

1.1. **Bergman kernels and emergent interfaces.** This article is a companion to our recent work [12] on the structure of planar orthogonal polynomials. We will make frequent use of methods developed there, and recommend that the reader keep that article available for ease of reference.

Recently, the asymptotic behaviour of Bergman kernels near emergent interfaces has attracted considerable attention. Such interfaces may occur, e.g., when the metric develops a singularity, or if it is negatively curved somewhere. Similar interfaces also appear in the context of partial Bergman kernels, that is, the kernel for the orthogonal projections of a weighted L^2-space onto a proper subspace of the full Bergman space. In this work, we intend to cover both situations in a unified manner, in the setting of exponentially varying weights on the complex plane. The partial Bergman kernels we consider here fall into one of two categories: polynomial kernels, and kernels for spaces defined by a prescribed degree of vanishing at a given point. In both cases, the transition of the Bergman kernel is described in terms of the error function, which is suggestive of an interpretation of this transition as the result of diffusion. We also obtain an extension of the orthogonal foliation flow developed in [12] in the context of the orthogonal polynomials, which immediately yields a perturbation result for orthogonal polynomials allowing for a twist of the base metric.

The key to our obtaining these results is the expansion of the Bergman kernel in the orthogonal basis provided by the normalized reproducing kernels $\mathrm{k}_n(z, w_0)$ with prescribed degree n of vanishing at an *off-spectral* point w_0, i.e., a point in the forbidden (asymptotically massless) region on one side of the interface, which

Date: May 25, 2018.

is globally valid in a neighbourhood the entire forbidden component. In particular, the analysis is not local.

A key ingredient is the *canonical positioning operator*, introduced in [12], which allows us to connect with the approach to the analysis of the planar orthogonal polynomials developed there. We now briefly recall the objects of study. The Bergman space A^2_{mQ} is defined as the collection of all entire functions f in with finite weighted L^2-norm

$$\|f\|^2_{mQ} := \int_{\mathbb{C}} |f(z)|^2 \mathrm{e}^{-2mQ(z)} \mathrm{dA}(z) < +\infty,$$

where dA denotes the planar area element normalized so that the unit disk $\mathbb{D}$ has unit area, and where Q is a potential with certain growth and regularity properties (see Definition 1.4.1). We denote the reproducing kernel for A^2_{mQ} by K_m, and for a given point $w_0 \in \mathbb{C}$, we consider the normalized kernel

$$\mathrm{k}_{m,w_0}(z) := K_m(w_0, w_0)^{-1/2} K_m(z, w_0), \tag{1.1.1}$$

which gets norm 1 in A^2_{mQ}. There is a notion of the *spectrum* $\mathcal{S}$, also called the *spectral droplet*. This set is closed, and defined in terms of an obstacle problem. Let $\mathrm{SH}(\mathbb{C})$ denote the cone of all subharmonic functions on the plane $\mathbb{C}$, and consider the function

$$\hat{Q}(z) := \sup\big\{ q(z) \,:\, q \in \mathrm{SH}(\mathbb{C}), \text{ and } q \le Q \text{ on } \mathbb{C} \big\}.$$

Given that Q is $C^{1,1}$-smooth and has some modest growth at infinity, it is known that $\hat{Q} \in C^{1,1}$ as well, and it is a matter of definition that $\hat{Q} \le Q$ pointwise (see, e.g., [9]). Here, $C^{1,1}$ denotes the standard smoothness class of differentiable functions with Lipschitz continuous first order partial derivatives. We define the *spectrum* (or the *spectral droplet*) as the contact set

$$\mathcal{S} := \big\{ z \in \mathbb{C} \,:\, \hat{Q}(z) = Q(z) \big\}. \tag{1.1.2}$$

We will need these notions in the context of partial Bergman kernels as well. For a non-negative integer n and a point $w_0 \in \mathbb{C}$, we consider the subspace A^2_{mQ,n,w_0} of A^2_{mQ}, consisting of those functions that vanish to order at least n at w_0. It may happen for some n that this space is trivial, for instance when the potential Q has logarithmic growth only, because then the space A^2_{mQ} consists of polynomials of a bounded degree. We denote its reproducing kernel by K_{m,n,w_0}, and observe that $K_{m,0,w_0} = K_m$. We shall need also the *root function of order* n *at* w_0, denoted k_{m,n,w_0}, which is the unique solution to the optimization problem

$$\max\big\{ \operatorname{Re} f^{(n)}(w_0) : \; f \in A^2_{mQ,n,w_0}, \; \|f\|_{mQ} \le 1 \big\},$$

provided the maximum is positive, in which case the optimizer has norm $\|f\|_{mQ} = 1$. In the remaining case, the maximum equals 0, and either only $f = 0$ is possible, or there are several competing optimizers, simply because we may multiply the function by unimodular constants and obtain alternative optimizers. In both remaining instances we declare that $\mathrm{k}_{m,n,w_0} = 0$. When nontrivial, the root function of order n at w_0 is connected with the reproducing kernel K_{m,n,w_0}:

$$\mathrm{k}_{m,n,w_0}(z) = \lim_{\zeta \to w_0} K_{m,n,w_0}(\zeta,\zeta)^{-1/2} K_{m,n,w_0}(z,\zeta), \tag{1.1.3}$$

where the point ζ should approach w_0 not arbitrarily but in a fashion such that the limit exists and has positive n-th derivative at w_0. The root function k_{m,n,w_0} will play a key role in our analysis, similar to that of the orthogonal polynomials in the context of polynomial Bergman kernels. As a result of the relation (1.1.3), we may alternatively call the root function k_{m,n,w_0} a *normalized partial Bergman kernel.* In this context, we note that for $n = 0$, $\mathrm{k}_{m,0,w_0} = \mathrm{k}_{m,w_0}$ is the normalized Bergman kernel of (1.1.1). The root functions k_{m,n,w_0} all have norm equal to 1 in A^2_{mQ}, except when they are trivial and have norm 0. The spectral droplet associated to a family of partial Bergman kernels of the above type is defined in Subsection 2.1 in terms of an obstacle problem, and we briefly outline how this is done. For $0 \le \tau < +\infty$, let $\mathrm{SH}_{\tau,w_0}(\mathbb{C})$ denote the convex set

$$\mathrm{SH}_{\tau,w_0}(\mathbb{C}) = \big\{q \in \mathrm{SH}\,(\mathbb{C}) \,:\, q(z) \le \tau \log|z - w_0| + \mathrm{O}(1) \;\text{ as }\; z \to w_0\big\},$$

so that for $\tau = 0$ we recover $\mathrm{SH}(\mathbb{C})$. We consider the corresponding obstacle problem

$$\hat{Q}_{\tau,w_0}(z) = \sup\big\{q(z) \;:\; q \in \mathrm{SH}_{\tau,w_0}(\mathbb{C}),\, q \le Q \text{ on } \mathbb{C}\big\}, \tag{1.1.4}$$

and observe that $\tau \mapsto \hat{Q}_{\tau,w_0}$ is monotonically decreasing pointwise. We define a family of spectral droplets as the coincidence sets

$$\mathcal{S}_{\tau,w_0} = \big\{z \in \mathbb{C} \,:\, Q(z) = \hat{Q}_{\tau,w_0}(z)\big\}. \tag{1.1.5}$$

Due to the monotonicity, the droplets $\mathcal{S}_{\tau,w_0}$ get smaller as τ increases, starting from $\mathcal{S}_{0,w_0} = \mathcal{S}$ for $\tau = 0$. The *partial Bergman density*

$$\rho_{m,n,w_0}(z) := m^{-1} K_{m,n,w_0}(z,z)\,\mathrm{e}^{-2mQ(z)}, \qquad z \in \mathbb{C},$$

may be viewed as the normalized local dimension of the space $A^2_{mQ,n,w_0}(\mathbb{C})$, and, in addition, it has the interpretation as the intensity of a corresponding (possibly infinite) Coulomb gas. In the case $n = 0$ we omit the word "partial" and speak of the *Bergman density.* It is known that in the limit as $m, n \to +\infty$ with $n = m\tau$,

$$\rho_{m,n,w_0}(z) \to 2\varDelta Q(z)\,1_{\mathcal{S}_{\tau,w_0}}(z),$$

in the sense of convergence of distributions. In particular, $\varDelta Q \ge 0$ holds a.e. on $\mathcal{S}$. Here, we write $\varDelta$ for differential operator $\partial\bar{\partial}$, which is one quarter of the usual Laplacian. The above convergence reinforces our understanding of the droplets $\mathcal{S}_{\tau,w_0}$ as spectra, in the sense that the Coulomb gas may be thought to model eigenvalues (at least in the finite-dimensional case). The *bulk* of the spectral droplet $\mathcal{S}_{\tau,w_0}$ is the set

$$\{z \in \mathrm{int}(\mathcal{S}_{\tau,w_0}) :\; \varDelta Q(z) > 0\},$$

where "int" stands for the operation of taking the interior.

Our motivation for the above setup originates with the theory of random matrices, specifically the *random normal matrix ensembles.* We should mention that an analogous situation occurs in the study of complex manifolds. The Bergman kernel then appears in the study of spaces of L^2-integrable global holomorphic sections of L^m, where L^m is a high tensor power of a holomorphic line bundle L over the manifold, endowed with an hermitian fiber metric h. If $\{e_i\}_i$ is a local basis for the fibers of L, then each section s may be written locally in the form $s_i\, e_i$ where the s_i are locally defined holomorphic functions. The pointwise norm of a section s may be then be written as $|s|_h^2 = |s|^2\mathrm{e}^{-m\phi}$, for some smooth real-valued function ϕ Along with the base metric, this defines an L^2 space which shares many characteristics with the spaces considered here.

The asymptotic behaviour of Bergman kernels has been the subject of intense investigation. However, the understanding has largely been limited to the analysis of the kernel inside the bulk of the spectrum, in which case the kernel enjoys a full *local* asymptotic expansion. The pioneering work on Bergman kernel asymptotics begins with the efforts by Hörmander [13] and Fefferman [7]. Developing further the microlocal approach of Hörmander, Boutet de Monvel and Sjöstrand [5] obtain an expansion of the Bergman kernel near the diagonal and near the boundary. Later, in the context of Kähler geometry, the influential *peak section method* was introduced by Tian [18]. His results were refined further by Catlin and Zelditch [6, 19], while the connection with microlocal analysis was greatly simplified in the more recent work by Berman, Berndtsson, and Sjöstrand [4]. A key element of all these methods is that the kernel is determined by the local geometry around the given point. This feature is absent when we consider the kernel near an off-spectral or a boundary point.

In recent work [12], we analyze the boundary behaviour of polynomial Bergman kernels, for which the corresponding spectral droplet is compact, connected, and has a smooth Jordan curve as boundary. The analysis takes the path via a full asymptotic expansion of the orthogonal polynomials, valid off a sequence of increasing compacts which eventually fill the droplet. By expanding the polynomial kernel in the orthonormal basis provided by the orthogonal polynomials, the error function asymptotics emerges at the spectral boundary.

The appearance of an interface for partial Bergman kernels in higher dimensional settings and in the context of complex manifolds has been observed more than once, notably in the work by Shiffman and Zelditch [17] and by Pokorny and Singer [14]. That the error function governs the transition behaviour across the interface was observed later but only in a simplified geometric context. For instance, in [15], Ross and Singer investigate the partial Bergman kernels associated to spaces of holomorphic sections vanishing along a divisor, and obtain error function transition behaviour under the assumption that the set-up is invariant under a holomorphic S^1-action. More recently, Zelditch and Zhou [20] obtain the same transition for partial Bergman kernels defined in terms of Toeplitz quantization of a general smooth Hamiltonian.

1.2. **Off-spectral asymptotics of normalized Bergman kernels.** The main contribution of the present work, put in the planar context, is a non-local asymptotic expansion of normalized (partial) Bergman kernels rooted at an off-spectral point. For ease of exposition, we begin with a version that requires as few prerequisites as possible for the formulation. We denote by $Q(z)$ an *admissible potential*, by which we mean the following:

(i) $Q : \mathbb{C} \to \mathbb{R}$ is C^2-smooth, and has sufficient growth at infinity:

$$\tau_Q := \liminf_{|z|\to+\infty} \frac{Q(z)}{\log|z|} > 0.$$

(ii) Q is real-analytically smooth and strictly subharmonic in a neighbourhood of $\partial\mathcal{S}$,

(iii) there exists a bounded component Ω of the complement $\mathcal{S}^c = \mathbb{C} \setminus \mathcal{S}$ which is simply connected, and has real-analytically smooth Jordan curve boundary.

We consider the case when there exists a non-trivial off-spectral component Ω which is bounded and simply connected, with real-analytic boundary, and pick a "root point" $w_0 \in \Omega$. To be precise, by an *off-spectral component* we mean a connectivity component of the complement $\mathcal{S}^c$. This situation occurs, e.g., if the

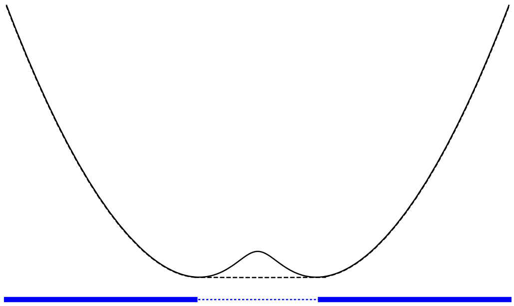

FIGURE 1.1. Illustration of the spectral droplet corresponding to the potential $Q(z) = |z|^2 - \log(a + |z|^2)$, with $a = 0.04$. The spectrum is illustrated with a thick line, and appears as the contact set between Q (solid) and the solution $\hat{Q}$ to the obstacle function (dashed).

potential is strictly superharmonic in a portion of the plane, as is illustrated in Figure 1.1. In terms of the metric, this means that there is a region where the curvature is negative. To formulate our first main result, we need the function $\mathcal{Q}_{w_0}$, which is bounded and holomorphic in Ω and whose real part equals Q along the boundary $\partial\Omega$. To fix the imaginary part, we require that $\mathcal{Q}_{w_0}(w_0) \in \mathbb{R}$. In addition, we need the conformal mapping φ_{w_0} which takes Ω onto the unit disk $\mathbb{D}$ with $\varphi_{w_0}(w_0) = 0$ and $\varphi'_{w_0}(w_0) > 0$. Since the boundary $\partial\Omega$ is assumed to be a real-analytically smooth Jordan curve, the function $\mathcal{Q}_{w_0}$ extends holomorphically across $\partial\Omega$, and the conformal mapping φ_{w_0} extends conformally across $\partial\Omega$.

Theorem 1.2.1. *Assuming that Q is an admissible potential, we have the following. Given a positive integer κ and a positive real A, there exist a neighbourhood $\Omega^{\langle\kappa\rangle}$ of the closure of Ω and bounded holomorphic functions $\mathcal{B}_{j,w_0}$ on $\Omega^{\langle\kappa\rangle}$ for $j = 0, \ldots, \kappa$, as well as domains $\Omega_m = \Omega_{m,\kappa,A}$ with $\Omega \subset \Omega_m \subset \Omega^{\langle\kappa\rangle}$ which meet*

$$\operatorname{dist}_{\mathbb{C}}(\partial\Omega_m, \partial\Omega) \geq A\, m^{-\frac{1}{2}}(\log m)^{\frac{1}{2}},$$

such that the normalized Bergman kernel at the point w_0 enjoys the asymptotic expansion

$$\mathrm{k}_m(z, w_0) = m^{\frac{1}{4}}(\varphi'_{w_0}(z))^{\frac{1}{2}}\mathrm{e}^{m\mathcal{Q}_{w_0}(z)}\bigg\{\sum_{j=0}^{\kappa} m^{-j}\mathcal{B}_{j,w_0}(z) + \mathrm{O}\big(m^{-\kappa-1}\big)\bigg\},$$

as $m \to +\infty$, where the error term is uniform on Ω_m. Here, the main term $\mathcal{B}_{0,w_0}$ is obtained as the unique zero-free holomorphic function on Ω which is smooth up to the boundary, positive at w_0, with prescribed modulus on the boundary

$$|\mathcal{B}_{0,w_0}(z)| = \pi^{-\frac{1}{4}}[\Delta Q(z)]^{\frac{1}{4}}, \qquad z \in \partial\Omega.$$

Remark 1.2.2. Using an approach based on Laplace's method, the functions $\mathcal{B}_{j,w_0}$ may be derived algorithmically, for $j = 1, 2, 3, \ldots$, see Theorem 3.2.2. The details of the algorithm are analogous with the case of the orthogonal polynomials presented in [12].

1.3. **Expansion of partial Bergman kernels in terms of root functions.** For a (τ, w_0)-admissible potential, the partial Bergman kernel K_{m,n,w_0} with the root point w_0 is well-defined and nontrivial. In analogy with Taylor's formula, it enjoys an expansion in terms of the root functions k_{m,n',w_0} for $n' \geq n$.

Theorem 1.3.1. *Under the above assumption of (τ, w_0)-admissibility of Q, we have that*

$$K_{m,n,w_0}(z,w) = \sum_{n'=n}^{+\infty} \mathrm{k}_{m,n',w_0}(z)\overline{\mathrm{k}_{m,n',w_0}(w)}, \qquad (z,w) \in \mathbb{C} \times \mathbb{C}.$$

Proof. For $n', n'' \geq n$ with $n'' < n'$, the functions k_{m,n',w_0} and k_{m,n'',w_0} are orthogonal in A^2_{mQ}. If one of them is trivial, orthogonality is immediate, while if both are nontrivial, we argue as follows. Let $\zeta', \zeta'' \in \mathbb{C}$ be close to w_0, and calculate that

$$\begin{aligned} &K_{m,n',w_0}(\zeta',\zeta')^{-\frac12} K_{m,n'',w_0}(\zeta'',\zeta'')^{-\frac12} \big\langle K_{m,n',w_0}(\cdot,\zeta'), K_{m,n'',w_0}(\cdot,\zeta'')\big\rangle_{mQ} \\ &= K_{m,n',w_0}(\zeta',\zeta')^{-\frac12} K_{m,n'',w_0}(\zeta'',\zeta'')^{-\frac12} K_{m,n',w_0}(\zeta'',\zeta') = \mathrm{O}(|\zeta''-w_0|^{n'-n''}), \end{aligned}$$

which tends to 0 as $\zeta'' \to w_0$. The claimed orthogonality follows. Moreover, since the root functions k_{m,n',w_0} have unit norm when nontrivial, the expression

$$\sum_{n'=n}^{+\infty} \mathrm{k}_{m,n',w_0}(z)\overline{\mathrm{k}_{m,n',w_0}(w)}$$

equals the reproducing kernel function for the Hilbert space with the norm of A^2_{mQ} spanned by the vectors k_{m,n',w_0} with $n' \geq n$. It remains to check that this is the whole partial Bergman space A^2_{mQ,n,w_0}. To this end, let $f \in A^2_{mQ,n,w_0}$ be orthogonal to all the the vectors k_{m,n',w_0} with $n' \geq n$. By the definition of the space A^2_{mQ,n,w_0}, this means that $f(z) = \mathrm{O}(|z-w_0|^n)$ near w_0. If f is notrivial, there exists an integer $N \geq n$ such that $f(z) = c\,(z-w_0)^N + \mathrm{O}(|z-w_0|^{N+1})$ near w_0, where $c \neq 0$ is complex. At the same time, the existance of such nontrivial f entails that the corresponding root functions k_{m,N,w_0} is nontrivial as well, and that $K_{m,N,w_0}(\zeta,\zeta) \asymp |\zeta - w_0|^{2N}$ for ζ near w_0. On the other hand, the orthogonality between f and k_{m,N,w_0} gives us that

$$\begin{aligned} 0 = \langle f, \mathrm{k}_{m,N,w_0}\rangle_{mQ} &= \lim_{\zeta\to w_0} K_{m,N,w_0}(\zeta,\zeta)^{-\frac12}\big\langle f, K_{m,N,w_0}(\cdot,\zeta)\big\rangle_{mQ} \\ &= \lim_{\zeta\to w_0} K_{m,N,w_0}(\zeta,\zeta)^{-\frac12} f(\zeta) \end{aligned}$$

where we approach w_0 only in an appropriate direction so that the limit exists. But this contradicts the given asymptotic behaviour of $f(\zeta)$ near w_0, since $c \neq 0$ tells us that any limit of the right-hand side would be nonzero. □

1.4. **Off-spectral asymptotics of partial Bergman kernels and interface transition.** Given a point $w_0 \in \mathbb{C}$ we recall the partial Bergman spaces A^2_{mQ,n,w_0}, and the associated spectral droplets $\mathcal{S}_{\tau,w_0}$ (see (1.1.5)), where we keep $n = \tau m$. Before we proceed with the formulation of the second result, let us fix some terminology.

FIGURE 1.2. Illustration of a compact spectral droplet (shaded) with two simply connected holes. In this case there are three off-spectral components: the two holes as well as the unbounded component. If we think of this in the context of the Riemann sphere we may allow for the point at infinity to be inside the spectrum.

Definition 1.4.1. A real-valued potential Q is said to be (τ, w_0)-*admissible* if the following conditions hold:

(i) $Q : \mathbb{C} \to \mathbb{R}$ is C^2-smooth and has sufficient growth at infinity:

$$\tau_Q := \liminf_{|z|\to+\infty} \frac{Q(z)}{\log|z|} > 0.$$

(ii) Q is real-analytically smooth and strictly subharmonic in a neighbourhood of $\partial\mathcal{S}_{\tau,w_0}$.

(iii) The point w_0 is an off-spectral point, i.e., $w_0 \notin \mathcal{S}_{\tau,w_0}$, and the component Ω_{τ,w_0} of the complement $\mathcal{S}^c_{\tau,w_0}$ containing the point w_0 is bounded and simply connected, with real-analytically smooth Jordan curve boundary.

If for an interval $I \subset [0, +\infty[$, the potential Q is (τ, w_0)-admissible for each $\tau \in I$ and $\{\Omega_{\tau,w_0}\}_{\tau\in I}$ is a smooth flow of domains, then Q is said to be (I, w_0)-admissible.

Generally speaking, off-spectral components may be unbounded. It is for reasons of simplicity that we focus on bounded off-spectral components in the above definition. We will assume in the sequel that Q is (I, w_0)-admissible for some non-trivial compact interval $I = I_0$. For an illustration of the situation, see Figure 1.2.

We let φ_{τ,w_0} denote the surjective Riemann mapping

$$\varphi_{\tau,w_0} : \Omega_{\tau,w_0} \to \mathbb{D}, \quad \varphi_{\tau,w_0}(0) = 0, \quad \varphi'_{\tau,w_0}(0) > 0, \tag{1.4.1}$$

which by our smoothness assumption on the boundary $\partial\Omega_{\tau,w_0}$ extends conformally across $\partial\Omega_{\tau,w_0}$. We denote by $\mathcal{Q}_{\tau,w_0}$ the bounded holomorphic function in Ω_{τ,w_0} whose real part equals Q on $\partial\Omega_{\tau,w_0}$ and is real-valued at w_0. It is tacitly assumed to extend holomorphically across the boundary $\partial\Omega_{\tau,w_0}$. We now turn to our second main result.

Theorem 1.4.2. *Assume that the potential Q is (I_0, w_0)-admissible, where the interval I_0 is compact. Given a positive integer κ and a positive real A, there exists a neighborhood $\Omega^{\langle\kappa\rangle}_{\tau,w_0}$ of the closure of Ω_{τ,w_0} and bounded holomorphic functions $\mathcal{B}_{j,\tau,w_0}$ on $\Omega^{\langle\kappa\rangle}_{\tau,w_0}$, as well as domains $\Omega_{\tau,w_0,m} = \Omega_{\tau,w_0,m,\kappa,A}$ with $\Omega_{\tau,w_0} \subset \Omega_{\tau,w_0,m} \subset \Omega^{\langle\kappa\rangle}_{\tau,w_0}$ which meet*

$$\mathrm{dist}_{\mathbb{C}}(\Omega^c_{\tau,w_0,m}, \Omega_{\tau,w_0}) \geq Am^{-\frac{1}{2}}(\log m)^{\frac{1}{2}},$$

such that the root function of order n at w_0 enjoys the expansion

$$\mathrm{k}_{m,n,w_0}(z) = m^{\frac{1}{4}}(\varphi'_{\tau,w_0}(z))^{\frac{1}{2}}[\varphi_{\tau,w_0}(z)]^n \mathrm{e}^{m\mathcal{Q}_{\tau,w_0}(z)}\left\{\sum_{j=0}^{\kappa} m^{-j}\mathcal{B}_{j,\tau,w_0}(z) + \mathrm{O}(m^{-\kappa-1})\right\}$$

on $\Omega_{\tau,m}$ as $n = \tau m \to +\infty$ while $\tau \in I_0$, where the error term is uniform. Here, the main term $\mathcal{B}_{0,\tau,w_0}$ is zero-free and smooth up to the boundary on Ω_{τ,w_0}, positive at w_0, with prescribed modulus

$$|\mathcal{B}_{0,\tau,w_0}(\zeta)| = \pi^{-\frac{1}{4}}[\varDelta Q(\zeta)]^{\frac{1}{4}}, \qquad \zeta \in \partial\Omega_\tau.$$

Remark 1.4.3. As in the case of the normalized Bergman kernels, the expressions $\mathcal{B}_{j,\tau,w_0}$ may be obtained algorithmically, for $j = 1, 2, 3, \ldots$ (see Theorem 3.2.2).

As a consequence of Theorem 1.4.2, we obtain the transition behaviour of the (partial) Bergman densities at emergent interfaces. To explain how this works, we fix a bounded simply connected off-spectral component Ω with real-analytic Jordan curve boundary, associated to either a sequence of partial Bergman kernels K_{m,n,w_0} of the space A^2_{mQ,n,w_0} with the ratio $\tau = \frac{n}{m}$ fixed, or, alternatively, a sequence K_m of full Bergman kernels. We think of the latter as the parameter choice $\tau = 0$. We assume that the potential Q is real-analytically smooth and strictly subharmonic near $\partial\Omega$. Let $z_0 \in \partial\Omega$, and denote by $\nu \in \mathbb{T}$ the inward unit normal to $\partial\Omega$ at z_0. We define the rescaled density $\varrho_m = \varrho_{m,\tau,w_0,z_0}$ by

$$\varrho_m(\xi) = \frac{1}{2m\varDelta Q(z_m(\xi))} K_{m,\tau m,w_0}\big(z_m(\xi), z_m(\xi)\big)\mathrm{e}^{-2mQ(z_m(\xi))}, \tag{1.4.2}$$

where the rescaled variable is defined implicitly by

$$z_m(\xi) = z_0 + \nu\frac{\xi}{\sqrt{2m\varDelta Q(z_0)}}.$$

Corollary 1.4.4. *The rescaled density ϱ_m in (1.4.2) has the limit*

$$\lim_{m\to+\infty} \varrho_m(\xi) = \mathrm{erf}\,(2\,\mathrm{Re}\,\xi),$$

where the convergence is uniform on compact subsets.

1.5. **Comments on the exposition.** In Section 2, the main results, Theorems 1.2.1 and 1.4.2 as well as Corollary 1.4.4, are obtained. The analysis follows closely that of our recent work on the orthogonal polynomials. In particular, the asymptotic expansion is a consequence of Lemma 4.1.2 in [12]. In Section 3, we obtain a more general version of this lemma, which allows for a change of the base metric. This has several applications, including a stability result for the root functions and the orthogonal polynomials under a $\frac{1}{m}$-perturbation of the potential Q (Theorems 3.2.1 and 3.4.2).

2. Off-spectral expansions of normalized kernels

2.1. **A family of obstacle problems and evolution of the spectrum.** The spectral droplets (1.1.2) and the partial analogues (1.1.5) were defined earlier. From that point of view, the spectral droplet $\mathcal{S}$ is the instance $\tau = 0$ of the partial spectral droplets $\mathcal{S}_{\tau,w_0}$. We should like to point out here that the partial spectral droplet $\mathcal{S}_{\tau,w_0}$ emerges as the full spectrum under a perturbation of the potential Q. To see this, we consider the perturbed potential

$$\tilde{Q}(z) = \tilde{Q}_{\tau,w_0}(z) := Q(z) - \tau \log|z - w_0|,$$

and observe that the coincidence set $\tilde{\mathcal{S}}$ for $\tilde{Q}$ is the same as the partial spectral droplet $\mathcal{S}_{\tau,w_0}$.

The following proposition summarizes some basic properties of the function $\hat{Q}_{\tau,w_0}$ given by (1.1.4). We refer to [9] for the necessary details.

Proposition 2.1.1. *Assume that $Q \in C^2(\mathbb{C})$ is real-valued with the logarithmic growth of condition (i) of Definition 1.4.1. Then for each τ with $0 \le \tau < \tau_Q$ and for each point $w_0 \in \mathbb{C}$, the function $\hat{Q}_{\tau,w_0}$ is a subharmonic function in the plane $\mathbb{C}$ which is $C^{1,1}$-smooth off w_0, and harmonic on $\mathbb{C}\setminus(\mathcal{S}_{\tau,w_0} \cup \{w_0\})$. Near the point w_0 we have*

$$\hat{Q}_{\tau,w_0}(z) = \tau \log|z - w_0| + \mathrm{O}(1).$$

The evolution of the free boundaries $\partial\mathcal{S}_{\tau,w_0}$, which is of fundamental importance for our understanding of the properties of the normalized reproducing kernels, is summarized in the following.

Proposition 2.1.2. *The continuous chain of off-spectral components Ω_{τ,w_0} for $\tau \in I_0$ deform according to weighted Laplacian growth with weight $2\Delta Q$, that is, for $\tau, \tau' \in I_0$ with $\tau' < \tau$, and for any bounded harmonic function h on Ω_{τ,w_0}, we have that*

$$\int_{\Omega_{\tau,w_0}\setminus\Omega_{\tau',w_0}} h\, 2\Delta Q \mathrm{dA} = (\tau - \tau')h(w_0).$$

Fix a point $\zeta \in \partial\Omega_{\tau,w_0}$, and denote for real ε by ζ_{ε,w_0} the point closest to ζ in the intersection

$$(\zeta + \nu_\tau(\zeta)\mathbb{R}_+) \cap \partial\Omega_{\tau-\varepsilon,w_0},$$

where $\nu_\tau(\zeta) \in \mathbb{T}$ points in the inward normal direction at ζ with respect to Ω_{τ,w_0}. Then we have that

$$\zeta_\varepsilon = \zeta + \varepsilon\, \nu_\tau(\zeta)\frac{|\varphi'_{\tau,w_0}(\zeta)|}{4\Delta Q(\zeta)} + \mathrm{O}(\varepsilon^2), \quad \varepsilon \to 0,$$

and the outer normal $\mathrm{n}_{\tau-\varepsilon,w_0}(\zeta_\varepsilon)$ satisfies

$$\mathrm{n}_{\tau-\varepsilon,w_0}(\zeta_\varepsilon) = \mathrm{n}_{\tau,w_0}(\zeta) + \mathrm{O}(\varepsilon).$$

Proof. That the domains deform according to Hele-Shaw flow is a direct consequence of the relation of Ω_{τ,w_0} to the obstacle problem. To see how it follows, assume that h is harmonic on Ω_{τ,w_0} and C^2-smooth up to the boundary, and apply

Green's formula to obtain

$$\int_{\Omega_{\tau,w_0}} h(z)\Delta Q(z)\mathrm{dA}(z) = \frac{1}{4\pi}\int_{\partial\Omega_{\tau,w_0}} \Big(h(z)\partial_{\mathrm{n}}Q(z) - Q(z)\partial_{\mathrm{n}}h(z)\Big)|\mathrm{d}z|$$
$$= \frac{1}{4\pi}\int_{\partial\Omega_{\tau,w_0}} \Big(h(z)\partial_{\mathrm{n}}\hat{Q}_{\tau,w_0}(z) - \hat{Q}_{\tau,w_0}(z)\partial_{\mathrm{n}}h(z)\Big)|\mathrm{d}z|$$
$$= \int_{\Omega_{\tau,w_0}} h(z)\Delta\hat{Q}_{\tau,w_0}\mathrm{dA}(z),$$

where the latter integral is understood in the sense of distribution theory. As $\hat{Q}_{\tau,w_0}$ is a harmonic perturbation of τ times the Green function for Ω_{τ,w_0}, the result follows by writing $\int_{\Omega_{\tau,w_0}\setminus\Omega_{\tau',w_0}} h\Delta Q\mathrm{dA}$ as the difference of two integrals of the above form, and by approximation of bounded harmonic functions by harmonic functions C^2-smooth up to the boundary.

The second part follows along the lines of [12, Lemma 2.3.1]. □

We turn next to an off-spectral growth bound for weighted holomorphic functions.

Proposition 2.1.3. *Assume that Q is admissible and denote by $\mathcal{K}_{\tau,w_0}$ a closed subset of the interior of $\mathcal{S}_{\tau,w_0}$. Then there exist constants c_0 and C_0 such that for any $f \in A^2_{mQ,n,w_0}(\mathbb{C})$ it holds that*

$$|f(z)| \le C_0\, m^{\frac{1}{2}}\mathrm{e}^{m\hat{Q}_{\tau,w_0}}\big\|1_{\mathcal{K}^{\mathrm{c}}_{\tau,w_0}} f\big\|_{mQ}, \qquad \operatorname{dist}(z,\mathcal{K}_{\tau,w_0}) \ge c_0 m^{-\frac{1}{2}}.$$

In case $\mathcal{K}_{\tau,w_0} = \emptyset$, the estimate holds globally.

Proof. This follows immediately by an application of the maximum principle, together with the result of Lemma 2.2.1 in [12], originating from [1]. □

2.2. **Some auxilliary functions.** There are a number of functions related to the potential Q that will be useful in the sequel. We denote by $\mathcal{Q}_{\tau,w_0}$ the bounded holomorphic function on Ω_{τ,w_0} whose real part on the boundary curve $\partial\Omega_{\tau,w_0}$ equals Q, uniquely determined by the requirement that $\operatorname{Im}\mathcal{Q}_{\tau,w_0}(w_0) = 0$. We also need the function $\breve{Q}_{\tau,w_0}$, which denotes the harmonic extension of $\hat{Q}_{\tau,w_0}$ across the boundary of the off-spectral component Ω_{τ,w_0}. These two functions are connected via

$$\breve{Q}_{\tau,w_0}(z) = \tau\log|\varphi_{\tau,w_0}(z)| + \operatorname{Re}\mathcal{Q}_{\tau,w_0}(z). \tag{2.2.1}$$

Since we work with (τ, w_0)-admissible potentials Q, the off-spectral component Ω_{τ,w_0} is a bounded simply connected domain with real-analytically smooth Jordan curve boundary. Without loss of generality, we may hence assume that $\mathcal{Q}_{\tau,w_0}$, $\breve{Q}_{\tau,w_0}$ as well as the conformal mapping φ_{τ,w_0} extend to a common domain Ω_0, containing $\bar{\Omega}_{\tau,w_0}$. By possibly shrinking the interval I_0, we may moreover choose the set Ω_0 to be independent of the parameter $\tau \in I_0$.

2.3. **Canonical positioning.** An elementary but important observation for the main result of [12] is that we may ignore a compact subset of the interior of the compact spectral droplet $\mathcal{S}_\tau$ associated to polynomial Bergman kernels when we study the asymptotic expansions of the orthogonal polynomials $P_{m,n}$ (with $\tau = \frac{n}{m}$). Indeed, only the behaviour in a small neighbourhood of the complement $\mathcal{S}^{\mathrm{c}}_\tau$ is of interest. The physical intuition behind this is the interpretation of the probability density $|P_{m,n}|^2\mathrm{e}^{-2mQ}$ as the net effect of adding one more particle to the system, and since the positions in the interior of the droplet are already occupied we would expect the net effect to occur near the boundary. The fact that we may restrict our attention to a simply connected proper subset of the Riemann sphere $\hat{\mathbb{C}}$ breaks up the rigidity and allows us to apply a conformal mapping to place ourselves in an appropriate model situation. At a technical level, this is accomplished by applying the *canonical positioning operator* $\mathbf{\Lambda}_{m,n}$ defined for functions f defined in a neighbourhood of the closure of the exterior disk

$$\mathbb{D}_{\mathrm{e}} := \{z \in \hat{\mathbb{C}} :\, 1 < |z| \le +\infty\}$$

in terms of the surjective conformal mapping $\phi_\tau : \mathcal{S}_\tau^c \to \mathbb{D}_{\mathrm{e}}$, which preserves the point at infinity and has $\phi_\tau'(\infty) > 0$, by

$$\mathbf{\Lambda}_{m,n}[f](z) = m^{\frac{1}{4}} \phi_\tau'(z)[\phi_\tau(z)]^n \mathrm{e}^{m\mathcal{Q}_\tau(z)} f \circ \phi_\tau(z).$$

We recall that for a compact set $\mathcal{K}$, a function F is called a *quasipolynomial of degree n on $\mathcal{K}^{\mathrm{c}}$* if F is holomorphic on $\mathcal{K}^{\mathrm{c}}$ and has the growth

$$|F(z)| \asymp |z|^n, \qquad |z| \to +\infty.$$

After canonical positioning, approximately orthogonal quasipolynomials with respect to the weight e^{-2mQ} are transformed into bounded holomorphic functions $f_{m,n}$ in a neighbourhood

$$\mathbb{D}_{\mathrm{e}}(0,\rho_0) = \{z \in \hat{\mathbb{C}} :\, \rho_0 < |z| \le +\infty\}$$

of the closed exterior disk, approximately orthogonal to all holomorphic functions on $\mathbb{D}_{\mathrm{e}}(0,\rho_0)$ vanishing at infinity with respect to an induced measure $\mathrm{e}^{-2mR_\tau}\mathrm{dA}$. The function R_τ is given by

$$R_\tau(z) = (Q - \breve{Q}_\tau) \circ \phi_\tau^{-1},$$

and is in particular quadratically flat along the unit circle $\mathbb{T}$, in the sense that both R_τ and its gradient ∇R_τ vanish on $\mathbb{T}$. The transformed problem is more tractable, and we perform the essential analysis after applying canonical positioning.

An important aspect of the analysis in [12] is the connection with a reproducing property. Indeed, after canonical positioning, the function $u_{m,n} = \mathbf{\Lambda}_{m,n}^{-1}[P_{m,n}]$ has the property of being essentially reproducing for the point at infinity. To see this, we observe that $u_{m,n}$ must be approximately orthogonal to the holomorphic functions that vanish at infinity, or, more accurately,

$$\int_{\mathbb{C}} \chi_1^2\, u_{m,n}\bar{q}\,\mathrm{e}^{-2mR_\tau}\mathrm{dA} = \frac{\bar{q}(\infty)}{u_{m,n}(\infty)} + \mathrm{O}\big(m^{-\kappa-1}\|\chi_1 q\|_{mR_\tau}\big),$$

for all bounded holomorphic q and any finite accuracy κ. Here, χ_1 is an appropriate cut-off function, which assumes the value 1 in a neighborhood of the closed exterior disk $\bar{\mathbb{D}}_{\mathrm{e}}$, and vanishes in the disk $\mathbb{D}(0,\rho_0)$ centered at 0 with radius ρ_0.

This point of view connects the orthogonal polynomials with the root functions at the point w_0. We let the canonical positioning operator for the point w_0 with respect to the off-spectral component Ω_{τ,w_0} be given by

$$\mathbf{\Lambda}_{m,n,w_0}[v] = \varphi_{\tau,w_0}'(z)[\varphi_{\tau,w_0}]^n \mathrm{e}^{m\mathcal{Q}_{\tau,w_0}} v \circ \varphi_{\tau,w_0}, \tag{2.3.1}$$

and make the corresponding ansatz $\mathrm{k}_{m,n,w_0} = \mathbf{\Lambda}_{m,n,w_0}[v_{m,n,w_0}]$. The assertion that k_{m,n,w_0} is the root function of order n at w_0 entails that v_{m,n,w_0} is approximately reproducing for the origin with respect to the corresponding weight R_{τ,w_0} given by

$$R_{\tau,w_0} := (Q - \breve{Q}_\tau) \circ \varphi_{\tau,w_0}^{-1}. \tag{2.3.2}$$

So, we should have that

$$\int_{\mathbb{C}} \chi_1^2 v_{m,n,w_0} \bar{q}\, \mathrm{e}^{-2mR_{\tau,w_0}} \mathrm{dA} = \frac{\bar{q}(0)}{v_{m,n,w_0}(0)} + \mathrm{O}\big(m^{-\kappa-1}\|\chi_1 q\|_{mR_{\tau,w_0}}\big), \tag{2.3.3}$$

where χ_1 an appropriate cut-off function, which assumes the value 1 in a neighborhood of the closed disk $\bar{\mathbb{D}}$, and vanishes in the exterior disk $\mathbb{D}_{\mathrm{e}}(0,\delta)$, for some $\delta > 1$.

This time around, we prefer to use the unit disk rather than the exterior disk as our model domain. The properties of the mapping $\mathbf{\Lambda}_{m,n,w_0}$ remain essentially the same, however, as summarized in the following proposition. For a potential V and a domain Ω, we denote by $A^2_{mV,n,w_0}(\Omega)$ the spaces of holomorphic functions on Ω which vanish to order n at $w_0 \in \Omega$, endowed with the topology of $L^2(\mathrm{e}^{-2mV},\Omega)$. In case $n = 0$ we simply denote the space by $A^2_{mV}(\Omega)$.

Proposition 2.3.1. *Let Q be a (τ, w_0)-admissible potential, and let Ω_{τ,w_0} denote the corresponding off-spectral component. Moreover, let R_{τ,w_0} be given by* (2.3.2)*. Then, for $\delta > 1$ sufficiently close to 1, the operator $\mathbf{\Lambda}_{m,n,w_0}$ defines an invertible isometry*

$$\mathbf{\Lambda}_{m,n,w_0} : A^2_{mR_{\tau,w_0}}\big(\mathbb{D}(0,\delta)\big) \to A^2_{mQ,n,w_0}(\Omega_0),$$

if Ω_0 is chosen to be of the form $\Omega_0 = \varphi^{-1}_{\tau,w_0}(\mathbb{D}(0,\delta))$. The isometry property remains valid in the context of weighted L^2-spaces as well.

Proof. The conclusion is immediate by the defining normalizations of the conformal mapping φ_{τ,w_0}. □

The following definition is an analogue of [12, Definition 3.1.2]. We denote by Ω_1 a domain containing the closure of the off-spectral component Ω_{τ,w_0}, and let $\chi_{0,\tau}$ denote a C^∞-smooth cut-off function which vanishes off Ω_1, and equals 1 in a neighbourhood of the closure of Ω_{τ,w_0}. As a remark before the formulation, we should point out that in (2.3.3), it is convenient to use one cut-off function for each of $v = v_{m,n,w_0}$ and q. However, if $\mathbf{\Lambda}_{m,n,w_0} q = f$ is an entire function, we can use the coordinates of the plane where f lives and make do with a single cut-off function.

Definition 2.3.2. Let κ be a positive integer. A sequence $\{F_{m,n,w_0}\}_{m,n}$ of holomorphic functions on Ω_0 is called a *sequence of approximate root vectors of order n at w_0 of accuracy κ* for the space A^2_{mQ,n,w_0} if the following conditions are met as $m \to +\infty$ while $\tau = \frac{n}{m} \in I_{w_0}$:
(i) For all $f \in A^2_{mQ,n+1,w_0}$, we have the approximate orthogonality

$$\int_{\mathbb{C}} \chi_{0,\tau} F_{m,n,w_0} \bar{f}\, \mathrm{e}^{-2mQ} \mathrm{dA} = \mathrm{O}(m^{-\kappa-\frac{1}{3}}\|p\|_{mQ}).$$

(ii) The approximate root functions have norm approximately equal to 1,

$$\int_{\mathbb{C}} \chi_{0,\tau}^2 |F_{m,n,w_0}(z)|^2 \mathrm{e}^{-2mQ}(z) \mathrm{dA}(z) = 1 + \mathrm{O}(m^{-\kappa-\frac{1}{3}}).$$

(iii) The functions F_{m,n,w_0} are approximately real and positive at w_0, in the sense that the leading coefficient $a_{m,n,w_0} = \lim_{z\to w_0}(z-w_0)^{-n}F_{m,n,w_0}(z)$ satisfies $\operatorname{Re} a_{m,n,w_0} > 0$ and

$$\frac{\operatorname{Im} a_{m,n,w_0}}{\operatorname{Re} a_{m,n,w_0}} = \mathrm{O}(m^{-\kappa-\frac{1}{12}}).$$

We remark that the exponents in the above error terms are chosen for reasons of convenience, related to the correction scheme of Subsection 2.5.

2.4. **The orthogonal foliation flow.** The orthogonal foliation flow $\{\gamma_{m,n,t}\}_t$ is a smooth flow of closed curves near the unit circle $\mathbb{T}$, originally formulated in [12] in the context of orthogonal polynomials. The defining property is that $P_{m,n}$ should be approximately orthogonal to the lower degree polynomials along the curves $\Gamma_{m,n,t} = \phi_\tau^{-1}(\gamma_{m,n,t})$ with respect to the induced measure $\mathrm{e}^{-2mQ}\nu_{\mathrm{n}}\mathrm{ds}$, where ν_{n} denotes the normal velocity of the flow $\{\Gamma_{m,n,t}\}_t$ and ds denotes normalized arc length measure.

We recall a slight variant of Definition 4.2.1 in [12], which fixes the class of weights considered. For the formulation, we need the notion of polarization, which applies to real-analytically smooth functions. If $R(z)$ is real-analytic, there exists a function of two complex variables, denoted by $R(z,w)$, which is holomorphic in $(z,\bar{w})$, whose diagonal restriction equals $R(z,z) = R(z)$. The function $R(z,w)$ is called *polarization* of $R(z)$. If $R(z,w)$ is such a polarization of a function $R(z)$ which is real-analytically smooth near the circle $\mathbb{T}$ and quadratically flat there, then $R(z,w)$ factors as $R(z,w) = (1-z\bar{w})^2 R_0(z,w)$, where $R_0(z,w)$ is holomorphic in $(z,\bar{w})$ in a neigborhood of the diagonal where both variables are near $\mathbb{T}$. The function $R_0(z,w)$ may be viewed as the polarization of the function R_0 given by $R_0(z) = (1-|z|^2)^{-2}R(z)$.

Definition 2.4.1. For real numbers $\delta > 1$ and $0 < \sigma$, we denote by $\mathcal{W}(\delta,\sigma)$ the class of non-negative C^2-smooth functions R on $\mathbb{D}(0,\delta)$ such that R is quadratically flat on $\mathbb{T}$ with $\Delta R|_{\mathbb{T}} > 0$ which satisfy

$$\inf_{z\in\mathbb{D}(0,\delta)} R_0(z) = \frac{R(z)}{(1-|z|^2)^2} = \alpha(R) > 0,$$

while on the annulus

$$\mathbb{A}(\delta^{-1},\delta) := \{z\in\mathbb{C} :\ \delta^{-1} < |z| < \delta\}$$

R is real-analytically smooth, and has a polarization $R(z,w)$ which is holomorphic in $(z,\bar{w})$ on the fattened diagonal annulus

$$\hat{\mathbb{A}}(\delta,\sigma) = \big\{(z,w)\in\mathbb{A}(\delta^{-1},\delta)\times\mathbb{A}(\delta^{-1},\delta)\ :\ |z-w|\le 2\sigma\big\},$$

which factors as $R(z,w) = (1-z\bar{w})^2R_0(z,w)$, where $R_0(z,w)$ is holomorphic $(z,\bar{w})$ on the set $\hat{\mathbb{A}}(\delta,\sigma)$, and bounded and bounded away from zero there. We say that a subset $S\subset\mathcal{W}(\delta,\sigma)$ is a *uniform family*, provided that for each $R\in S$, the corresponding $R_0(z,w)$ is uniformly bounded on $\hat{\mathbb{A}}(\delta,\sigma)$ while the $\alpha(R)$ is uniformly bounded away from 0.

The significance of the above definition is that it helps us encode uniformity properties in the parameter τ as well as the point w_0.

We recall from Proposition 4.2.2 of [12] that if $f(z,w)$ is holomorphic in $(z,\bar{w})$ on the 2σ-fattened diagonal annulus $\hat{\mathbb{A}}(\delta,\sigma)$, then the function $f_{\mathbb{T}}(z)$, which equals the diagonal restriction $f(z,z)$ for $z\in\mathbb{T}$, extends holomorphically to $\mathbb{A}((\delta')^{-1},\delta')$, where

$$\delta' := \min\{\delta, \sqrt{1+\sigma^2}+\sigma\} > 1 \tag{2.4.1}$$

depends only on δ and σ. One may actually need to restrict the numbers ρ and δ further. Indeed, it turns out that we need for the functions $\log\Delta Q$, $\hat{R} = \sqrt{R}$ (which is positive inside and negative outside the unit circle) as well as $\log(z\partial_z\hat{R})$ have polarizations which are holomorphic in $(z,\bar{w})$ for $(z,w)\in\hat{A}(\delta,\sigma)$ and uniformly bounded there aswell. If R belongs to a uniform family of $\mathcal{W}(\delta_0,\sigma_0)$, then there exist (δ_1,σ_1) such that these properties hold for the polarizations with $\delta=\delta_1$ and $\sigma=\sigma_1$ (See Proposition 4.2.3 of [12]). For fixed (δ_0,σ_0), we let $\delta'=\delta_1'$ be the number defined in (2.4.1), where we use $(\delta,\sigma)=(\delta_1,\sigma_1)$.

Lemma 2.4.2. *Let $\mathcal{K}$ be a compact subset of each of the domains Ω_{τ,w_0}, where $\tau\in I_0$. Then there exist constants $\delta>1$ and $\sigma>0$ such that the collection of weights R_{τ,w_0} with $w_0\subset\mathcal{K}$ and $\tau\in I_0$ is a uniform family in $\mathcal{W}(\delta,\sigma)$.*

This is completely analogous to the corresponding claim in of [12], which was expressed in the context of an exterior conformal mapping.

The existence of the orthogonal foliation flow around the circle $\mathbb{T}$ and the asymptotic expansion of the root functions after canonical positioning are stated in the following lemma (compare with Lemma 4.1.2 in [12]).

Lemma 2.4.3. *Fix an accuracy parameter κ and let $R\in\mathcal{W}(\delta_0,\sigma_0)$. Then, if δ' is as in (2.4.1) with respect to (δ_1,σ_1), there exist a radius δ'' with $1<\delta''<\delta'$, bounded holomorphic functions h_s on $\mathbb{D}(0,\delta')$ of the form*

$$h_s=\sum_{0\le j\le\kappa}s^jB_j,\qquad z\in\mathbb{D}(0,\delta'),$$

and conformal mappings $\psi_{s,t}$ from $\mathbb{D}_{\mathrm{e}}(0,\delta'')$ into the plane given by

$$\psi_{s,t}=\psi_{0,t}+\sum_{\substack{(j,l)\in\mathcal{I}_{2\kappa+1}\\ j\ge1}}s^jt^l\hat{\psi}_{j,l}$$

such that for s,t small enough, the domains $\psi_{s,t}\big(\mathbb{D}\big)$ grow with t, while they remain contained in $\mathbb{D}(0,\delta')$. Moreover, for $\zeta\in\mathbb{T}$ we have that

$$\begin{aligned}(2.4.2)\qquad |h_s\circ\psi_{s,t}(\zeta)|^2\,\mathrm{e}^{-2s^{-1}R\circ\psi_{s,t}}\,\mathrm{Re}\,\big(\bar{\zeta}\partial_t\psi_{s,t}(\zeta)\overline{\psi_{s,t}'(\zeta)}\big)\\ =\mathrm{e}^{-s^{-1}t^2}\Big\{(4\pi)^{-\frac12}+\mathrm{O}\big(|s|^{\kappa+\frac12}+|t|^{2\kappa+1}\big)\Big\}.\end{aligned}$$

For small positive s, when t varies in the interval $[-\beta_s,\beta_s]$ with $\beta_s:=s^{1/2}\log\frac{1}{s}$, the flow of loops $\{\psi_{s,t}(\mathbb{T})\}_t$ cover a neighborhood of the circle $\mathbb{T}$ of width proportional to β_s smoothly. In addition, the first term B_0 is zero-free, positive at the origin, and has modulus $|B_0|=\pi^{-\frac14}(\Delta R)^{\frac14}$ on $\mathbb{T}$. The other terms B_j are all real-valued at the origin. The implied constant in (2.4.2) is uniformly bounded, provided that R is confined to a uniform family of $\mathcal{W}(\delta_0,\sigma_0)$.

2.5. **$\bar\partial$-corrections and asymptotic expansions of root functions.** In this section, we supply a proof of the main result, Theorem 1.4.2. The proof consists of two parts. First, we construct a family of approximate root function of a given order and accuracy, after which we apply Hörmander-type $\bar\partial$-estimates to correct these approximate kernels to entire functions. The precise result needed for the correction scheme runs as follows.

Proposition 2.5.1. *Let $f\in L^\infty(\mathcal{S}_{\tau,w_0})$, where $\tau=\frac{n}{m}$, and denote by $u=u_{m,n,w_0}$ the norm-minimal solution in $L^2_{m\hat{Q}_\tau}$ to the problem*

$$\bar\partial u=f,$$

in the sense of distributions, which vanishes at w_0 *to order* n*:* $|u(z)| = \mathrm{O}(|z-w_0|^n)$ *around* w_0*. Then* u *meets the bound*

$$\int_{\mathbb{C}} |u|^2 \mathrm{e}^{-2m\hat{Q}_{\tau,w_0}} \mathrm{dA} \le \frac{1}{2m} \int_{\mathcal{S}_{\tau,w_0}} |f|^2 \frac{\mathrm{e}^{-2mQ}}{\Delta Q} \mathrm{dA}.$$

This is an immediate consequence of Corollary 2.4.2 in [12], and essentially amounts to Hörmander's classical bound for the $\bar\partial$-equation in the given setting.

We turn to the proof of Theorem 1.4.2.

Sketch of proof of Theorem 1.4.2. As the proof is analogous to that of Theorems 1.3.3 and 1.3.4 in [12], we supply only an outline of the proof.

The construction of approximate root functions. We apply Lemma 2.4.3 with $s = m^{-1}$ and $R = R_{\tau,w_0}$ to obtain a smooth flow $\gamma_{s,t} = \gamma_{m,n,t,w_0}$ of curves, as well as bounded holomorphic functions $h_s = h^{\langle\kappa\rangle}_{m,n,w_0}$ such that the flow equation (2.4.2) is met. The sequence $\{B_j\}_j$ of bounded holomorphic functions produced by the lemma actually depend (smoothly) on the parameter τ and the root point w_0, so we put $B_j = B_{j,\tau,w_0}$ and define

$$h^{\langle\kappa\rangle}_{m,n,w_0} = \sum_{j=0}^{\kappa} m^{-j} B_{j,\tau,w_0}. \tag{2.5.1}$$

If we write

$$\mathcal{B}_{j,\tau,w_0} := (\varphi'_{\tau,w_0})^{\frac12} B_{j,\tau,w_0} \circ \varphi_{\tau,w_0},$$

it follows that

$$\mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0} := m^{\frac14} \mathbf{\Lambda}_{m,n,w_0}[h^{\langle\kappa\rangle}_{m,n,w_0}]$$

has the claimed form. It remains to show that $\mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0}$ is a family of approximate root functions of order n at w_0 with the stated uniformity property, and to show that it is close to the true normalizing reproducing kernel.

We denote by $\mathcal{D}_{m,n,w_0}$ the domain covered by the foliation flow, over the parameter range $-\delta_m \le t \le \delta_m$, where $\delta_m := m^{-\frac12}\log m$. Moreover, we define the domain $\mathcal{E}_{\tau,w_0}$ as the image of $\mathbb{D}(0,\delta'')$ under φ^{-1}_{τ,w_0}, and let $\chi_0 = \chi_{0,\tau,w_0}$ denote an an appropriately chosen smooth cut-off function, which takes the value 1 on a neighborhood of $\bar\Omega_{\tau,w_0}$ and vanishes off $\mathcal{E}_{\tau,w_0}$. If we let $\chi_1 := \chi_0 \circ \varphi^{-1}_{\tau,w_0}$ denote the corresponding cut-off extended to vanish off $\mathbb{D}(0,\delta'')$, we may show that

$$\int_{\mathbb{C}\setminus\mathcal{D}_{m,n,w_0}} \chi_1^2 |h^{\langle\kappa\rangle}_{m,n,w_0}|^2 \mathrm{e}^{-2mR_{\tau,w_0}} \mathrm{dA}(z) = \mathrm{O}(m^{-\alpha_0\log m+\frac12}) \tag{2.5.2}$$

holds for some $\alpha_0 > 0$. Let $g \in A^2_{mQ,n,w_0}$ be given, and put $q = \mathbf{\Lambda}^{-1}_{m,n,w_0}[g]$. Then, by the isometric property of $\mathbf{\Lambda}_{m,n,w_0}$ and the estimate (2.5.2), it follows that

$$\int_{\mathbb{C}} \chi_0\, \mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0}(z)\bar g(z)\,\mathrm{e}^{-2mQ}\mathrm{dA}(z) = m^{\frac14}\int_{\mathbb{C}} \chi_1 h^{\langle\kappa\rangle}_{m,n,w_0}(z)\bar q(z)\,\mathrm{e}^{-2mR_{\tau,w_0}}\mathrm{dA}(z)$$
$$= m^{\frac14}\int_{\mathcal{D}_{m,n,w_0}} \chi_1 h^{\langle\kappa\rangle}_{m,n,w_0}(z)\bar q(z)\,\mathrm{e}^{-2mR_{\tau,w_0}}\mathrm{dA}(z) + \mathrm{O}\big(m^{-\frac{\alpha_0}{2}\log m+\frac34}\|g\|_{mQ}\big).$$

The function $h^{\langle\kappa\rangle}_{m,n,w_0}$ may be assumed to be zero-free in $\mathcal{D}_{m,n}$, as the main term is bounded away from 0 in modulus, and consecutive terms are much smaller. For large enough m, $\chi_1 = 1$ holds on $\mathcal{D}_{m,n}$. We now introduce the function $q_{m,n} := q/h^{\langle\kappa\rangle}_{m,n}$, and integrate along the flow:

$$
\begin{aligned}
(2.5.3)\quad & m^{\frac14}\int_{\mathcal{D}_{m,n}} q_{m,n}(z)|h^{\langle\kappa\rangle}_{m,n,w_0}(z)|^2 \mathrm{e}^{-2mR_{\tau,w_0}(z)}\mathrm{dA}(z) \\
&= 2m^{\frac14}\int_{-\delta_m}^{\delta_m}\int_{\mathbb{T}} q_{m,n}\circ\psi_{m,n,t}(\zeta)\big|h^{\langle\kappa\rangle}_{m,n,w_0}\circ\psi_{m,n,t}(\zeta)\big|^2\mathrm{e}^{-2m\,R_{\tau,w_0}\circ\psi_{m,n,t}(\zeta)} \\
&\qquad\qquad \times \operatorname{Re}\big\{\bar\zeta\partial_t\psi_{m,n,t}(\zeta)\overline{\psi'_{m,n,t}(\zeta)}\big\}\mathrm{ds}(\zeta)\mathrm{d}t \\
&= 2m^{\frac14}\int_{-\delta_m}^{\delta_m}\int_{\mathbb{T}} q_{m,n}\circ\psi_{m,n,t}(\zeta)\Big\{(4\pi)^{-\frac12}\mathrm{e}^{-mt^2}+\mathrm{O}\big(m^{-\kappa-\frac13}\mathrm{e}^{-mt^2}\big)\Big\}\mathrm{ds}(\zeta)\mathrm{d}t \\
&= m^{-\frac14}\frac{q(0)}{h^{\langle\kappa\rangle}_{m,n}(0)}\big(1+\mathrm{O}(m^{-\log m})\big) \\
&\qquad + \mathrm{O}\bigg(m^{-\kappa-\frac{1}{12}}\int_{-\delta_m}^{\delta_m}\int_{\mathbb{T}}|q_{m,n}\circ\psi_{m,n,t}(\zeta)|\mathrm{ds}(\zeta)\,\mathrm{e}^{-mt^2}\mathrm{d}t\bigg) \\
&= m^{-\frac14}\frac{q(0)}{h^{\langle\kappa\rangle}_{m,n}(0)}+\mathrm{O}\big(m^{-\kappa-\frac13}\|g\|_{mQ}\big).
\end{aligned}
$$

Here, the crucial observation in the last step is that for fixed t, the composition $q_{m,n}\circ\psi_{m,n,t}$ is holomorphic, so that we may apply the mean value property:

$$
\int_{\mathbb{T}} q_{m,n}\circ\psi_{m,n,t}\mathrm{ds} = q_{m,n}\circ\psi_{m,n,t}(0) = q_{m,n}(0) = \frac{q(0)}{h^{\langle\kappa\rangle}_{m,n}(0)}.
$$

In particular, if $q(0)=0$, that is, if g vanishes to order $n+1$ or higher at w_0, then $\chi_0\mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0}$ and g are approximately orthogonal in L^2_{mQ}. Moreover, the same calculation also shows that up to a small error, $\chi_0\mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0}$ has norm 1 in L^2_{mQ}. The estimate in the error term comparing the integral in the flow parameters with the norm of g is analogous to the calculation for the orthogonal polynomials in Subsection 4.8 of [12].

The $\bar\partial$-correction scheme. The approximate normalized reproducing kernels are not globally defined, and are consequently not elements of our Bergman spaces of entire functions. However, by applying the Hörmander-type $\bar\partial$-estimate of Proposition 2.5.1, we obtain a solution $u = u_{m,n,w_0}$ to the equation

$$
\bar\partial u = \mathrm{k}^{\langle\kappa\rangle}_{m,n,q_0}\bar\partial\chi_0.
$$

By the proposition, it vanishes to order n at the root point w_0, and has exponentially small norm in $L^2_{mQ} = L^2(\mathbb{C},\mathrm{e}^{-2mQ})$. The function

$$
\mathrm{k}^\star_{m,n,w_0} := \chi_0\mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0} - u_{m,n,w_0}
$$

is also an approximate normalized partial Bergman reproducing kernel of the correct accuracy, but this time it at least is an element of the right space,

$$
\mathrm{k}^\star_{m,n,w_0} \in A^2_{mQ,n,w_0}.
$$

We denote by $\mathbf{P}_{m,n+1,w_0}$ the orthogonal projection onto the subspace $A^2_{mQ,n+1,w_0}$ of functions vanishing to order at least $n+1$, and put

$$
\tilde{\mathrm{k}}_{m,n,w_0} = \mathrm{k}^\star_{m,n,w_0} - \mathbf{P}_{m,n+1,w_0}\mathrm{k}^\star_{m,n,w_0}.
$$

By construction, $\mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0}$ vanishes precisely to the order n at the root point w_0, and the same is true for $\mathrm{k}^{\star}_{m,n,w_0}$ as well because the perturbation is very small. It follows that $\tilde{\mathrm{k}}_{m,n,w_0}$ inherits this property, that is to say,

$$\tilde{\mathrm{k}}_{m,n,w_0}(z) = C(z-w_0)^n + \mathrm{O}(|z-w_0|^{n+1}),$$

holds near w_0 for some complex constant $C \neq 0$, which would be positive from Lemma 2.4.3 except that we correct with the small $\bar{\partial}$-solution u_{m,n,w_0}. This gives that $C = (1+\mathrm{O}(\mathrm{e}^{-\alpha_1 m}))\, C_1$ where $C_1 > 0$ may depend on all the parameters but $\alpha_1 > 0$ is a uniform constant. Since the function $\tilde{\mathrm{k}}_{m,n,w_0}$ is orthogonal to all functions in A^2_{mQ} that vanish to order at least $n+1$ at the root point w_0, it follows that $\tilde{\mathrm{k}}_{m,n,w_0}$ equals a scalar multiple of the true root function k_{m,n,w_0}:

$$\tilde{\mathrm{k}}_{m,n,w_0} = c\, \mathrm{k}_{m,n,w_0},$$

for some complex constant $c \neq 0$. In view of the above, we conclude that $c = (1+\mathrm{O}(\mathrm{e}^{-\alpha_1 m}))\, c_1$, where $c_1 > 0$ may depend on all the parameters. As $\tilde{\mathrm{k}}_{m,n,w_0}(w_0)$ is approximately real, it follows that $c = c'\gamma$, where c' is real and positive, while $\gamma = 1+\mathrm{O}(m^{-\kappa-\frac{1}{2}})$. From the approximate orthogonality discussed earlier, and the smallness of u_{m,n,w_0}, we find that the orthogonal projection $\mathbf{P}_{m,n+1,w_0}\mathrm{k}^{\star}_{m,n,w_0}$ has small norm:

$$\|\mathbf{P}_{m,n+1,w_0}\mathrm{k}^{\star}_{m,n,w_0}\|_{mQ} = \mathrm{O}(m^{-\kappa-\frac{1}{12}}).$$

This leads to the estimate

$$\|\tilde{\mathrm{k}}_{m,n,w_0} - \chi_0 \mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0}\|_{mQ} = \mathrm{O}(m^{-\kappa-\frac{1}{12}})$$

and since

$$\|\chi_0 \mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0}\|_{mQ} = 1 + \mathrm{O}(m^{-\kappa-\frac{1}{3}})$$

as a consequence of the computation in (2.5.3), we obtain that positive constant c_1 has the asymptotics $c_1 = 1+\mathrm{O}(m^{-\kappa-\frac{1}{12}})$, which allows to say that $\tilde{\mathrm{k}}_{m,n,w_0}$ and the true root function k_{m,n,w_0}, which differ by a multiplicative constant, are very close. It now follows that

$$\|\mathrm{k}_{m,n,w_0} - \chi_0 \mathrm{k}^{\langle\kappa\rangle}_{m,n,w_0}\|_{mQ} = \mathrm{O}(m^{-\kappa-\frac{1}{12}}),$$

so that k_{m,n,w_0} has the desired asymptotic expansion in norm. In view of Proposition 2.1.3, the pointwise expansion is essentially immediate, at least in the region $\Omega_{\tau,w_0,m}$ where

$$\mathrm{dist}_{\mathbb{C}}(z, \Omega_{\tau,w_0}) \leq A\, m^{-\frac{1}{2}}(\log m)^{\frac{1}{2}},$$

which is where the functions $\breve{Q}_{\tau,w_0}$ and $\hat{Q}_{\tau,w_0}$ are comparable in the sense that

$$0 \leq m(\hat{Q}_{\tau,w_0} - \breve{Q}_{\tau,w_0}) \leq A^2 D \log m$$

for some fixed positive constant D depending only on Q. The only remaining issue is that the error terms are slightly worse than claimed. However, by replacing κ with $\kappa + 2 + A^2 D$ and deriving the expansion with the indicated higher accuracy, we conclude that they hold as well. This completes the outline of the proof. □

2.6. **Interface asymptotics of the Bergman density.** In this section we show how to obtain the error function transition behaviour of Bergman densities at interfaces, where the interface may occurs as a result of a region of negative curvature (understood as where $\Delta < 0$ holds in terms of the potential Q) or as a consequence of dealing with partial Bergman kernels. Here, we focus on the the partial Bergman kernel analysis. In fact, we may think of the first instance of the full Bergman kernel as a special case and maintain that it is covered by the presented material.

The following Corollary of the main theorem summarizes the asymptotics of normalized off-spectral partial Bergman kernels in a suitable form. The domains $\Omega_{\tau,w_0,m}$ are as in Theorem 1.4.2, for a given positive parameter A chosen suitably large.

Corollary 2.6.1. *Under the assumptions of Theorem 1.4.2, we have the asymptotics*

$$|\mathrm{k}_{m,n,w_0}(z)|^2 \mathrm{e}^{-2mQ(z)} \\ = \pi^{-\frac{1}{2}} m^{\frac{1}{2}} |\varphi'_{\tau,w_0}(z)|\, \mathrm{e}^{-2m(Q-\check{Q}_{\tau,w_0})(z)} \big\{ \mathrm{e}^{2\mathrm{Re}\,\mathcal{H}_{Q,\tau,w_0}(z)} + \mathrm{O}(m^{-1}) \big\},$$

on the domain $\Omega_{\tau,w_0,m}$, as $n = \tau m \to +\infty$ while $\tau \in I_0$, where $\mathcal{H}_{Q,\tau,w_0}$ is the bounded holomorphic function on Ω_{τ,w_0} whose real part equals $\frac{1}{4}\log(2\Delta Q)$ on the boundary, and is real-valued at the root point w_0.

Proof. In view of the decomposition (2.2.1), this is just the assertion of Theorem 1.4.2 with accuracy $\kappa = 1$. □

We proceed with a sketch of the error function asymptotics at interfaces, in particular we point out why we may proceed exactly as is done in the proof of Theorem 1.4.1 of [12].

Proof sketch of Corollary 1.4.4. We expand the partial Bergman kernel K_{m,n,w_0} along the diagonal in terms of the root functions k_{m,n',w_0}, for $n' \geq n$. We keep $\tau = \frac{n}{m}$ throughout. In view of Theorem 1.3.1, we have

(2.6.1)
$$K_{m,n,w_0}(z_m(\xi), z_m(\xi))\, \mathrm{e}^{-2mQ(z_m(\xi))} = \sum_{n'=n}^{+\infty} |\mathrm{k}_{m,n',w_0}(z_m(\xi))|^2 \mathrm{e}^{-2mQ(z_m(\xi))},$$

where $z_0 \in \partial\Omega_{\tau,w_0}$ and where $z_m(\xi)$ gives the rescaled coordinate implicitly by

$$z_m(\xi) = z_0 + \nu \frac{\xi}{\sqrt{2m\Delta Q(z_0)}}.$$

The rescaled Bergman density is then obtained by

$$\rho_m(\xi) = \frac{1}{2m\Delta Q(z_0)} \sum_{n \geq \epsilon m} |\mathrm{k}_{m,n,w_0}(z_m(\xi))|^2 \mathrm{e}^{-2mQ(z_m(\xi))}.$$

In view of the assumed (I_0, w_0)-admissibility, we may apply the asymptotic expansion in the main result, specifically in the form of Corollary 2.6.1. Since Proposition 2.1.2 tells us how the smooth Jordan curves $\partial\Omega_{\tau,w_0}$ propagate, a Taylor series expansion of the function $Q - \check{Q}_{\tau,w_0}$ allows us to write the partial Bergman density approximately as a sum of translated Gaussians

$$\rho_m(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{j \geq 0} \frac{\gamma_0}{\sqrt{m}} \mathrm{e}^{-\frac{1}{2}(2\,\mathrm{Re}\,\xi + j\frac{\gamma_0}{\sqrt{m}})^2} + \mathrm{O}\big(m^{-\frac{1}{2}}(\log m)^3\big), \tag{2.6.2}$$

where $\gamma_0 = \gamma_{z_0,w_0,Q}$ is a positive constant. As in the proof of Theorem 1.4.1 of [12], we proceed to interpret the above sum (2.6.2) as a Riemann sum for the integral formula for the error function:

$$\operatorname{erf}(2\mathrm{Re}\,\xi) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} \mathrm{e}^{-\frac{1}{2}(2\mathrm{Re}\,(\xi)+t)^2}\mathrm{d}t.$$

This proof is complete. □

3. The foliation flow for a twisted base metric

3.1. Twists of the base metric and inversion invariance. It will be desirable to obtain some flexibility on the part of the weight e^{-2mQ} in the expansion of Theorem 1.4.2. In particular, in the following subsections we will discuss various situations in which one needs asymptotics for root functions and orthogonal polynomials with respect to measures

$$\mathrm{e}^{-2mQ}V\,\mathrm{dA},$$

where V is non-negative C^2-smooth function which is real-analytic and non-zero in a neighbourhood of the fixed smooth spectral interface of interest, which meet the polynomial growth bound

$$V(z) = \mathrm{O}(|z|^N), \qquad |z| \to +\infty, \tag{3.1.1}$$

for some number $N < +\infty$. In particular, this covers working with the spherical area measure $\mathrm{dA}_{\mathbb{S}}(z) := (1+|z|^2)^{-2}\mathrm{dA}(z)$ in place of planar area measure simply by putting $V(z) = (1+|z|^2)^{-2}$. Working with the spherical area measure has the advantage of invariance with respect to rotations and inversion. For a more general twist V, we factor $V\mathrm{dA} = V_{\mathbb{S}}\mathrm{dA}_{\mathbb{S}}$, where $V_{\mathbb{S}}(z) = (1+|z|^2)^2V(z)$, and see that our weighted measure is

$$\mathrm{e}^{-2mQ}V_{\mathbb{S}}\,\mathrm{dA}_{\mathbb{S}},$$

which has a more invariant appearance. If we write $\iota(z) = z^{-1}$, the spaces of polynomials of degree at most n with respect to the L^2-space with measure $\mathrm{e}^{-2mQ}V_{\mathbb{S}}\,\mathrm{dA}_{\mathbb{S}}$ becomes isometrically isomorphic to the L^2-space of rational functions on the sphere $\mathbb{S}$ with a simple pole of order at most n at the origin, with respect to the L^2-space with measure $\mathrm{e}^{-2mQ\circ\iota}V_{\mathbb{S}}\circ\iota\,\mathrm{dA}_{\mathbb{S}}$. This provides an extension of the scale of root functions to zeros of negative order (i.e. poles), and the apparent similarities between orthogonal polynomials and root functions may be viewed in this light. This analogy goes even deeper than that. Assuming that 0 is an off-spectral point for the weighted L^2-space with measure $\mathrm{e}^{-2mQ\circ\iota}V_{\mathbb{S}}\circ\iota\,\mathrm{dA}_{\mathbb{S}}$, we may multiply by a suitable power of the conformal mapping from the off-spectral region to the unit disk $\mathbb{D}$, which preserves the origin, to obtain a space of functions holomorphic in a neighborhood of the off-spectral region. Hörmander-type estimates for the $\bar\partial$-equation then permit us to correct the functions so that they are entire, with small cost in norm.

We note that twists appear naturally from working with perturbations of the potential Q. Indeed, if we consider $\tilde{Q} = Q - m^{-1}h$ for some smooth function h of modest growth, we have that

$$\mathrm{e}^{-2m\tilde{Q}}\mathrm{dA} = \mathrm{e}^{-2mQ}\mathrm{e}^{2h}\mathrm{dA},$$

which corresponds precisely to the twist weight $V = \mathrm{e}^{2h}$.

3.2. **The asymptotics of root functions and orthogonal polynomials for twisted base metrics.** Our analysis will show that the root function asymptotics of Theorem 1.4.2 holds also in the context of a twisted base metric, with only a slight change in the structure of the coefficients $\mathcal{B}_{j,\tau,w_0}$. Let $A^2_{mQ,V}$ denote the weighted Bergman space of entire functions with respect to the Hilbert space norm

$$\|f\|^2_{mQ,V} := \int_{\mathbb{C}} |f|^2 \mathrm{e}^{-2mQ} V \mathrm{dA} < +\infty.$$

The corresponding Bergman kernel is denoted by $K_{m,V}$. We also need the partial Bergman spaces A^2_{mQ,V,n,w_0}, consisting of the functions in $A^2_{mQ,V}$ that vanish at w_0 to order n or higher. These are closed subspaces of $A^2_{mQ,V}$ which get smaller as n increases: $A^2_{mQ,V,n+1,w_0} \subset A^2_{mQ,V,n,w_0}$. The successive difference spaces $A^2_{mQ,V,n,w_0} \ominus A^2_{mQ,V,n+1,w_0}$ have dimension at most 1. If the dimension equals 1, we single out an element $\mathrm{k}_{m,n,w_0,V} \in A^2_{mQ,V,n,w_0} \ominus A^2_{mQ,V,n+1,w_0}$ of norm 1, which has positive derivative of order n at w_0. In the remaining case when the dimension equals 0 we put $\mathrm{k}_{m,n,w_0,V} = 0$. We call $\mathrm{k}_{m,n,w_0,V}$ *root functions*, and observe that these are the same objects as defined previously for $V = 1$ in terms of an extremal problem.

Theorem 3.2.1. *Under the assumptions of Theorem 1.4.2 and the above-mentioned assumptions on V, with respect to the interface $\partial\Omega_{\tau,w_0}$, we have, using the notation of the same theorem, for fixed accuracy and a given positive real A, the asymptotic expansion of the root function*

$$\mathrm{k}_{m,n,w_0,V}(z) \\ = m^{\frac{1}{4}} (\varphi'_{\tau,w_0}(z))^{\frac{1}{2}} (\varphi_{\tau,w_0}(z))^n \mathrm{e}^{m\mathcal{Q}_{\tau,w_0}} \Big\{ \sum_{j=0}^{\kappa} m^{-j} \mathcal{B}_{j,\tau,w_0,V}(z) + \mathrm{O}\big(m^{-\kappa-1}\big) \Big\},$$

on the domain $\Omega_{\tau,w_0,m}$ which depends on A, where $\tau = \frac{n}{m}$, and the implied constant is uniform. Here, the main term $\mathcal{B}_{0,\tau,w_0}$ is zero-free and smooth up to the boundary on Ω_{τ,w_0}, positive at w_0, with prescribed modulus

$$[\mathcal{B}_{0,\tau,w_0,V}(z)| = \pi^{-\frac{1}{4}} (\Delta Q(z))^{\frac{1}{4}} V(z)^{-\frac{1}{2}}, \qquad z \in \partial\Omega_{\tau,w_0}.$$

The proof of this theorem is analogous to that of Theorem 1.4.2, given that we have explained how to modify the orthogonal foliation flow with respect to the twist in Lemma 3.3.2. The lemma is applied with $s = m^{-1}$. We omit the necessary details.

We turn next to the computation of the coefficients $\mathcal{B}_{j,\tau,w_0,W}$ in the above expansion. We recall that R_{τ,w_0} is the potential induced by Q in the canonical positioning procedure, and we put analogously

$$W_{\tau,w_0}(z) = V \circ \varphi^{-1}_{\tau,w_0}(z), \qquad z \in \mathbb{D}(0,\delta).$$

For the formulation, we need the orthogonal projection $\mathbf{P}_{H^2_0}$ of $L^2(\mathbb{T})$ onto the Hardy space H^2_0 of functions f in the Hardy space H^2 that vanish at the origin.

Theorem 3.2.2. *In the asymptotic expansion of root functions in Theorem 3.2.1, the coefficient functions $\mathcal{B}_{j,\tau,w_0,V}$ are obtained by*

$$\mathcal{B}_{j,\tau,w_0,V} = (\varphi'_{\tau,w_0})^{\frac{1}{2}} B_{j,\tau,w_0,V} \circ \varphi_{\tau,w_0}, \qquad j = 1, 2, 3, \ldots.$$

If $H_{\tau,w_0,V}$ denotes the unique bounded holomorphic function on $\mathbb{D}$, whose real part meets

$$\operatorname{Re} H_{\tau,w_0,V} = \frac{1}{4}\log(4\Delta R_{\tau,w_0}) + \frac{1}{2}\log(W_{\tau,w_0}), \qquad \text{on } \mathbb{T},$$

with $\operatorname{Im} H_{\tau,w_0,V}(0) = 0$*, the functions $B_{j,\tau,w_0,V}$ may be obtained algorithmically as*

$$B_{j,\tau,w_0,V} = c_j \mathrm{e}^{H_{\tau,w_0,V}} - \mathrm{e}^{H_{\tau,w_0,V}} \mathbf{P}_{H^2_0}\left[\mathrm{e}^{\bar{H}_{\tau,w_0,V}} F_j\right]$$

for some real constants $c_j = c_{j,\tau,w_0,V}$ and real-analytically smooth functions $F_j = F_{j,\tau,w_0,V}$ on the unit circle $\mathbb{T}$. Here, both the constants c_j and the functions F_j may be computed iteratively in terms of $B_{0,\tau,w_0,V}, \ldots B_{j-1,\tau,w_0,V}$.

One may further derive concrete expressions for the constants c_j and the real-analytic functions F_j in the above result, in terms of the rather complicated explicit differential operators $\mathbf{L}_k$ and $\mathbf{M}_k$ as defined in equation (1.3.4) and Lemma 3.2.1 in [12]. We should mention that the definition of the operator $\mathbf{M}_k$ contains a parameter l, which is allowed to assume only non-negative values. However, the same definition works also for $l < 0$, which is necessary for the present application. In terms of the operators $\mathbf{M}_k$ and $\mathbf{L}_k$, we have

$$F_j(\theta) = \sum_{k=1}^{j} \mathbf{M}_k[B_{j-k,\tau,w_0,V} W_{\tau,w_0}]$$

and

$$c_j = -\frac{(4\pi)^{-\frac{1}{4}}}{2} \sum_{(i,k,l)\in\mathcal{J}_j} \int_{\mathbb{T}} \frac{\mathbf{L}_k[rB_{i,\tau,w_0,V}(r\mathrm{e}^{\mathrm{i}\theta})\bar{B}_{l,\tau,w_0,V}(r\mathrm{e}^{\mathrm{i}\theta})W_{\tau,w_0}(r\mathrm{e}^{\mathrm{i}\theta})]}{(4\Delta R_{\tau,w_0}(r\mathrm{e}^{\mathrm{i}\theta}))^{\frac{1}{2}}}\Big|_{r=1} \mathrm{ds}(\mathrm{e}^{\mathrm{i}\theta}).$$

Here, the index set $\mathcal{J}_j$ is defined as

$$\mathcal{J}_j = \{(i,k,l) \in \mathbb{N}^3 : i, l < j, \;\; i + k + l = j\},$$

where we use the convention that the natural numbers $\mathbb{N}$ includes 0. This theorem is obtained in the same fashion as Theorem 1.3.7 in [12] in the context of orthogonal polynomials, and we do not write down a proof here.

3.3. **The flow modified by a twist.** We proceed first to modify the book-keeping slightly by formulating an analogue of Definition 2.4.1, which applies to weights after canonical positioning.

Definition 3.3.1. Let δ and σ be given positive numbers, with $\delta > 1$. A pair (R, W) of non-negative C^2-smooth weights defined on $\mathbb{D}(0,\delta)$ is said to belong to the class $\mathcal{W}_1(\delta,\sigma)$ if $R \in \mathcal{W}(\delta,\sigma)$ and if the weight W meets the following conditions:

(i) W is real-analytic and zero-free in the neighbourhood $\mathbb{A}(\delta^{-1},\delta)$ of the unit circle $\mathbb{T}$,

(ii) The polarization $W(z,w)$ of W extends to a bounded holomorphic function of $(z,\bar{w})$ on the 2σ-fattened diagonal annulus $\hat{\mathbb{A}}(\sigma,\delta)$, which is also bounded away from 0.

A collection S of pairs (R, W) is said to be a uniform family in $\mathcal{W}_1(\delta,\sigma)$ if the weights R with $(R,W) \in S$ are confined to a uniform family in $\mathcal{W}(\delta,\sigma)$, while $W(z,w)$ is uniformly bounded and bounded away from 0 in $\hat{\mathbb{A}}(\delta,\sigma)$.

Fix a pair (δ_0,σ_0). Just as before, we let (δ_1,σ_1) denote a possibly more restrictive pair of positive reals with $\delta_1<1$ such that the relevant polarizations are hermitian-holomorphic and uniformly bounded on $\hat{A}(\delta_1,\sigma_1)$. In connection with this definition, we recall the number $\delta'=\delta'(\delta,\sigma)$ given by

$$\delta'=\min\left\{\delta_1,\sqrt{1+\sigma_1^2}+\sigma_1\right\}>1,$$

obtained from the defining property that if $f(z,w)$ is holomorphic in $(z,\bar{w})$ on the set $\hat{\mathbb{A}}(\sigma,\delta)$, then the function $f_{\mathbb{T}}(z)=f(z,\bar{z}^{-1})$ may be continued holomorphically to the annulus $\mathbb{A}((\delta')^{-1},\delta')$.

We proceed with the main result of this section.

Lemma 3.3.2. *Fix an accuracy parameter κ and let $(R,W)\in\mathcal{W}_1(\delta_0,\sigma_0)$. Then there exist a radius δ'' with $\delta'>\delta''>1$, bounded holomorphic functions h_s on $\mathbb{D}(0,\delta')$ of the form*

$$h_s=\sum_{j=0}^{\kappa}s^jB_j,\qquad z\in\mathbb{D}(0,\delta')$$

and normalized conformal mappings $\psi_{s,t}$ on $\mathbb{D}(0,\delta'')$ given by

$$\psi_{s,t}=\psi_{0,t}+\sum_{\substack{(j,l)\in\mathcal{I}_{2\kappa+1}\\ j\geq1}}s^jt^l\hat{\psi}_{j,l}$$

such that for s,t small enough it holds that the domains $\psi_{s,t}(\mathbb{D})$ increase with t, while they remain contained in $\mathbb{D}(0,\delta')$. Moreover, for $\zeta\in\mathbb{T}$, we have

$$\begin{aligned}(3.3.1)\qquad |h_s\circ\psi_{s,t}(\zeta)|^2\,\mathrm{e}^{-2s^{-1}R\circ\psi_{s,t}}\operatorname{Re}\big(\bar{\zeta}\partial_t\psi_{s,t}(\zeta)\overline{\psi'_{s,t}(\zeta)}\big)W\circ\psi_{s,t}(\zeta)\\ =\mathrm{e}^{-s^{-1}t^2}\Big\{(4\pi)^{-\frac12}+\mathrm{O}\big(|s|^{\kappa+\frac12}+|t|^{2\kappa+1}\big)\Big\}.\end{aligned}$$

For small positive s, when t varies in the interval $[-\beta_s,\beta_s]$ with $\beta_s:=s^{1/2}\log\frac{1}{s}$, the flow of loops $\{\psi_{s,t}(\mathbb{T})\}_t$ cover a neighborhood of the circle $\mathbb{T}$ of width proportional to β_s smoothly. In addition, the main term B_0 is zero-free, positive at the origin, and has modulus $|B_0|=\pi^{-\frac14}(\Delta R)^{\frac14}W^{-\frac12}$ on $\mathbb{T}$, and the other terms B_j are all real-valued at the origin. The implied constant in (3.3.1) *is uniformly bounded, provided that (R,W) is confined to a uniform family of $\mathcal{W}_1(\delta_0,\sigma_0)$*

In order to obtain this lemma, we need to modify the algorithm which gives the original result. We proceed to sketch the outlines of this modification. The omitted details are avaiable in [12], and we intend to guide the reader for easy reading.

We recall the following index sets from [12]. For an integer n, we introduce

$$(3.3.2)\qquad \mathcal{I}_n=\big\{(j,l)\in\mathbb{N}^2\,:\,2j+l\leq n\big\}.$$

We endow the set $\mathcal{I}_n$ with the ordering $\prec$ induced by the lexicographic ordering, so we agree that $(j,l)\prec(a,b)$ if $j<a$ or if $j=a$ and $l<b$.

Proof of Lemma 3.3.2. The conformal mappings $\psi_{s,t}$ are assumed to have the form

$$\psi_{s,t}=\psi_{0,t}+\sum_{\substack{(j,l)\in\mathcal{I}_{2\kappa+1}\\ j\geq1}}s^jt^l\hat{\psi}_{j,l}$$

for some bounded holomorphic coefficients $\hat{\psi}_{j,l}$ and a conformal mapping

$$\psi_{0,t} = \sum_{l=0}^{+\infty} t^l \hat{\psi}_{0,l}.$$

We make the following initial observation. In the limit case $s = 0$, the flow equation (3.3.1) of the lemma forces $\psi_{0,t}$ to be a mapping from $\mathbb{D}$ onto the interior of suitably chosen level curves of R, and from this we may obtain the coefficients $\hat{\psi}_{0,l}$. Indeed, if we take logarithms of both sides of the equation and multiply by s we obtain

$$s \log|h_s \circ \psi_{s,t}|^2 - 2R \circ \psi_{s,t} + s \log(1-t) \log J_\Psi + s \log W \circ \psi_{s,t} = -t^2 + \mathrm{O}(s), \tag{3.3.3}$$

where the J_Ψ denotes a Jacobian, of the form

$$J_{\Psi_s}((1+t)\zeta) = \mathrm{Re}\left(\bar{\zeta}\partial_t\psi_{s,t}\overline{\psi'_{s,t}(\zeta)}\right), \qquad \zeta \in \mathbb{T}.$$

Assuming some reasonable stability with respect to the variable s as $s \to 0$ in (3.3.3), we obtain in the limit

$$2R \circ \psi_{0,t}(\zeta) = t^2, \qquad \zeta \in \mathbb{T}. \tag{3.3.4}$$

It follows that the loop $\psi_{0,t}(\mathbb{T})$ is a part of the level set where $R = t^2/2$. This level set consists of two disjoint simple closed curves, one on either side of $\mathbb{T}$, at least for small enough t. For $t > 0$, we choose the curve outside the unit circle, while for $t < 0$ we choose the other one. We normalize the mapping $\psi_{0,t}$ so that it preserves the origin and has positive derivative there. In this fashion, the coefficients $\hat{\psi}_{0,l}$ are now uniquely determined by the level set condition. The smoothness of the level curves was worked out in some detail in Proposition 4.2.5 in [12]. The details of the determination of the coefficients $\hat{\psi}_{0,l}$ are given in terms of Herglotz integrals as in Proposition 4.6.1 of [12].

Our next task is to obtain iteratively the coefficients B_j for $j = 0, 1, 2, \ldots$ and the higher order corrections to the conformal mapping, given in terms of the coefficients $\hat{\psi}_{j,l}$ for $j = 1, \ldots, \kappa$. These coefficient functions are obtained by differentiating the flow equation (3.3.1). To set things up correctly, we write

$$\omega_{s,t}(\zeta) = |h_s \circ \psi_{s,t}|^2 \mathrm{e}^{-2s^{-1}(R\circ\psi_{s,t} - \frac{t^2}{2})} W \circ \psi_{s,t} \,\mathrm{Re}\left(\bar{\zeta}\partial_t\psi_{s,t}\overline{\psi'_{s,t}(\zeta)}\right)$$

for $\zeta \in \mathbb{T}$, where we have used that $R \circ \psi_{0,t} = \frac{t^2}{2}$. We need to show that

$$\omega_{s,t}(\zeta) = (4\pi)^{-\frac{1}{2}} + \mathrm{O}\left(|s|^{\kappa+\frac{1}{2}} + |t|^{2\kappa+1}\right). \tag{3.3.5}$$

Since we believe that $\omega_{s,t}$ should be smooth in both parameters s and t, we aim to deduce (3.3.5) from demonstrating the solvability of the system of equations

$$\partial_s^j \partial_t^l \omega_{s,t}(\zeta)\big|_{s=t=0} = \begin{cases} (4\pi)^{-\frac{1}{2}} & \text{for } \zeta \in \mathbb{T} \text{ and } (j,l) = (0,0), \\ 0 & \text{for } \zeta \in \mathbb{T} \text{ and } (j,l) \in \mathcal{I}_{2\kappa} \setminus (0,0), \end{cases} \tag{3.3.6}$$

in terms of the unknown coefficients. If, in addition, we can show that the functions B_j and $\hat{\psi}_{j,l}$ remain holomorphic and uniformly bounded in the appropriate domains provided that (R, W) remains confined to a uniform family of $\mathcal{W}_1(\delta_0, \sigma_0)$, the result follows.

To proceed to solve the system (3.3.6), we express the partial derivatives of $\omega_{s,t}$ in terms of our unknowns. We determine these unknowns by an iterative procedure. At the given step in the iteration, some of the coefficient functions will be already found. We split the equation (3.3.6) into a term containing an unknown coefficient function and a second term which contains only already determined coefficient functions. Here, we refrain from giving a complete account, which is available with minor modifications in [12]. The higher order partial derivatives of the function $\omega_{s,t}$ with respect of s and t are given by

$$(3.3.7)\quad 0 = \frac{1}{(j-1)!l!}\partial_s^{j-1}\partial_t^l \omega_{s,t}(\zeta)\big|_{s=t=0}$$
$$= 4(4\pi)^{-\frac{1}{2}}\Delta R(\zeta)\ \mathrm{Re}(\bar\zeta\hat\psi_{j,l-1}(\zeta))\,\mathrm{Re}(\bar\zeta\hat\psi_{0,1}(\zeta))W(\zeta) + \mathfrak{F}_{j-1,l,W}(\zeta), \qquad \zeta\in\mathbb{T},$$

when $j, l > 0$, and

$$(3.3.8)\quad 0 = \frac{1}{j!}\partial_s^j \omega_{s,t}(\zeta)\big|_{s=t=0}$$
$$= 2\,\mathrm{Re}\left(\bar B_0(\zeta)B_j(\zeta)\right)W(\zeta)(4\Delta R(\zeta))^{-\frac{1}{2}} + \mathfrak{F}_{j,0,W}(\zeta), \qquad \zeta\in\mathbb{T},$$

when $l = 0$ and $j > 0$. Here, we have inserted the equation (3.3.6) for convenience. The expressions $\mathfrak{F}_{j,l,W}$ are real-valued real-analytic functions which are uniformly bounded while (R, W) remains in a uniform family in $\mathcal{W}_1(\delta_0, \sigma_0)$, and may be explicitly written down using the Faà di Bruno's formula. The crucial point for us is the dependence structure of the functions $\mathfrak{F}_{j,l,W}$, which remains the same as in the algorithm for the orthogonal polynomials:

The function $\mathfrak{F}_{j-1,l,W}$, $l \geq 1$ is an expression in terms of $B_0, \ldots, B_{j-1,0}$ and $\hat\psi_{p,q}$ for indices $(p,q) \in \mathcal{I}_{2\kappa+1}$ with $(p,q) \prec (j, l-1)$

and

The function $\mathfrak{F}_{j,0,W}$ depends only on $B_0, \ldots, B_{j-1}$ and $\hat\psi_{p,q}$ with $(p,q) \prec (j+1, 0)$.

That this is so follows immediately from noticing that Propositions 4.2.5 and 4.6.1 in [12] and remain unchanged, while Proposition 4.6.2-4.6.4 in [12] require the obvious modifications related to our replacing the weight e^{-2mR} on the exterior disk $\mathbb{D}_{\mathrm{e}}(0,\rho)$ by a weight $\mathrm{e}^{-2mR}W$ on the disk $\mathbb{D}(0,\delta)$. In particular, it is paramount that $W(z)$ is strictly positive in a fixed neighbourhood of the unit circle $\mathbb{T}$. A natural approach to the computations is to write

$$\omega_{s,t}^I = |h_s\circ\psi_{s,t}|^2 \mathrm{e}^{-2s^{-1}(R\circ\psi_{s,t}-\frac{t^2}{2})}\,\mathrm{Re}\left(\bar\zeta\partial_t\psi_{s,t}\overline{\psi_{s,t}'(\zeta)},\right)$$

which is essentially the expression which gets expanded in [12], and introduce the modification $\omega_{s,t}^{II} = W\circ\psi_{s,t}$, since then $\omega_{s,t} = \omega_{s,t}^I\omega_{s,t}^{II}$. By Leibnitz' formula, it is now easy to express the partial derivatives of $\omega_{s,t}$ in terms of derivatives of $\omega_{s,t}^I$ and of $\omega_{s,t}^{II}$. The partial derivatives of $\omega_{s,t}^I$ are as in Proposition 4.6.4 of [12], while the latter may be computed explicitly using the Faà di Bruno formula. The leading behaviour of $\partial_s^j\partial_t^l\omega_{s,t}$ comes from the contribution of $\partial_s^j\partial_t^l\omega_{s,t}^I$, which is as in [12]. It is easily verified that the remainders $\mathfrak{F}_{j,l,W}$ have the indicated properties.

We sketch the solution algorithm below, and indicate where the necessary modifications to the corresponding steps in [12] are required. We need the notation Γ_t for the component curve of the level set where

$$R(z) = \tfrac{t^2}{2}$$

chosen as before to expand as t increases.

Step 1. Take as above $\psi_{0,t}$ to be the conformal mapping $\psi_{0,t}:\mathbb{D}\to D_t$, where $\psi_{0,t}(0)=0$ and $\psi'_{0,t}(0)>0$, where D_t denotes the domain bounded by the curve Γ_t. It follows from the smoothness of the flow of the level curves Γ_t (for details see Proposition 4.2.5 in [12]) that we have an expansion

$$\psi_{0,t}=\sum_{l=0}^{+\infty}t^l\hat{\psi}_{0,l}$$

of $\psi_{0,t}$, which determines the coefficient functions $\hat{\psi}_{0,l}$ for all $l=0,1,2,\dots$ (see Proposition 4.6.1 [12]). In particular, we have

$$\mathrm{Re}(\bar{\zeta}\hat{\psi}_{0,1}(\zeta))=[4\mathit{\Delta}R(\zeta)]^{-\frac{1}{2}},\qquad \zeta\in\mathbb{T}.$$

Step 2. The equation (3.3.6) for $(j,l)=(0,0)$ gives that

$$|B_0(\zeta)|^2\,\mathrm{Re}(\bar{\zeta}\hat{\psi}_{0,1}(\zeta))\,W(\zeta)=(4\pi)^{-\frac{1}{2}},\qquad \zeta\in\mathbb{T},$$

which in its turn determines B_0. Indeed, the only outer function which is positive at the origin and meets this equation is

$$B_0(\zeta)=(4\pi)^{-\frac{1}{2}}\exp\Big\{\frac{1}{2}\mathbf{H}_{\mathbb{D}}\Big[\log[4\mathit{\Delta}R]^{\frac{1}{2}}-\log W(\zeta)\Big]\Big\},$$

where the Herglotz operator $\mathbf{H}_{\mathbb{D}}$ is given by

$$\mathbf{H}_{\mathbb{D}}f(z):=\int_{\mathbb{T}}\frac{1+\bar{w}z}{1-\bar{w}z}\,f(w)\,\mathrm{ds}(w),\qquad z\in\mathbb{D},$$

extended to the boundary via nontangential boundary values, and, when possible, also by analytic continuation.

As W is real-analytic and positive in a neighbourhood of $\mathbb{T}$, and moreover meets the regularity requirements of Definition 3.3.1, nothing is essentially different from the situation described in [12]. For instance, we may conclude that B_0 extends as a bounded, zero-free holomorphic function to $\mathbb{D}(0,\delta')$ where $\delta'>1$.

We proceed to Step 3 with $j_0=1$.

Step 3. At the outset, we have an integer $j_0\geq 1$ such that we have already successfully determined the coefficient functions $B_0,\dots,B_{j_0-1}$ as well as $\hat{\psi}_{j,l}$ for all $(j,l)\prec(j_0,0)$. In this step, we intend to determine all the coefficient functions $\hat{\psi}_{j_0,l}$ such that $(j_0,l)\in\mathcal{I}_{2\kappa+1}$. This is done in inductively in the parameter l, starting with $\hat{\psi}_{j_0,0}$. Assume that it has been carried out for all $l=0,\dots,l_0-1$. The coefficient $\hat{\psi}_{j_0,l_0}$ which we are looking for appears as the leading term in the equation (3.3.7) corresponding to $j=j_0$ and $l=l_0+1$. We solve for the coefficient function $\hat{\psi}_{j_0,l_0}$ in terms of the Herglotz operator:

$$\hat{\psi}_{j_0,l_0}(\zeta)=-(4\pi)^{\frac{1}{4}}\zeta\mathbf{H}_{\mathbb{D}}\Big[\frac{\mathfrak{F}_{j_0-1,l_0+1,W}}{(4\mathit{\Delta}R)^{\frac{1}{2}}W}\Big](\zeta).$$

The first step $l_0=0$ of the induction is carried out in a similar fashion. This completes Step 3.

Step 4. After completing Step 3, we find ourselves in the following situation: the coefficients B_j are known for $j<j_0$, while $\hat{\psi}_{j,l}$ are known for all $(j,l)\in\mathcal{I}_{2\kappa+1}$ with

$(j,l) \prec (j_0+1,0)$. We proceed to determine B_{j_0} using the equation (3.3.8) with index $(j,l)=(j_0,0)$. The equation says that on $\mathbb{T}$,

$$2W(4\varDelta R)^{-\frac{1}{2}}\operatorname{Re}\big(\bar{B}_0 B_{j_0}\big)+\mathfrak{F}_{j_0,0,W}=0,$$

where $\mathfrak{F}_{j_0,0,W}$ depends on the known data. So we may simply solve for B_{j_0} using the formula

$$B_{j_0}=-\tfrac{1}{2}B_0\,\mathbf{H}_{\mathbb{D}_{\mathrm{e}}}\Big[\frac{(4\varDelta R)^{\frac{1}{2}}\mathfrak{F}_{j_0,0,W}}{|B_0|^2W}\Big]=-\pi^{\frac{1}{2}}\mathbf{H}_{\mathbb{D}}\big[\mathfrak{F}_{j_0,0,W}\big],$$

where we have used that $|B_0|^2W=(4\pi)^{-\frac{1}{2}}[\operatorname{Re}(\bar{\zeta}\hat{\psi}_{0,1})]^{-1}$ and that $[\operatorname{Re}(\bar{\zeta}\hat{\psi}_{0,1})]^{-1}=(4\varDelta R)^{\frac{1}{2}}$ on $\mathbb{T}$. This completes Step 4, and we have extended the set of known data so that we may proceed to Step 3 with j_0 replaced with j_0+1.

The above algorithm continues until all the unknowns have been determined, up to the point where the whole index set $\mathcal{I}_{2\kappa+1}$ has been exhausted. In the process, we have in fact solved the equations (3.3.6) for all $(j,l)\in\mathcal{I}_{2\kappa}$. This means that if we form h_s and $\psi_{s,t}$ in terms of the functions B_j and $\hat{\psi}_{j,l}$ obtained with the above algorithm, a Taylor series expansion of $\omega_{s,t}$ in the parameters s and t along with the equations (3.3.6) shows that (3.3.5) holds. This completes the sketch of the proof. □

3.4. **Changing the base metric for orthogonal polynomials.** The results concerning twisting of the base metric for root functions apply also to the setting of orthogonal polynomials. Although many things are pretty much the same, we make an effort to explain what the precise result is in this context.

We need an appropriate notion of admissibility of the potential Q which applies to the setting of orthogonal polynomials. We recall that the spectral droplet $\mathcal{S}_\tau$ is the contact set

$$\mathcal{S}_\tau=\{z\in\mathbb{C}:\,\hat{Q}_\tau(z)=Q(z)\},$$

which is typically compact, where $\hat{Q}_\tau$ is the function

$$\hat{Q}_\tau(z):=\sup\big\{q(z):\,q\in\mathrm{SH}(\mathbb{C}),\ q\le Q\ \text{ on }\ \mathbb{C},\ q(z)\le\tau\log(|z|+1)+\mathrm{O}(1)\big\}.$$

Definition 3.4.1. We say that the potential Q is τ-admissible if the following conditions are met:

(i) $Q:\mathbb{C}\to\mathbb{R}$ is C^2-smooth,

(ii) Q meets the growth bound

$$\tau_Q:=\liminf_{|z|\to+\infty}\frac{Q(z)}{\log|z|}>\tau>0,$$

(iii) The unbounded component Ω_τ of the complement of the spectral droplet $\mathcal{S}_\tau$ is simply connected on the Riemann sphere $\hat{\mathbb{C}}$, with real-analytic Jordan curve boundary,

(iv) Q is strictly subharmonic and real-analytically smooth in a neighbourhood of the boundary $\partial\Omega_\tau$.

As before, we consider measures $\mathrm{e}^{-2mQ}V\mathrm{dA}$, where the twist function V is assumed to be nonnegative, positive near the curve $\partial\Omega_\tau$, and real-analytically smooth in a neighbourhood of $\partial\Omega_\tau$ with at most polynomial growth at infinity (3.1.1). We denote by ϕ_τ the surjective conformal mapping

$$\phi_\tau:\Omega_\tau\to\mathbb{D}_{\mathrm{e}},$$

which preserves the point at infinity and has $\phi'_\tau(\infty) > 0$. The function $\mathcal{Q}_\tau$ is defined as the bounded holomorphic function on Ω_τ whose real part equals Q on the boundary $\partial\Omega_\tau$, and whose imaginary part vanishes at infinity.

The orthogonal polynomials $P_{m,n,V}$ have degree n, positive leading coefficient, and unit norm in $A^2_{mQ,V}$. They have the additional property that

$$\langle P_{m,n,V}, P_{m,n',V}\rangle_{mQ,V} = 0, \qquad n \neq n'.$$

We will work with $\tau = \frac{n}{m}$.

Theorem 3.4.2. *Suppose Q is τ-admissible for $\tau \in I_0$, where I_0 is a compact interval of the positive half-axis. Suppose in addition that V meets the above regularity requirements. Given a positive integer κ and a positive real A, there exists a neighborhood $\Omega_\tau^{\langle\kappa\rangle}$ of the closure of Ω_τ and bounded holomorphic functions $\mathcal{B}_{j,\tau,V}$ on $\Omega_\tau^{\langle\kappa\rangle}$, as well as domains $\Omega_{\tau,m} = \Omega_{\tau,m,\kappa,A}$ with $\Omega_\tau \subset \Omega_{\tau,m} \subset \Omega_\tau^{\langle\kappa\rangle}$ which meet*

$$\mathrm{dist}_{\mathbb{C}}(\Omega^c_{\tau,m}, \Omega_\tau) \geq A m^{-\frac12}(\log m)^{\frac12},$$

such that the orthogonal polynomials enjoy the expansion

$$P_{m,n,V}(z) = m^{\frac14}(\phi'_\tau(z))^{\frac12}(\phi_\tau(z))^n \mathrm{e}^{m\mathcal{Q}_\tau}\Big\{\sum_{j=0}^{\kappa} m^{-j}\mathcal{B}_{j,\tau,V}(z) + \mathrm{O}\big(m^{-\kappa-1}\big)\Big\},$$

on $\Omega_{\tau,m}$ as $n = \tau m \to +\infty$ while $\tau \in I_0$, where the error term is uniform. Here, the main term $\mathcal{B}_{0,\tau,V}$ is zero-free and smooth up to the boundary on Ω_τ, positive at infinity, with prescribed modulus

$$|\mathcal{B}_{0,\tau,V}(\zeta)| = \pi^{-\frac14}[\Delta Q(\zeta)]^{\frac14} V(\zeta)^{-\frac12}, \qquad \zeta \in \partial\Omega_\tau.$$

In view of Lemma 3.3.2, the construction of approximately orthogonal quasipolynomials may be carried out in the same way as in the case when $V = 1$. The same can be said of the $\bar\partial$-correction scheme, since the solution of the corresponding $\bar\partial$-problem only requires properties of the weight in a neighborhood of the closure of the off-spectral component. For instance, if $F_{m,n,V}$ denotes an appropriate sequence of approximately orthogonal quasipolynomials, we may obtain a solution u to the problem

$$\bar\partial u = F_{m,n,W}\bar\partial\chi_0,$$

with polynomial growth $|u(z)| = \mathrm{O}(|z|^n)$ which enjoys the estimate

$$\int_{\mathbb{C}} |u|^2 \mathrm{e}^{-2mQ} V \mathrm{dA} \leq \frac{1}{2m}\int_{\mathrm{supp}\,\bar\partial\chi_0} |F_{m,n,W}|^2\,|\bar\partial\chi_0|^2 \frac{\mathrm{e}^{-2mQ}V}{\Delta Q - \frac{1}{2m}\Delta\log V}\mathrm{dA},$$

where we point out that $\bar\partial\chi_0$ ought to be supported in the region where V is real-analytic and non-vanishing, and in addition where $\Delta\hat{Q}_\tau - \frac{1}{2m}\Delta\log V > 0$. From this it is clear how to proceed as we did without the twist.

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Paper C

Scaling limits of random normal matrix processes at singular boundary points
(joint with Y. Ameur, N.-G. Kang and N. Makarov)
Preprint: arXiv (2015), 17 pages.

SCALING LIMITS OF RANDOM NORMAL MATRIX PROCESSES AT SINGULAR BOUNDARY POINTS

YACIN AMEUR, NAM-GYU KANG, NIKOLAI MAKAROV, AND ARON WENNMAN

Abstract. We study scaling limits for eigenvalues of random normal matrices near certain types of singular points of the droplet. In particular, we prove existence of new types of determinantal point fields, which differ from those which can appear at a regular boundary point.

The method of rescaled Ward identities was introduced in the paper [3], where the main focus was on scaling limits at a regular boundary point of the droplet associated to a random normal matrix process. In this note, we will apply the same method to obtain results about scaling limits near *singular* boundary points, which may be either cusps or double points.

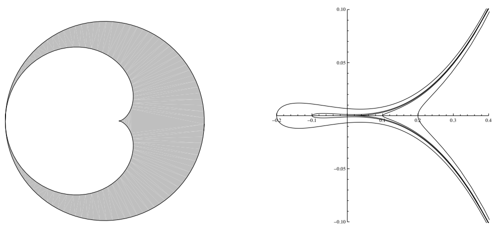

Figure 1. The left figure shows a droplet (shaded) exhibiting both kinds of singular points; a cusp and a double point. The picture on the right shows a $(5, 2)$-cusp under Laplacian growth/Hele-Shaw evolution.

Recall that, in the random normal matrix model, we are given a suitable real-valued function Q, called the "potential". We consider configurations (or "systems") $\{\zeta_j\}_1^n$ of points in $\mathbb{C}$, having the interpretation of identical point charges subject to the external field Q. The energy of the system is defined to be

$$H_n(\zeta_1, \ldots, \zeta_n) = \sum_{j \neq k} \log \frac{1}{|\zeta_j - \zeta_k|} + n \sum_{j=1}^{n} Q(\zeta_j),$$

and we consider the ensemble of systems picked randomly with respect to the Boltzmann-Gibbs law,

$$d\mathbb{P}_n(\zeta) = \frac{1}{Z_n} e^{-H_n(\zeta)} dV_n(\zeta), \quad \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n. \tag{0.1}$$

Here dV_n is Lebesgue measure in $\mathbb{C}^n$ divided by π^n, and $Z_n = \int e^{-H_n} dV_n$.

2010 *Mathematics Subject Classification.* Primary: 60B20. Secondary: 60G55; 81T40; 30C40; 30D15; 35R09.
Key words and phrases. Random normal matrix; Ward's equation; Droplet; Cusp; Double point.
Nam-Gyu Kang was supported by Samsung Science and Technology Foundation (SSTF-BA1401-01).

As $n \to \infty$, the random samples $\{\zeta_j\}_1^n$ tend to condensate on a compact set S known as the droplet, the boundary of which is a finite union of real-analytic arcs, possibly containing finitely many singular points where the arcs meet. The main problem considered in this note is to understand the microscopic properties of the system near such a singular boundary point $p \in \partial S$, as $n \to \infty$.

We prove that nothing of interest is to be found in the vicinity of the singular point itself, but instead we shall find new point fields located somewhat inside the droplet, close to the singular point. This is accomplished by zooming about a "moving point", which approaches the singular point at a proper rate. Cf. Figure 2.

1. Introduction and main results

Notational conventions. We write $D(p;r)$ for the open disk centered at p of radius r. $\mathbb{C}_+$ denotes the upper half-plane. The characteristic function of a set E will be denoted χ_E. We use the notation $\Delta = \partial\bar\partial$, so Δ is 1/4 of the usual Laplacian. We write $dA(z) = d^2z/\pi$ for two-dimensional Lebesgue measure in $\mathbb{C}$, normalized so that the unit disk has unit area. A continuous function $f : \mathbb{C}^2 \to \mathbb{C}$ is termed *Hermitian* if $f(z,w) = \overline{f(w,z)}$. We say that f is *Hermitian-entire* if f is Hermitian and entire as a function of z and $\bar w$. A Hermitian-entire function is uniquely determined by its diagonal values $f(z,z)$ by polarization. Finally, a Hermitian function c is called a *cocycle* if $c(z,w) = g(z)\overline{g(w)}$ for a continuous unimodular function g.

1.1. **External potential and droplet.** Our basic setup parallells that of [3]. In short, let $Q : \mathbb{C} \to \mathbb{R} \cup \{\infty\}$ is a suitable "external potential" of sufficient growth; requiring

$$\liminf_{\zeta\to\infty} \frac{Q(\zeta)}{\log|\,\zeta\,|^2} > 1$$

will do. If μ is a positive, compactly supported Borel measure we define its weighted logarithmic Q-energy by

$$I_Q[\mu] = \int_{\mathbb{C}} Q\,d\mu + \int_{\mathbb{C}^2} \log\frac{1}{|\,\zeta-\eta\,|}\,d\mu(\zeta)\,d\mu(\eta). \tag{1.1}$$

A theorem of Frostman asserts that there is a unique *equilibrium measure* σ of total mass 1, which minimizes $I_Q[\mu]$ over all compactly supported Borel probability measures μ on $\mathbb{C}$. The support of the measure σ is called the *droplet* in external field Q, and is denoted

$$S = S[Q] := \operatorname{supp}\sigma.$$

It is well-known (see e.g. [16]) that S is a compact set and σ is absolutely continuous and given by the formula

$$d\sigma(z) = \Delta Q(z)\,\chi_S(z)\,dA(z), \tag{1.2}$$

where we recall that we adhere to the convention that $\Delta := \partial\bar\partial$ is 1/4 of the standard Laplacian while $dA(z) := d^2z/\pi$ is two-dimensional Lebesgue measure divided by π.

We make the standing assumptions that there is a neighbourhood Ω of S such that

(1) Q is real-analytic in Ω,
(2) $\Delta Q > 0$ in Ω.

Under these conditions, the complement S^c has a local Schwarz function, and one can apply Sakai's regularity theorem [17] to conclude that the boundary points $p \in \partial S$ can be characterized as follows.

The most common type of boundary point is a *regular* boundary point. This is a point p such that there exists a neighbourhood $D = D(p;\epsilon)$ such that $D \setminus S$ is a Jordan domain and $D \cap (\partial S)$

is a simple real-analytic arc. By Sakai's theorem, all but finitely many boundary points of S are regular. The finitely many exceptional points are called *singular*.

When analyzing a singular point, we can without loss of generality assume that it is located on the *outer* boundary of S, i.e. on the boundary of the unbounded component U of $\hat{\mathbb{C}} \setminus S$. If there are other boundary components, they can be treated in the same way.

There are two kinds of singular boundary points. A point at the outer boundary $p \in \partial U$ is a (conformal) *cusp* if there is $D = D(p;\epsilon)$ such that $D \setminus S$ is a Jordan domain and every conformal map $\Phi : \mathbb{C}_+ \to D \setminus U$ with $\Phi(0) = p$ extends analytically to a neighbourhood of 0 and satisfies $\Phi'(0) = 0$; p is a *double point* if there is a disk D about p such that $D \setminus S$ is a union of two Jordan domains, and p is a regular boundary point of each of them. Note that the cusps which appear on the boundary of a droplet S always point out from S.

One can further classify singular points according to degrees of tangency; we briefly recall how this works for cusps.

Thus assume that ∂S has a cusp at the outer boundary ∂U at $p = 0$. We can assume that a conformal map $\Phi : \mathbb{C}_+ \to U$ satisfies

$$\Phi'(z) = z + a_2 z^2 + \cdots + (a_{\nu-1} + ib) z^{\nu-1} + \cdots$$

where a_j and b are real and $b \neq 0$. This means that

$$\Phi(z) = \frac{1}{2} z^2 + \frac{a_2}{3} z^3 + \cdots + \frac{a_{\nu-1} + ib}{\nu} z^\nu + \cdots .$$

If we write $\Phi = u + iv$, this gives

$$u(x) = \frac{1}{2} x^2 + \cdots , \quad v(x) = \frac{b}{\nu} x^\nu + \cdots , \quad (x \in \mathbb{R}).$$

By definition, this means that the cusp at 0 is of type $(\nu, 2)$.

Some cusps, in particular $(3, 2)$-cusps, which are generic in Sakai's theory can not appear on a free boundary such as those appearing in the present context. For technical reasons, the methods in this paper do not apply to $(3, 2)$-cusps, but we are able to treat $(\nu, 2)$ cusps where ν is odd and $\nu \geq 5$, provided that our standing assumptions (1) and (2) hold[1]. We will also obtain results for double points. For a brief discussion of these matters, which may be regarded folklore, see Subection 2.1 below.

Droplets with singular boundary points have been studied e.g. in the papers [7, 14, 21], the book [17], and the thesis [8]. We refer to those sources and the references there for more detailed information on various types of singular boundary points.

1.2. **Rescaled ensembles.** Let $\{\zeta_j\}_1^n$ be a random sample from the Boltzmann-Gibbs distribution (0.1). As in [3], we will denote by boldface characters the objects pertaining to this ensemble. We shall for example use the k-point function

$$\mathbf{R}_{n,k}(\eta_1, \ldots, \eta_k) = \lim_{\varepsilon \to 0} \left[\varepsilon^{-2k} \cdot \mathbb{P}_n \left(\cap_{j=1}^k \{ N_{D(\eta_j;\varepsilon)} \geq 1 \} \right) \right],$$

where N_B is the number of points ζ_j which fall in the set B. It is well-known that we can write

$$\mathbf{R}_{n,k}(\zeta_1, \ldots, \zeta_k) = \det(\mathbf{K}_n(\zeta_i, \zeta_j))_{i,j=1}^k,$$

where $\mathbf{K}_n$ is a Hermitian function, which we call a correlation kernel of the process. The correlation kernel $\mathbf{K}_n$ is obtained as the reproducing kernel for the space of weghted polynomials

$$f(z) e^{-nQ(z)/2},$$

where f is a polynomial of degree at most $n - 1$, endowed with the topology of $L^2(\mathbb{C}, dA)$.

[1]In fact it is enough to require that they hold locally, near the singular point in question.

Now consider a moving point $p_n \in S$ and a sequence of angles $\theta_n \in \mathbb{R}$. We shall consider rescaled point processes $\Theta_n = \{z_j\}_1^n$ where

$$z_j = e^{-i\theta_n}\sqrt{n\Delta Q(p_n)}(\zeta_j - p_n). \tag{1.3}$$

The law of the process Θ_n is defined as the image of the Boltzmann-Gibbs law under the map $\{\zeta_j\}_1^n \mapsto \{z_j\}_1^n$.

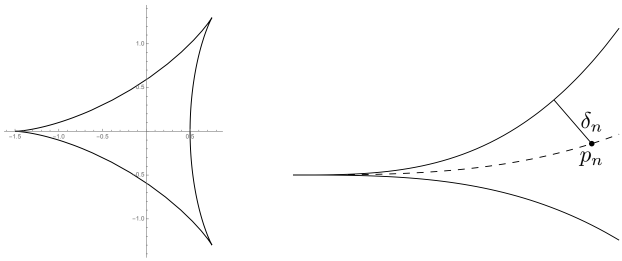

FIGURE 2. The deltoid (left) has three maximal $(3,2)$-cusps. The figure on the right shows a moving point p_n on the bisectrix of a cusp point. Here, $\delta_n = Tn^{-\frac{1}{2}}$ for some positive T.

Objects pertaining to the rescaled system $\{z_j\}_1^n$ are denoted by plain symbols. For example, the k-point function of the rescaled system will be written

$$R_{n,k}(z_1, \ldots, z_k) := \frac{1}{(n\Delta Q(p_n))^k}\mathbf{R}_{n,k}(\zeta_1, \ldots, \zeta_k).$$

The rescaled process is determinantal with correlation kernel

$$K_n(z, w) = \frac{1}{n\Delta Q(p_n)}\mathbf{K}_n(\zeta, \eta),$$

where

$$z = e^{-i\theta_n}\sqrt{n\Delta Q(p_n)}(\zeta - p_n), \quad w = e^{-i\theta_n}\sqrt{n\Delta Q(p_n)}(\eta - p_n).$$

We write $R_n(z) = K_n(z, z)$.

Lemma 1.1. *There is a sequence of cocycles c_n such that every subsequence of $c_n K_n$ has a subsequence converging to $G\Psi$ where*

$$G(z, w) = e^{z\bar{w} - |z|^2/2 - |w|^2/2}$$

is the Ginibre kernel and Ψ is a Hermitian entire function satisfying the mass-one inequality

$$\int_{\mathbb{C}} e^{-|z-w|^2} |\Psi(z, w)|^2 \, dA(w) \le \Psi(z, z). \tag{1.4}$$

Further, the function $R(z) := K(z, z) = \Psi(z, z)$ is either identically zero, or else it is everywhere strictly positive.

Proof. The first statement follows from a normal families argument; see Theorem A in [3]. The last statement is proven in [3], Theorem B. $\square$

A limit point K in Lemma 1.1 will be called a *limiting kernel*, and

$$R(z) := K(z, z)$$

is the corresponding *limiting* 1*-point function*. The general theory of point fields implies that a limiting kernel K is the correlation kernel of a "limiting infinite point field" Θ with k-point function $R_k(z_1, \ldots, z_k) = \det(K(z_i, z_j))_{i,j=1}^k$. See [3] and the references there.

There is nothing which prevents the limiting kernel K from being trivial, i.e., we may well have $K = 0$. In this case the point process Θ_n degenerates to the trivial point field, all of whose k-point functions vanishes identically. As we shall see, this happens when we rescale about certain types of singularities. However, when we shift focus and rescale about a suitable moving point, approaching the singular point from the inside of the droplet at a suitable distance away, we will obtain new, non-degenerate, limiting point fields.

If R is non-trivial, we can go on to define the Berezin kernel

$$B(z, w) = \frac{|K(z, w)|^2}{R(z)}$$

and its Cauchy transform

$$C(z) = \int_{\mathbb{C}} \frac{B(z, w)}{z - w}\, dA(w).$$

By Theorem B in [3], the function C is smooth and we have Ward's equation

$$\bar{\partial} C = R - 1 - \Delta \log R. \tag{1.5}$$

Since K (and therefore B and C) is uniquely determined by R by polarization, it makes sense to say that R *satisfies Ward's equation* if (1.5) holds.

1.3. **Main results.** Let p be a singular boundary point, i.e., a cusp or a double point.

Theorem I. *Let $e^{i\theta}$ be one of the normal directions to ∂S at p and rescale about p according to*

$$z_j = e^{-i\theta} \sqrt{n \Delta Q(p)}(\zeta_j - p), \quad j = 1, \ldots, n.$$

Then any limiting 1*-point function R vanishes identically. As a consequence, any limiting point field at p is trivial.*

Let us remark that the choice of angle θ above is merely made for convenience. Any other angle, or sequence θ_n of angles, would produce the same result.

Since Theorem I says that nothing of interest is to be found in the vicinity of the singular point p, we shift focus and look a bit to the inside of the droplet. We need to consider the case of a cusp and a double point separately.

Definition. Fix a positive parameter T.

(i) If ∂S has a cusp at p, we consider the moving point $p_n \in S$ of distance $T/\sqrt{n \Delta Q(p)}$ from the boundary ∂S, which is closest to the singular point p. (See Fig. 1.)

(ii) If S has a double point, there are instead two distinct points p_n', p_n'' in S of distance $T/\sqrt{n \Delta Q(p)}$ to ∂S, of minimal distance to p.

Theorem II. *Suppose that S has a $(\nu, 2)$-cusp at p where $\nu \geq 5$. Rescale about p_n according to*

$$z_j = e^{-i\theta_n} \sqrt{n \Delta Q(p_n)}(\zeta_j - p_n), \qquad j = 1, \ldots, n, \tag{1.6}$$

where the angle θ_n is chosen so that the image of the cusp, i.e. the point $e^{-i\theta_n}(p - p_n)$, is on the positive imaginary axis. Then, if T is sufficiently large, each limiting 1*-point function*

$R(z) = K(z,z)$ is everywhere positive and satisfies Ward's equation. Moreover, R satisfies the estimate

$$R(z) \le Ce^{-2(|x|-T)^2}, \qquad (x = \operatorname{Re} z). \tag{1.7}$$

The assumption that the parameter $T > 0$ be sufficiently large is made for technical reasons of the proof; it should really not be needed. See the concluding remarks for related comments.

Our result for double points is similar.

Theorem III. *If S has a double point at p, we rescale as in* (1.6) *with p_n replaced by p_n' or p_n''. The conclusions of Theorem II then hold also for the limiting* 1*-point function R rescaled about p_n' or p_n''.*

We remark that the limiting point fields, whose existence is guaranteed by Theorems II and III are necessarily different from those which can appear at a fixed regular boundary point. Indeed, as was observed in [3], a limiting 1-point function rescaled in the outer normal direction about a regular boundary point will satisfy the estimate

$$\left| R(z) - \chi_{(-\infty,0)}(x) \right| \le Ce^{-cx^2}$$

where c is some positive constant. The latter estimate is not consistent with (1.7), which is seen by letting $x \to -\infty$.

The concluding remarks in Section 5 contains further comments about the limiting point fields in theorems II and III.

1.4. **Organization of the paper.** Our approach uses Bergman space techniques and some rather careful estimates for the limiting 1-point function in the *exterior* of the droplet. It is convenient to start with the latter estimates.

In Section 2, we prove an estimate for the decay of a limiting 1-point function in the exterior of the droplet, near a cusp. In Section 3, we extend the estimate to rather general (moving) boundary points. In Section 4 we prove our main results, Theorems I, II, and III. Section 5 contains concluding remarks.

2. Exterior estimate for the 1-point function at a cusp

In this section, we assume that the droplet S has a cusp at the point $p \in \partial S$. To simplify the discussion, we will assume that p on the *outer* boundary of S.

Now let $e^{i\theta}$ be one of the normal directions to ∂S and rescale about p in the usual manner:

$$z_j = e^{-i\theta}\sqrt{n\Delta Q(p)}(\zeta_j - p), \qquad j = 1, \ldots, n.$$

Let $K = G\Psi$ denote a limiting kernel. Write $R(z) = K(z,z)$ and $R_n(z) = K_n(z,z)$. We have the following result.

Theorem 2.1. *With the above assumptions, we have*

$$R(z) \le Ce^{-2x^2}, \qquad (x = \operatorname{Re} z). \tag{2.1}$$

To prove the estimate (2.1) we shall use the well-known estimate (e.g. [3], Section 3)

$$\mathbf{R}_n(\zeta) \le Cne^{-n(Q-\check{Q})(\zeta)}. \tag{2.2}$$

Here $\check{Q}$ is the "obstacle function", i.e. the $C^{1,1}$-smooth function on $\mathbb{C}$ which coincides with Q on S and is harmonic in $\mathbb{C} \setminus S$ and increases like $\check{Q}(\zeta) = \log|\zeta|^2 + O(1)$ as $\zeta \to \infty$.

Now assume *w.l.o.g.* that $p = 0$ and that the cusp at p points in the positive real direction. The rescaling is then simply given by

$$z = i\sqrt{n\Delta Q(p)}\,\zeta, \tag{2.3}$$

and in the z-plane, the droplet appears as a narrow neighbourhood of the ray $(-i\infty, 0)$.

Write U for the unbounded component of $\mathbb{C} \setminus S$ and let $\Phi : \mathbb{C}_+ \to U$ be a conformal map such that $\Phi(0) = 0$ and $\Phi(i) = \infty$. Since 0 is a conformal cusp, Φ extends analytically to some neighbourhood N of the origin. Likewise, the harmonic function $\check{Q} \circ \Phi : \mathbb{C}_+ \to \mathbb{R}$ extends harmonically across $\mathbb{R}$ to a harmonic function V. Referring to (2.3), we shall put

$$\zeta = \Phi(\lambda).$$

We can assume that Φ' has the Taylor expansion

$$\Phi'(\lambda) = \lambda + a_2\lambda^2 + a_3\lambda^3 \ldots, \qquad (\lambda \in N).$$

We form the functions

$$Q_\Phi := Q \circ \Phi, \quad \check{Q}_\Phi := \check{Q} \circ \Phi.$$

The function $\check{Q}_\Phi$ is harmonic in $\mathbb{C}_+$ and extends across $\mathbb{R}$ to a harmonic function V. Write

$$M(\lambda) := (Q_\Phi - V)(\lambda), \qquad \lambda = \sigma + i\tau.$$

Thus $M = (Q - \check{Q}) \circ \Phi$ in $\mathbb{C}_+$.

Lemma 2.2. *For $\lambda = \sigma + i\tau$, we have*

$$M(\lambda) = 2\Delta Q(p)\tau^2\sigma^2 + O\left(|\,\lambda\,|^5\right), \qquad (\lambda \to 0). \tag{2.4}$$

Before we prove the lemma, we proceed to prove the estimate (2.1). To this end, note that the estimate (2.2) gives (with a new C depending on $\Delta Q(p)$)

$$R_n(z) \le Ce^{-nM(\lambda_n(z))}, \qquad \text{where} \quad \lambda_n(z) := \Phi^{-1}\left(-iz/\sqrt{n\Delta Q(p)}\right). \tag{2.5}$$

If $z = x + iy$, then, since $\Phi(\lambda) = \lambda^2/2 + O(\lambda^3)$ as $\lambda \to 0$,

$$-x = \sqrt{n\Delta Q(p)}\,\mathrm{Im}\left(\lambda^2/2 + O(\lambda^3)\right) = \sqrt{n\Delta Q(p)}\left(\sigma\tau + O(|\,\lambda\,|^3)\right), \quad (\lambda = \sigma + i\tau \to 0). \tag{2.6}$$

The estimates (2.4) and (2.6) now give that

$$nM(\lambda_n(z)) = 2x^2 + O\left(n\,|\,\lambda_n(z)\,|^5\right), \quad (n \to \infty).$$

Choosing, for example, $|\,z\,| \le \log n$, we see via (2.5) that the estimate (2.1) holds.

It remains to prove Lemma 2.2.

Recalling that $\check{Q}(\lambda) \sim \log|\,\lambda\,|^2$ as $\lambda \to \infty$, it is now seen that the Poisson representation of the harmonic function $\check{Q}_\Phi|_{\mathbb{C}_+}$ takes the form

$$\check{Q}_\Phi(\lambda) := \int_{\mathbb{R}} Q_\Phi(t)P(\lambda, t)\,dt + G(\lambda), \quad \lambda \in \mathbb{C}_+, \tag{2.7}$$

where $G(\lambda) = \log\left|\frac{\lambda+i}{\lambda-i}\right|^2$ is (twice) the Green's function for $\mathbb{C}_+$ with pole at i, and

$$P(\lambda, t) = \frac{1}{\pi}\frac{\tau}{(\sigma - t)^2 + \tau^2}, \quad \lambda = \sigma + i\tau$$

the Poisson kernel for $\mathbb{C}_+$.

Let us write C^ω for the real-analytic class. For a function $f \in C^\omega(\mathbb{R})$ we write $P_f(\lambda) = \int_{\mathbb{R}} f(t)P(\lambda,t)\,dt$ for the Poisson integral and $\mathrm{p.v.} = \mathrm{p.v.}^{(\sigma,\infty)}$ for the double principal value of integrals, defined by

$$\mathrm{p.v.}\int_{\mathbb{R}} f(t)dt = \lim_{\epsilon\downarrow 0}\int_{\epsilon<|\sigma-t|<\epsilon^{-1}} f(t)dt,$$

when this limit exists. If f is absolutely integrable near either the point σ or at infinity, then the principal value integral agrees with the usual (Lebesgue) integral. Noting that, for $\lambda = \sigma + i\tau \in \mathbb{C}_+$,

$$(\sigma - t)^2 P(\lambda,t) - \tau/\pi = -\tau^2 P(\lambda,t),$$

we compute

(2.8)

$$\begin{aligned}P_f(\lambda) - f(\sigma) &= \mathrm{p.v.}\int_{\mathbb{R}} \left[f(t) - f(\sigma) - (t-\sigma)f'(\sigma)\right] P(\lambda,t)\,dt \\ &= \mathrm{p.v.}\int_{\mathbb{R}} \left[\frac{f(t) - f(\sigma) - (t-\sigma)f'(\sigma)}{(\sigma-t)^2}\right](\sigma-t)^2 P(\lambda,t)\,dt \\ &= \frac{\tau}{\pi}\cdot \mathrm{p.v.}\int_{\mathbb{R}} \frac{f(t) - f(\sigma) - (t-\sigma)f'(\sigma)}{(\sigma-t)^2}\,dt - \tau^2\int_{\mathbb{R}} \frac{f(t) - f(\sigma) - (t-\sigma)f'(\sigma)}{(\sigma-t)^2} P(\lambda,t)\,dt.\end{aligned}$$

Note that the last integral is absolutely convergent, and approaches $\frac{1}{2}f''(\sigma)$ as $\tau \downarrow 0$.

Let us denote by $S_\sigma : C^\omega(\mathbb{R}) \to C^\omega(\mathbb{R})$ the backward shift by σ: $S_\sigma f(t) = \frac{f(t)-f(\sigma)}{t-\sigma}$. Write

$$I_f^k(\sigma) := \mathrm{p.v.}\int_{\mathbb{R}} S_\sigma^{2k} f(t)\,dt, \quad k = 1, 2, \ldots.$$

A repetition of the calculation in (2.8) gives that P_f has the asymptotic expansion

$$P_f(\sigma + i\tau) = f(\sigma) + I_f^1(\sigma)\cdot\tau - \frac{1}{2}f''(\sigma)\cdot\tau^2 - I_f^2(\sigma)\cdot\tau^3 + \frac{1}{4!}f^{(4)}(\sigma)\cdot\tau^4 + \ldots. \tag{2.9}$$

Choosing $f = Q_\Phi$ and using identity (2.7), we find

$$I^1_{Q_\Phi}(\sigma) = \partial_\tau P_{Q_\Phi}(\sigma) = \partial_\tau(Q_\Phi - G)(\sigma), \quad \sigma \in \mathbb{R}.$$

More generally, it is easy to verify by induction that

$$(2k-1)!\cdot I^k_{Q_\Phi}(\sigma) = \partial_\sigma^{2k-2}\partial_\tau(Q_\Phi - G)(\sigma). \tag{2.10}$$

Since $\partial\check{Q}_\Phi$ is continuous on $\mathrm{cl}\,\mathbb{C}_+$, while $Q_\Phi = \check{Q}_\Phi$ in the lower half plane, we can replace "Q_Φ" by "$\check{Q}_\Phi$" in the right side of (2.10), and $M = Q_\Phi - P_{Q_\Phi} - G$ satisfies $M = \partial_\tau M = 0$ on $\mathbb{R}$. Inserting the expansions (2.9) and (2.10), we thus find that

$$\begin{aligned}M(\sigma+i\tau) &= \frac{1}{2}\partial_\tau^2 M(\sigma)\cdot\tau^2 + \frac{1}{3!}\partial_\tau^3 M(\sigma)\cdot\tau^3 + \frac{1}{4!}\partial_\tau^4 M(\sigma)\cdot\tau^4 + \cdots \\ &= \frac{1}{2}\partial_\tau^2 Q_\Phi(\sigma)\cdot\tau^2 + \frac{1}{3!}\partial_\tau^3 Q_\Phi(\sigma)\cdot\tau^3 + \frac{1}{4!}\partial_\tau^4 Q_\Phi(\sigma)\cdot\tau^4 + \cdots \\ &+ \frac{1}{2}\partial_\sigma^2 Q_\Phi(\sigma)\cdot\tau^2 + \frac{1}{3!}\partial_\sigma^2\partial_\tau Q_\Phi(\sigma)\cdot\tau^3 - \frac{1}{4!}\partial_\sigma^4 Q_\Phi(\sigma)\cdot\tau^4 + \cdots \\ &- \frac{1}{2}\partial_\tau^2 G(\sigma)\cdot\tau^2 - \frac{1}{3!}\partial_\tau(\partial_\tau^2 + \partial_\sigma^2)G(\sigma)\cdot\tau^3 - \frac{1}{4!}\partial_\tau^4 G(\sigma)\cdot\tau^4 + \cdots.\end{aligned}$$

The last line vanishes because all coefficients are derivatives of ΔG evaluated at σ. This implies that

$$(2.11)\quad M(\sigma+i\tau)=2\Delta Q_\Phi(\sigma)\cdot\tau^2+\frac{4}{3!}\partial_\tau\Delta Q_\Phi(\sigma)\cdot\tau^3+\frac{4}{4!}(\partial_\tau^2-\partial_\sigma^2)\Delta Q_\Phi(\sigma)\cdot\tau^4 + \frac{4}{5!}(\partial_\tau^3-\partial_\tau\partial_\sigma^2)\Delta Q_\Phi(\sigma)\tau^5+\cdots.$$

But $\Delta Q_\Phi(\sigma+i\tau)=\Delta Q(\Phi(\sigma+i\tau))\,|\,\Phi'(\sigma+i\tau)\,|^2=\Delta Q(\Phi(\sigma+i\tau))\cdot(\sigma^2+\tau^2+\ldots)$. We have shown that

$$M(\sigma+i\tau)\geq 2\Delta Q(0)\cdot\sigma^2\tau^2+O\left(|\,\lambda\,|^5\right),\qquad(\lambda=\sigma+i\tau\to 0).$$

The proof of Lemma 2.2 is complete. □

2.1. **Classification of cusps.** Consider a droplet $\mathcal{S}$ for which a cusp appears at $p\in\partial\mathcal{S}$. Since Q is assumed to be real-analytically smooth and strictly subharmonic in a neighbourhood of $\mathcal{S}$, it follows from Sakai's regularity theorem that the conformal mapping $\Phi:\mathbb{H}^+\to\mathcal{S}^c$ which maps the origin to p extends holomorphically across $\mathbb{R}$, and hence admits a power series expansion at p. After possibly applying an affine transformation, we may deduce that the boundary admits a local approximate parameterization

$$(2.12)\qquad \partial\mathcal{S}\cap\mathbb{D}(0,\delta)=\left\{x+iy\,:\,x=-\frac{1}{2}t^2+O(t^3),\;\;y=c_\nu t^\nu+O(t^{\nu+1}),\;\;t\in(-\epsilon,\epsilon)\right\},$$

for some positive numbers ϵ and δ and some constant c_ν. Here, we may immediately observe that the above parameterization says that $y=x^{\frac{\nu}{2}}+O(x^{\frac{\nu}{2}+\frac{1}{2}})$ on the boundary near the cusp point, and from this it follows that ν is necessarily odd. Indeed, otherwise (2.12) parameterizes a curve which is differentiable across the cusp.

However, more is true. Only a cusp where $\nu=4k+1$ for some $k=1,2,3,\ldots$ may appear on a free boundary such as $\mathcal{S}$. Indeed, it may be shown that if $\nu=4k+3$ for some k, then it must necessarily hold that $Q-\check{Q}$ assumes negative values arbitrarily close to p, see e.g. [13, pp. 388-390]. It appears to be difficult to find a good reference for this general classification result, which is perhaps to be regarded as folklore. For us, it is of importance to rule out the occurrence of $(3,2)$-cusps, and we present proof of this fact below for completeness.

Proposition 2.3. *A cusp of type $(3,2)$ cannot occur on the boundary of the droplet $\mathcal{S}$.*

Proof. Assume without loss of generality that a cusp of type $(3,2)$ occurs at the origin, and moreover that it points in the positive real direction. In order to reach a contradiction, we intend to compute $M(i\tau)$ using (2.11), and show that $M(i\tau)$ must take on negative values arbitrarily close to 0. Since the cusp is assumed to be of type $(3,2)$ the conformal mapping Φ takes the form

$$\Phi(z)=-\frac{1}{2}z^2+\frac{a+ib}{3}z^3+O(|z|^4)$$

where $b\neq 0$, from which it follows that

$$\Delta Q_\Phi(z)=\Delta Q(\Phi(z))\;|\Phi'(z)|^2=\sigma^2+\tau^2+2a(\sigma^3+\sigma\tau^2)-2b(\tau^3+\sigma^2\tau)+O(|z|^4)$$

when $z=\sigma+i\tau\to 0$. A small computation using (2.11) shows that

$$M(i\tau)=-\frac{32b}{5!}\tau^5+O(\tau^6),\qquad y\to 0$$

from which the assertion follows. □

3. A general exterior estimate for the 1-point function

In this section, we consider a (possibly moving) boundary point p and rescale about p in the outer normal direction $e^{i\theta}$ in the usual way,

$$z_j = e^{-i\theta}\sqrt{n\Delta Q(p)}(\zeta_j - p), \qquad j = 1, \dots, n.$$

We can then form the rescaled 1-point function $R_{n,p}(z)$, and also sequential limiting 1-point functions

$$R_p(z) = \lim_{k\to\infty} R_{n_k,p}(z), \qquad z \in \mathbb{C}.$$

In the situations relevant to this paper, we find that the decay of $R_p(z)$ is at least like e^{-2x^2} where $x = \operatorname{Re} z$. Our estimates are however not quite uniform; there is a critical case of boundary points at distance about $n^{-1/3}$ to the closest singular boundary point when our method is inconclusive. Since such points play no role to our analysis anyway, we will simply disregard them here.

Theorem 3.1. *Let δ_n be a sequence of positive numbers with $\delta_n \to 0$ and $n^{1/3}\delta_n \to \infty$ as $n \to \infty$. Then there is a constant C such that if the distance from a boundary point p to the nearest singular point is at least δ_n, then $R_{n,p}(z) \le Ce^{-2x^2}$, when $|z| \le \log n$, $x = \operatorname{Re} z$. As a consequence, any limiting 1-point function R_p satisfies the estimate*

$$R_p(z) \le Ce^{-2x^2}.$$

As a simple corollary, we will obtain the estimate (1.7) in Theorem II.

The proof will be carried out in steps, in the following three paragraphs. We will first treat the simple case of an "ordinary" fixed boundary point p. After that we consider cases of moving points near singular boundary points.

3.1. **Fixed boundary points.** Let p be an arbitrary fixed boundary point of S such that $\Delta Q(p) > 0$. We rescale about p according to

$$z = \sqrt{n\Delta Q(p)}e^{-i\theta}(\zeta - p)$$

where $e^{i\theta}$ is chosen as an outer normal direction to ∂S. The case when p is a cusp was already treated in Theorem 2.1. Double points can be treated similarly to regular points, so we give complete details only in the regular case.

Thus suppose that p is a regular boundary point at distance at least δ from all singular boundary points, where $\delta > 0$ is independent of n. It will suffice to prove that there is a constant $C = C(\delta)$ such that

$$\zeta \in S^c, \quad |z| \le \log n \quad \Rightarrow \quad R_n(z) \le Ce^{-2x^2}, \qquad (z = x + iy). \tag{3.1}$$

As before, we shall use the estimate $\mathbf{R}_n(\zeta) \le Cne^{-n(Q-\check{Q})(\zeta)}$. Let N_p be the outer normal direction at p and let V be the harmonic continuation of $\check{Q}|_{S^c}$ to a neighbourhood of p. We write $\delta_n = C\log n/\sqrt{n}$.

By Taylor's formula we have, for $x > 0$, with $M := Q - V$,

$$M\left(p + \frac{x}{\sqrt{n\Delta Q(p)}}N_p\right) = \frac{1}{2n\Delta Q(p)}\frac{\partial^2 M}{\partial n^2}(p)\,x^2 + \frac{1}{6(n\Delta Q(p))^{3/2}}\frac{\partial^3 M}{\partial n^3}(p+\theta)\,x^3,$$

where "$\partial/\partial n$" is exterior normal derivative and $\theta = \theta(n,p)$ is some number between 0 and δ_n. However, since p is a regular point and $Q = V$ on ∂S we have $\frac{\partial^2 M}{\partial s^2}(p) = 0$ where $\partial/\partial s$ denotes differentiation in the tangential direction. Adding this to the above Taylor expansion, using that $(\partial_s^2+\partial_n^2)M = 4\Delta M = 4\Delta Q$, we obtain, when $|z| \le C\log n$,

$$nM\left(p + \frac{z}{\sqrt{n\Delta Q(p)}}N_p\right) = 2\,x^2 + O\left(\frac{\log^3 n}{\sqrt{n}}\right), \qquad 0 \le x \le \delta_n. \tag{3.2}$$

This proves the desired implication (3.1).

3.2. **Boundary points near cusps.** Assume now that the droplet S has a $(\nu,2)$-cusp at the origin, pointing in the positive real direction. Here $\nu \ge 5$ is an odd integer.

Fix $T>0$ and a large integer n. Let p_n be the unique point such that the disk

$$D_n := D\left(p_n; T/\sqrt{n\Delta Q(0)}\right)$$

is inside S and is tangent to ∂S at two points; one on each arc terminating in the cusp. Let q_n be the point in $(\partial D_n)\cap(\partial S)$ in the upper half-plane. It is easy to check that $|\,p_n\,| \sim n^{-1/\nu}$ and $|\,q_n\,| \sim n^{-1/\nu}$. (Here "$a_n \sim b_n$" means that there is a constant $C>0$ such that $C^{-1}a_n \le b_n \le Ca_n$.)

We rescale about q_n as follows. Let $e^{i\theta_n}$ be the outer normal to ∂S at q_n and put

$$z = e^{-i\theta_n}\sqrt{n\Delta Q(0)}\;(\zeta - q_n)\,. \tag{3.3}$$

The rescaled droplet near q_n looks roughly like the infinite strip

$$0 < \operatorname{Re} z < 2T,$$

and the image of the cusp is far away on the positive imaginary axis at a distance $\sim n^{1/2-1/\nu}$ from the origin.

Let Φ be a conformal map $\mathbb{C}_+ \to U$ where $\mathbb{C}_+$ is the upper half-plane and U is the complement of the component of S containing 0. We assume that $\Phi(0)=0$ and

$$\Phi(\lambda) = -\frac{1}{2}\lambda^2 + O(\lambda^3), \qquad \lambda \to 0.$$

Since the cusp is conformal, Φ extends analytically across $\mathbb{R}$.

Now let σ_n be the point on $\mathbb{R}$ such that $\Phi(\sigma_n) = q_n$ and let $\varepsilon_n = \alpha_n + i\beta_n$ be the point in $\mathbb{C}$ such that

$$\Phi(\sigma_n+\varepsilon_n) = q_n + \frac{ze^{i\theta_n}}{\sqrt{n\Delta Q(0)}}.$$

We will be dealing with the Taylor expansion about σ_n given by

$$\Phi(\sigma_n+\varepsilon_n) = q_n + \Phi'(\sigma_n)\varepsilon_n + \Phi''(\sigma_n)\frac{\varepsilon_n^2}{2} + \cdots\,.$$

We now note the simple approximation

$$|\,\Phi'(\sigma_n)\,| \asymp |\,\sigma_n\,| \asymp |\,q_n\,|^{1/2} \asymp n^{-1/2\nu}, \tag{3.4}$$

which implies that the condition $|z| \le \log n$ in (3.3) corresponds to

$$|\,\varepsilon_n\,| \le C_0\frac{\log n}{n^{1/2-1/(2\nu)}}. \tag{3.5}$$

Observe first that it follows from (3.5) that

$$n\,|\,\varepsilon_n\,|^3 \le C_0 n\frac{\log^3 n}{n^{3/2-3/(2\nu)}} \le C_0\frac{\log^3 n}{n^{(\nu-3)/(2\nu)}} \to 0, \quad (n\to\infty) \tag{3.6}$$

where the last assertion holds true since $\nu \ge 5$.

After these observations, we can prove the required decay about the moving point q_n.

Lemma 3.2. *Let $R_n(z)$ be the rescaled one-point function according to the rescaling* (3.3)*. There is then a constant C such that $R_n(z) \le Ce^{-2x^2}$ when $|z| \le \log n$ and $x = \operatorname{Re} z < 0$.*

Proof. We know that $\mathbf{R}_n(\zeta) \le Cne^{-n(Q-\check{Q})(\zeta)}$. We will compare $2x^2$ with $n(Q-\check{Q})(\zeta)$. We shall be done if we can prove that

$$2\,x^2 = n(Q_\Phi - \check{Q}_\Phi)(\sigma_n + \varepsilon_n) + o(1). \tag{3.7}$$

However, by the estimate (2.11) we have

$$n(Q_\Phi - \check{Q}_\Phi)(\sigma_n + \varepsilon_n) = 2n\Delta Q(\Phi(\sigma_n + \alpha_n))|\Phi'(\sigma_n + \alpha_n)|^2 \beta_n^2 + O(n\beta_n^3).$$

It follows from (3.4) – (3.6) that

$$n(Q_\Phi - \check{Q}_\Phi)(\sigma_n + \varepsilon_n) = 2n\Delta Q(0)\,|\Phi'(\sigma_n)|^2\,\beta_n^2 + o(1).$$

Indeed, by (3.4) and (3.5), we have

$$n\Delta Q(\Phi(\sigma_n + \alpha_n))|\Phi'(\sigma_n + \alpha_n)|^2\beta_n^2 - n\Delta Q(0)\,|\Phi'(\sigma_n + \alpha_n)|^2\,\beta_n^2 = O(n|\sigma_n + \alpha_n|^3\beta_n^2) = o(1)$$

and

$$n\Delta Q(0)\,|\Phi'(\sigma_n + \alpha_n)|^2\,\beta_n^2 - n\Delta Q(0)\,|\Phi'(\sigma_n)|^2\,\beta_n^2 = O(n|\sigma_n\alpha_n|\beta_n^2) = o(1).$$

Inserting here the Taylor expansion of Φ about σ_n we find that

$$z = e^{-i\theta_n}\sqrt{n\Delta Q(0)}\left(\Phi'(\sigma_n)(\alpha_n + i\beta_n) + O(|\varepsilon_n|^2)\right), \qquad e^{i\theta_n} = i\frac{\Phi'(\sigma_n)}{|\Phi'(\sigma_n)|}.$$

It follows from (3.4) – (3.6) that

$$2x^2 = 2n\Delta Q(0)\,|\Phi'(\sigma_n)|^2\,\beta_n^2 + o(1),$$

which leads to

$$2x^2 - n(Q - \check{Q})\Big(q_n + \frac{ze^{i\theta_n}}{\sqrt{n\Delta Q(0)}}\Big) = o(1).$$

The proof of the lemma is finished. □

Let us now change the rescaling so that p_n is mapped to the origin,

$$z = e^{-i\theta_n}\sqrt{n\Delta Q(0)}(\zeta - p_n),$$

and write R_n for the rescaled 1-point function. Since

$$|\,p_n - q_n\,| = \operatorname{Re}(q_n + p_n) + o(1) = T/\sqrt{n\Delta Q(0)}$$

we obtain the estimate

$$R_n(z) \le Ce^{-2(\,|\,x\,|\,-\,T\,)^2}, \qquad |z| \le \log n, \quad x = \operatorname{Re} z. \tag{3.8}$$

3.3. **Boundary points near double points.** The case of double points is a rather routine application of (2.11), which applies also in the case when the conformal mapping Φ extends univalently to a neighbourhood of the origin.

Of course, in this situation one could perform a more direct analysis using the fact that M extends real-analytically across each of the two arcs of $\partial\mathcal{S}$ which meet at the double point. Hence, one may apply Taylor's formula directly at the point q_n of rescaling and proceed as in the case of a regular point.

We have now completely proved Theorem 3.1. Since the estimate (3.8) holds also for the case of double points, we obtain as a corollary the estimate (1.7) in Theorem II. q.e.d.

4. Proofs of the main results

4.1. **The triviality theorem.** We now prove Theorem I. Suppose that p is either a double point or a cusp of type $(\nu, 2)$ where $\nu > 3$ and that $\Delta Q(p) > 0$ and rescale about p according to

$$z = e^{-i\theta}\sqrt{n\Delta Q(p)}(\zeta - p)$$

where $e^{i\theta}$ is one of the normal directions to ∂S at p.

Let $K = G\Psi$ be a limiting kernel. We must prove that the limiting 1-point function $R(z) = K(z, z) = \Psi(z, z)$ vanishes identically. To this end, we shall use the corresponding *holomorphic kernel*

$$L(z, w) = e^{z\bar{w}}\Psi(z, w).$$

We can now finish the proof of Theorem I.

The function $S(z) := |z|^2 + \log R(z)$ is subharmonic, see e.g. Lemma 4.3 in [3] for an argument. Next recall the estimate $R(z) \le Ce^{-2x^2}$ for some constant C, obtained in Theorem 2.1 for cusps and in §3.1 in the case of double points. This gives the bound

$$S(z) \le \log C + y^2 - x^2.$$

But $y^2 - x^2$ is harmonic, so the function $\tilde{S} = S - (y^2 - x^2)$ is subharmonic and bounded above by $\log C$. Hence it is constant, i.e.,

$$R(z) = Ce^{-2x^2}$$

for a (new) constant C. If R is nontrivial we can assume that $C = 1$. By polarization, then

$$\Psi(z, w) = e^{-(z+\bar{w})^2/2},$$

so the kernel $L(z, w) = e^{z\bar{w}}\Psi(z, w)$ must satisfy

$$\int |L(0, w)|^2 e^{-|w|^2}\, dA(w) = \int |\Psi(0, w)|^2 e^{-|w|^2}\, dA(w) = \int e^{-2x^2}\, dA = \infty.$$

This contradicts Lemma 4.9 in [3]. The contradiction shows that we must have $C = 0$. q.e.d.

We are grateful to Håkan Hedenmalm for helpful communication in connection with the above proof, [11].

4.2. **Proof of the existence theorems.** We now prove the existence theorems, Theorem II and III.

Let p be either a $(\nu, 2)$-cusp with $\nu \ge 5$ or a double point. In both cases we assume that $\Delta Q(p) > 0$. Also fix a number $T > 0$. For a given $n \in \mathbb{Z}_+$, we let p_n be a point in S whose distance to the boundary is $T/\sqrt{n\Delta Q(p)}$ and whose distance to p is minimal.

We rescale about p_n,

$$z_j = e^{-i\theta_n}\sqrt{n\Delta Q(p)}(\zeta_j - p_n), \qquad j = 1, \ldots, n,$$

where the angle θ_n is chosen so that the image of the point p lies on the positive imaginary axis.

Note that as $n \to \infty$, the image of S near p_n looks approximately like the strip

$$\Sigma_T : \quad -T < \operatorname{Re} z < T. \tag{4.1}$$

Let K_n be the kernel of the rescaled system $\Theta_n = \{z_j\}_1^n$. We write $R_n(z) = K_n(z, z)$. By Lemma 1.1 we know that there is a sequence of cocycles c_n such that every subsequence of c_nK_n has a subsequence converging to $G\Psi$ where Ψ is some Hermitian entire function. It remains only to show that the function $R(z) = \Psi(z, z)$ does not vanish identically if T is large enough.

To this end, we shall use the estimate in [3], Theorem 5.4,

$$|\mathbf{R}_n(\zeta) - n\Delta Q(\zeta)| \le C\left(1 + ne^{-n\ell\Delta Q(\zeta)\cdot\delta(\zeta)^2}\right), \quad \zeta \in S,$$

where ℓ is a positive constant and $\delta(\zeta) = \operatorname{dist}(\zeta, \partial S)$. If we choose $\zeta = p_n$ where $\delta(p_n) = T/\sqrt{n\Delta Q(p_n)}$, we obtain for the rescaled 1-point function R_n that

$$|\, R_n(0) - 1\,| \le Ce^{-\ell T^2}.$$

choosing T sufficiently large that the right hand side is < 1, we obtain that $R(0) > 0$. An application on Lemma 1.1 now shows that $R(z) > 0$ for all $z \in \mathbb{C}$. q.e.d.

5. Concluding remarks

We shall here discuss a family of natural candidates of the limiting point fields whose existence is guaranteed by theorems II and III.

Let p be a singular point; for definiteness, let us say it is a cusp.

For a "large" T, consider the point $p_n \in S$ at distance $T/\sqrt{n\Delta Q(p)}$ from ∂S being closest to p and rescale about p_n as in Theorem II. Let $K = G\Psi$ be a limiting kernel in Lemma 1.1; then K is non-trivial.

Recall that a Hermitian-entire function Ψ is called translation invariant (in short: *t.i.*) if it takes the form $\Psi(z, w) = \Phi(z + \bar{w})$ for some entire function Φ, which entails that $\Psi(z + it, w - it)$ is independent of $t \in \mathbb{R}$. We do not know that a limiting kernel must be *t.i.* but it seems to be a reasonable assumption, since the rescaled droplet looks like the strip

$$\Sigma_T = \{z;\, -T \le \operatorname{Re} z \le T\}.$$

In any case, we shall now use theory from [3] to narrow down the set of possible limiting kernels, under the extra hypothesis of translation invariance.

First of all, Theorem E in [3] implies that a *t.i.* limiting kernel $K(z, w) = G(z, w)\Phi(z + \bar{w})$ has the "Gaussian representation"

$$\Phi(z) = \gamma * f(z) = \int_{-\infty}^{+\infty} \gamma(z - t) f(t)\, dt,$$

where $\gamma(z) = (2\pi)^{-1/2} e^{-z^2/2}$ is the Gaussian kernel and $f(t)$ is some Borel function with $0 \le f \le 1$. Secondly, Theorem F says that the function Φ above gives rise to a solution to Ward's equation (1.5) if and only if $f = \chi_I$ is the characteristic function of some *interval* I.

Finally, by the estimate (1.7), we know that the function $R(z) = \Phi(z + \bar{z})$ must satisfy

$$\Phi(2x) \le Ce^{-2(|x| - T)^2}, \qquad x \in \mathbb{R}.$$

In order for this to be consistent with the identity

$$\Phi(x) = \gamma * \chi_I(x) = \frac{1}{\sqrt{2\pi}} \int_I e^{-(x-t)^2/2}\, dt,$$

we must have $I \subset [-2T, 2T]$.

Thus if there exists a limiting *t.i.* kernel, it necessarily has the structure $\Phi = \gamma * \chi_I$ where I is an interval contained in $[-2T, 2T]$.

For reasons on symmetry, it is natural to assume that an interval I as above can be chosen to be symmetric, i.e. $I = [-s/2, s/2]$ for some number $s = S(T)$ between 0 and $4T$.

Let us denote by Φ_s the function

$$\Phi_s(z) := \gamma * \chi_{(-s/2, s/2)}(z) = \frac{1}{\sqrt{2\pi}} \int_{-s/2}^{s/2} e^{-(z-t)^2/2}\, dt.$$

We know from Lemma 7.7 in [3] that the kernel $K_s(z, w) := G(z, w)\Phi_s(z + \bar{w})$ appears as the correlation kernel of a unique point field.

It is close at hand to guess that one of the K_s $(0 < s \leq 4T)$ will appear as a limiting kernel in Theorem II.

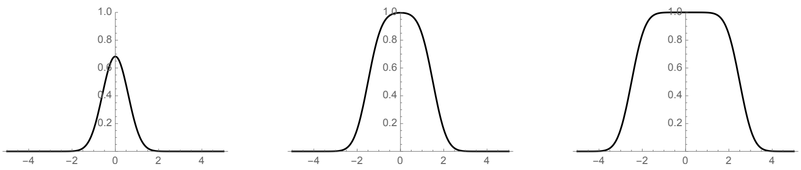

FIGURE 3. The graph of $R_s(x) := \gamma * \chi_{(-s/2,s/2)}(2x)$ for $s = 2$, $s = 5$, and $s = 8$.

To further corroborate our point, we may observe that the kernel K_s interpolates in a natural way between the trivial kernel $K_0 = 0$ (at the singular point) and the Ginibre kernel $K_\infty = G$ (the bulk regime). See Figure 3.

Now fix a large enough number T. The conditions shown for a limiting 1-point function R, i.e., that it be non-trivial, satisfy Ward's equation, and have the decay $R(x) \leq Ce^{-2(|x|-T)^2}$, are not enough to fix a solution uniquely, even under the assumption of translation invariance.

In the paper [3], the corresponding question was solved for regular boundary points, by using the 1/8-formula ([3], Theorem D)

$$\int_{\mathbb{R}} t \cdot (\chi_{(-\infty,0)}(t) - R(t))\, dt = \frac{1}{8}. \tag{5.1}$$

This suggests that, in order to single out a unique solution, one could seek some additional condition similar to (5.1), which is to hold for a limiting kernel near a singular boundary point. One of the main challenges with this is that the boundary fluctuation theorem from [2], which was used to prove the condition (5.1), is so far only known for connected droplets with everywhere smooth boundary. Thus, it would seem natural to try to extend the methods in [2] to domains with singular boundary points.

We also mention that the reason that we had to insist that the parameter T be sufficiently large is that we have used the technique of Hörmander estimates to ensure non-triviality of the 1-point function, and Hörmander estimates require a certain distance to the boundary. On the other hand, if one could prove a generalized boundary fluctuation theorem, as suggested above, one would probably obtain non-triviality also for small $T > 0$.

There is a parallel "hard-edge" theory, where one confines the system $\{\zeta_j\}_1^n$ to the droplet by redefining Q outside the droplet to be $+\infty$ there. Similar results to those studied in this paper (the "free boundary" case) can be obtained; cf. [3] and [4]. In the following we will freely use results from the forthcoming paper [4].

If one rescales near a singular boundary point p in the hard-edge case, at distance $T/\sqrt{n\Delta Q(p)}$ from the boundary, one obtains limiting kernels of the form

$$K(z, w) = G(z, w)\, \Psi(z, w)\, \chi_{(-T,T)}(\operatorname{Re} z)\, \chi_{(-T,T)}(\operatorname{Re} w)$$

where Ψ is some Hermitian-entire function. In the $t.i.$ case $\Psi(z, w) = \Phi(z + \bar{w})$, we anticipate that Φ necessarily has a representation

$$\Phi = \gamma * \left(\frac{\chi_I}{F_T}\right)$$

where $F_T = \gamma * \chi_{(-2T,2T)}$ and I is some interval. In this case, the natural candidates for limiting point fields again correspond to symmetric intervals $I = (-2T, 2T)$ (or perhaps $I = (-s, s)$ for some suitable value of $s = s(T)$). We are thus led to study the functions of the type

$$H_T(z) := \frac{1}{\sqrt{2\pi}} \int_{-2T}^{2T} \frac{e^{-(z-t)^2/2}}{F_T(t)}\, dt.$$

The corresponding "1-point function" is then $R_T^h(z) := H_T(z+\bar{z})\chi_{(-T,T)}(\operatorname{Re} z)$.

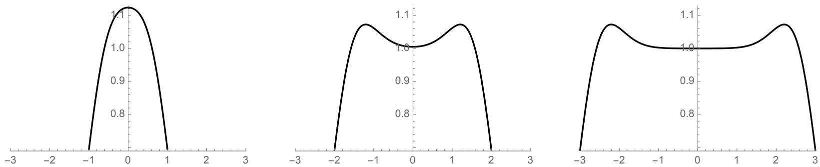

FIGURE 4. The graph of R_T^h restricted to the reals, for $T = 2$, $T = 5$, and $T = 8$.

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Paper D

A central limit theorem for polyanalytic Ginibre ensembles
(joint with A. Haimi)
Int. Math. Res. Not. IMRN, advance online publ., (2017).

A. Haimi and A. Wennman (2017) "A Central limit theorem for fluctuations in Polyanalytic Ginibre ensembles,"
International Mathematics Research Notices, Vol. 2017, No. 00, pp. 1–23
doi: 10.1093/imrn/rnx147

A Central limit theorem for fluctuations in Polyanalytic Ginibre ensembles

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We study fluctuations of linear statistics in polyanalytic Ginibre ensembles, a family of point processes describing planar free fermions in a uniform magnetic field at higher Landau levels. Our main result is asymptotic normality of fluctuations, extending a result of Rider and Virág. As in the analytic case, the variance is composed of independent terms from the bulk and the boundary. Our methods rely on a structural formula for polyanalytic polynomial Bergman kernels which separates out the different pure q-analytic kernels corresponding to different Landau levels. The fluctuations with respect to these pure q-analytic Ginibre ensembles are also studied, and a central limit theorem is proved. The results suggest a stabilizing effect on the variance when the different Landau levels are combined together.

1 Introduction

The Ginibre ensemble is one of the major point processes in random matrix theory and mathematical physics. The model has at least three possible interpretations: in terms of eigenvalues of random matrices, Coulomb gas of charged particles in an external field, or ground state free fermions in a magnetic field perpendicular to the plane. In this paper, we study *polyanalytic Ginibre ensembles* [12], a family of point process which

Received March 8, 2017; Revised May 19, 2017; Accepted May 23, 2017
Communicated by Prof. Misha Sodin

generalizes the last of these three notions so that the particles are allowed to occupy more general energy levels.

From the point of view of quantum mechanics, we can arrive at the polyanalytic Ginibre ensembles in the following way. It is a consequence of the Pauli exclusion principle that the wavefunction of the N-body system of free (i.e., non-interacting) particles is given by the Slater determinant

$$\det[\psi_i(z_j)]_{1\le i,j\le N} \tag{1.1}$$

where ψ_j are orthonormal single particle wave functions. We take these to be eigenstates of the *Landau Hamiltonian*

$$H_B := \frac{1}{2}\left(\left(\frac{\partial}{\partial x}-\frac{B}{2}y\right)^2+\left(i\frac{\partial}{\partial y}-\frac{B}{2}x\right)^2\right),$$

which is known to describe a single electron in the plane subjected to a perpendicular magnetic field of strength B. It is known ([2, 10]) that the spectrum of this operator, acting on $L^2(\mathbb{R}^2)$, consists of eigenvalues

$$e_q^B = (q+1/2)B.$$

These eigenvalues are referred to as *Landau levels*, and the eigenspace corresponding to e_q^B consists of functions of the form

$$f(z)\mathrm{e}^{-B|z|^2/2},$$

where f belongs to the *pure q-analytic Bargmann-Fock space* $A^2_{\delta;B,q}$, defined as the orthogonal difference $A^2_{B,q}\ominus A^2_{B,q-1}$ between two consequtive *q-analytic Bargmann-Fock spaces*

$$A^2_{B,q} := \left\{f : \int_{\mathbb{C}} |f(z)|^2\mathrm{e}^{-B|z|^2}\mathrm{d}A(z) < \infty,\ \bar{\partial}^q f(z) = 0\right\}.$$

The case of the Ginibre ensemble (i.e. $q=1$) corresponds to the classical Bargmann–Fock space, which is associated with the orthonormal basis

$$\psi_j(z) = \frac{B^{1/2}}{\sqrt{j!}}(B^{1/2}z)^j\mathrm{e}^{-B|z|^2/2},\quad 0\le j.$$

According to a well-known computation in point process theory, the probability density corresponding to the many-body wavefunction in (1.1) can be written (up to a constant) in the form:

$$\left|\det[\psi_i(z_j)]_{1\le i,j\le N}\right|^2 \sim \frac{1}{N!}\det[K(z_i,z_j)]_{1\le i,j\le N}\mathrm{e}^{-B|z_1|^2-\ldots-B|z_N|^2},$$

where

$$K(z,w)=\sum_{j=0}^{N-1}\psi_j(z)\overline{\psi_j(w)}.$$

In the Ginibre case, this coincides with the reproducing kernel of the space of analytic polynomials of degree $\le N-1$ in $L^2(\mathrm{e}^{-B|z|^2}\mathrm{d}A(z))$.

To obtain the polyanalytic Ginibre ensembles, we allow wavefunctions from general eigenspaces of H_B, not just from the lowest level. The point processes that we will consider fall into to two categories: *full type* and *pure type*. In the first case, we have n particles at each level up to q and and in the second n particles at q'th level only. So, the processes of full type contain nq points, and those of the pure type consist of n points. It is natural to take the field B to be equal to the number of particles n at each level; physically, this corresponds to each level being completely filled. The corresponding reproducing kernels are formed by choosing the appropriate wavefunctions from $A^2_{B,q}$ and $A^2_{\delta;B,q}$ and will be denoted by $K_{n,q}$ and $K_{\delta;n,q}$. They correspond to spaces

$$\mathrm{Pol}_{n,q}=\mathrm{span}\left\{\bar{z}^r z^j : 0\le j\le n-1, 0\le r\le q-1\right\}\subset L^2(\mathrm{e}^{-n|z|^2}\mathrm{d}A)$$

and

$$\delta\mathrm{Pol}_{n,q}:=\mathrm{Pol}_{n,q}\ominus\mathrm{Pol}_{n,q-1},$$

respectively. We see that when we allow higher Landau levels in the process, the function spaces do not only consist of analytic functions as in the Ginibre case, but rather more general *polyanalytic functions*. The study of these functions has attracted increasing attention recently, and interesting connections between signal analysis, quantum mechanics and complex analysis have been found. For an introduction, see [3] or [2].

The study of processes of this type was initiated in mathematics literature in [12], where precise estimates and scaling limits for the kernels $K_{\delta;n,q}$ and $K_{n,q}$ were obtained when $n\to\infty$ and q is fixed. In particular, it follows from the results there that the circular law will appear in the limit also when higher Landau levels are included, see Figure 1. Our main theorem shows asymptotic normality of fluctuations around this mean, generalising a result of Rider and Virág [16] from the analytic case. For a continuous

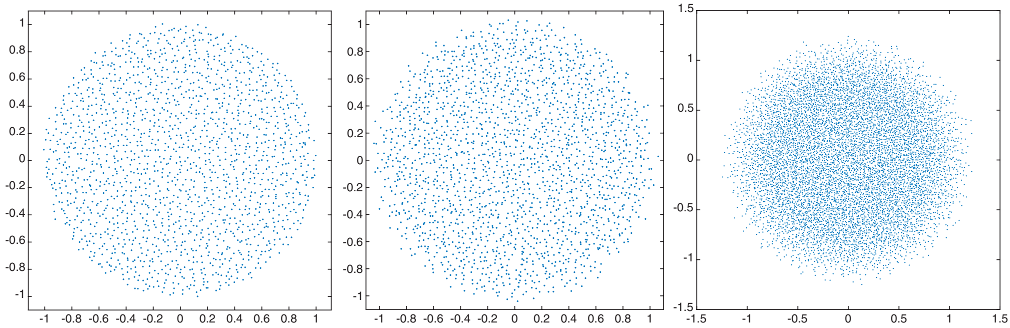

Fig. 1. Polyanalytic Ginibre ensembles defined by the kernel $K_{n,q}$ with different values of q and n. Notice that when q is relatively large compared to n, the density is less uniform.

test function, we define the linear statistics associated with processes of the pure type by

$$\mathrm{trace}_{\delta;n,q} g = \sum_{j=1}^{n} g(\lambda_j),$$

where the vector $(\lambda_j)_{j=1}^{n}$ is picked from the determinantal process defined by the kernel $K_{\delta;n,q}$. The random variables $\mathrm{trace}_{n,q}$ are defined similarly:

$$\mathrm{trace}_{n,q} g = \sum_{j=1}^{nq} g(\lambda_j).$$

Here we have taken nq points from the process defined by $K_{n,q}$. The corresponding fluctuations are defined by

$$\mathrm{fluct}_{\delta;n,q} g = \mathrm{trace}_{\delta;n,q} g - \mathbb{E}(\mathrm{trace}_{\delta;n,q} g), \qquad \mathrm{fluct}_{n,q} g = \mathrm{trace}_{n,q} g - \mathbb{E}(\mathrm{trace}_{n,q} g).$$

Here, we adopt the convention that the fluctuations are measured about the mean. This is in line with the definition in [16], but slightly different from that in [6] and [5]. In the latter, the fluctuations are measured as deviations from the limiting mean, which we compute in (2.6). Our main theorem would be the same also with respect to the other possible definition.

We denote by $N(a, \sigma^2)$ a normal variable with mean a and variance σ^2. We write

$$\|g\|_{H^1(\mathbb{D})}^2 := \int_{\mathbb{D}} |\bar{\partial} g|^2 \mathrm{d}A(z)$$

for the Dirichlet semi-norm of g on $\mathbb{D}$ and

$$\|g\|^2_{H^{1/2}(\partial\mathbb{D})} = \sum_{k\in\mathbb{Z}} |k||\hat{g}(k)|^2,$$

for the $H^{1/2}$ semi-norm of g on the unit circle. The numbers

$$\hat{g}(k) := \frac{1}{2\pi}\int_0^{2\pi} g(\mathrm{e}^{i\theta})\mathrm{e}^{-ik\theta}\mathrm{d}\theta$$

are the Fourier coefficients of g restricted to $\{|z| = 1\}$.

Theorem 1.1. Let $g \in C_0^\infty(\mathbb{C})$ be real valued. We have

$$\mathrm{fluct}_{\delta;q,n}g \to N\big(0, (2q-1)\|g\|^2_{H^1(\mathbb{D})} + \frac{1}{2}\|g\|^2_{H^{1/2}(\partial\mathbb{D})}\big) \tag{1.2}$$

and

$$\mathrm{fluct}_{q,n}g \to N\big(0, q\big(\|g\|^2_{H^1(\mathbb{D})} + \frac{1}{2}\|g\|^2_{H^{1/2}(\partial\mathbb{D})}\big)\big), \tag{1.3}$$

in distribution as $n \to \infty$. □

We observe that the variance consists of essentially the same two terms as in the case $q = 1$, the bulk term and the boundary term. For an interpretation in terms of Gaussian fields, see concluding remarks in [6] and [14] as a background reading. In the present polyanalytic setting, the coefficients in front of the two terms bring in new phenomena which are not present with the Ginibre ensemble. In the formula (1.3), the variance from the analytic case is just multiplied by q. In the pure polyanalytic case (1.2), only the bulk term involves a factor depending on q and the boundary is the same as in the analytic setting. Moreover, if we concentrate only on test functions which are supported in the bulk, the variance in (1.2) is higher than in (1.3). In fact, because $\frac{1}{q}\sum_{r=1}^{q}(2r-1) = q$, the variance in (1.3) is obtained by averaging the variances from each of the q Landau levels. This suggests that adding several Landau levels together has a certain smoothening effect on the variance. Interestingly, for the boundary terms the situation is different, because the boundary contribution in (1.3) is just the sum of boundary contributions from the individual levels. We do not have a physical explanation for these facts at the moment.

In the analytic case, $q = 1$, one may use Green's furmula to rewrite the limiting variance as a single Dirichlet integral [5]

$$\|h\|^2_{H^1(\mathbb{D})} + \frac{1}{2}\|h\|^2_{H^{1/2}(\partial\mathbb{D})} = \|h^{\mathbb{D}}\|^2_{H^1(\mathbb{C})},$$

where $h^{\mathbb{D}}$ is the bounded harmonic extension to $\mathbb{C}$ of the restriction $h|_{\mathbb{D}}$. For the full polyanalytic process, (1.3), such a rewriting is also possible. Because of the coefficients appearing in front of the boundary and bulk terms in (1.2), it is not clear how to do this with processes of pure type.

As in [16] and [6], the proof is based on the cumulant method introduced by Costin and Lebowitz [9]. However, instead of using explicit expression for the correlation kernel or estimates of it directly, our proof starts with a partial integration procedure (Proposition 3.1), based on expressing the kernels in terms of quantum mechanical raising operators which act isometrically between Landau level eigenspaces. As a result, we can rewrite the cumulants in terms of the Ginibre kernel only. This representation combined with rather general estimates of integrals involving cyclic products of kernels allows us to make a reduction to the analysis in [6] and [16]. We want to emphasize that even though precise estimates and an explicit expressions for our polyanalytic kernels are known, applying formulas for the cumulants directly would most likely lead to very complicated calculations.

In this article, the function spaces are defined with a Gaussian weight only. In the analytic case, the result of Rider and Virág has been generalized to more general weights by *Ameur et al* (in [6] for test funtions with support in the bulk, and in [5] for general test functions). While it is reasonable to expect that results of the former type can be extended to polyanalytic setting by using estimates from [11], it remains an interesting question what happens with general test functions. The technique in [5] seems not to generalize to our polyanalytic case. On the other hand, our approach is based on expressing the polyanalytic kernels in terms of the Landau level raising operators and these are closely tied to the Gaussian case.

Our methods could be used to study other configurations of particles as well, not just those where either one level or all levels up to a given level are filled. However, our techniques do require the highest level q to be fixed as we let $n \to \infty$. In a forthcoming paper we will study asymptotic behaviour of the kernel $K_{n,q}$ when both q and n tend to infinity. Some preliminary observations about this setting can be found in [12].

The article is organized as follows. In Section 2, we provide basic facts about the function spaces and point processes involved. In Section 3, we introduce cumulants and our main technique, the partial integration procedure in Proposition 3.1. In Section 4, we prove certain estimates for integrals involving cyclic products of kernels that allow us to estimate the cumulants. The main theorem is then proved in Section 5.

2 Preliminaries

2.1 Notation

We write $z = x + iy$ and let $\partial = \frac{1}{2}(\partial_x - i\partial_y)$ and $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ denote the standard Wirtinger differential operators. We will write $\mathbb{D}$ for the open unit disk. We will write $\mathrm{d}A(z) = \frac{1}{\pi}\mathrm{d}x\mathrm{d}y$ and let $\mathrm{d}\mu_n(z) = \mathrm{e}^{-n|z|^2}\mathrm{d}A(z)$ be the Gaussian measure on $\mathbb{C}$. We will use the notation $\mathrm{d}\mu_n^k(z_1, \dots, z_k) = \mathrm{d}\mu_n(z_1)\dots\mathrm{d}\mu_n(z_k)$.

2.2 Spaces of polyanalytic polynomials

A function f defined on a subset of the complex plane is called q-analytic if it satisfies the equation $\bar{\partial}^q f = 0$ in the sense of distributions. Equivalently, a function f is q-analytic if it can be decomposed as

$$f(z) = \sum_{j=0}^{q-1} \bar{z}^j f_j(z),$$

where the functions f_j are analytic. A function that is q-analytic for some q is called *polyanalytic*. We define the *Bargmann–Fock space of q-analytic functions* as

$$A^2_{n,q} := \left\{ f(z) : \int_{\mathbb{C}} |f(z)|^2 \mathrm{d}\mu_n(z) < \infty, \bar{\partial}^q f = 0 \right\}.$$

The spaces of *q-analytic polynomials* are defined by

$$\mathrm{Pol}_{n,q} = \mathrm{span}\left\{ \bar{z}^r z^j : 0 \le j \le n-1, 0 \le r \le q-1 \right\}.$$

We equip $A^2_{n,q}$ and $\mathrm{Pol}_{n,q}$ with the inner product from $L^2(d\mu_n)$, and consider $A^2_{n,q}$ as a subspace of $\mathrm{Pol}_{n,q}$. The *Bargmann–Fock space of pure q-analytic functions* is defined as the orthogonal difference

$$A^2_{\delta;n,q} := A^2_{n,q} \ominus A^2_{n,q-1}.$$

The spaces of *pure q-analytic polynomials* are defined analogously by

$$\delta\mathrm{Pol}_{n,q} = \mathrm{Pol}_{n,q} \ominus \mathrm{Pol}_{n,q-1}.$$

Any function space H encountered here possesses a reproducing kernel, i.e. a function $K(z, w)$ such that for any $w \in \mathbb{C}$, $K_w := K(\cdot, w)$ is an element of the space, and for any function $f \in H$ it holds that

$$f(w) = \langle f, K_w \rangle, \qquad w \in \mathbb{C}.$$

We denote by $K_{n,q}(z,w)$ the reproducing kernel for the space $\mathrm{Pol}_{n,q}$ and by $K_{\delta;n,q}(z,w)$ the kernel for the space $\delta\mathrm{Pol}_{n,q}$. The kernel $K_{n,1}(z,w)$ is simply written $K_n(z,w)$, and we will denote by P_n the corresponding projection operator from $L^2(s\mu_n)$ onto $\mathrm{Pol}_{n,1}$. Clearly, $K_{n,q}$ can be expressed in terms of the pure q-analytic kernels as follows:

$$K_{n,q}(z,w) = \sum_{r=1}^{q} K_{\delta;n,r}(z,w). \tag{2.1}$$

In [12], the following explicit expression for the kernel $K_{n,q}$ was given:

$$\begin{aligned} K_{n,q}(z,w) = n \sum_{r=0}^{q-1} \sum_{i=0}^{n-r-1} \frac{r!}{(r+i)!} (nz\bar{w})^i L_r^i(n|z|^2) L_r^i(n|w|^2) \\ + n \sum_{j=0}^{q-2} \sum_{k=1}^{q-j-1} \frac{j!}{(k+j)!} (n\bar{z}w)^k L_j^k(n|z|^2) L_j^k(n|w|^2). \end{aligned} \tag{2.2}$$

Here,

$$L_r^k := \sum_{j=0}^{r} (-1)^j \binom{r+k}{r-j} \frac{1}{j!} x^j \tag{2.3}$$

denotes the associated Laguerre polynomial of index k and degree r. The kernel $K_{\delta;n,q}$ can then be obtained from this and the relation

$$K_{\delta;n,q} = K_{n,q}(z,w) - K_{n,q-1}(z,w). \tag{2.4}$$

We will not need to use the explicit expressions for these kernels in this article. We just observe that

$$K_n(z,w) = K_{n,1}(z,w) = n \sum_{j=0}^{n-1} \frac{(nz\bar{w})^j}{j!}$$

is the standard Ginibre kernel from random matrix theory. Rather than use (2.2) and (2.4), we will express $K_{n,q}$ and $K_{\delta;n,q}$ in terms of K_n and the *raising operators*

$$(T_{n,r} f)(z) = \frac{n^{-r/2}}{\sqrt{r!}} \mathrm{e}^{n|z|^2} \partial^r \left\{ f(z) \mathrm{e}^{-n|z|^2} \right\}.$$

Frequently, it will be convenient to use the notation $T_n := -n^{1/2} T_{n,1}$. As explained in [12], $T_{n,r}$ is an isometric isomorphism from $\mathrm{Pol}_{n,1}$ to $\delta\mathrm{Pol}_{n,r+1}$. Thus, $T_{n,r}$ maps orthonormal bases to orthonormal bases. It thus implies that the pure polyanalytic kernels may be obtained from the analytic kernel $K_n(z,w)$ by

$$K_{\delta;n,r}(z,w) = [T_{n,r-1}]_z \overline{[T_{n,r-1}]}_w K_n(z,w).$$

Consequently,

$$K_{n,q}(z,w) = \sum_{r=0}^{q-1} [T_{n,r}]_z \overline{[T_{n,r}]}_w K_n(z,w) = \sum_{r=0}^{q-1} \frac{n^{-r}}{r!} [T_n]_z^r \overline{[T_n]}_w^r K_n(z,w). \tag{2.5}$$

Polyanalytic functions appear naturally in quantum mechanics and time–frequency analysis. The reader who is interested in more background can consult [3], [1], [20], or [8].

2.3 Polyanalytic Ginibre ensembles

Our aim is to study determinantal processes associated with the kernels $K_{n,q}$ and $K_{\delta;n,q}$. These are given by the following probability measures on $\mathbb{C}^{nq}$ and $\mathbb{C}^n$:

$$\mathrm{d}\mathbb{P}_{n,q} := \frac{1}{(nq)!} \det[K_{n,q}(z_j,z_k)]_{1\le j,k\le nq} \mathrm{e}^{-n|z_1|^2-\ldots-n|z_{nq}|^2} \mathrm{d}A(z_1)\ldots \mathrm{d}A(z_{nq})$$

and

$$\mathrm{d}\mathbb{P}_{\delta;n,q} := \frac{1}{n!} \det[K_{\delta;n,q}(z_j,z_k)]_{1\le j,k\le n} \mathrm{e}^{-n|z_1|^2-\ldots-n|z_n|^2} \mathrm{d}A(z_1)\ldots \mathrm{d}A(z_n).$$

The fact that these are probability measures is a standard result in theory of determinantal point processes (see e.g., [13] for an introduction). We will identify all copies of $\mathbb{C}$ and interpret $\mathrm{d}\mathbb{P}_{n,q}$ and $\mathrm{d}\mathbb{P}_{\delta;n,q}$ as densities for random configurations of nq and n unlabelled points in $\mathbb{C}$, respectively. It is well known ([15, 19]) that any locally trace class projection kernel defines a determinantal process. Because our spaces are finite dimensional, we do not need this general definition here. We just note that an infinite dimensional counterpart of polyanalytic Ginibre ensembles has been studied by Shirai [18] and it belongs to a more general class of point processes called *Weyl–Heisenberg ensembles*, recently introduced in [4].

2.4 Linear statistics

Recall from the introduction that given a bounded, continuous function g, the linear statistics are defined by

$$\mathrm{trace}_{n,q} g := \sum_{j=1}^{nq} g(\lambda_j),$$

where $(\lambda_1, \ldots, \lambda_{nq})$ is picked from the measure $\mathbb{P}_{n,q}$. The variables $\mathrm{trace}_{\delta;n,q} g$ are defined in a similar way, but one picks a random vector $(\lambda_1, \ldots, \lambda_n)$ from $\mathbb{P}_{\delta;n,q}$ instead.

Asymptotic behaviour of the expectation of linear statistics can be analysed as follows:

$$\frac{1}{nq}\mathbb{E}\mathrm{trace}_{n,q}(f) = \mathbb{E}f(z_1) = \frac{1}{nq}\int_{\mathbb{C}} f(z) K_{n,q}(z,z) \mathrm{e}^{-n|z|^2} \mathrm{d}A(z) \to \int_{\mathbb{D}} f(z) \mathrm{d}A. \tag{2.6}$$

Here, the second equality is a general fact about determinantal point processes. The limit follows from the weak convergence $\frac{1}{nq} K_{n,q}(z,z)\mathrm{e}^{-n|z|^2} \to 1_{\mathbb{D}} dA$ which follows from the results in [12]. A similar statement is true for pure polyanalytic kernels $K_{\delta;n,q}$, but this time one just divides with n instead of nq. This generalizes the well-known circular law about the Ginibre ensemble (the case $q = 1$). The interpretation is that the particles tend to accumulate uniformly on the unit disk. We use standard terminology and refer to the open unit disk as the *bulk*.

The goal of this article is to understand how the linear statistics fluctuate around the mean. We let $\mathrm{fluct}_{n,q} g$ and $\mathrm{fluct}_{\delta;n,q} g$ be the mean-zero variables

$$\mathrm{fluct}_{n,q} g = \sum_{\lambda_j} g(\lambda_j) - \mathbb{E}g(\lambda_j),$$

where $\lambda_1, \ldots, \lambda_{nq}$ is picked from the density $d\mathbb{P}_{n,q}$. The variables $\mathrm{fluct}_{\delta;n,q} g$ have an analogous definition.

3 Cumulants

For any real-valued random variable X, the cumulants $C_k(X)$ are defined implicitly by

$$\log \mathbb{E}\left[\mathrm{e}^{tX}\right] = \sum_{j=1}^{\infty} \frac{t^k}{k!} C_k(X).$$

The first cumulant is equal to the expectation and the second to the variance of X. For linear statistics of determinantal processes, the cumulants are given by an explicit formula introduced by Costin and Lebowitz [9]. We will use a formulation from [6]. For $X = \mathrm{trace}_{n,q}(g)$, we have

$$C_k(X) = \int_{\mathbb{C}^k} G_k(z_1, \ldots, z_k) K_{n,q}(z_1, z_2) K_{n,q}(z_2, z_3) \cdots K_{n,q}(z_k, z_1) \mathrm{d}\mu_n^k(z_1, \ldots z_k), \tag{3.1}$$

where

$$G_k(z_1, \ldots z_k) = \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} \sum_{\substack{k_1+k_2+\ldots+k_j=k \\ k_1,\ldots,k_j \geq 1}} \frac{k!}{k_1!k_2! \cdots k_j!} \prod_{l=1}^{j} g(z_l)^{k_l}.$$

For $X = \mathrm{trace}_{\delta;n,q}(g)$, one just replaces each occurrence of $K_{n,q}$ by $K_{\delta;n,q}$ in (3.1). For future reference, note that G_k is a sum of products of functions of one variable.

It is known that asymptotic normality holds if the cumulants convergence to those of the normal distribution. Moreover, it is known that a random variable is normally distributed iff all cumulants of order $k \geq 3$ vanish. Therefore, to prove the main theorem, we need to show that for $k \geq 3$, $C_k(\mathrm{trace}_{n,q})$ and $C_k(\mathrm{trace}_{\delta;n,q})$ tend to zero and compute the limits of the second cumulants as $n \to \infty$.

The following proposition gives an alternative way to represent the cumulants. In this new representation, the cumulants are written as integrals involving a cyclic product with the analytic Ginibre kernel K_n.

For non-negative integers α, β and n, we define the differential operators

$$\mathcal{D}_{\alpha,\beta,n} = \sum_{j=0}^{\min(\alpha,\beta)} \binom{\min(\alpha, \beta)}{j} (\max(\alpha, \beta))_j n^j \bar{\partial}^{\alpha-j} \partial^{\beta-j},$$

where $(r)_j = r(r-1) \cdots (r-j+1)$. Assuming $\beta \geq \alpha$, this can be written in a slightly more compact form as

$$\mathcal{D}_{\alpha,\beta,n} = \alpha! n^\alpha \partial^{\beta-\alpha} L_\alpha^{\beta-\alpha}(-n^{-1}\Delta),$$

where L_r^k denotes the associated Laguerre polynomial of index k and degree r. Similarly, when $\alpha \geq \beta$, we have

$$\mathcal{D}_{\alpha,\beta,n} = \beta! n^\beta \bar{\partial}^{\alpha-\beta} L_\beta^{\alpha-\beta}(-n^{-1}\Delta).$$

In particular, in the special case $\alpha = \beta = r$ we have

$$\mathcal{D}_{\alpha,\beta,n} = n^r r! L_r^0(-n^{-1}\Delta). \tag{3.2}$$

Proposition 3.1. Let $F : \mathbb{C}^k \to \mathbb{C}$ be such that all derivatives are up to order $2q-2$ are bounded and continuous. Then for $0 \le i_1, \dots, i_k \le q-1$, we have

$$\int_{\mathbb{C}^k} F(z_1,\dots,z_k)[T_n]_{z_1}^{i_1}\overline{[T_n]_{z_2}^{i_1}}K_n(z_1,z_2)\cdots[T_n]_{z_k}^{i_k}\overline{[T_n]_{z_1}^{i_k}}K_n(z_k,z_1)\mathrm{d}\mu_n^k(z_1,\dots,z_k)$$
$$= \int_{\mathbb{C}^k} \mathcal{D}_{i_1,\dots,i_k,n}F(z_1,\dots,z_k)K_n(z_1,z_2)\cdots K_n(z_k,z_1)d\mu_n^k(z_1,\dots,z_k)$$

where

$$\mathcal{D}_{i_1,\dots,i_k,n} = \left[\mathcal{D}_{i_k,i_1,n}\right]_{z_1}[\mathcal{D}_{i_1,i_2,n}]_{z_2}\cdots[\mathcal{D}_{i_{k-1},i_k,n}]_{z_k}. \qquad \square$$

Proof. We will use that

$$\int f(z)T_n g(z)\mathrm{d}\mu_n(z) = \int \partial f(z)g(z)\mathrm{d}\mu_n(z), \tag{3.3}$$

where $f, g \in C^1$. In our applications, f and g will have at most polynomial growth in z, so boundary terms do not appear in the partial integration. We will also need the following: for $j \le r$,

$$\bar{\partial}^j T_n^r f = (r)_j n^j T_n^{r-j} f \tag{3.4}$$

for an analytic function f. This can be seen as follows:

$$\begin{aligned}\bar{\partial}^j T_n^r f &= (-1)^r \bar{\partial}^j \sum_{k=0}^{r}\binom{r}{k} f^{(k)}(z)(-n\bar{z})^{r-k}\\ &= (-1)^{r+j} n^j \sum_{k=0}^{r-j}\binom{r}{k}(r-k)_j(-n)^{r-k-j}\bar{z}^{r-j-k}f^{(k)}(z)\\ &= (-1)^{r+j} n^j (r)_j \sum_{k=0}^{r-j}\binom{r-j}{k}(-n\bar{z})^{r-j-k}f^{(k)}(z) = n^j(r)_j T_n^{r-j}f.\end{aligned}$$

We integrate by parts in the variable z_1. Suppose that $i_1 \le i_k$. Using (3.3) and (3.4), we obtain

$$\begin{aligned}&\int_{\mathbb{C}^k} F(z_1,\dots,z_k)[T_n]_{z_1}^{i_1}\overline{[T_n]_{z_2}^{i_1}}K_n(z_1,z_2)\cdots[T_n]_{z_k}^{i_k}\overline{[T_n]_{z_1}^{i_k}}K_n(z_k,z_1)\mathrm{d}\mu_n^k\\ &= \sum_{j=0}^{i_1}\binom{i_1}{j} n^j (i_k)_j \partial_{z_1}^{i_1-j}F(z_1,\dots,z_k)\overline{[T_n]_{z_2}^{i_1}}K_n(z_1,z_2)\cdots[T_n]_{z_k}^{i_k}\overline{[T_n]_{z_1}^{i_k-j}}K_n(z_k,z_1)\mathrm{d}\mu_n^k\\ &= \sum_{j=0}^{i_1}\binom{i_1}{j} n^j (i_k)_j \bar{\partial}_{z_1}^{i_k-j}\partial_{z_1}^{i_1-j}F(z_1,\dots,z_k)\overline{[T_n]_{z_2}^{i_1}}K_n(z_1,z_2)\cdots[T_n]_{z_k}^{i_k}K_n(z_k,z_1)\mathrm{d}\mu^k.\end{aligned}$$

The last equality holds because $\overline{[T_n]_{z_2}^{i_1}K_n(z_1,z_2)}$ is analytic in z_1. If $i_1 \geq i_k$ instead, the operator acting on F is $\sum_{j=0}^{i_k}\binom{i_k}{j}n^j(i_1)_j\bar{\partial}_{z_1}^{i_k-j}\partial_{z_1}^{i_1-j}$. In any case, the operator acting on F is $[\mathcal{D}_{i_k,i_1,n}]_{z_1}$.

The claim now follows by performing the same procedure in the other variables. ■

Taking $F = G_k$, it follows directly from this proposition and (2.5) that

$$C_k(\mathrm{trace}_{n,q}g) \tag{3.5}$$

$$= \sum_{0\leq i_1,\dots,i_k\leq q-1}\frac{n^{-i_1-\dots-i_k}}{i_1!\cdots i_k!}\int_{\mathbb{C}^k}\mathcal{D}_{i_1,\dots,i_k,n}G_k(z_1,\dots,z_k)K_n(z_1,z_2)\cdots K_n(z_k,z_1)\mathrm{d}\mu_n^k(z_1,\dots,z_k). \tag{3.6}$$

In the pure polyanalytic case, we have the following, slightly simpler version. Using (3.2), we obtain

$$C_k(\mathrm{trace}_{\delta;n,q}g) \tag{3.7}$$

$$= \int_{\mathbb{C}^k}\left(L_{q-1}^0(-n^{-1}\Delta_{z_1})\cdots L_{q-1}^0(-n^{-1}\Delta_{z_k})G_k(z_1,\dots,z_k)\right)K_n(z_1,z_2)\cdots K_n(z_k,z_1)\mathrm{d}\mu_n^k(z_1,\dots,z_k).$$

4 Estimation of the Cumulants

In the previous section, we saw that the cumulants $C_k(\mathrm{trace}_{q,n}g)$ and $C_k(\mathrm{trace}_{\delta;q,n})$ can both be written as a finite series

$$\sum_j n^{-j}\int_{\mathbb{C}^k}H_j(z_1,\dots,z_k)K_n(z_1,z_2)K_n(z_2,z_3)\cdots K_n(z_k,z_1)\mathrm{d}\mu_n^k(z_1,\dots,z_k), \tag{4.1}$$

where the functions H_j are of the form

$$H_j(z_1,\dots,z_k) = \sum_\alpha\prod_{m=1}^k f_{j,\alpha,m}(z_m).$$

Here, the sum over α is finite and the functions $f_{j,\alpha,m}$ are bounded and continuous and each of them is either compactly supported or identically equal to 1. In addition, we know that for each α and j, the former is the case for at least one $f_{j,\alpha,m}$.

In this section, we will show that for every $j \geq 2$, the corresponding term in the sum (4.1) tends to zero as $n \to \infty$. Furthermore, we will show that this is the case also for the $j = 1$ term if H_j vanishes on the diagonal, that is, if $H_j(z, \dots, z) = 0$. These results are then applied in Section 5 to prove Theorem 1.1.

We will start by applying well-known estimates of the Ginibre kernel K_n. In fact, essentially the same estimates are also valid for kernels which are defined with more general than Gaussian weights.

We have the following off-diagonal decay estimate for K_n (Theorem 8.1 in [7]):

$$|K_n(z,w)|^2 \mathrm{e}^{-n\left(|z|^2+|w|^2\right)} \le Cn^2 \mathrm{e}^{-c\sqrt{n}\min\{|z-w|, d(z,\partial\mathbb{D})\}}, \quad z \in \mathbb{D}, w \in \mathbb{C} \tag{4.2}$$

where the constants C and c are absolute constants, in particular independent of n. We will also need (Proposition 3.6 and p. 1541 in [7])

$$|K_n(z,w)|^2 \mathrm{e}^{-n|z|^2 - n|w|^2} \le Cn^2 \mathrm{e}^{-n(Q(z)-\hat{Q}(z)) - n(Q(w)-\hat{Q}(w))}, \tag{4.3}$$

where $Q(z) = |z|^2$ and

$$\hat{Q}(z) = \begin{cases} |z|^2, & |z| \le 1 \\ \log|z|^2 + 1, & |z| \ge 1. \end{cases}$$

According to this estimate, $|K_n(z,w)|^2 \mathrm{e}^{-n|z|^2 - n|w|^2}$ decays quickly to zero as $n \to \infty$ if either z or w is outside the unit disk.

Let

$$\varepsilon_n = \varepsilon_{n,k} = M_k \frac{\log n}{\sqrt{n}},$$

where M_k is some large constant to be specified later. We split $\mathbb{C}^k$ into three different regions Λ_n, Γ_n and Ω_n as follows:

$$\Lambda_n = \{z_1 \in (1-\varepsilon_n)\mathbb{D}, \max_{2\le j\le k}|z_j - z_1| \le \varepsilon_n/2\},$$

$$\Gamma_n = \{d(z_1, \partial\mathbb{D}) \le \varepsilon_n, (z_2, \dots, z_k) \in \mathbb{C}^{k-1}\},$$

$$\Omega_n = \Omega_n^1 \cup \Omega_n^2 = \{z_1 \in (1-\varepsilon_n)\mathbb{D}, |z_j - z_1| \ge \varepsilon_n/2 \text{ for some } j\} \cup \{z_1 \in \mathbb{C}\setminus\mathbb{D}, d(z_1,\partial\mathbb{D}) \ge \varepsilon_n\}.$$

The following estimate on the domain Ω_n is a straigthforward application of the kernel estimates (4.3) and (4.2). We will write

$$R_{k,n}(z_1, \dots, z_k) := K_n(z_1, z_2) \cdots K_n(z_k, z_1).$$

Proposition 4.1. For any $F \in L^\infty(\mathbb{C}^k)$, we have that

$$\int_{\Omega_n} F(z_1,\ldots,z_k)R_{k,n}(z_1,\ldots,z_k)\mathrm{d}\mu_n^k(z_1,\ldots,z_k) = O(n^{-1}),$$

whenever $M_k \geq 2(k+1)/c$, where c is the constant in (4.2). □

Proof. Defining $f(x) = x - \log x - 1$, we can use (4.3) to write

$$\left| \int_{\Omega_n^2} F(z_1,\ldots,z_k)R_{k,n}(z_1,\ldots,z_k)\mathrm{d}\mu_n^k(z_1,\ldots,z_k) \right|$$
$$\leq C^{k/2}n^k\|F\|_\infty \int_{|z|^2\geq(1+\varepsilon_n)^2} \mathrm{e}^{-nf(|z|^2)}\mathrm{d}A(z)\left[\int_{\mathbb{C}} \mathrm{e}^{-n(Q(z)-\hat{Q}(z))}\mathrm{d}A(z)\right]^{k-1}.$$

For $x \geq (1+\varepsilon_n)^2$, we estimate by convexity

$$f(x) \geq f((1+\varepsilon_n)^2) + f'((1+\varepsilon_n)^2)(x-(1+\varepsilon_n)^2) \geq \varepsilon_n^2 + \frac{2\varepsilon_n+\varepsilon_n^2}{(1+\varepsilon_n)^2}[x-(1+\varepsilon_n)^2],$$

so

$$\int_{|z|^2\geq(1+\varepsilon_n)^2} \mathrm{e}^{-nf(|z|^2)}\mathrm{d}A(z) \leq \mathrm{O}(\mathrm{e}^{-n\varepsilon_n^2}) = \mathrm{O}(n^{-M_k^2\log n}).$$

Because

$$\int_{\mathbb{C}} \mathrm{e}^{-n(Q(z)-\hat{Q}(z))}\mathrm{d}A(z) = \mathrm{O}(1),$$

we obtain

$$\int_{\Omega_n^1} F(z_1,\ldots,z_k)R_{k,n}(z_1,\ldots,z_k)d\mu_n^k(z_1,\ldots,z_k) = \mathrm{O}(n^{-\kappa})$$

for any desired $\kappa > 0$.

Proceeding to the bulk term, we note that at each point $(z_1,\ldots,z_k) \in \Omega_n^1$, there is some index j for which $|z_j - z_{j+1}| \geq \frac{\varepsilon_n}{2k}$. Choosing $j_0 \geq 1$ to be the smallest such index, we observe that $d(z_{j_0},\partial\mathbb{D}) \geq \varepsilon_n/2$. Now we use the off-diagonal estimate (4.2), and obtain

$$|K_n(z_{j_0},z_{j_0+1})|\mathrm{e}^{-n(|z_{j_0}|^2+|z_{j_0+1}|^2)/2} \leq Cn\mathrm{e}^{-\frac{1}{4k}cM_k\log n} = Cn^{1-\frac{1}{4k}cM_k}.$$

We thus get

$$|R_{k,n}(z_1,\ldots,z_k)\mathrm{e}^{-n(|z_1|^2+\ldots+|z_k|^2)}| \leq Cn^{k-\frac{1}{4k}cM_k}. \tag{4.4}$$

Because

$$\int_{\mathbb{C}\setminus 2\mathbb{D}} \mathrm{e}^{-n(|z|^2-\log|z|^2-1)}\mathrm{d}A(z) = \mathrm{O}(\mathrm{e}^{-an})$$

for some $a > 0$, we see by using (4.3) that

$$\int_{\Omega_n^1} F(z_1,\ldots,z_k)R_{k,n}(z_1,\ldots,z_k)\mathrm{d}\mu_n^k(z_1,\ldots,z_k)$$
$$= \int_{\Omega_n^1\cap(2\mathbb{D})^k} F(z_1,\ldots,z_k)R_{k,n}(z_1,\ldots,z_k)d\mu_n^k(z_1,\ldots,z_k) + \mathrm{O}(\mathrm{e}^{-bn})$$

for some $b > 0$. Applying this with (4.4) gives

$$\int_{\Omega_n^1} F(z_1,\ldots,z_k)R_{k,n}(z_1,\ldots,z_k)\mathrm{d}\mu_n^k(z_1,\ldots,z_k) = O\left(n^{k-\frac{cM_k}{4k}}\right) = O(n^{-1}),$$

where the last equality follows by the choice $M_k \geq 4k(k+1)/c$. ■

The terms $j \geq 2$ in (4.1) are negligible because of the following proposition.

Proposition 4.2. Assume that $f_1 \in L^1(\mathbb{C})$, and $f_2,\ldots f_k \in L^\infty(\mathbb{C})$. Then

$$\left|\int_{\mathbb{C}^k} \prod_{j=1}^{n} f_j(z_j)R_{k,n}(z_1,\ldots,z_k)\mathrm{d}\mu_n^k(z_1,\ldots,z_k)\right| \leq n\|f_1\|_{L^1(\mathbb{C})}\prod_{j=2}^{k}\|f_j\|_\infty. \qquad \square$$

Proof. We write the integral on the left hand side as

$$\int_{\mathbb{C}^k} \prod_{j=1}^{n} f_j(z_j)R_{k,n}(z_1,\ldots,z_k)\mathrm{d}\mu_n^k(z_1,\ldots,z_k) = \int_{\mathbb{C}} f(z_1)F(z_1)\mathrm{d}A(z_1),$$

where

$$F(z_1) = \int_{\mathbb{C}^{k-1}} \prod_{j=2}^{k} f_j(z_j)K_n(z_1,z_2)\cdots K_n(z_k,z_1)\mathrm{d}\mu_n^{k-1}(z_2,\ldots,z_k)\mathrm{e}^{-n|z_1|^2}$$
$$= P_n[f_2P_n[\ldots P_n[f_kK_{n,z_1}]\ldots]](z_1)\mathrm{e}^{-n|z_1|^2}.$$

Introducing the function

$$F(z_1,z) = P_n[f_2P_n[\ldots P_n[f_kK_{n,z_1}]\ldots]](z)\mathrm{e}^{-n|z|^2},$$

we may write $F(z_1) = F(z_1, z_1)$. Now,

$$|F(z_1, z)| \leq \sqrt{K_n(z,z)}\|F(z_1,\cdot)\|_{L^2(d\mu_n)}\mathrm{e}^{-n|z|^2} \leq \sqrt{K_n(z,z)}\sqrt{K_n(z_1,z_1)}\left[\prod_{j=2}^{k}\|f_j\|_\infty\right]\mathrm{e}^{-n|z|^2}$$

where the first inequality is by Cauchy–Schwarz, and the second follows since projections have norm one. It follows that

$$\left|\int_{\mathbb{C}} f_1(z_1)F(z_1)\mathrm{d}A(z_1)\right| \leq \prod_{j=2}^{k}\|f_j\|_\infty \int_{\mathbb{C}} |f_1(z_1)|K_n(z_1,z_1)\mathrm{e}^{-n|z_1|^2}\mathrm{d}A(z_1),$$

which by the trivial estimate $K_n(z,z)\mathrm{e}^{-n|z|^2} \leq n$ proves the assertion. ■

A similar statement can be found in [17] where an L^1 off-diagonal decay assumption is used. Our result does not use such an assumption. For simplicity, we have formulated the result only for the Ginibre kernel K_n but the same proof strategy works in greater generality.

The following lemma deals with the term corresponding to $j = 1$ in (4.1).

Lemma 4.3. Let $F(z_1, \ldots, z_k)$ function of the form

$$F(z_1,\ldots,z_k) = \sum_{\alpha=1}^{M}\prod_{j\leq k} f_{\alpha,j}(z_j), \quad M \geq 1$$

where $f_{\alpha,j} \in C^2(\mathbb{C})$. Assume furthermore that

$$F(z, z, \ldots, z) = 0, \quad z \in \mathbb{C}.$$

Then

$$n^{-1}\int_{\mathbb{C}^k} F(z_1,\ldots,z_k)R_{k,n}(z_1,\ldots,z_k)\mathrm{d}\mu_n^k(z_1,\ldots,z_k) = o(1). \qquad \square$$

Proof. We split the integral over $\mathbb{C}^k$ as

$$\int_{\mathbb{C}^k} FR_{n,k}\mathrm{d}\mu_n^k = \int_{\Lambda_n} FR_{n,k}\mathrm{d}\mu_n^k + \int_{\Gamma_n} FR_{n,k}\mathrm{d}\mu_n^k + O(n^{-1}).$$

Turning to the integral over Γ_n, we may assume without loss of generality that $F(z_1,\ldots,z_k) = f_1(z_1)\cdots f_k(z_k)$ for some bounded continuous functions f_j, since vanishing of F on the diagonal will not be used in this case. It follows immediately from

Proposition 4.2 that

$$\int_{\Gamma_n} FR_{k,n}\mathrm{d}\mu_n = \int_{\mathbb{C}^k} f_1(z_1)\,1_{\{\mathrm{dist}(z_1,\partial\mathbb{D})\le M_k\log n/\sqrt{n}\}}(z_1)\prod_{j=2}^{k} f_j(z_j)R_{k,n}(z_1,\dots,z_k)\mathrm{d}\mu_n^k(z_1,\dots,z_k)$$
$$\le O\big(n\|f_1 1_{\{\mathrm{dist}(z,\partial\mathbb{D})\le M_k\log n/\sqrt{n}\}}\|_{L^1}\big) = O(\sqrt{n}\log n).$$

For the integral over Λ_n, we estimate F near Λ_n by a Taylor expansion, and obtain

$$\delta_n := \|F_{|\Lambda_n}\|_\infty = (k-1)\max_{2\le j\le k}(\|\partial_{z_j}F\|_{L^\infty(\Lambda_n)} + \|\bar\partial_{z_j}F\|_{L^\infty(\Lambda_n)})\frac{\varepsilon_n}{2} + O(\varepsilon_n^2) = O(\varepsilon_n).$$

The crude estimate $|R_{k,n}|\mathrm{e}^{-n|z_1|^2-\dots-n|z_k|^2} \le n^k$ and $Vol(\Lambda_n) = O(\varepsilon_n^{2(k-1)})$, where Vol denotes the volume in $\mathbb{C}^k$, then gives

$$\int_{\Lambda_n} |F|R_{k,n}\mathrm{d}\mu_n^k \le \int_{\Lambda_n} C\delta_n n^k \mathrm{d}A^k = O\left(\delta_n n(M_k\log n)^{2k-2}\right) = O\left(n\frac{(M_k\log n)^{2k-1}}{\sqrt{n}}\right) = o(n),$$

which completes the proof. ■

5 Proof of the Main Theorem

Proof of Theorem 1.1. We will start by proving (1.2). By (3.7), we know that $C_k(\mathrm{trace}_{\delta;n,q}g)$ can be written as a finite series of the form (4.1). By Proposition 4.2, it is enough pick the terms corresponding to the powers n^0 and n^{-1} from the expression

$$L^0_{q-1}(n^{-1}\Delta_{z_1})\cdots L^0_{q-1}(n^{-1}\Delta_{z_k})G_k(z_1,\dots,z_k).$$

We get

$$C_k(\mathrm{trace}_{\delta,n,q}g) \tag{5.1}$$
$$= \int_{\mathbb{C}^k}\left(I + n^{-1}(q-1)\sum_{j=1}^{k}\Delta_{z_j}\right)G_k(z_1,\dots,z_k)R_{k,n}(z_1,\dots,z_k)d\mu_n^k(z_1,\dots,z_k) + O(n^{-1}).$$

Suppose first that $k\ge 3$. The term corresponding to the identity operator I in this formula is just the cumulant of order k in the analytic case $q=1$. By the result of Rider and Virág [16], this expression vanishes as $n\to\infty$. For the n^{-1} term, it was shown in Lemma 3.3 of [6] that $\sum_{j=1}^{k}\Delta_{z_j}G_k(z_1,\dots,z_k)$ vanishes on the diagonal $z_1=\dots=z_k$. The claim now follows by Lemma 4.3.

It remains to calculate the second cumulant, which is equal to the variance. We use (5.1) again. Recall that $G_2(z,w) = \frac{1}{2}(f(z) - f(w))^2$. By the result of [16], we have

$$\int_{\mathbb{C}^k} G_2(z_1,z_2)R_{2,n}(z_1,z_2)\mathrm{d}\mu_n^k(z_1,z_2) \to \|\bar\partial g\|^2_{L^2(\mathbb{D})} + \frac{1}{2}\|g\|^2_{H^{1/2}(\mathbb{T})}. \tag{5.2}$$

To analyse the terms involving the Laplacians in (5.1), we calculate

$$(\Delta_z + \Delta_w)(g(z) - g(w))^2 = 2(\Delta g(z) - \Delta g(w))(g(z) - g(w)) + 2|\bar\partial g(z)|^2 + 2|\bar\partial g(w)|^2.$$

We now observe that $2(\Delta g(z) - \Delta g(w))(g(z) - g(w))$ vanishes when $z = w$. With Lemma 4.3 we now have

$$\begin{aligned} &C_k(\mathrm{trace}_{\delta;n,q} g) \\ &= \|\bar\partial g\|^2_{L^2(\mathbb{D})} + \frac{1}{2}\|g\|^2_{H^{1/2}(\mathbb{T})} + \frac{1}{2n}(q-1)\int_{\mathbb{C}^2}(2|\bar\partial g(z_1)|^2 + 2|\bar\partial g(z_2)|^2|K_n(z_1,z_2)|^2\mathrm{d}\mu_n^2(z_1,z_2) + \mathrm{o}(1) \\ &= \|\bar\partial g\|^2_{L^2(\mathbb{D})} + \frac{1}{2}\|g\|^2_{H^{1/2}(\mathbb{T})} + \frac{2(q-1)}{n}\int_{\mathbb{C}^2}|\bar\partial g(z_1)|^2|K_n(z_1,z_2)|^2\mathrm{d}\mu_n^2(z_1,z_2) + \mathrm{o}(1) \\ &= \|\bar\partial g\|^2_{L^2(\mathbb{D})} + \frac{1}{2}\|g\|^2_{H^{1/2}(\mathbb{T})} + \frac{2(q-1)}{n}\int_{\mathbb{C}}|\bar\partial g(z)|^2 K_n(z,z)\mathrm{d}\mu_n(z) + \mathrm{o}(1) \\ &\to (2q-1)\int_{\mathbb{D}}|\bar\partial g|^2\mathrm{d}A + \frac{1}{2}\|g\|^2_{H^{1/2}(\mathbb{T})}. \end{aligned} \tag{5.3}$$

To obtain the last limit, we used that $\frac{1}{n}K_n(z,z)\mathrm{d}\mu_n(z) \to 1_{\mathbb{D}}\mathrm{d}A(z)$ weakly.

We will now turn to the proof of (1.3). We have

$$\begin{aligned} &C_k(\mathrm{trace}_{q,n} g) \\ &= \sum_{1\le i_1,\cdots,i_k\le q-1} \frac{n^{-i_1-\ldots-i_k}}{i_1!\cdots i_k} \int_{\mathbb{C}^k} G_k(z_1,\ldots,z_k)[T_{n,1}]^{i_1}_{z_1}\overline{[T_{n,1}]^{i_1}_{z_2}}K_n(z_1,z_2)\cdots[T_{n,1}]^{i_k}_{z_k}\overline{[T_{n,1}]^{i_k}_{z_1}}K_n(z_k,z_1)\mathrm{d}\mu_n^k. \end{aligned}$$

As with pure polyanalytic kernels, we assume first that $k \ge 3$. Let us now fix a multi-index $(i_1,\ldots,i_k)$ and estimate the corresponding term in the previous expression. By Proposition 3.1, we get

$$\begin{aligned} &\frac{n^{-i_1-\ldots-i_k}}{i_1!\cdots i_k!}\int_{\mathbb{C}^k} G_k(z_1,\ldots,z_k)[T_{n,1}]^{i_1}_{z_1}\overline{[T_{n,1}]^{i_1}_{z_2}}K_n(z_1,z_2)\cdots[T_{n,1}]^{i_k}_{z_k}\overline{[T_{n,1}]^{i_k}_{z_1}}K_n(z_k,z_1)\mathrm{d}\mu_n^k \\ &\quad= \frac{n^{-i_1-\ldots-i_k}}{i_1!\cdots i_k!}\int_{\mathbb{C}^k}\mathcal{D}_{i_1,\ldots,i_k,n}G_k(z_1,\ldots,z_k)K_n(z_1,z_2)\cdots K_n(z_k,z_1)\mathrm{d}\mu_n^k(z_1,\ldots,z_k). \end{aligned}$$

We have

$$n^{-i_1-\ldots-i_k}\mathcal{D}_{i_1,\ldots,i_k,n}G_k(z_1,\ldots,z_k) = \mathrm{O}(n^{I(i_1,\ldots,i_k)}),$$

where

$$I(i_1,\dots,i_k) := -i_1 - \dots - i_k + \min(i_1,i_k) + \min(i_1,i_2) + \dots + \min(i_{k-1},i_k).$$

Clearly, $I(i_1,\dots,i_k) \le 0$. We have that $I(i_1,\dots,i_k) = 0$ if and only if $i_1 = \dots = i_k$, and this case is treated above. The terms where $I(i_1,\dots,i_k) \le -2$ are negligible by proposition 4.2, so the only case that needs further analysis is $I(i_1,\dots,i_k) = -1$. This happens exactly when there exists a unique $l \in \{1,2,\dots,k\}$ such that $i_l < i_{l+1}$ and for this l it holds $i_l = i_{l+1} - 1$. Here and later, we identify i_{k+j} with i_j. This condition on the multi-index $(i_1,\dots,i_k)$ can be rephrased as follows: $(i_1,\dots,i_k) = \mathbf{d}(j,r,i)$ for some $1 \le j \le k$, $0 \le r \le k-2$ and $1 \le i \le q-1$, where $\mathbf{d}(j,r,i)$ is defined to be the multi-index $(i_1,\dots,i_k)$ which satisfies $i_m = i$ for $m \in \{j,\dots,j+r\}$ and $i_m = i-1$ for the remaining indices i_m. If $(i_1,\dots,i_k) = \mathbf{d}(j,r,i)$, we have

$$\begin{aligned} &\frac{n^{-i_1-\dots-i_k}}{i_1!\cdots i_k!}\mathcal{D}_{i_1,\dots,i_k,n} G_k(z_1,\dots,z_k) \qquad (5.4)\\ &= \frac{n^{-1} i_1!\cdots i_{j+r}!(i_{j+r+1}+1)!\cdots i_k!}{i_1!\cdots i_k} \partial_{z_j}\bar\partial_{z_{j+r+1}} G_k(z_1,\dots,z_k) + \mathrm{O}(n^{-2})\\ &= n^{-1} i \partial_{z_j}\bar\partial_{z_{j+r+1}} G_k(z_1,\dots,z_k) + \mathrm{O}(n^{-2}), \end{aligned}$$

where we collected from each operator $\mathcal{D}_{i_j,i_{j+1},n}$ the term which has the highest degree in n. The $\mathrm{O}(n^{-2})$ are negligible by Proposition 4.2. Summing up the contributions from all such $(i_1,\dots,i_k)$, we can write the k'th cumulant as

$$C_k(\mathrm{trace}_{n,q} g) \qquad (5.5)$$

$$= \sum_{j=1}^{q} \int_{\mathbb{C}^k} G_k(z_1,\dots,z_k) K_{\delta;n,j}(z_1,z_2)\cdots K_{\delta;j,n}(z_k,z_1)\mathrm{d}\mu_n^k(z_1,\dots,z_k) \qquad (5.6)$$

$$+ \int_{\mathbb{C}^k} \Big[\sum_{j=1}^{k}\sum_{r=0}^{k-2}\sum_{i=1}^{q-1} i\partial_j\bar\partial_{j+r+1}\Big] G_k(z_1,\dots,z_k) R_{k,n}(z_1,\dots,z_k) d\mu_n^k(z_1,\dots,z_k) + \mathrm{O}(n^{-1}) \qquad (5.7)$$

$$= \sum_{j=1}^{q} \int_{\mathbb{C}^k} G_k(z_1,\dots,z_k) K_{\delta;n,j}(z_1,z_2)\cdots K_{\delta;j,n}(z_k,z_1)\mathrm{d}\mu_n^k(z_1,\dots,z_k) \qquad (5.8)$$

$$+ \frac{(q-1)q}{2} \int_{\mathbb{C}^k} \sum_{j\neq l} \partial_j\bar\partial_l G_k(z_1,\dots,z_k)\big] R_{k,n}(z_1,\dots,z_k)\mathrm{d}\mu_n^k(z_1,\dots,z_k) + \mathrm{O}(n^{-1}). \qquad (5.9)$$

By Lemma 3.4 in [6], $\sum_{l\neq j} \partial_j\bar\partial_l G_k(z_1,\dots,z_k)$ vanishes on $z_1 = \dots = z_k$. This together with Lemma 4.3 and the result in the pure polyanalytic case proves that the cumulants of orders $k \ge 3$ vanish in the limit.

Next, we calculate the limiting variance. Suppose $0 \le i_1 \le i_2 \le q-1$. Again, we only need to analyse the case $I(i_1, i_2) = -1$, and this happens exactly when $i_2 = i_1 + 1$. Now, using (5.4) again,

$$\begin{aligned}
&\frac{n^{-2i_1-1}}{i_1!(i_1+1)!}\int_{\mathbb{C}^2} G_2(z_1,z_2)[T_n]_{z_1}^{i_1}\overline{[T_n]_{z_2}^{i_1}}K_n(z_1,z_2)[T_n]_{z_2}^{i_1+1}\overline{[T_n]_{z_1}^{i_1+1}}K_n(z_2,z_1)\mathrm{d}\mu_n^2(z_1,z_2)\\
&= n^{-1}(i_1+1)\int_{\mathbb{C}^2}\bar{\partial}_{z_1}\partial_{z_2}G_2(z_1,z_2)K_n(z_1,z_2)K_n(z_2,z_1)\mathrm{d}\mu_n^2(z_1,z_2)+\mathrm{O}(n^{-1})\\
&= -n^{-1}(i_1+1)\int_{\mathbb{C}^2}\bar{\partial}_{z_1}g(z_1)\partial_{z_2}g(z_2)|K_n(z_1,z_2)|^2\mathrm{d}\mu^2(z_1,z_2)+\mathrm{O}(n^{-1})\\
&= -n^{-1}(i_1+1)\int_{\mathbb{C}^2}\left[\bar{\partial}_{z_1}g(z_1)(\partial_{z_2}g(z_2)-\partial_{z_1}g(z_1))+|\bar{\partial}_{z_1}g(z_1)|^2\right]K_n(z_1,z_2)|^2\mathrm{d}\mu_n^2(z_1,z_2)+\mathrm{O}(n^{-1})\\
&= -(i_1+1)\int_{\mathbb{D}}|\bar{\partial}g(z)|^2K_n(z,z)\mathrm{d}\mu_n(z)+\mathrm{o}(1)\to-(i_1+1)\int_{\mathbb{D}}|\bar{\partial}g(z)|^2\mathrm{d}A(z).
\end{aligned}$$

Here, the second last equality followed from Lemma 4.3 and integrating out the second variable. For the last limit, we used again the weak convergence $K_n(z,z)\mathrm{d}\mu_n(z) \to 1_{\mathbb{D}}dA(z)$. Summing up all such contributions gives

$$\begin{aligned}
&\int_{\mathbb{C}^2} G_2(z_1,z_2)|K_{n,q}(z_1,z_2)|^2\mathrm{d}\mu_n^2(z_1,z_2)\\
&\quad= \int_{\mathbb{C}^2} G_2(z_1,z_2)\left[\sum_{j=1}^{q}|K_{\delta;n,j}(z_1,z_2)|^2+2\mathrm{Re}\sum_{1\le j<l\le q}K_{\delta;n,j}(z_1,z_2)K_{\delta;n,l}(z_2,z_1)\right]\mathrm{d}\mu_n^2(z_1,z_2)\\
&\quad\to \sum_{j=1}^{q}(2j-1)\|\bar{\partial}g\|_{L^2(\mathbb{D})}^2+\frac{q}{2}\|g\|_{H^{1/2}(\partial\mathbb{D})}^2-2\sum_{j=0}^{q-2}(j+1)\|\bar{\partial}g\|_{L^2(\mathbb{D})}^2 = q(\|\bar{\partial}g\|_{L^2(\mathbb{D})}^2+\frac{1}{2}\|g\|_{H^{1/2}(\partial\mathbb{D})}^2).
\end{aligned}$$

Here, the limit follows from the variance computation in the pure polyanalytic case (5.3) and (5.5), using that the terms where $l - j > 1$ in the second sum of the second line are negligible. ■

Funding

A.H. was supported by Lise Meitner grant of Austrian Science Fund (FWF), Grant no. M1821 and A.W. was supported by Swedish Research Council Grant no. 2012-3122 and by foundations managed by the Royal Swedish Academy of Sciences.

Acknowledgements

We would like to thank Joaquim Ortega-Cerdà for sharing the code which was used to produce the simulations in Figure 1 and Seung-Yeop Lee for interesting remarks concerning the main theorem. We would also like thank the referee for many helpful suggestions.

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Paper E

Discrepancy densities for planar and hyperbolic zero packing
J. Funct. Anal **272** (2017) 5284-5306.

Journal of Functional Analysis 272 (2017) 5282–5306

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Discrepancy densities for planar and hyperbolic zero packing

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ARTICLE INFO

Article history:
Received 10 November 2016
Accepted 30 January 2017
Available online 16 February 2017
Communicated by K. Seip

MSC:
primary 30C62
secondary 30C70, 30H20

Keywords:
Geometric zero packing
$\bar\partial$-Estimates
Asymptotic variance

ABSTRACT

We study the problem of *geometric zero packing*, recently introduced by Hedenmalm [7]. There are two natural densities associated with this problem: the *discrepancy density* $\rho_{\mathbb{H}}$, given by

$$\rho_{\mathbb{H}} = \liminf_{r\to 1^-} \inf_f \frac{\int_{\mathbb{D}(0,r)} \left((1-|z|^2)|f(z)|-1\right)^2 \frac{dA(z)}{1-|z|^2}}{\int_{\mathbb{D}(0,r)} \frac{dA(z)}{1-|z|^2}}$$

which measures the discrepancy in optimal approximation of $(1-|z|^2)^{-1}$ with the modulus of polynomials f, and its relative, the *tight discrepancy density* $\rho^*_{\mathbb{H}}$, which will trivially satisfy $\rho_{\mathbb{H}} \le \rho^*_{\mathbb{H}}$. These densities have deep connections to the boundary behaviour of conformal mappings with k-quasiconformal extensions, which can be seen from Hedenmalm's result that the universal asymptotic variance Σ^2 is related to $\rho^*_{\mathbb{H}}$ by $\Sigma^2 = 1-\rho^*_{\mathbb{H}}$. Here we prove that in fact $\rho_{\mathbb{H}} = \rho^*_{\mathbb{H}}$, resolving a conjecture by Hedenmalm in the positive. The natural planar analogues $\rho_{\mathbb{C}}$ and $\rho^*_{\mathbb{C}}$ to these densities make contact with work of Abrikosov on Bose–Einstein condensates. As a second result we prove that also $\rho_{\mathbb{C}} = \rho^*_{\mathbb{C}}$. The methods are based on Ameur, Hedenmalm and Makarov's Hörmander-type $\bar\partial$-estimates with polynomial growth control [2]. As a consequence we obtain

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[1] The author was supported by the Swedish Research Council, Grant no. 2012-3122, and by foundations managed by the Royal Swedish Academy of Sciences.

http://dx.doi.org/10.1016/j.jfa.2017.01.022

sufficiency results on the degrees of approximately optimal polynomials.

1. Introduction

1.1. Hyperbolic discrepancy densities

Let $0 < r < 1$ and let f be a holomorphic function defined on the unit disk $\mathbb{D}$. We shall be concerned with the *hyperbolic discrepancy function* $\Phi_f(z,r)$, defined by

$$\Phi_f(z,r) = \big((1-|z|^2)|f(z)| - 1_{\mathbb{D}(0,r)}(z)\big)^2, \qquad z \in \mathbb{D}.$$

The intuition is that Φ_f measures the discrepancy between f and the hyperbolic metric $\vartheta(z) = (1-|z|^2)^{-1}$. Since f is holomorphic, $\Delta \log|f(z)|$ is, considered as a distribution, a sum of point masses, while $\Delta \log \vartheta(z)$ is a smooth positive density. This constitutes a clear obstruction to obtain a perfect approximation with holomorphic f. The term *zero packing*, introduced by Hedenmalm [7], comes from the realization that this problem can be phrased in terms of optimally discretizing the smooth positive mass $\Delta \log \vartheta$ as a sum of point masses – corresponding to the zeros of the holomorphic function f.

Our main interest lies in the *hyperbolic discrepancy density* $\rho_{\mathbb{H}}$, and in a related object called the *tight hyperbolic discrepancy density* $\rho^*_{\mathbb{H}}$. Without further delay we proceed to define these. For polynomials f we consider the functionals

$$\rho_{\mathbb{H},r}(f) = \frac{\int_{\mathbb{D}(0,r)} \Phi_f(z,r)\frac{dA(z)}{1-|z|^2}}{\int_{\mathbb{D}(0,r)} \frac{dA(z)}{1-|z|^2}} = \frac{\int_{\mathbb{D}(0,r)} \Phi_f(z,r)\frac{dA(z)}{1-|z|^2}}{\log\frac{1}{1-r^2}}$$

and

$$\rho^*_{\mathbb{H},r}(f) = \frac{\int_{\mathbb{D}} \Phi_f(z,r)\frac{dA(z)}{1-|z|^2}}{\int_{\mathbb{D}(0,r)} \frac{dA(z)}{1-|z|^2}} = \frac{\int_{\mathbb{D}} \Phi_f(z,r)\frac{dA(z)}{1-|z|^2}}{\log\frac{1}{1-r^2}}.$$

In terms of these, the two densities are obtained as

$$\rho_{\mathbb{H}} = \liminf_{r\to 1^-} \inf_f \rho_{\mathbb{H},r}(f), \tag{1.1}$$

and

$$\rho^*_{\mathbb{H}} = \liminf_{r\to 1^-} \inf_f \rho^*_{\mathbb{H},r}(f), \tag{1.2}$$

where in both cases the infimum is taken over the set of all polynomials $\mathrm{Pol}(\mathbb{C})$.

Remark 1.1. Let $A^2_{1,r}$ denote the standard weighted Bergman space on $\mathbb{D}(0,r)$, i.e. the intersection of the set $A(\mathbb{D}(0,r))$ of analytic functions on $\mathbb{D}(0,r)$ with $L^2(\mathbb{D}(0,r),(1-|z|^2)dA(z))$. We may equally well take the above infimums in the definitions of $\rho_{\mathbb{H}}$ and $\rho^*_{\mathbb{H}}$ over $A^2_{1,r}$ and $A^2_{1,1}$, respectively. This follows since polynomials are dense in both spaces, and since the functionals $\rho_{\mathbb{H},r}$ and $\rho^*_{\mathbb{H},r}$ are continuous with respect to the norms $\|\cdot\|_{A^2_{1,r}}$ and $\|\cdot\|_{A^2_{1,1}}$, respectively.

As a side note, observe that $A^2_{1,r}$ coincides with the corresponding unweighted space $A^2(\mathbb{D}(0,r))$ as sets for $0<r<1$, since the weight has a strictly positive minimum.

The exact values of these are unknown. The only available quantitative result is due to Hedenmalm, which in particular shows that the indicated obstacle to perfect approximation is real, in the sense that $\rho_{\mathbb{H}}>0$.

Theorem 1.2 *(Hedenmalm, [7]). The hyperbolic discrepancy densities enjoy the estimate*

$$2\times 10^{-8}\le \rho_{\mathbb{H}}\le \rho^*_{\mathbb{H}}\le 0.12087.$$

For an illustration of the importance of this theorem, in particular of the property that $\rho^*_{\mathbb{H}}>0$, see Subsection 1.2.

That the densities satisfy the inequality $\rho_{\mathbb{H}}\le\rho^*_{\mathbb{H}}$ is immediate. The density $\rho^*_{\mathbb{H}}$ differs from $\rho_{\mathbb{H}}$ in that it adds an L^2-punishment near the boundary:

$$\rho^*_{\mathbb{H},r}(f)=\rho_{\mathbb{H},r}(f)+\frac{1}{\log\frac{1}{1-r^2}}\int\limits_{\mathbb{D}\setminus\mathbb{D}(0,r)}|f(z)|^2(1-|z|^2)dA(z),$$

and in light of this, it is clear that $\rho_{\mathbb{H}}\le\rho^*_{\mathbb{H}}$. Hedenmalm has conjectured that equality holds, which is what our main theorem concerns.

Theorem 1.3. *It holds that $\rho_{\mathbb{H}}=\rho^*_{\mathbb{H}}$.*

In the process we obtain the following corollary, which gives a sufficiency result regarding the degree of approximately optimal polynomials. We let $\lceil x\rceil$ denote the smallest integer n with $n\ge x$.

Corollary 1.4. *The densities $\rho_{\mathbb{H}}$ and $\rho^*_{\mathbb{H}}$ may as well be calculated as*

$$\rho_{\mathbb{H}}=\liminf_{r\to 1^-}\inf_{f\in\mathrm{Pol}_{n(r)}}\rho_{\mathbb{H},r}(f),\qquad \rho^*_{\mathbb{H}}=\liminf_{r\to 1^-}\inf_{f\in\mathrm{Pol}_{n(r)}}\rho^*_{\mathbb{H},r}(f),$$

where $n(r)=\left\lceil\frac{r^2}{1-r^2}\right\rceil$.

Note that $r^2/(1-r^2)$ is the hyperbolic area of the disc $\mathbb{D}(0,r)$. Ideally, one would want to show that the $n(r)$ zeros of approximating polynomials are uniformly spread out with respect to the hyperbolic metric. This, however, remains out of reach at present.

The proof of Theorem 1.3 follows the route suggested by Hedenmalm in [7]. We employ the machinery of Hörmander-type $\bar{\partial}$-estimates with polynomial growth control developed by Ameur, Hedenmalm and Makarov in [2], and an array of variational arguments. The difficulty is to control the size of minimizers of $\rho_{\mathbb{H},r}(f)$ near the boundary. The key ingredient in the solution to this problem is an L^2-non-concentration estimate (see Theorem 4.5), which asserts that for minimizers f we have an estimate

$$\int_{\mathbb{A}((1-\delta)r,r)} |f(z)|^2(1-|z|^2)\,dA(z) = o(1),$$

along certain sequences of radii $r \to 1^-$.

1.2. Quasiconformal mappings: the integral means spectrum and quasicircles

The number $\rho^*_{\mathbb{H}}$ has turned out to play a significant role in the theory of quasiconformal mappings, due to its relation to the universal asymptotic variance Σ^2. The number Σ^2 was introduced in [4], and is defined in terms of McMullen's asymptotic variance [11]

$$\sigma^2(g) = \limsup_{r\to 1^-} \frac{\int_{\partial\mathbb{D}}|g(r\zeta)|^2 d\sigma(\zeta)}{\log\frac{1}{1-r^2}},$$

by

$$\Sigma^2 = \sup\{\sigma^2(g) : g = \mathbf{P}\mu,\ \|\mu\|_{L^\infty(\mathbb{D})} = 1\}$$

where $\mathbf{P}$ denotes the Bergman projection

$$\mathbf{P}f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-z\bar{w})^2}dA(w), \qquad f \in L^1(\mathbb{D}).$$

For details we refer to e.g. [4,8,7]. Here we mention a couple of recent developments: A well-known conjecture by Prause and Smirnov [13] (see also [9]) for the quasiconformal integral means spectrum $B(k,t)$ stated that

$$B(k,t) = \begin{cases} \frac{1}{4}k^2|t|^2, & |t| \le \frac{2}{k} \\ k|t|-1, & |t| > \frac{2}{k}. \end{cases}$$

Ivrii recently proved [8] that $B(k,t)$ satisfies $B(k,t) \sim \frac{1}{4}\Sigma^2 k^2|t|^2$ in the sense that

$$\lim_{k\to 0^+}\lim_{t\to 0}\frac{B(k,t)}{k^2|t|^2} = \frac{\Sigma^2}{4}$$

In [4], Astala, Ivrii, Perälä and Prause obtained the bounds $0.879 \le \Sigma^2 \le 1$, and it was conjectured that $\Sigma^2 = 1$, which would be implied by the above conjecture. However, in addition to Theorem 1.2, Hedenmalm [7] has recently proven that

$$\Sigma^2 = 1 - \rho^*_{\mathbb{H}}, \tag{1.3}$$

and taken together, these facts refute the conjecture.

The same family of objects is also relevant to work by Ivrii on the dimension of k-quasicircles: If $D(k)$ is the maximal Hausdorff dimension of a k-quasicircle, a theorem of Smirnov (see the book [5] for an exposition) says that

$$D(k) \le 1 + k^2, \qquad 0 \le k < 1. \tag{1.4}$$

Astala conjectured this result [3], and furthermore suggested that this bound is sharp. Ivrii [8] proved that

$$D(k) = 1 + \Sigma^2 k^2 + O(k^{8/3-\epsilon}), \tag{1.5}$$

which together with Theorem 1.2 and the identity (1.3) effectively disproves the latter part of the conjecture.

1.3. The planar discrepancy densities

We are also interested in planar analogues of the densities $\rho_{\mathbb{H}}$ and $\rho^*_{\mathbb{H}}$. For $R > 0$ and an entire function $f(z)$, we consider the *planar discrepancy function*

$$\Psi_f(z,R) = \left(|f(z)|e^{-|z|^2} - 1_{\mathbb{D}(0,R)}(z)\right)^2,$$

and set

$$\rho_{\mathbb{C}} = \liminf_{R\to\infty} \inf_f \frac{\int_{\mathbb{D}(0,R)} \Psi_f(z,R)dA(z)}{\int_{\mathbb{D}(0,R)} dA(z)} = \liminf_{R\to\infty} \inf_f \frac{1}{R^2} \int\limits_{\mathbb{D}(0,R)} \Psi_f(z,R)dA(z), \tag{1.6}$$

and correspondingly

$$\rho^*_{\mathbb{C}} = \liminf_{R\to\infty} \inf_f \frac{\int_{\mathbb{C}} \Psi_f(z,R)dA(z)}{\int_{\mathbb{D}(0,R)} dA(z)} = \liminf_{R\to\infty} \inf_f \frac{1}{R^2} \int\limits_{\mathbb{C}} \Psi_f(z,R)dA(z), \tag{1.7}$$

where the infimum is taken over all polynomials. The next result corresponds completely to Theorem 1.3.

Theorem 1.5. *It holds that* $\rho_{\mathbb{C}} = \rho^*_{\mathbb{C}}$.

Also analogously to the hyperbolic setting, we may say something about the degree of approximately minimal polynomials.

Corollary 1.6. *The densities $\rho_{\mathbb{C}}$ and $\rho^*_{\mathbb{C}}$ are unchanged if the infimum in (1.6) and (1.7) are taken over $\mathrm{Pol}_{n(R)}(\mathbb{C})$ instead of over $\mathrm{Pol}(\mathbb{C})$, where*

$$n(R) = \lceil 2R^2 \rceil .$$

It thus suffices to consider polynomials of degrees that are essentially proportional to the area of the disk $\mathbb{D}(0, R)$. Here the factor 2 is natural, since each zero carries a mass of $1/2$.

By a change of variables, we may perform the calculation

$$\frac{1}{R^2} \int\limits_{\mathbb{D}(0,R)} \Psi_f(z, R) dA(z) = \int\limits_{\mathbb{D}} \left(|f(Rw)| e^{-R^2|w|^2} - 1 \right)^2 dA(w).$$

Since the dilation $f \mapsto f_R$ where $f_R(z) = f(Rz)$ will not affect holomorphicity of f, we may as well use the functionals

$$\rho_{\mathbb{C},\gamma}(f) = \int\limits_{\mathbb{D}} \left(|f(z)| e^{-\gamma|z|^2} - 1 \right)^2 dA(z), \tag{1.8}$$

$$\rho^*_{\mathbb{C},\gamma}(f) = \int\limits_{\mathbb{C}} \left(|f(z)| e^{-\gamma|z|^2} - 1_{\mathbb{D}}(z) \right)^2 dA(z), \tag{1.9}$$

and instead obtain the densities by

$$\rho_{\mathbb{C}} = \liminf_{\gamma \to \infty} \inf_f \rho_{\mathbb{C},\gamma}(f), \qquad \rho^*_{\mathbb{C}} = \liminf_{\gamma \to \infty} \inf_f \rho^*_{\mathbb{C},\gamma}(f).$$

For the purpose of this paper, it turns out to be more convenient to work with this formulation.

Theorem 1.5 could be seen as a toy problem for Theorem 1.3, in that the $\bar{\partial}$-estimates are slightly more readily applicable in this setting. Since the proof of this illustrates the methods used very transparently, we present it first.

1.4. Relation to Bose–Einstein condensates

The planar density $\rho_{\mathbb{C}}$ is part of a bigger family of densities, $\rho^\beta_{\mathbb{C}}$, defined for $\beta > 0$ by

$$\rho^\beta_{\mathbb{C}} = \liminf_{R \to \infty} \inf_f \rho^\beta_{\mathbb{C},R}(f),$$

where

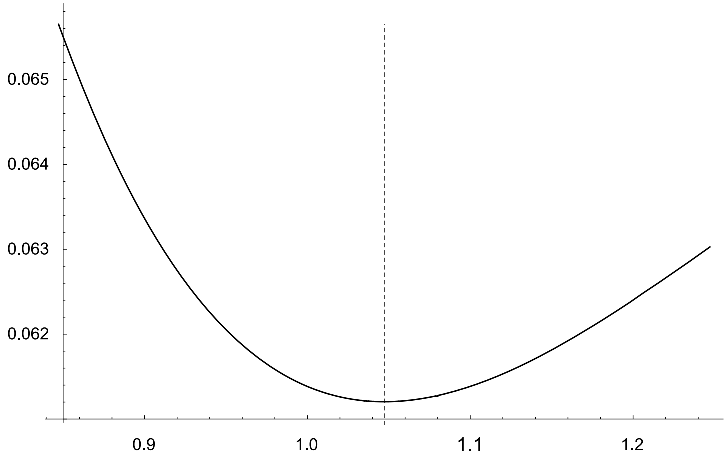

Fig. 1. $\rho^1_{\mathbb{C}}(f_{0,\theta})$ for θ near $\pi/3$ (dashed line), plotted using MATHEMATICA

$$\rho^{\beta}_{\mathbb{C},R}(f) = \frac{1}{R^2} \int_{\mathbb{D}(0,R)} \left(|f(z)|^{\beta} e^{-|z|^2} - 1\right)^2 dA(z).$$

The case $\beta = 1$ is the density $\rho_{\mathbb{C}}$. Also of particular interest is the case $\beta = 2$, which can be traced back to work by Abrikosov [1] on Bose–Einstein Condensates. Abrikosov suggested that it should be enough to look for minimizers among functions that are quasiperiodic with respect to lattices. The conjecture, which is attributed to Abrikosov in [7], is that the equilateral triangular lattice should be the correct choice for any β.

Consider the triangular lattices $2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z}$, where $\omega_1 \in \mathbb{R}^+$ and $\omega_2 = \omega_1 e^{i\theta}$. For each lattice, good candidates $f_0 = f_{0,\beta,\theta}$ for minimizers of $\rho^{\langle\beta\rangle}_{\mathbb{C}}(f)$ are given explicitly in terms of Weierstrass' σ-function, see [7]. A numerical computation using this choice yields the value

$$\rho 1_{\mathbb{C}}(f_0) = \lim_{R\to\infty} \frac{1}{R^2} \int_{\mathbb{D}(0,R)} \left(|f_0(z)| e^{-|z|^2} - 1\right)^2 dA(z) = 0.061203\ldots,$$

which in particular gives a numerical bound $\rho_{\mathbb{C}} \le 0.061203$.

In Fig. 1, we have plotted $\rho^{\langle 1\rangle}_{\mathbb{C}}(f_{0,\theta})$ for triangular lattices with different angles. The minimum appears to be at $\theta = \pi/3$, in support of the conjecture that the equilateral triangular lattice is optimal.

1.5. Notation and special conventions

By $\mathbb{D}(z_0, r)$ we mean the open disk centred at $z_0 \in \mathbb{C}$ with radius $r > 0$, and by $\mathbb{A}(z_0, r, R)$ we mean an open annulus $\mathbb{D}(z_0, R) \setminus \overline{\mathbb{D}(z_0, r)}$, where $R > r > 0$. When $z_0 = 0$ we simply denote the annulus by $\mathbb{A}(r, R)$.

By dA we mean the normalized area measure,

$$dA(z) = \frac{dxdy}{\pi}, \qquad z = x + iy \in \mathbb{C}.$$

We shall make frequent use of the Cauchy–Riemann operators

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

We use the Laplacian Δ, which is a quarter of the usual Laplacian, where this normalization is chosen so that it factorizes as

$$\Delta = \partial\bar{\partial}.$$

We will frequently consider $\bar{\partial}$-equations of the kind

$$\bar{\partial}u = f, \qquad f \in L^2(\mathscr{T}, d\mu), \tag{1.10}$$

where $\mathscr{T}$ is some compact subset of $\mathbb{C}$ and μ is a measure. By a solution u to (1.10), we mean an element $u \in W^{1,2}(d\mu, \mathscr{T})$ such that $\bar{\partial}u = f$ in $L^2(d\mu, \mathscr{T})$.

2. Preliminaries

2.1. Function spaces with polynomial growth

By $\mathrm{Pol}_n(\mathbb{C})$ we mean the space of polynomials of degree at most $n-1$, and we denote by $\mathrm{Pol}(\mathbb{C})$ the space of all polynomials.

Let ϕ denote a real-valued function defined on some domain Ω, possibly the entire plane. By $L^2_\phi = L^2_\phi(\Omega)$ we mean the usual L^2-space with inner product

$$\langle f, g\rangle_\phi := \int_\Omega f(z)\overline{g(z)}e^{-\phi(z)}dA(z).$$

We denote by $A(\Omega)$ the set of holomorphic functions on Ω, and let $A^2_\phi = A^2_\phi(\Omega)$ be the intersection

$$A^2_\phi = A(\Omega) \cap L^2_\phi,$$

endowed with the inner product inherited from L^2_ϕ.

When $\Omega = \mathbb{C}$, we also consider the spaces

$$L^2_{n,\phi} = \left\{f \in L^2_\phi : |f(z)| \le |z|^{n-1} + O(1) \text{ as } z \to \infty\right\},$$

with a polynomial growth restriction at infinity, and the space

$$A^2_{n,\phi} = L^2_\phi(\mathbb{C}) \cap \mathrm{Pol}_n(\mathbb{C}) = A^2_\phi(\mathbb{C}) \cap L^2_{n,\phi}(\mathbb{C}).$$

We will be especially concerned with the spaces A^2_ϕ and $A^2_{n,\phi}$ in the cases when $\phi = \log\frac{1}{1-|z|^2}$ and $\phi = c|z|^2$ for some constant $c > 0$. In the literature these are often referred to as (polynomial) Bergman and Fock spaces, respectively.

2.2. Cut-off functions

We will find the need to make use of cut-off functions $\chi = \chi_{\delta,r}$ that are identically one on a disk $\mathbb{D}(0, r(1-\delta))$, and vanish off the slightly bigger disk $\mathbb{D}(0, r)$. These can be chosen so as to satisfy the estimates

$$|\bar\partial\chi_{\delta,r}|^2 \le \frac{C}{\delta^2 r^2} 1_{\mathbb{A}((1-\delta)r,r)}, \qquad \|\bar\partial\chi_{\delta,r}\|^2_{L^2} \le \frac{4}{\delta},$$

for $0 < r < 1$ and $0 < \delta < 1$. An example of such a function is given by

$$\chi_{\delta,r}(z) = \begin{cases} 1, & |z| \le (1-\delta)r \\ \left(\frac{1}{\delta} - \frac{|z|}{\delta r}\right)^2, & (1-\delta)r < |z| \le r \\ 0, & |z| > r \end{cases} \tag{2.1}$$

Note that this function is merely Lipschitz. In case one requires more regularity, it suffices to note that the above properties should be stable under smoothing procedures, such as convolution.

2.3. A $\bar\partial$-estimate with polynomial growth control

We rely on methods from the work of Ameur, Hedenmalm and Makarov [2]. They prove a version of Hörmander's classical $\bar\partial$-estimates, which gives polynomial growth control at infinity. Here we only need the following direct special case of [2, Theorem 4.1].

Theorem 2.1. *Let $\mathscr{T}$ be a compact subset of $\mathbb{C}$, and denote by $\phi, \widehat\phi$ two real-valued functions on $\mathbb{C}$ of class $\mathscr{C}^{1,1}$, such that*

- $\phi(z) = \widehat\phi(z)$ *for* $z \in \mathscr{T}$ *and* $\widehat\phi(z) \le \phi(z)$ *for* $z \in \mathbb{C}$,
- $\Delta\widehat\phi > 0$ *on* $\mathscr{T}$, *and* $\Delta\widehat\phi \ge 0$ *on* $\mathbb{C}$.
- $\widehat\phi(z) = \tau\log|z|^2 + O(1)$ *as* $z \to \infty$.

Then, for any integer $n \ge \tau$ and $f \in L^\infty(\mathscr{T})$, the $L^2_{n,\phi}$-minimal solution $u_{0,n}$ to $\bar\partial u = f$ exists and satisfies

$$\int_{\mathbb{C}} |u_{0,n}|^2 e^{-\phi} dA \le \int_{\mathscr{T}} |f|^2 \frac{e^{-\phi}}{\Delta\widehat\phi} dA.$$

Remark 2.2. a) We remark that we may allow the function ϕ to take on the value $+\infty$ on $\mathscr{T}^c$. That this is the case can be seen by applying the theorem to the pair $(\widehat{\phi}, \widehat{\phi})$ to obtain that the $L^2_{n,\widehat{\phi}}$-minimal solution v_0 to $\bar{\partial} v = f$ satisfies

$$\int_{\mathbb{C}} |v_0|^2 e^{-\widehat{\phi}} dA \le \int_{\mathscr{T}} |f|^2 \frac{e^{-\phi}}{\Delta\phi} dA.$$

Since $L^2_{n,\widehat{\phi}} \subset L^2_{n,\phi}$, it follows that $\int |u_0|^2 e^{-\phi} dA \le \int |v_0|^2 e^{-\phi} dA$. We may therefore infer the desired result from the fact that $\widehat{\phi} \le \phi$.

b) The original theorem pertains to a wider class of $\widehat{\phi}$ in terms of growth at infinity than considered here, but requires that $\Delta\widehat{\phi} > 0$ on the entire plane. For the specific form of $\widehat{\phi}$ considered here, this requirement may easily be removed by approximation, as is done in [2, Section 4.4]. Indeed, by letting

$$\widehat{\phi}_\epsilon(z) = \left(1 - \frac{\epsilon}{\tau}\right) \widehat{\phi}(z) + \epsilon \log(1 + |z|^2)$$

for $\epsilon > 0$ and applying the theorem to $(\phi, \widehat{\phi}_\epsilon)$, the desired inequality follows by letting $\epsilon \to 0$.

3. The planar case: proof of Theorem 1.5

3.1. The fundamental $\bar{\partial}$-estimate

Recall the functionals $\rho_{\mathbb{C},\gamma}(f)$ and $\rho^*_{\mathbb{C},\gamma}(f)$ from (1.8) and (1.9). To prove that $\rho_{\mathbb{C}} = \rho^*_{\mathbb{C}}$, we follow the approach suggested in [7], which is to modify minimizers f of $\rho_{\mathbb{C},\gamma}(f)$ outside $\mathbb{D}$ so as to make sure that $\rho^*_{\mathbb{C},\gamma}(f)$ is close to $\rho_{\mathbb{C},\gamma}(f)$. This is done in two steps: first one multiplies f by a cut-off function χ that vanishes outside $\mathbb{D}$. Secondly, one must correct χf so that it once again becomes a polynomial. This is done using the Hörmander-type $\bar{\partial}$-techniques of [2], i.e. Theorem 2.1 above.

Denote by $\chi = \chi_{\delta,1}$ the cut-off function from Subsection 2.2. We will apply Theorem 2.1 to the equation $\bar{\partial} u = \bar{\partial}(\chi f) = f \bar{\partial}\chi$, where f is a minimizer of $\rho_{\mathbb{C},\gamma}(f)$. One observes that u is then the desired correction: $\bar{\partial}(\chi f - u) = f\bar{\partial}\chi - u = 0$, so $\chi f - u$ is holomorphic. We let $\mathscr{T} = \mathbb{D}$, $\phi(z) = 2\gamma|z|^2$ and we define $\widehat{\phi}$ to be the unique function satisfying

- $\widehat{\phi} = \phi$ on $\mathbb{D}$,
- $\widehat{\phi} \in \mathscr{C}^{1,1}$ and $\Delta\phi \ge 0$.
- $\widehat{\phi}$ is pointwise minimal with these conditions satisfied.

Since ϕ is radial, this reversed obstacle problem is easy, and we can give an explicit formula for $\widehat{\phi}$:

$$\widehat{\phi}(z) = \begin{cases} 2\gamma|z|^2, & |z| < 1 \\ 2\gamma\log|z|^2 + 2\gamma, & |z| \geq 1. \end{cases}$$

To verify that this formula is correct, we note that $\widehat{\phi} = \phi$ on $\mathbb{D}$ and $\partial_n\widehat{\phi} = \partial_n\phi$ on $\partial\mathbb{D}$, so since $\widehat{\phi}$ is smooth away from $\partial\mathbb{D}$, it inherits the $\mathscr{C}^{1,1}$-regularity from ϕ. That $\widehat{\phi}$ is harmonic for $|z| > 1$ ensures minimality by use of the maximum principle. It is also easy to verify that $\widehat{\phi} \leq \phi$.

Thus all conditions of Theorem 2.1 are satisfied, and we may infer that the $L^2_{n,\phi}$-minimal solution u to the $\bar{\partial}$-equation satisfies

$$\int_{\mathbb{C}} |u(z)|^2 e^{-2\gamma|z|^2} dA(z) \leq \int_{\mathscr{T}} |\bar{\partial}(\chi f)|^2 \frac{e^{-\phi}}{\Delta\phi} dA(z) = \frac{1}{2\gamma} \int_{\mathbb{A}(1-\delta,1)} |f(z)|^2 |\bar{\partial}\chi(z)|^2 e^{-2\gamma|z|^2} dA(z).$$

We have thus arrived at the following result, which controls the L^2-norm of the correction to non-holomorphicity of χf.

Theorem 3.1. *Let $\gamma > 0$ and let f be a bounded holomorphic function on the unit disk. Then there exists a solution u to $\bar{\partial}u = \bar{\partial}(\chi f)$ that satisfies the estimate*

$$\int_{\mathbb{C}} |u(z)|^2 e^{-2\gamma|z|^2} dA(z) \leq \frac{1}{2\gamma} \int_{\mathbb{A}((1-\delta),1)} |f(z)|^2 \, |\bar{\partial}\chi(z)|^2 e^{-2\gamma|z|^2} dA(z),$$

with polynomial growth control

$$|u(z)| = O\left(|z|^{n-1}\right), \qquad z \to \infty$$

where $n = n(\gamma) = \lceil 2\gamma \rceil$.

3.2. Existence and a priori control of minimizers

We first observe that for a fixed $\gamma < \infty$, there exists a holomorphic function f_0 on $\mathbb{D}$ that minimizes $\rho_{\mathbb{C},\gamma}(f)$. Indeed, let $f_n \in \mathrm{Pol}_n(\mathbb{C})$ be a sequence of polynomials, for which $\rho_{\mathbb{C},\gamma}(f_n) \to \inf \rho_{\mathbb{C},\gamma}(f)$. We may assume that they are absolute minimizers within their respective spaces $Pol_n(\mathbb{C})$. Let $\phi = 2\gamma|z|^2$. A simple variational argument, see the proof of Lemma 3.2 below, shows that the L^2-norms $\|f_n\|_{A^2_\phi(\mathbb{D})}$ are uniformly bounded. Denote by $K_\phi(z,w)$ the reproducing kernel for the space $A^2_\phi(\mathbb{D})$. By Cauchy–Schwarz inequality, one finds the pointwise bound

$$|f(z)|^2 \leq \|K_\phi(\,\cdot\,,z)\|^2_\phi \, \|f\|^2_{A^2_\phi}, \qquad z \in \mathbb{D}$$

for $f \in A^2_\phi$, which yields a uniform bound $\|f_{n\,|K}\|_\infty \leq M_K$ independently of n, for each fixed compact subset K of $\mathbb{D}$. By a normal families argument, there exists a holomorphic function f_0 and a subsequence $\{n_k\}$ along which $f_{n_k} \to f_0$ uniformly on compact subsets. By Fatou's Lemma we find that

$$\rho_{\mathbb{C},\gamma}(f_0) \le \liminf_{k\to\infty} \rho_{\mathbb{C},\gamma}(f_{n_k}),$$

so f_0 is indeed a minimizer.

These minimizers turn out to have good properties, even uniformly in the parameter γ.

Lemma 3.2. *Assume that $f = f_\gamma$ is a minimizer of $\rho_{\mathbb{C},\gamma}(f)$. Then*

$$\int_{\mathbb{D}} |f(z)|^2 e^{-2\gamma|z|^2} dA(z) = \int_{\mathbb{D}} |f(z)| e^{-\gamma|z|^2} dA(z),$$

and both expressions are bounded as $\gamma \to \infty$.

Proof. Define $V(\alpha)$ for $\alpha > 0$ by

$$V(\alpha) = \rho_{\mathbb{C},\gamma}(\alpha f)$$

Since αf is holomorphic whenever f is, it is clear that we may vary α within the class of admissible functions for the infimum. It follows that $V'(1) = 0$, which after expanding the square reads

$$\int_{\mathbb{D}} |f(z)|^2 e^{-2\gamma|z|^2} dA(z) = \int_{\mathbb{D}} |f(z)| e^{-\gamma|z|^2} dA(z),$$

which is exactly the first assertion. Using this property, one finds that

$$\rho_{\mathbb{C},\gamma}(f) = 1 - \int_{\mathbb{D}} |f(z)| e^{-\gamma|z|^2} dA(z)$$

and since $\rho_{\mathbb{C},\gamma}(f) > 0$, this implies the boundedness assertion. $\square$

The next results controls the L^1-norm of minimizers near $\partial\mathbb{D}$, and will be referred to as the L^1-*non-concentration estimate.*

Lemma 3.3. *Assume that $\delta = \delta_\gamma = o(1)$ as $\gamma \to \infty$. Let $\{f_\gamma\}$ be a sequence of minimizers of $\rho_{\mathbb{C},\gamma}(f)$. Then*

$$\int_{\mathbb{A}(1-\delta,1)} |f(z)| e^{-\gamma|z|^2} dA(z) = o(1), \qquad \gamma \to \infty.$$

Proof. By Cauchy–Schwarz inequality,

$$\int_{\mathbb{A}(1-\delta,1)} |f(z)|e^{-\gamma|z|^2} dA(z) \le \left(\int_{\mathbb{A}(1-\delta,1)} |f(z)|^2 e^{-2\gamma|z|^2} dA(z) \right)^{1/2} \left(\int_{\mathbb{A}(1-\delta,1)} dA(z) \right)^{1/2}$$

The first integral is uniformly bounded as $n \to \infty$ by Lemma 3.2, and the area of annuli with radii $(1-\delta, 1)$ tends to zero as $\delta \to 0$. □

It turns out to be beneficial to introduce one more parameter in the functionals. For $\alpha > 0$ we consider

$$\rho_{\mathbb{C},\gamma,\alpha}(f) = \int_{\mathbb{D}} \left(|f(z)|e^{-\alpha\gamma|z|^2} - 1 \right)^2 dA(z).$$

Proposition 3.4. *For any sequence $\alpha_\gamma \to 1$ it holds that*

$$\lim_{\gamma\to\infty} \inf \inf_f \rho_{\mathbb{C},\gamma,\alpha}(f) = \rho_{\mathbb{C}}.$$

Proof. This is immediate after the change of variables $w = \alpha^{1/2}\gamma^{1/2}z$, by the fact $\mathrm{Pol}(\mathbb{C})$ is invariant under dilations and the original definition (1.6) of $\rho_{\mathbb{C}}$. □

3.3. An L^2-non-concentration estimate

The point of this section is to control the growth of minimizers f_γ near $\partial\mathbb{D}$. This is done effectively via the following theorem, which we will refer to as the planar L^2-non-concentration estimate.

Theorem 3.5. *Let f be a minimizer of $\rho_{\mathbb{C},\gamma}(f)$ and let $\delta = \delta_\gamma \to 0$. Then*

$$\int_{\mathbb{A}(1-\delta,1)} |f(z)|^2 e^{-2\gamma|z|^2} dA(z) = o(1),$$

as $\gamma \to \infty$ along a subsequence $\Gamma = \{\gamma_k\}$ for which there exist polynomials f_k such that $\rho_{\mathbb{C},\gamma}(f_k) \to \rho_{\mathbb{C}}$.

Proof. Fix such a sequence Γ, and let f_γ denote a sequence of minimizers. Let $\alpha = (1-\delta)^2$ and $g_\gamma(z) = f_\gamma(\alpha^{1/2}z)$. Computing $\rho_{\mathbb{C},\gamma,\alpha}(g)$ we find that

$$\begin{aligned}\rho_{\mathbb{C},\gamma,\alpha}(g) &= \int\limits_{\mathbb{D}(0,1-\delta)} |f(z)|^2 e^{-2\gamma|z|^2} dA(z) - 2 \int\limits_{\mathbb{D}(0,1-\delta)} |f(z)| e^{-\gamma|z|^2} dA(z) + 1 \\ &= \int\limits_{\mathbb{D}} |f(z)|^2 e^{-2\gamma|z|^2} dA(z) - 2 \int\limits_{\mathbb{D}} |f(z)| e^{-\gamma|z|^2} dA(z) + 1 \\ &\quad - \int\limits_{\mathbb{A}(1-\delta,1)} |f(z)|^2 e^{-2\gamma|z|^2} dA(z) + 2 \int\limits_{\mathbb{A}(1-\delta,1)} |f(z)| e^{-\gamma|z|^2} dA(z).\end{aligned}$$

From the L^1-non-concentration estimate (Lemma 3.3), it follows that we may write

$$\rho_{\mathbb{C},\gamma,\alpha}(g) = \rho_{\mathbb{C},\gamma}(f) - \int\limits_{\mathbb{A}(1-\delta,1)} |f(z)|^2 e^{-2\gamma|z|^2} dA + o(1).$$

Since the remaining integral is positive;

$$\liminf_{\gamma\to\infty,\gamma\in\Gamma} \rho_{\mathbb{C},\gamma,\alpha}(g) = \rho_{\mathbb{C}} - \limsup_{\gamma\to\infty,\gamma\in\Gamma} \int\limits_{\mathbb{A}(1-\delta,1)} |f(z)|^2 e^{-2\gamma|z|^2} dA.$$

Proposition 3.4 tells us that $\liminf \rho_{\mathbb{C},\gamma,\alpha}(g) \geq \rho_{\mathbb{C}}$. If the above equation is to refrain from violating this, it must hold that

$$\int\limits_{\mathbb{A}(1-\delta,1)} |f(z)|^2 e^{-2\gamma|z|^2} dA = o(1)$$

which is the desired conclusion. □

3.4. Proof of Theorem 1.5

Let Γ be a sequence of numbers $\gamma \to \infty$, along which

$$\lim_{k\to\infty} \inf_f \rho_{\mathbb{C},\gamma_k}(f) = \rho_{\mathbb{C}}.$$

We will take all subsequent limits along this sequence.

Let $\delta \geq \gamma^{-1/2}$, so that by (2.1) we have the bound

$$|\bar{\partial}\chi_{\delta,1}|^2 \leq C\gamma.$$

For each $\gamma \in \Gamma$, let $f = f_\gamma$ be a minimizer of $\rho_{\mathbb{C},\gamma}(f)$, and let u be a solution to $\bar{\partial}u = \bar{\partial}(\chi f)$, as in Theorem 3.1. Then

$$\int_{\mathbb{C}} |u|^2 e^{-2\gamma|z|^2} dA \le \frac{1}{2\gamma} \int_{\mathbb{A}(1-\delta,1)} |\bar\partial\chi|^2 |f|^2 e^{-2\gamma|z|^2} dA \le C \int_{\mathbb{A}(1-\delta,1)} |f|^2 e^{-2\gamma|z|^2} dA = o(1) \tag{3.1}$$

where the asymptotics follows from the L^2-non-concentration estimate in Theorem 3.5.

Let $\nu = \chi f - u$. Then ν is holomorphic, and since χ has compact support and $|u(z)| = O(|z|^{n-1})$, it follows by Liouville's theorem that $\nu \in \mathrm{Pol}_n(\mathbb{C})$. We calculate the functional $\rho^*_{\mathbb{C},\gamma}(\nu)$ as

$$\rho^*_{\mathbb{C},\gamma}(\nu) = \int_{\mathbb{C}\setminus\mathbb{D}} |u|^2 e^{-2\gamma|z|^2} dA + \int_{\mathbb{D}} |\nu|^2 e^{-2\gamma|z|^2} dA - 2\int_{\mathbb{D}} |\nu| e^{-\gamma|z|^2} dA + 1 \tag{3.2}$$

The first term in (3.2) is $o(1)$ by (3.1).

We turn to the L^p-norms of $\nu e^{-\gamma|z|^2}$. By the L^1-non-concentration estimates, we have that

$$\int_{\mathbb{D}} |\nu| e^{-\gamma|z|^2} dA = \int_{\mathbb{D}} |f| e^{-\gamma|z|^2} dA + \int_{\mathbb{D}} (|\nu| - |f|) e^{-\gamma|z|^2} dA$$

and

$$\left| \int_{\mathbb{D}} (|\nu| - |f|) e^{-\gamma|z|^2} dA \right| \le 2 \int_{\mathbb{A}(1-\delta,1)} |f| e^{-\gamma|z|^2} dA + \int_{\mathbb{D}} |u| e^{-\gamma|z|^2} dA = o(1),$$

where the last assertion follows from the $\bar\partial$-estimate (3.1) and the L^1-non-concentration estimate. Thus

$$\int_{\mathbb{D}} |\nu| e^{-\gamma|z|^2} dA = \int_{\mathbb{D}} |f| e^{-\gamma|z|^2} dA + o(1).$$

Turning to the L^2-norms,

$$\int_{\mathbb{D}} |\nu|^2 e^{-2\gamma|z|^2} dA = \int_{\mathbb{D}} |\chi f|^2 e^{-2\gamma|z|^2} dA + \int_{\mathbb{D}} (|u|^2 - 2\Re(\chi f \overline{u})) e^{-2\gamma|z|^2} dA.$$

The latter integral is $o(1)$ by the $\bar\partial$-estimate (3.1) and an application of the Cauchy–Schwarz inequality. The former satisfies

$$\int_{\mathbb{D}} |\chi f|^2 e^{-2\gamma|z|^2} dA = \int_{\mathbb{D}} |f|^2 e^{-2\gamma|z|^2} dA + o(1),$$

in light of Theorem 3.5. It follows from this that

$$\rho^*_{\mathbb{C},\gamma}(\nu) = \rho_{\mathbb{C},\gamma}(f) + o(1), \qquad \gamma \to \infty, \gamma \in \Gamma.$$

Since ν are admissible polynomials, this implies that

$$\rho^*_{\mathbb{C}} \le \lim_{\gamma\to\infty,\gamma\in\Gamma} \rho^*_{\mathbb{C},\gamma}(\nu) = \lim_{\gamma\to\infty,\gamma\in\Gamma} \rho_{\mathbb{C},\gamma}(f) = \rho_{\mathbb{C}}.$$

The reversed inequality is known to hold, so it follows that $\rho_{\mathbb{C}} = \rho^*_{\mathbb{C}}$. □

4. The hyperbolic case: proof of Theorem 1.3

4.1. Application of the $\bar\partial$-estimate

We begin by applying Theorem 2.1 in our setting to get the following theorem, which gives the crucial L^2-control of solutions to $\bar\partial u = \bar\partial(\chi f)$ on the entire disk. We let $0 < r < 1$ and denote by χ a cut-off function $\chi_{\delta,r}$, as in (2.1).

Theorem 4.1. *Let $0 < r < 1$ and let f be a bounded holomorphic function on $\mathbb{D}$. There exists a solution $u = u_{0,r}$ to $\bar\partial u = \bar\partial(\chi f)$ that enjoys the estimate*

$$\int_{\mathbb{D}} |u(z)|^2(1-|z|^2)dA(z) \le \int_{\mathbb{A}((1-\delta)r,r)} |\bar\partial\chi(z)|^2|f(z)|^2(1-|z|^2)^3 dA(z).$$

Moreover,

$$|u(z)| = O\left(|z|^{n-1}\right), \qquad z \to \infty$$

where $n = n(r) = \lceil r^2(1-r^2)^{-1}\rceil$.

Proof. Let $\phi(z) = \log 1/(1-|z|^2)$ for $z \in \mathbb{D}$, and extend it to the entire plane by defining $\phi(z) = +\infty$ for $z \in \overline{\mathbb{D}^e}$. The compact set $\mathscr{T}$ is taken to be $\overline{\mathbb{D}(0,r)}$, for a fixed $0 < r < 1$.

Define the function $\widehat{\phi}$ as the minimal subharmonic function of class $\mathscr{C}^{1,1}$, that agrees with ϕ on $\mathbb{D}(0,r)$. Since ϕ is radial, the function $\widehat{\phi}$ is readily found; indeed since

$$\phi_{|\partial\mathbb{D}(0,r)} = \log\frac{1}{1-r^2}, \qquad \partial_n\phi_{|\partial\mathbb{D}(0,r)} = \frac{2r}{1-r^2},$$

one easily checks that

$$\widehat{\phi}(z) := \frac{r^2}{1-r^2}\log\frac{|z|^2}{r^2} + \log\frac{1}{1-r^2}$$

is a candidate, in that it agrees with ϕ in the right sense. Since it is harmonic in the exterior disk $\mathbb{D}(0,r)^e$, it follows by the maximum principle for subharmonic functions that it is the correct choice. With this pair $(\phi, \widehat{\phi})$, all assumptions of Theorem 2.1 are satisfied.

Applying the theorem, we obtain a solution $u \in L^2_{n,\phi}$ for which the estimate

$$\int_{\mathbb{C}} |u|^2 e^{-\phi} dA \leq \int_{\mathscr{T}} |\bar{\partial}(\chi f)|^2 \frac{e^{-\phi}}{\Delta\widehat{\phi}},$$

holds true. Since $\Delta\widehat{\phi} = (1-|z|^2)^{-2}$ on $\mathscr{T} = \mathbb{D}(0,r)$, and since f is holomorphic, it follows that

$$\int_{\mathbb{D}} |u(z)|^2(1-|z|^2)dA \leq \int_{\mathbb{A}((1-\delta)r,r)} |\bar{\partial}\chi|^2|f|^2(1-|z|^2)^3 dA,$$

which completes the proof. $\square$

4.2. Non-concentration estimates for minimizers

Just as in the planar case, it is clear that for each $0 < r < 1$ there exists a holomorphic function f_0 which attains the value $\inf_f \rho_{\mathbb{H},r}(f)$, taken over all polynomials. The estimates of $L^2_{n,\phi}$-minimal solutions to $\bar{\partial}u = \bar{\partial}(\chi f)$, where $f = f_r$ is such a minimizer, control the norm of u in terms of the behaviour of f near $\partial\mathbb{D}$, as $r \to 1^-$. In this section we aim to understand this behaviour of f better. We begin with the following simple variational identity.

Lemma 4.2. *Let $f = f_r$ be a minimizer of $\rho_{\mathbb{H},r}(f)$. Let*

$$\ell_{k,r} := \frac{1}{\log\frac{1}{1-r^2}} \int_{\mathbb{D}(0,r)} |f(z)|^k(1-|z|)^{k-1}dA(z), \quad k \in \{1,2\}, \quad 0 < r < 1.$$

Then $\ell_{1,r} = \ell_{2,r}$, and both sequences are bounded.

Proof. Consider the variation

$$V(\alpha) = \rho_{\mathbb{H},r}(\alpha f), \qquad 0 < \alpha < \infty.$$

Since αf is admissible for any α, it follows that if f is a minimizer then $V'(1) = 0$. By expanding the square, one observes that this says that $\ell_{2,r} = \ell_{1,r}$. Calculating $\rho_{\mathbb{H},r}(f)$ using this equality, we see that

$$\rho_{\mathbb{H},r}(f) = 1 - \ell_{1,r}.$$

Since $0 < \rho_{\mathbb{H},r}(f) \leq 1$ and since the integrals are positive, it follows that $\ell_{1,r}$ is uniformly bounded, and thus the same holds for $\ell_{2,r}$. $\square$

In the following, we will consider limiting procedures as $r \to 1^-$. Many objects will depend on r, and sometimes on subsequences $\mathcal{R} = \{r_k\}_{k\geq 1}$. In order not to obscure the notation, we will often suppress indices when no confusion should occur.

Lemma 4.3. *Let δ be a sequence tending to zero as $r \to 1^-$ and denote by f a minimizer of $\rho_{\mathbb{H},r}(f)$. We have that*

$$\frac{1}{\log \frac{1}{1-r^2}} \int\limits_{\mathbb{A}(r(1-\delta),r)} |f(z)| dA(z) = O\left(\left(1 - \frac{\log \frac{1}{1-r^2(1-\delta)^2}}{\log \frac{1}{1-r^2}}\right)^{1/2}\right).$$

In particular, if $\delta = (1-r)$, the O-expression is $o(1)$.

Proof. Cauchy–Schwarz inequality gives that

$$\int\limits_{\mathbb{A}(r(1-\delta),r)} |f(z)| dA(z)$$

$$\leq \left(\int\limits_{\mathbb{A}(r(1-\delta),r)} |f(z)|^2(1-|z|^2) dA(z)\right)^{1/2} \left(\int\limits_{\mathbb{A}(r(1-\delta),r)} \frac{dA(z)}{1-|z|^2}\right)^{1/2}.$$

Using that $(\log \frac{1}{1-r^2})^{-1} \int_{\mathbb{D}(0,r)} |f|^2(1-|z|^2) dA$ is bounded, we may estimate further

$$\frac{1}{\log \frac{1}{1-r^2}} \int\limits_{\mathbb{A}(r(1-\delta),r)} |f(z)| dA(z) \leq C \left(\frac{\int_{\mathbb{A}(r(1-\delta),r)} \frac{dA}{1-|z|^2}}{\log \frac{1}{1-r^2}}\right)^{1/2}.$$

Calculating the integrals, we find that

$$\frac{\int_{\mathbb{A}(r(1-\delta),r)} \frac{dA}{1-|z|^2}}{\log \frac{1}{1-r^2}} = \frac{\log \frac{1}{1-r^2} - \log \frac{1}{1-r^2(1-\delta)^2}}{\log \frac{1}{1-r^2}} = 1 - \frac{\log \frac{1}{1-r^2(1-\delta)^2}}{\log \frac{1}{1-r^2}},$$

which proves the first assertion.

Next, if $\delta = (1-r)$, then we note that

$$\begin{aligned} 1 - \frac{\log(1-r^2(1-\delta)^2)}{\log(1-r^2)} &= 1 - \frac{\log(1-r+r(1-r)) + O(1)}{\log(1-r) + O(1)} = 1 - \frac{\log(1-r) + O(1)}{\log(1-r) + O(1)} \\ &= o(1), \end{aligned}$$

which completes the proof. $\square$

Consider the functional $\rho_{\mathbb{H},r,\alpha}(f)$, defined for $\alpha < 1$ by

$$\rho_{\mathbb{H},r,\alpha}(f) = \frac{\alpha^2}{\log\frac{1}{1-r^2}} \int_{\mathbb{D}(0,r)} \left((1-|\alpha z|^2)|f(z)|-1\right)^2 \frac{dA(z)}{1-|\alpha z|^2}.$$

We have the following lemma, allowing for the freedom of an extra parameter.

Proposition 4.4. *Let α_r be a sequence of numbers $\alpha_r \to 1^-$, such that $\alpha_r \geq r^k$ for some k. Then*

$$\liminf_{r\to 1^-} \inf_f \rho_{\mathbb{H},r,\alpha}(f) \geq \rho_{\mathbb{H}},$$

where the infimum is taken over the Bergman space $A^2(\mathbb{D}(0,r))$.

The condition $\alpha \geq r^k$ is by no means meant to be sharp. It illustrates some flexibility compared to the restrictions on δ, while it is clearly compatible with $\alpha = r$, which corresponds to the choice $\delta = 1-r$ which will be made shortly.

Proof. For $f \in A^2(\mathbb{D}(0,r))$ we have that

$$\begin{aligned}\rho_{\mathbb{H},r,\alpha}(f) &= \frac{\alpha^2}{\log\frac{1}{1-r^2}} \int_{\mathbb{D}(0,r)} \left((1-|\alpha z|^2)|f(z)|-1\right)^2 \frac{dA(z)}{1-|\alpha z|^2} \\ &= \frac{\log\frac{1}{1-\alpha^2 r^2}}{\log\frac{1}{1-r^2}} \rho_{\mathbb{H},\alpha r,1}\left(f\left(\frac{z}{\alpha}\right)\right).\end{aligned}$$

If $\alpha \geq r^k$, it follows that

$$1 \geq \frac{\log\frac{1}{1-\alpha^2 r^2}}{\log\frac{1}{1-r^2}} \geq \frac{\log\frac{1}{1-r^{k+1}} + O(1)}{\log\frac{1}{1-r} + O(1)} = 1 + o(1).$$

Since $f_{\alpha^{-1}}(z) = f(z/\alpha) \in A^2(\mathbb{D}(0,\alpha r))$, it follows by Remark 1.1 that $\rho_{\mathbb{H},r}(f_{\alpha^{-1}})$ is an upper bound for $\inf_{f\in \mathrm{Pol}(\mathbb{C})} \rho_{\mathbb{H},r}(f)$. The result follows. $\square$

The following theorem is the key ingredient to the proof of our main result, and will be referred to as the hyperbolic L^2-non-concentration estimate.

Theorem 4.5. *Let f be a minimizer of $\rho_{\mathbb{H},r}(f)$, and let $\delta = (1-r)$. Then*

$$\frac{1}{\log\frac{1}{1-r^2}} \int_{\mathbb{A}(r(1-\delta),r)} |f(z)|^2(1-|z|^2)dA(z) = o(1)$$

as $r \to 1^-$ along a sequence $\mathcal{R} = \{r_k\}$ for which $\inf_f \rho_{\mathbb{H},r_k}(f) \to \rho_{\mathbb{H}}$.

Proof. Let $\mathcal{R}$ be a sequence of indices along which $\inf_f \rho_{\mathbb{H},r}(f) \to \rho_{\mathbb{H}}$, and let $f = f_r$ denote a minimizer. Denote by c_0 the number

$$c_0 = \limsup_{r\to 1^-, r\in\mathcal{R}} \frac{1}{\log\frac{1}{1-r^2}} \int_{\mathbb{A}(r(1-\delta),r)} |f(z)|^2(1-|z|^2)dA(z).$$

Let $\alpha = 1-\delta$. Consider the function $g(z) = f(\alpha z) \in A^2(\mathbb{D}(0,\alpha^{-1}r)) \subset A^2(\mathbb{D}(0,r))$, and the functionals $\rho_{\mathbb{H},r,\alpha}(g)$. These satisfy

$$\begin{aligned}\rho_{\mathbb{H},r,\alpha}(g) &= \frac{\alpha^2}{\log\frac{1}{1-r^2}} \int_{\mathbb{D}(0,r)} \left((1-|\alpha z|^2)|f(\alpha z)|-1\right)^2 \frac{dA(z)}{1-|\alpha z|^2} \\ &= \frac{1}{\log\frac{1}{1-r^2}} \int_{\mathbb{D}(0,(1-\delta)r)} \left((1-|z|^2)|f(z)|-1\right)^2 \frac{dA(z)}{1-|z|^2}.\end{aligned}$$

Computing the functionals by expanding the squares, we see that

$$\begin{aligned}\rho_{\mathbb{H},r,\alpha}(g) = \rho_{\mathbb{H},r}(f) - \frac{1}{\log\frac{1}{1-r^2}}\Bigg(&\int_{\mathbb{A}((1-\delta)r,r)} |f(z)|^2(1-|z|^2)dA(z) \\ &- 2\int_{\mathbb{A}((1-\delta)r,r)} |f(z)|dA(z)\Bigg) \qquad (4.1)\\ = \rho_{\mathbb{H},r}(f) - \frac{1}{\log\frac{1}{1-r^2}} &\int_{\mathbb{A}((1-\delta)r,r)} |f(z)|^2(1-|z|^2)dA(z) + o(1), \qquad (4.2)\end{aligned}$$

where the last equality follows from Lemma 4.3. By Proposition 4.4, it follows that $\liminf_{r\to 1^-, r\in\mathcal{R}} \rho_{\mathbb{H},r,\alpha}(g) \geq \rho_{\mathbb{H}}$. Taking the lower limit in (4.2) as $r \to 1^-$ along $\mathcal{R}$, noting that $\rho_{\mathbb{H},r}(f)$ converges, we find that

$$\rho_{\mathbb{H}} \leq \liminf_{r\to 1^-} \rho_{\mathbb{H},r,\alpha}(g) = \rho_{\mathbb{H}} - c_0.$$

Since $c_0 \geq 0$, clearly it follows that $c_0 = 0$. □

Proposition 4.6. *Let f be a minimizer of $\rho_{\mathbb{H},r}(f)$ and let $\delta = 1-r$. Let u be the solution to $\bar\partial u = \bar\partial(\chi f)$ from Theorem 4.1. Then*

$$\frac{1}{\log\frac{1}{1-r^2}} \int_{\mathbb{D}} |u_r|^2(1-|z|^2)dA(z) = o(1)$$

as $r \to 1^-$ along a sequence $\mathcal{R}$ along which $\inf_f \rho_{\mathbb{H},r}(f) \to \rho_{\mathbb{H}}$.

Proof. From Theorem 4.1 we have the estimate

$$\frac{1}{\log\frac{1}{1-r^2}}\int_{\mathbb{D}}|u(z)|^2(1-|z|^2)dA(z)\le\frac{1}{\log\frac{1}{1-r^2}}\int_{\mathbb{A}((1-\delta)r,r)}|\bar\partial\chi|^2|f(z)|^2(1-|z|^2)^3dA(z).$$

We estimate this by

$$\begin{aligned}&\frac{1}{\log\frac{1}{1-r^2}}\int_{\mathbb{A}((1-\delta)r,r)}|\bar\partial\chi|^2|f(z)|^2(1-|z|^2)^3dA(z)\\&\le\frac{\left\|\bar\partial\chi\,(1-|z|^2)\right\|_\infty^2}{\log\frac{1}{1-r^2}}\int_{\mathbb{A}((1-\delta)r,r)}|f(z)|^2(1-|z|^2)dA(z).\end{aligned}$$

The supremum norm may be estimated by

$$|\bar\partial\chi(z)|^2(1-|z|^2)^2\le C\frac{(1-r^2(1-\delta)^2)^2}{\delta^2}=O\left(\left(\frac{1-r+r\delta}{\delta}\right)^2\right).$$

Since $\delta=1-r$, the latter expression is $O(1)$. Invoking Theorem 4.5 and using again that $\delta=1-r$ completes the proof. □

Remark 4.7. In order to control $\|\bar\partial\chi\,(1-|z|^2)\|_\infty$, the parameter δ needs to be controlled from below, to avoid χ dropping off too steeply. On the other hand, in order to apply Theorem 4.5 we need instead an upper bound on the same quantity. The choice $\delta=1-r$ balances these very well (but is probably not sharp).

4.3. Proof of the main theorem

Let $r\to1^-$, along a subsequence $\mathcal{R}=\{r_k\}$ such that there are admissible f_k for which $\rho_{\mathbb{H},r_k}(f_k)\to\rho_{\mathbb{H}}$. Let $\delta=(1-r)$, ensuring that all estimates from the previous results come into play. Let f be minimizers of $\rho_{\mathbb{H},r}(f)$, and let u be the $L^2_{n(r),\phi}$-minimal solutions to $\bar\partial u=\bar\partial(\chi f)$. Put $\nu=\chi f-u$. Then by Liouville's Theorem, ν is a polynomial of degree at most $n=\lceil r^2/(1-r^2)\rceil$. As such, it is admissible for $\rho^*_{\mathbb{H},r}$. Calculating this functional, we obtain

$$\rho^*_{\mathbb{H},r}(\nu)=\rho_{\mathbb{H},r}(\nu)+\frac{1}{\log\frac{1}{1-r^2}}\int_{\mathbb{D}\setminus\mathbb{D}(0,r)}|u(z)|^2(1-|z|^2)dA(z)=:\rho_{\mathbb{H},r}(\nu)+I_{\text{ext}}.$$

By Proposition 4.6, the term I_{ext} is $o(1)$, so disappears as we take the lower limit.

We focus on the term $\rho_{\mathbb{H},r}(\nu)$. Expanding the square, we find that

$$\rho_{\mathbb{H},r}(\nu)=\frac{1}{\log\frac{1}{1-r^2}}\int_{\mathbb{D}(0,r)}|\nu|^2(1-|z|^2)dA-\frac{2}{\log\frac{1}{1-r^2}}\int_{\mathbb{D}(0,r)}|\nu|dA+1=:I_{L^2}-2I_{L^1}+1.$$

Turning first to I_{L^2}, we see that

$$I_{L^2} = \frac{1}{\log \frac{1}{1-r^2}} \int\limits_{\mathbb{D}(0,r)} \left(|\chi f|^2 - 2\Re[\chi f \overline{u}] + |u|^2\right)(1-|z|^2)dA.$$

We estimate the three terms separately: the main contribution comes from the first term;

$$\frac{1}{\log \frac{1}{1-r^2}} \left| \int\limits_{\mathbb{D}(0,r)} (|\chi_\delta f|^2 (1-|z|^2)dA - \int\limits_{\mathbb{D}(0,r)} |f|^2(1-|z|^2)dA \right| = o(1)$$

by the L^2-non-concentration estimate. The middle term is handled as follows: $\Re[\chi f\overline{u}] \leq |\chi f|\,|u|$, and

$$\frac{1}{\log \frac{1}{1-r^2}} \int\limits_{\mathbb{D}(0,r)} |\chi f|\,|u|(1-|z|^2)dA(z) \leq \left(\frac{1}{\log \frac{1}{1-r^2}} \int\limits_{\mathbb{D}(0,r)} |f|^2(1-|z|^2)dA \right)^{1/2} \times \left(\frac{1}{\log \frac{1}{1-r^2}} \int\limits_{\mathbb{D}(0,r)} |u|^2(1-|z|^2)dA \right)^{1/2}.$$

Since the L^2-norm of f is bounded independently of r, it follows by applying Proposition 4.6 to the second factor that the expression is $o(1)$. The third term is also $o(1)$, in light of the $\bar{\partial}$-estimate Proposition 4.6. In summary:

$$I_{L^2} = \int\limits_{\mathbb{D}(0,r)} |f|^2(1-|z|^2)dA + o(1).$$

Next, turning to I_{L^1} we find that

$$\left| I_{L^1} - \frac{1}{\log \frac{1}{1-r^2}} \int\limits_{\mathbb{D}(0,r)} |f|dA \right| \leq \frac{1}{\log \frac{1}{1-r^2}} \left(2 \int\limits_{\mathbb{A}(r(1-\delta),r)} |f|dA + \int\limits_{\mathbb{D}(0,r)} |u|dA \right).$$

The first term on the right is $o(1)$ by Proposition 4.6, and, using Cauchy–Schwarz inequality and Proposition 4.6 we find that the second term also vanishes in the limit. It follows that

$$I_{L^1} = \frac{1}{\log \frac{1}{1-r^2}} \int\limits_{\mathbb{D}(0,r)} |f|dA + o(1).$$

Thus, considering the sequence $\mathcal{R} = \{r_k\}$ along which $\rho_{\mathbb{H},r}(f)$ tends to $\rho_{\mathbb{H}}$, we may write

$$\rho^*_{\mathbb{H},r_k}(\nu) = I_{L^2} - 2I_{L^1} + 1 + I_{\text{ext}}$$
$$= \frac{1}{\log\frac{1}{1-r^2}} \int\limits_{\mathbb{D}(0,r)} |f|^2(1-|z|^2)dA - \frac{2}{\log\frac{1}{1-r^2}} \int\limits_{\mathbb{D}(0,r)} |f| + 1 + o(1)$$
$$= \rho_{\mathbb{H},r}(f) + o(1).$$

It thus follows that

$$\rho^*_{\mathbb{H}} \le \lim_{k\to\infty} \rho^*_{\mathbb{H},r_k}(\nu_{r_k}) = \rho_{\mathbb{H}}.$$

Put together with the trivial inequality $\rho^*_{\mathbb{H}} \ge \rho_{\mathbb{H}}$, this concludes the proof.

Proof of Corollaries 1.4 and 1.6. We begin with Corollary 1.4. Let $\mathcal{R}$ be the subsequence of the previous proof. The statement regarding $\rho^*_{\mathbb{H}}$ follows immediately from the fact that the polynomials ν_r used in the proof of Theorem 1.3 are elements of $\mathrm{Pol}_{n(r)}(\mathbb{C})$. However, since $\rho_{\mathbb{H},r}(\nu) = \rho^*_{\mathbb{H},r}(\nu) + o(1)$ as $r \to 1^-$ while $r \in \mathcal{R}$, the result follows for $\rho_{\mathbb{H}}$ as well.

The analogous planar result, Corollary 1.6, follows in exactly the same fashion. □

5. Concluding remarks

5.1. General functionals and $L^{2\beta}$-estimates for the $\bar\partial$-equations

One can also consider the more general discrepancy density, $\rho^\beta_{\mathbb{C}}$, discussed in Subsection 1.4, and its canonical counterpart $\rho^{\beta,*}_{\mathbb{C}}$. It is natural to ask the same question, i.e. whether or not $\rho^\beta_{\mathbb{C}} = \rho^{\beta,*}_{\mathbb{C}}$.

One can follow the line of proof of Theorem 1.5 verbatim, up until the point where the L^2-estimate of Theorem 2.1, from [2], is used. The strategy remains to modify minimizers $f = f_{\beta,R}$ of the original functional $\rho^\beta_{\mathbb{C},R}(f)$ by multiplying with a suitable cut-off function χ, and then find a 'minimal' solution to

$$\bar\partial u(z) = \bar\partial(f(z)\,\chi(z))$$

in order for the candidate approximate minimizer $\nu := \chi f - u$ to be holomorphic. In order to control the functional $\rho^{\beta,*}_{\mathbb{C},R}(\nu)$ we need an estimate of the kind

$$\frac{1}{R^2}\int\limits_{\mathbb{C}} |u(z)|^{2\beta} e^{-2|z|^2} dA(z) \le \frac{C}{R^2} \int\limits_{\mathbb{A}(R(1-\delta),R)} |\bar\partial\chi(z)|^{2\beta} |f(z)|^{2\beta} e^{-2|z|^2} dA(z).$$

This involves L^p-control of solutions to the $\bar\partial$-equation, and is not covered by the classical theory. At present, the author is not aware of any results that may replace Theorem 2.1 in that it offers polynomial growth control, but there are some L^p-estimates at hand. The following is Proposition 1.4 in [10]. See also [12] and [6]. For a subharmonic weight ϕ, we let μ be the measure $\mu = \Delta\phi$, and define $\rho(z)$ be the function such that

$$\mu(\mathbb{D}(z, \rho(z))) = 1.$$

One may think of ρ as a regularized version of $(\Delta\phi)^{-1/2}$.

Theorem 5.1. *Let ϕ be a subharmonic function on $\mathbb{C}$, such that $\Delta\phi$ is a doubling measure. Then the $L^2_{2\phi}$-minimal solution to $\bar{\partial}u = f$ satisfies*

$$\|ue^{-\phi}\|_{L^p(\mathbb{C})} \leq C\|fe^{-\phi}\rho\|_{L^p(\mathbb{C})}, \qquad 1 \leq p \leq \infty$$

for some constant $C = C(p, \phi)$.

We will not do this here, but by applying Theorem 5.1 one should be able to conclude that $\rho^{\beta}_{\mathbb{C}} = \rho^{\beta,*}_{\mathbb{C}}$. In order to obtain information on the degree of approximate minimizers of $\rho^{\beta,*}_{\mathbb{C},R}(f)$, substantial analysis would have to be performed.

Acknowledgments

I want to thank Håkan Hedenmalm for his generosity with his insights, for useful comments while preparing the manuscript and not least for suggesting the study of this problem. I'm grateful to Oleg Ivrii for numerous suggestions on the manuscript, and for him sharing his insights about intriguing remaining questions. Thanks also to Simon Larson for inspiring discussions and proof-reading, and to the referee for carefully reading the manuscript and suggesting clarifying changes in Section 4.

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Paper F

A critical topology for L^p-Carleman classes with $0 < p < 1$
(joint with H. Hedenmalm)
Math. Annalen, advance online publ., (2018).

Math. Ann.
https://doi.org/10.1007/s00208-018-1654-3

Mathematische Annalen

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A critical topology for L^p-Carleman classes with $0 < p < 1$

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Received: 29 August 2017 / Revised: 30 January 2018

Abstract In this paper, we explore a sharp phase transition phenomenon which occurs for L^p-Carleman classes with exponents $0 < p < 1$. These classes are defined as for the standard Carleman classes, only the L^∞-bounds are replaced by corresponding L^p-bounds. We study the quasinorms

$$\|u\|_{p,\mathcal{M}} = \sup_{n\geq 0} \frac{\|u^{(n)}\|_p}{M_n},$$

for some weight sequence $\mathcal{M} = \{M_n\}_n$ of positive real numbers, and consider as the corresponding L^p-Carleman space the completion of a given collection of smooth test functions. To mirror the classical definition, we add the feature of dilatation invariance as well, and consider a larger soft-topology space, the L^p-Carleman class. A particular degenerate instance is when $M_n = 1$ for $0 \leq n \leq k$ and $M_n = +\infty$ for $n > k$. This would give the L^p-Sobolev spaces, which were analyzed by Peetre, following an initial insight by Douady. Peetre found that these L^p-Sobolev spaces are highly degenerate for $0 < p < 1$. Indeed, the canonical map $W^{k,p} \to L^p$ fails to be injective, and there is even an isomorphism

Communicated by Loukas Grafakos.

The research of both authors was supported by Vetenskapsrådet (VR) (Grant no. 2016-04912).

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$$W^{k,p} \cong L^p \oplus L^p \oplus \cdots \oplus L^p,$$

corresponding to the canonical map $f \mapsto (f, f', \ldots, f^{(k)})$ acting on the test functions. This means that e.g. the function and its derivative lose contact with each other (they "disconnect"). Here, we analyze this degeneracy for the more general L^p-Carleman classes defined by a weight sequence $\mathcal{M}$. If $\mathcal{M}$ has some regularity properties, and if the given collection of test functions is what we call (p, θ)-*tame*, then we find that there is a sharp boundary, defined in terms of the weight $\mathcal{M}$: on the one side, we get Douady–Peetre's phenomenon of "disconnexion", while on the other, the completion of the test functions consists of C^∞-smooth functions and the canonical map $f \mapsto (f, f', f'', \ldots)$ is correspondingly well-behaved in the completion. We also look at the more standard second phase transition, between non-quasianalyticity and quasianalyticity, in the L^p setting, with $0 < p < 1$.

1 Introduction

1.1 The Sobolev spaces for $0 < p < 1$

We first survey the properties of the Sobolev spaces with exponents in the range $0 < p < 1$. These were considered by Peetre [18] after Adrien Douady suggested that they behave very differently from the standard $1 \le p \le +\infty$ case. For an exponent p with $0 < p < 1$, and an integer $k \ge 0$, *we define the Sobolev space* $W^{k,p}(\mathbb{R})$ *on the real line* $\mathbb{R}$ *to be the abstract completion of the space* C_0^k *of compactly supported* k *times continuously differentiable functions*, *with respect to the quasinorm*

$$\|f\|_{k,p} = \left(\|f\|_p^p + \|f'\|_p^p + \cdots \|f^{(k)}\|_p^p\right)^{1/p}. \tag{1.1}$$

Here, as a matter of notation, $\|\cdot\|_p$ denotes the quasinorm of L^p. Defined in this manner, $W^{k,p}$ becomes a quasi-Banach space whose elements consist of equivalence classes of Cauchy sequences of test functions. At first glance, this definition is very natural, and other approaches to define the same object in the classical case $p \ge 1$ do not generalize. For instance, we cannot use the usual notion of weak derivatives to define these spaces, as functions in L^p need not be locally L^1 for $0 < p < 1$.

Some observations suggest that things are not the way we might expect them to be. For instance, as a consequence of the failure of local integrability, for a given point $a \in \mathbb{R}$, the formula for the primitive

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

cannot be expected to make sense. A related difficulty is the invisibility of Dirac "point masses" in the quasinorm of L^p. Indeed, if $0 < p < 1$ and $u_\epsilon := \epsilon^{-1} 1_{[0,\epsilon]}$, then as $\epsilon \to 0^+$, we have the convergence $u_\epsilon \to 0$ in L^p while $u_\epsilon \to \delta_0$ in the sense of distribution theory. Here, we use standard notational conventions: δ_0 denotes the unit point mass at 0, and 1_E is the characteristic function of the subset E, which equals 1 on E and vanishes off E.

1.2 Independence of the derivatives in Sobolev spaces

For $1 \le p \le +\infty$, we think of the Sobolev space $W^{k,p}$ as a subspace of L^p, consisting of functions of a specified degree of smoothness. As such, the identity mapping $\boldsymbol{\alpha}: W^{k,p} \to L^p$ defines a canonical injection. Douady observed that we cannot have this picture in mind when $0 < p < 1$, as the corresponding canonical map $\boldsymbol{\alpha}$ fails to be injective. Peetre built on Douady's observation and showed that this uncoupling (or disconnexion) between the derivatives goes even deeper. In fact, the standard map $f \mapsto (f, f', \ldots, f^{(k)})$ on test functions f defines a topological isomorphism of the completion $W^{k,p}$ onto the direct sum of $k+1$ copies of L^p. For convenience, we state the theorem here.

Theorem 1.1 [18] *Let $0 < p < 1$ and $k = 1, 2, 3, \ldots$. Then $W^{k,p}$ is isometrically isomorphic to $k+1$ copies of L^p:*

$$W^{k,p} \cong L^p \oplus L^p \oplus \cdots \oplus L^p. \tag{1.2}$$

This decoupling occurs as a result of the availability of approximate point masses which are barely visible in the quasinorm. As a consequence, if we define Sobolev spaces as completions with respect to the Sobolev quasinorm, we obtain highly pathological (and rather useless) objects.

1.3 A bootstrap argument to control the L^∞-norm in terms the L^p-quasinorms of the higher derivatives

From the Douady–Peetre analysis of the Sobolev spaces $W^{k,p}$ for exponents $0 < p < 1$, we might be inclined to believe that for such small p, we always run into pathology. However, there is in fact an argument of bootstrap type which can save the situation if we simultaneously control *the derivatives of all orders*. To explain the bootstrap argument, we take a (weight) sequence $\mathcal{M} = \{M_k\}_{k=0}^{+\infty}$ of positive reals, and define the quasinorm

$$\|f\|_{p,\mathcal{M}} = \sup_{k \ge 0} \frac{\|f^{(k)}\|_p}{M_k}, \quad f \in \mathscr{S}, \tag{1.3}$$

on some appropriate linear space $\mathscr{S} \subset C^\infty(\mathbb{R})$ of test functions that we choose to begin with. In the analogous setting of periodic functions (i.e., with the unit circle $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ in place of $\mathbb{R}$), it would be natural to work with the linear span of the periodic complex exponentials as $\mathscr{S}$. In the present setting of the line $\mathbb{R}$, there is no such canonical choice. Short of a natural explicit linear space of functions, we ask instead that $\mathscr{S}$ should satisfy a property.

Definition 1.1 ($0 < p \leq 1$, $\theta \in \mathbb{R}$) We say that $f \in C^\infty(\mathbb{R})$ is (p, θ)*-tame* if $f^{(n)} \in L^\infty(\mathbb{R})$ for each $n = 0, 1, 2, \ldots$, and

$$\limsup_{n \to +\infty} (1-p)^n \log \|f^{(n)}\|_\infty \leq \theta. \tag{1.4}$$

Moreover, for subsets $\mathscr{S} \subset C^\infty(\mathbb{R})$, we say that $\mathscr{S}$ is (p, θ)*-tame* if every element $f \in \mathscr{S}$ is (p, θ)-tame.

Remark 1.1 (a) For $\theta < 0$, only the constant functions $f = 0$ are (p, θ)-tame, unless if $p = 1$, in which case no (p, θ)-tame function exists (the limsup vanishes and the right-hand side $\theta < 0$). The claim that only the constant functions are (p, θ)-tame for $0 < p < 1$ and $\theta < 0$ may be obtained using an argument involving entire functions, Liouville's theorem, and an application of the Phragmen–Lindelöf principle. For this reason, in the sequel, *we shall restrict our attention to* $\theta \geq 0$.

(b) Loosely speaking, for $\theta \geq 0$, the requirement (1.4) asks that the L^∞-norms of the higher order derivatives do not grow too wildly. We note that for p close to one, (p, θ)-tameness is a very weak requirement; indeed, at the endpoint value $p = 1$, it is void.

A natural suitable choice of a (p, θ)-tame collection of test functions might be the following Hermite class:

$$\mathscr{S}^{\mathrm{Her}} := \left\{ f : f(x) = \mathrm{e}^{-x^2} q(x), \text{ where } q \text{ is a polynomial} \right\}.$$

Indeed, a rather elementary argument shows that (1.4) holds with $\theta = 0$ for $\mathscr{S} = \mathscr{S}^{\mathrm{Her}}$ (for the details, see Lemma 4.1 below). However, it might be the case that not all $f \in \mathscr{S}^{\mathrm{Her}}$ have finite quasinorm $\|f\|_{p,\mathcal{M}} < +\infty$ (this depends on the choice of weight sequence $\mathcal{M}$). In that case, we would then replace $\mathscr{S}^{\mathrm{Her}}$ by its linear subspace

$$\mathscr{S}^{\mathrm{Her}}_{p,\mathcal{M}} := \left\{ f \in \mathscr{S}^{\mathrm{Her}} : \|f\|_{p,\mathcal{M}} < +\infty \right\},$$

and hope that this collection of test functions is not too small.

The bootstrap argument. We proceed with the bootstrap argument. We assume that our parameters are confined to the intervals $0 < p \leq 1$ and $0 \leq \theta < +\infty$. Moreover, we assume that the collection of test functions $\mathscr{S}$ is (p, θ)-tame and that $\|f\|_{p,\mathcal{M}} < +\infty$ holds for each $f \in \mathscr{S}$. We pick a normalized element $f \in \mathscr{S}$ with $\|f\|_{p,\mathcal{M}} = 1$. Since $1 = p + (1-p)$, it follows from the fundamental theorem of Calculus that for $x \leq y$,

$$\begin{aligned} |f(y) - f(x)| = \left| \int_x^y f'(t)\mathrm{d}t \right| &\leq \int_x^y |f'(t)|\mathrm{d}t \leq \|f'\|_\infty^{1-p} \int_x^y |f'(t)|^p \mathrm{d}t \\ &\leq \|f'\|_\infty^{1-p} \|f'\|_p^p. \end{aligned} \tag{1.5}$$

As $f \in L^p(\mathbb{R})$, the function f must assume values arbitrarily close to 0 on rather big subsets of $\mathbb{R}$. By taking the limit of such y in (1.5), we arrive at

$$|f(x)| \le \|f'\|_\infty^{1-p}\|f'\|_p^p, \quad x \in \mathbb{R},$$

which gives that

$$\|f\|_\infty \le \|f'\|_\infty^{1-p}\|f'\|_p^p. \tag{1.6}$$

By iteration of the estimate (1.6), we obtain that for $n = 1, 2, 3, \dots$,

$$\begin{aligned}\|f\|_\infty &\le \|f^{(n)}\|_\infty^{(1-p)^n}\|f^{(n)}\|_p^{p(1-p)^{n-1}}\cdots\|f'\|_p^p \\ &= \|f^{(n)}\|_\infty^{(1-p)^n}\prod_{j=1}^n \|f^{(j)}\|_p^{(1-p)^{j-1}p}.\end{aligned} \tag{1.7}$$

As it is given that $\|f\|_{p,\mathcal{M}} = 1$, we have that $\|f^{(j)}\|_p \le M_j$, which we readily implement into (1.7):

$$\|f\|_\infty \le \|f^{(n)}\|_\infty^{(1-p)^n}\prod_{j=1}^n M_j^{(1-p)^{j-1}p} \quad \text{if } \|f\|_{p,\mathcal{M}} = 1. \tag{1.8}$$

Finally, we let n approach infinity, so that in view of the (p,θ)-tameness assumption (1.4) and homogeneity, we obtain that

$$\|f\|_\infty \le \mathrm{e}^\theta \|f\|_{p,\mathcal{M}} \limsup_{n\to+\infty}\prod_{j=1}^n M_j^{(1-p)^{j-1}p}, \quad f \in \mathscr{S}. \tag{1.9}$$

The estimate (1.9) tells us that under the requirement

$$\kappa(p,\mathcal{M}) := \limsup_{n\to+\infty}\sum_{j=1}^n (1-p)^j \log M_j < +\infty, \tag{1.10}$$

we may control the sup-norm of a test function $f \in \mathscr{S}$ in terms of its quasinorm $\|f\|_{p,\mathcal{M}}$. We will refer to the quantity $\kappa(p,\mathcal{M})$ as the *p-characteristic of the sequence* $\mathcal{M}$. It follows that the Douady–Peetre disconnexion phenomenon does not occur if we simultaneously control all the higher derivatives under (1.10) (provided the test function space $\mathscr{S}$ is (p,θ)-tame). But the condition (1.10) achieves more. To see this, we first observe that as the inequality (1.7) applies to an arbitrary smooth function f, and in particular to a derivative $f^{(k)}$, for $k = 0, 1, 2, \dots$:

$$\|f^{(k)}\|_\infty \le \|f^{(n+k)}\|_\infty^{(1-p)^n}\prod_{j=1}^n \|f^{(j+k)}\|_p^{(1-p)^{j-1}p}, \quad n = 1, 2, 3, \dots. \tag{1.11}$$

Next, we let n tend to infinity in (1.11) and use homogeneity and (p,θ)-tameness as in (1.9), and arrive at

$$\|f^{(k)}\|_\infty \leq \|f\|_{p,\mathcal{M}} \limsup_{n\to+\infty} \|f^{(k+n)}\|_\infty^{(1-p)^n} \limsup_{n\to+\infty} \prod_{j=1}^{n} M_{j+k}^{(1-p)^{j-1}p}, \quad f \in \mathscr{S}. \tag{1.12}$$

Moreover, since

$$\sum_{j=1}^{n}(1-p)^j \log M_{j+k} = (1-p)^{-k}\sum_{j=1}^{n+k}(1-p)^j \log M_j - (1-p)^{-k}\sum_{j=1}^{k}(1-p)^j \log M_j,$$

and

$$(1-p)^n \log \|f^{(k+n)}\|_\infty = (1-p)^{-k}(1-p)^{n+k} \log \|f^{(k+n)}\|_\infty$$

it follows from (1.12) and (1.10) that

$$\|f^{(k)}\|_\infty \leq \|f\|_{p,\mathcal{M}}\, \mathrm{e}^{\theta(1-p)^{-k}+p(1-p)^{-k-1}\kappa(p,\mathcal{M})} \prod_{j=1}^{k} M_j^{-(1-p)^{j-k-1}p}, \quad f \in \mathscr{S},\ k = 0, 1, 2, \ldots. \tag{1.13}$$

It is immediate from (1.13) that under the summability condition (1.10), we may in fact control the sup-norm of all the higher order derivatives, which guarantees that the elements of the completion of the test function class $\mathscr{S}$ under the quasinorm $\|\cdot\|_{p,\mathcal{M}}$ consists of C^∞ functions, and the failure of the Douady–Peetre disconnexion phenomenon is complete. We refer to the argument leading up to (1.13) as a "bootstrap" because we were able to rid ourselves of the sup-norm control on the right-hand side by diminishing its contribution in the preceding estimate and taking the limit.

Remark 1.2 The above argument is inspired by an argument which goes back to work of Hardy and Littlewood on Hardy spaces of harmonic functions. The phenomenon is coined *Hardy–Littlewood ellipticity* in [7]. To explain the background, we recall that for a function $u(z)$ harmonic in the unit disk $\mathbb{D}$, the function $z \mapsto |u(z)|^p$ is subharmonic if $p \geq 1$. As such, it enjoys the mean value estimate

$$|u(0)|^p \leq \frac{1}{\pi}\int_{\mathbb{D}} |u(z)|^p \mathrm{d}A(z).$$

A remarkable fact is that this inequality survives even for $0 < p < 1$ (with a different constant) even though subharmonicity fails. See, e.g., [7, Lemma 4.2], [13, Lemma 3.7], and the original work of Hardy and Littlewood [14]. The similarity with the bootstrap argument used here is striking if we compare with e.g. [13, Lemma 3.7].

1.4 The L^p-Carleman spaces and classes

The Carleman class $\mathscr{C}_{\mathcal{M}}$ associated with the weight sequence $\mathcal{M}$ is defined as the linear subspace of $f \in C^\infty(\mathbb{R})$ for which

$$\frac{\|f^{(k)}\|_\infty}{M_k} \le C A^k,$$

for some positive constant $A = A_f$ (which may depend on f). The theory for Carleman classes was developed in order to understand for which classes of functions the (formal) Taylor series at a point uniquely determines the function. Denjoy [12] provided an answer under regularity assumptions on the weight sequence, and Carleman [8–10] proved what has since become known as the Denjoy–Carleman Theorem: if $\mathcal{N}$ denotes the largest logarithmically convex minorant of $\mathcal{M}$, then $\mathscr{C}_{\mathcal{M}}$ has this uniqueness property if and only if

$$\sum_{j\ge 0} \frac{N_j}{N_{j+1}} = +\infty.$$

Bang later found a simplified proof of this result [1]. Bang also found numerous other remarkable results for the Carleman classes. To name one, he proved that the Bang degree $\mathfrak{n}_f$ of $f \in \mathscr{C}_{\mathcal{M}}([0,1])$, defined as the maximal integer $N \ge 0$ such that

$$\sum_{\log\|f\|_\infty^{-1} < j \le N} \frac{M_{j-1}}{M_j} < \mathrm{e},$$

is an upper bound for the number of zeros of f on the interval $[0, 1]$. For an account of several of the interesting results in [1] as well as in Bang's thesis [2], together with some further developments in the theory of quasianalytic functions, we refer to the work of Borichev, Nazarov, Sodin, and Volberg [6,17].

The present work is devoted to the study of the analogous classes defined in terms of L^p-norms, mainly for $0 < p < 1$. In view of the preceding subsection, it is natural to select the biggest possible collection of test functions so that the bootstrap argument has a chance to apply under the assumption (1.10):

$$\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}} := \left\{ f \in C^\infty(\mathbb{R}) : \|f\|_{p,\mathcal{M}} < +\infty \text{ and } \limsup_{n\to+\infty} (1-p)^n \log\|f^{(n)}\|_\infty \le \theta \right\}.$$

Here, we keep our standing assumptions that $0 < p < 1$ and $\theta \ge 0$ (many assertion will hold also at the endpoint value $p = 1$). Then $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$ automatically meets the asymptotic growth condition (1.4), but for (p, θ)-tameness to hold we also need for each individual derivative to be bounded. However, the condition

$$\limsup_{n\to+\infty}(1-p)^n \log \|f^{(n)}\|_\infty \le \theta$$

guarantees that $f^{(n)} \in L^\infty$ at least for big positive integers n, say for $n \ge n_0$. But then, since $\|f\|_{p,\mathcal{M}} < +\infty$ and in particular $f^{(k)} \in L^p$ for all $k = 0, 1, 2, \ldots$, the estimate (1.6) gives that $f^{(n-1)} \in L^\infty$ as well. Proceeding iteratively, we find that $f^{(n)} \in L^\infty$ for all $n = 0, 1, 2, \ldots$ if $f \in \mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$. This means that all the estimates of the preceding subsection are sound for $f \in \mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$. Of course, for very degenerate weight sequences $\mathcal{M}$, it might unfortunately happen that $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}} = \{0\}$. We proceed to define the L^p-Carleman spaces.

Definition 1.2 $(0 < p < 1)$ The L^p-Carleman space $W^{p,\theta}_{\mathcal{M}}$ is the completion of the test function class $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$ with respect to the quasinorm $\|\cdot\|_{p,\mathcal{M}}$.

In the standard textbook presentations, the Carleman classes are defined for a regular weight sequence $\mathcal{M}$ in the same way, only the L^p quasinorm is replaced by the L^∞ norm, and the class is to be minimal given the following two requirements: (i) the space is contained in the class, and (ii) the class is invariant with respect to dilatation [10,15,16]. It is easy to see that claiming that $f \in \mathscr{C}_{\mathcal{M}}$ is the same as saying that $f(t) = g(at)$ for some positive real a, where

$$\frac{\|g^{(k)}\|_\infty}{M_k} \le C, \quad k = 0, 1, 2, \ldots,$$

for some constant C, which we understand as the requirement $\|g\|_{\infty,\mathcal{M}} < +\infty$ (with exponent $p = +\infty$). This allows us to extend the notion of the Carleman classes for exponents $0 < p < 1$ as follows.

Definition 1.3 $(0 < p < 1)$ The L^p-Carleman class $\mathscr{C}^{p,\theta}_{\mathcal{M}}$ consists of all the dilates $f_a(t) := f(at)$ (with $a > 0$) of functions $f \in W^{p,\theta}_{\mathcal{M}}$.

Here, to avoid unnecessary abstraction, we need to understand that each element in $W^{p,\theta}_{\mathcal{M}}$ gives rise to an element of the Cartesian product space $L^p \times L^p \times \cdots$ via the lift of the map $f \mapsto (f, f', f'', \ldots)$ initially defined on test functions $f \in \mathscr{S}^{\circledast}_{p,\mathcal{M}}$ (for more details, see the next subsection). Moreover, it is easy to see that $f \in W^{p,\theta}_{\mathcal{M}}$ is uniquely determined by the corresponding element in $L^p \times L^p \times \cdots$. It is important to note that in the Cartesian product space, the dilatation operation is well-defined, so that the L^p-Carleman class $\mathscr{C}^{p,\theta}_{\mathcal{M}}$ can be understood as a submanifold of $L^p \times L^p \times \cdots$ in the above sense. Moreover, $\mathscr{C}^{p,\theta}_{\mathcal{M}}$ is actually a linear subspace, as the quasinorm criterion for being in the class is (analogously)

$$\frac{\|f^{(k)}\|_p}{M_k} \le C A^k,$$

for some positive constants C and A, and this kind of bound is closed under linear combination.

1.5 Classes of weight sequences

From this point onward, we will restrict attention to positive, logarithmically convex weight sequences, i.e. sequences $\mathcal{M} = \{M_j\}_j$ of positive numbers such that the function $j \mapsto \log M_j$ is convex.

We will consider the following notions of regularity for weight sequences.

Definition 1.4 A logarithmically convex sequence $\mathcal{M} = \{M_n\}_{n=0}^{+\infty}$ with infinite p-characteristic $\kappa(p,\mathcal{M}) = +\infty$ is called *p-regular* if one of the following conditions (i)–(ii) holds:

(i) $\liminf_{n\to+\infty}(1-p)^n \log M_n > 0$, or
(ii) $\log M_n = \mathrm{o}((1-p)^{-n})$ and $n\log n \le (\delta + \mathrm{o}(1)) \log M_n$ as $n \to +\infty$, for some δ with $0 < \delta < 1$.

In view of the Denjoy–Carleman theorem, either of the conditions (i) or (ii) implies that the standard Carleman class with the weight sequence $\mathcal{M}$ is non-quasianalytic. Using this observation, it can be shown that the class of test functions $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$ contains nontrivial compactly supported functions.

It is a simple observation that p-regular sequences are stable under the process of shifts and under replacing M_n for a finite number of indices, as long as log-convexity is kept.

Remark 1.3 (a) We note in the context of Definition 1.4 that if $\mathcal{M}$ grows so fast that $\kappa(p,\mathcal{M}) = +\infty$ holds, then in particular the asymptotic estimate $\log M_n = \mathrm{O}(q^{-n})$ fails as $n \to +\infty$ for any given q with $1-p < q < 1$. The second condition in (ii), which says that $n\log n \le (\delta + \mathrm{o}(1)) \log M_n$, is a rather mild lower bound on $\log M_n$ compared with this exponential growth along subsequences.

(b) For logarithmically convex sequences $\mathcal{M}$, the sum

$$\kappa(p,\mathcal{M}) = \sum_{j=0}^{+\infty} (1-p)^j \log M_j \tag{1.14}$$

is actually convergent to an extended real number in $\mathbb{R} \cup \{+\infty\}$.

We conclude with a notion of regularity that applies in the regime with finite p-characteristic.

Definition 1.5 A logarithmically convex weight sequence $\mathcal{M}$ for which $\kappa(p, \mathcal{M}) < +\infty$ is said to be *decay-regular* if $M_n/M_{n+1} \geq \epsilon^n$ holds for some positive real ϵ, or alternatively, if $\mathcal{M}$ meets the nonquasianalyticity condition

$$\sum_{n=1}^{+\infty} \frac{M_n}{M_{n+1}} < +\infty.$$

1.6 The three phases

We mentioned already the phenomenon that the Carleman classes exhibit the phase transition associated with quasianalyticity. Here, the concept of quasianalyticity is usually defined in terms of the unique continuation property that the (formal) Taylor series at any given point determines the function uniquely. Under the regularity condition that the sequence $\mathcal{M} = \{M_n\}_n$ is logarithmically convex, it is known classically that the *Carleman class* $\mathscr{C}_{\mathcal{M}} = \mathscr{C}_{\mathcal{M}}^{\infty,0}$ *is quasianalytic if and only if*

$$\sum_{n=0}^{+\infty} \frac{M_n}{M_{n+1}} = +\infty.$$

In the small exponent range $0 < p < 1$ considered here, it turns out that we have actually *two phase transitions*:

(i) the Douady–Peetre disconnexion barrier, and
(ii) the quasianalyticity barrier.

Here, we shall attempt to explore both phenomena.

In the degenerate case when $M_n = 1$ for $n = 0, \ldots, k$ and $M_n = +\infty$ for $n > k$, the Carleman space $W_{\mathcal{M}}^{p,\theta}$ does not depend on the parameter $\theta \geq 0$, and is the same as the Sobolev space $W^{p,k}$, except that it is equipped with another (but equivalent) quasinorm. So in this instance, we get the Douady–Peetre disconnexion phenomenon (1.2) for $W_{\mathcal{M}}^{p,\theta}$, while under the bounded p-characteristic condition (1.10), the following result shows that it does not happen.

To properly formulate the result, we consider the canonical mapping $\boldsymbol{\pi} : W_{\mathcal{M}}^{p,\theta} \to L^p \times L^p \times \cdots$, defined initially on the test function space $\mathscr{S}_{p,\theta,\mathcal{M}}^{\circledast}$ by

$$\boldsymbol{\pi} f = (f, f', f'', \ldots).$$

It is natural to replace here the product space $L^p \times L^p \times \cdots$ by its linear subspace $\ell^\infty(L^p, \mathcal{M})$ supplied with the standard quasinorm:

$$\|(f_0, f_1, f_2, \ldots)\|_{\ell^\infty(L^p,\mathcal{M})} := \sup_{n\geq 0} \frac{\|f_n\|_p}{M_n}. \tag{1.15}$$

Indeed, the linear mapping $\boldsymbol{\pi} : W_{\mathcal{M}}^{p,\theta} \to \ell^\infty(L^p, \mathcal{M})$ then becomes an isometry. This is obvious for test functions in $\mathscr{S}_{p,\theta,\mathcal{M}}^{\circledast}$, and, then automatically holds for elements of the abstract completion as well. We denote the n-th component projection of $\boldsymbol{\pi}$ by $\boldsymbol{\pi}_n$: $\boldsymbol{\pi}_n(f_0, f_1, f_2, \ldots) := f_n$.

Theorem 1.2 $(0 < p < 1)$ *Suppose that the weight sequence $\mathcal{M}$ is logarithmically convex and meets the finite p-characteristic condition* (1.10). *Then, for each $n = 0, 1, 2\ldots$, $\boldsymbol{\pi}_n$ maps $W^{p,\theta}_{\mathcal{M}}$ into $C^\infty(\mathbb{R})$, and the space $W^{p,\theta}_{\mathcal{M}}$ is coupled, in the sense that*

$$\partial \boldsymbol{\pi}_n f = \boldsymbol{\pi}_{n+1} f, \quad f \in W^{p,\theta}_{\mathcal{M}},$$

where ∂ stands for the differentiation operation. Moreover, the projection $\boldsymbol{\pi}_0$ is injective, and, in the natural sense, the space equals the collection of test functions: $W^{p,\theta}_{\mathcal{M}} = \mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$.

The proof of this theorem is supplied in Sect. 4.2.

The conclusion of Theorem 1.2 is that under the strong finite p-characteristic condition, $W^{p,\theta}_{\mathcal{M}}$ is a space of smooth functions, and, indeed, it is identical with the test function class $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$. The situation is drastically different when $\kappa(p, \mathcal{M}) = +\infty$. For reasons of convenience, in this regime we work with the smaller space of compactly supported test functions $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}} \cap C^\infty_0$ and its closure $W^{p,\theta}_{\mathcal{M},0}$ in the space $W^{p,\theta}_{\mathcal{M}}$. It remains a possibility that the spaces $W^{p,\theta}_{\mathcal{M},0}$ and $W^{p,\theta}_{\mathcal{M}}$ actually coincide.

Theorem 1.3 $(0 < p < 1)$ *Suppose $\mathcal{M}$ is a p-regular sequence with infinite p-characteristic $\kappa(p, \mathcal{M}) = +\infty$. Then $\boldsymbol{\pi}_0 \colon W^{p,\theta}_{\mathcal{M}} \to L^p$ is such that already its restriction $\boldsymbol{\pi}_0 \colon W^{p,\theta}_{\mathcal{M},0} \to L^p$ is surjective onto L^p, and, moreover, $\boldsymbol{\pi}$ supplies an isomorphism*

$$W^{p,\theta}_{\mathcal{M},0} \cong L^p(\mathbb{R}) \oplus W^{p,\theta_1}_{\mathcal{M}_1,0},$$

where $\mathcal{M}_1$ denotes the shifted sequence $\mathcal{M}_1 := \{M_{n+1}\}_n$ and $\theta_1 = \theta/(1-p)$. In addition, $\mathcal{M}_1$ inherits the assumed properties of $\mathcal{M}$.

A sketch of the proof of Theorem 1.3 is supplied in Sect. 5.3. As for the formulation, the summand $W^{p,\theta_1}_{\mathcal{M}_1,0}$ on the right-hand side arises as the space of "derivatives" of functions in $W^{p,\theta}_{\mathcal{M},0}$. Here, the reason why $\mathcal{M}$ gets replaced by $\mathcal{M}_1$ is due to a one-unit shift in the sequence space $\ell^\infty(L^p, \mathcal{M})$. Moreover, the reason why θ gets replaced by $\theta_1 = \theta/(1-p)$ is the corresponding shift in the space of test functions $\mathscr{S}_{p,\theta,\mathcal{M}}$ when we take the derivative.

Remark 1.4 Since $\mathcal{M}_1$ inherits all relevant properties from $\mathcal{M}$, the theorem may be applied iteratively to obtain an isomorphism

$$W^{p,\theta}_{\mathcal{M},0} \cong L^p \oplus L^p \oplus \cdots \oplus L^p \oplus W^{p,\theta_k}_{\mathcal{M}_k,0}, \quad k \in \mathbb{N},$$

where $\theta_k = \theta(1-p)^{-k}$ and $\mathcal{M}_k$ denotes the k-shifted sequence $\{M_{k+n}\}_n$. Note that in principle, the space $W^{p,\theta}_{\mathcal{M}}$ may be even bigger than $W^{p,\theta}_{\mathcal{M},0}$.

We briefly comment on the remaining transition, between non-quasianalyticity and quasianalyticity. Here, for a general linear space of smooth functions, if for each point in the domain of definition, the Borel map is injective, we say that the space is *quasianalytic*. We recall that the Borel map for the given point a is $f \mapsto \{f(a), f'(a), f''(a), \ldots\}$. If the given linear space is not quasianalytic we call it *non-quasianalytic*. For the classical Carleman classes the structure is well understood, but we need a clearcut definition in the setting of the new L^p-Carleman classes $\mathscr{C}_{\mathcal{M}}^{p,\theta}$. Now, for $\theta = 0$, we characterize quasianalyticity for $\mathscr{C}_{\mathcal{M}}^{p,\theta} = \mathscr{C}_{\mathcal{M}}^{p,0}$ in terms of the quasianalyticity of the standard Carleman class $\mathscr{C}_{\mathcal{N}}$ for a certain related sequence $\mathcal{N}$. Moreover, the classical Denjoy–Carleman theorem supplies criteria for when the class $\mathscr{C}_{\mathcal{N}}$ is quasianalytic or non-quasianalytic. The associated sequence $\mathcal{N} = \{N_n\}_n$ is given by

$$N_n := \prod_{j=1}^{\infty} M_{n+j}^{(1-p)^{j-1}p}, \quad k = 0, 1, 2, \ldots,$$

which we recognize as coming from the L^∞-bound of the higher order derivatives in (1.13).

The result runs as follows.

Theorem 1.4 $(0 < p < 1)$ *Assume that $\mathcal{M}$ is logarithmically convex and bounded away from zero. If $\kappa(p, \mathcal{M}) < +\infty$, the following holds*:

(i) *If $\theta > 0$, then $\mathscr{C}_{\mathcal{M}}^{p,\theta}$ is never quasianalytic.*
(ii) *If $\theta = 0$ and $\mathcal{M}$ is decay-regular in the sense of Definition 1.5, then $\mathscr{C}_{\mathcal{M}}^{p,0}$ is quasianalytic if and only if $\mathscr{C}_{\mathcal{N}}$ is quasianalytic.*

Finally, we comment on the dependence of the classes on the parameter θ in the smooth context of Theorem 1.2.

Theorem 1.5 $(0 < p < 1)$ *Let $\mathcal{M}$ be an increasing, log-convex sequence such that $\kappa(p, \mathcal{M}) < +\infty$. Let $0 \le \theta < \theta'$. Then the inclusion $W_{\mathcal{M}}^{p,\theta} \subset W_{\mathcal{M}}^{p,\theta'}$ is strict: $W_{\mathcal{M}}^{p,\theta} \neq W_{\mathcal{M}}^{p,\theta'}$.*

We do not know whether such a strict inclusion holds in the uncoupled regime when $\kappa(p, \mathcal{M}) = +\infty$. It remains a possibility that the spaces are then so large that the parameter θ is not felt. In any case, we are able to show that (Proposition 5.3)

$$c_0(L^p, \mathcal{M}) \subset \boldsymbol{\pi} W_{\mathcal{M},0}^{p,\theta} \subset \boldsymbol{\pi} W_{\mathcal{M}}^{p,\theta} \subset \ell^\infty(L^p, \mathcal{M}). \tag{1.16}$$

Here, $c_0(L^p, \mathcal{M})$ denotes the subspace of $\ell^\infty(L^p, \mathcal{M})$ consisting of sequences $(f_0, f_1, f_2, \ldots)$ with

$$\lim_{n \to +\infty} \frac{\|f_n\|_p}{M_n} = 0.$$

Remark 1.5 (a) In view of the Douady–Peetre disconnexion phenomenon, the fact that for $0 < p < 1$, L^p functions fail to define distributions is a serious obstruction. An alternative approach is to consider the real Hardy spaces H^p in place of L^p, since H^p functions automatically define distributions. The drawback of that approach is that for $p = 1$, H^1 is substantially smaller than L^1. Our theme here is to keep L^p and to let a bootstrap argument (involving infinitely many higher order derivatives) take care of the smoothness, and to explore what happens when the bootstrap argument fails to supply appropriate bounds.

(b) A word on the title. The term *a critical topology* used here is borrowed from Beurling's work [5], where another phase transition is the object of study.

2 Sobolev spaces: Peetre's proof and failure of embedding

2.1 Sobolev spaces for $0 < p < 1$

We fix a number p with $0 < p < 1$ and an integer $k \geq 0$. Following Peetre [18] we consider the Sobolev space $W^{k,p} = W^{k,p}(\mathbb{R})$, defined as the abstract completion of $C_0^k(\mathbb{R})$ with respect to the quasinorm

$$\|f\|_{k,p} = \left(\|f\|_p^p + \|f'\|_p^p + \cdots + \|f^{(k)}\|_p^p\right)^{1/p}. \tag{2.1}$$

The resulting space $W^{k,p}$ is then a quasi-Banach space. Here, $C_0^k(\mathbb{R})$ denotes the space of compactly supported functions in $C^k(\mathbb{R})$.

Remark 2.1 In this paper we shall mostly work on the entire line. If at some place we consider spaces on bounded intervals, this will be explicitly mentioned. The definition of $W^{k,p}(I)$ for a general interval I is entirely analogous.

The space $W^{k,p}$ comes with two canonical mappings, $\boldsymbol{\alpha} = \boldsymbol{\alpha}_k : W^{k,p} \to L^p$, and $\boldsymbol{\delta} = \boldsymbol{\delta}_k : W^{k,p} \to W^{k-1,p}$. These are both initially defined for test functions $f \in C_0^k(\mathbb{R})$ by

$$\boldsymbol{\alpha} f = f, \quad \text{and} \quad \boldsymbol{\delta} f = f'.$$

The mappings $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$ are bounded and densely defined, and hence extend to bounded operators on the entire space $W^{k,p}$.

2.2 Douady–Peetre disconnexion

We begin this section by considering a simple example which explains how a crucial feature differs in the setting of $0 < p < 1$ as compared to the classical Sobolev space case.

We are used to thinking of $W^{k,p}$ as being a certain subspace of L^p, consisting of functions that are sufficiently smooth. As mentioned in the introduction, the first observation, made by Douady, is that this is not the right way to think when $0 < p < 1$. Indeed, the canonical map $\boldsymbol{\alpha}$ is not injective.

Proposition 2.1 (Douady) *There exists* $f \in W^{1,p}([0,1])$ *such that*

$$\boldsymbol{\alpha} f = 0, \quad \boldsymbol{\delta} f = 1.$$

This would suggest that there ought to exist functions that vanish identically but nevertheless the derivative equals the nonzero constant 1. This is of course absurd, and the right way to think about it is to realize that in the completion, the function and its derivative lose contact, they disconnect.

Proof A small argument (see [18, Lemma 2.1]) shows that we are allowed to work with functions whose derivatives have jumps. We let $\{\epsilon_j\}_j$ be a sequence of positive reals, such that $j\epsilon_j \to 0$ as $j \to +\infty$, and define f_j on the interval $[0, \frac{1}{j} + \epsilon_j]$ by

$$f_j(x) = \begin{cases} x, & 0 \le x \le \frac{1}{j}, \\ (\frac{1}{j} + \epsilon_j - x)/(j\epsilon_j), & \frac{1}{j} < x \le \frac{1}{j} + \epsilon_j, \end{cases}$$

and extend it *periodically* to $\mathbb{R}$ with period $\frac{1}{j} + \epsilon$. The resulting function f_j will be a skewed saw-tooth function that rises slowly with slope 1 and then drops steeply. By differentiating f_j, we have that

$$1 - f_j'(x) = \begin{cases} 0, & 0 \le x < \frac{1}{j}, \\ 1 + \frac{1}{j\epsilon_j}, & \frac{1}{j} < x < \frac{1}{j} + \epsilon_j. \end{cases}$$

Since f_j assumes values between 0 and $\frac{1}{j}$, it is clear that $f_j \to 0$ as $j \to +\infty$ in L^∞ and hence in L^p. Within the interval $[0, 1]$, there are at most j full periods of the function f_j, which allows us to estimate

$$\begin{aligned} \int_{[0,1]} |1 - f_j'(x)|^p \mathrm{d}x &\le j\epsilon_j \left(1 + \frac{1}{j\epsilon_j}\right)^p \le j\epsilon_j (j\epsilon_j)^{-p} \\ &= (j\epsilon_j)^{1-p} \to 0 \quad \text{as } j \to +\infty. \end{aligned}$$

In view of the above observations, $f_j \to 0$ while $f_j' \to 1$, both in the quasinorm of L^p, as $j \to +\infty$. In particular, $\{f_j\}_j$ is a Cauchy sequence, and if we let f denote the abstract limit in the completion, we find that $\boldsymbol{\alpha} f = 0$ while $\boldsymbol{\delta} f = 1$. □

2.3 The isomorphism and construction of the canonical lifts

We fix an integer $k = 1, 2, 3, \ldots$ and an exponent $0 < p < 1$.

The space $L^p \oplus W^{k-1,p}$ consists of pairs (g, h), where $g \in L^p$ and $h \in W^{k-1,p}$, and we equip it with the quasinorm

$$\|(g, h)\|^p = \|g\|_p^p + \|h\|_{k-1,p}^p, \quad g \in L^p, \; h \in W^{k-1,p}.$$

From the definition of the norm (2.1), we see that the operator $\mathbf{A}\colon W^{k,p} \to L^p \oplus W^{k-1,p}$ given by $\mathbf{A}f := (\boldsymbol{\alpha} f, \boldsymbol{\delta} f)$ is an *isometry*. Indeed, for $f \in C_0^k(I)$, we have that

$$\|\mathbf{A}f\|^p = \|(\boldsymbol{\alpha} f, \boldsymbol{\delta} f)\|^p = \|\boldsymbol{\alpha} f\|_p^p + \|\boldsymbol{\delta} f\|_{k-1,p}^p = \|f\|_p^p + \|f'\|_{k-1,p}^p = \|f\|_{k,p}^p, \tag{2.2}$$

and this property survives the completion process. If $\mathbf{A}$ can be shown to be surjective, then it is an isometric isomorphism $\mathbf{A}\colon W^{k,p} \to L^p \oplus W^{k-1,p}$. Proceeding iteratively with $W^{k-1,p}$, we obtain the desired decomposition, since clearly $W^{0,p} = L^p$.

To obtain the surjectivity of $\mathbf{A}$, we shall construct two canonical lifts, $\boldsymbol{\beta}\colon L^p \to W^{k,p}$ and $\boldsymbol{\gamma}\colon W^{k-1,p} \to W^{k,p}$ of $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$, respectively. These are injective mappings, from L^p and $W^{k-1,p}$ to $W^{k,p}$, respectively, satisfying certain relations with $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$. The properties of these are summarized in the following lemma (the notation id_X stands for the identity mapping on the space X). The details of the construction are postponed until Sect. 3.2.

Lemma 2.1 *For each $k = 1, 2, 3, \ldots$, there exist bounded linear mappings $\boldsymbol{\beta}\colon L^p \to W^{k,p}$ and $\boldsymbol{\gamma}\colon W^{k-1,p} \to W^{k,p}$, such that*

$$\boldsymbol{\alpha}\boldsymbol{\beta} = \mathrm{id}_{L^p}, \quad \boldsymbol{\delta}\boldsymbol{\gamma} = \mathrm{id}_{W^{k-1,p}}, \quad \boldsymbol{\delta}\boldsymbol{\beta} = 0, \quad \boldsymbol{\alpha}\boldsymbol{\gamma} = 0.$$

With this result at hand, the proof of the main theorem about the $W^{k,p}$-spaces becomes a simple exercise.

Proof of Theorem 1.1 As noted above, it will be enough to show that the isometry $\mathbf{A}\colon W^{k,p} \to L^p \oplus W^{k-1,p}$ given by $\mathbf{A}f := (\boldsymbol{\alpha} f, \boldsymbol{\delta} f)$ is surjective. To this end, we pick $(g, h) \in L^p \oplus W^{k-1,p}$. Then $\boldsymbol{\beta} g$ and $\boldsymbol{\gamma} h$ are both elements of $W^{k,p}$, and so is their sum

$$f = \boldsymbol{\beta} g + \boldsymbol{\gamma} h \in W^{k,p}.$$

It now follows from Lemma 2.1 that

$$\mathbf{A}f = (\boldsymbol{\alpha}(\boldsymbol{\beta} g + \boldsymbol{\gamma} h), \boldsymbol{\delta}(\boldsymbol{\beta} g + \boldsymbol{\gamma} h)) = (\boldsymbol{\alpha}\boldsymbol{\beta} g + \boldsymbol{\alpha}\boldsymbol{\gamma} h, \boldsymbol{\delta}\boldsymbol{\beta} g + \boldsymbol{\delta}\boldsymbol{\gamma} h) = (g, h).$$

As a consequence, $\mathbf{A}$ is surjective, and hence $\mathbf{A}$ induces an isometric isomorphism

$$W^{k,p} \cong L^p \oplus W^{k-1,p}.$$

By iteration of the same argument, the claimed decomposition of $W^{k,p}$ follows. □

Remark 2.2 The lift $\boldsymbol{\gamma}$ does not appear in Peetre's work [18]. Reading between the lines one can discern its role, but here we fill in the blanks and treat it explicitly.

3 Construction of lifts, and invisible mollifiers

3.1 A collection of smooth functions by iterated convolution

For an integer $k \geq 0$ and a real $0 \leq \alpha \leq 1$ let $C^{k,\alpha}$ denote the class of k times continuously differentiable functions, whose derivative of order k is Hölder continuous with exponent α. Given two functions $f, g \in L^1(\mathbb{R})$, their convolution $f * g \in L^1(\mathbb{R})$ is as usual given by

$$(f * g)(x) = \int_{\mathbb{R}} f(x-t)g(t)\mathrm{d}t, \quad x \in \mathbb{R}.$$

For $a > 0$, we let the function H_a denote the normalized characteristic function $H_a = a^{-1}1_{[0,a]}$. For a decreasing sequence of positive reals $a_1, a_2, a_3, \ldots$, consider the associated repeated convolutions (for $j \leq k$)

$$\Phi_{j,k} := H_{a_j} * \cdots * H_{a_k}. \tag{3.1}$$

The function $\Phi_{j,k}$ then has compact support $[0, a_j + \cdots a_k]$ and belongs to the smoothness class $C^{k-j-1,1}$ which means that the derivative of order $k - j - 1$ is Lipschitz continuous. We will assume that the sequence $a_1, a_2, a_3, \ldots$ decreases to 0 at least fast enough for $(a_j)_{j\geq 1} \in \ell^1$ to hold. Then we may form the limits

$$\Phi_{j,\infty} := \lim_{k\to+\infty} \Phi_{j,k}, \quad j = 1, 2, 3, \ldots,$$

and see that each such limit $\Phi_{j,\infty}$ is C^∞-smooth with support $[0, a_j + a_{j+1} + \ldots]$. Moreover, we have the sup-norm controls

$$\|\Phi_{j,k}\|_\infty \leq \frac{1}{a_j}, \quad \|\Phi_{j,\infty}\|_\infty \leq \frac{1}{a_j}. \tag{3.2}$$

Next, since for $l < k$

$$\Phi_{j,k} = \Phi_{j,l} * \Phi_{l+1,k} \quad \text{and} \quad \Phi_{j,\infty} = \Phi_{j,l} * \Phi_{l+1,\infty},$$

we may calculate the higher order derivatives by the formula

$$\Phi_{j,k}^{(n)} = \Phi_{j,j+n-1}^{(n)} * \Phi_{j+n,k} \quad \text{and} \quad \Phi_{j,\infty}^{(n)} = \Phi_{j,j+n-1}^{(n)} * \Phi_{j+n,\infty},$$

interpreted when needed in the sense of distribution theory. Here, we should ask that $j + n \leq k + 1$ in the first formula. By calculation,

$$\Phi_{j,j+n-1}^{(n)} = \frac{1}{a_j \cdots a_{j+n-1}}(\delta_{a_j} - \delta_0) * \cdots * (\delta_{a_{j+n-1}} - \delta_0),$$

which when expanded out is the sum of delta masses at 2^n (generically distinct) points, each with mass $(a_j \cdots a_{j+n-1})^{-1}$. By the convolution norm inequality $\|f * g\|_\infty \leq \|f\|_1 \|g\|_\infty$, where the L^1 norm may be extended to the finite Borel measures, we have that

$$\|\Phi_{j,k}^{(n)}\|_\infty = \|\Phi_{j,j+n-1}^{(n)}\|_1 \|\Phi_{j+n,k}\|_\infty \leq \frac{2^n}{a_j \cdots a_{j+n}},$$

where we used the estimate (3.2). The analogous estimate holds for $k = \infty$ as well:

$$\|\Phi_{j,\infty}^{(n)}\|_\infty \leq \frac{2^n}{a_j \cdots a_{j+n}}. \tag{3.3}$$

We need to estimate the L^p-norm of the function $\Phi_{j,\infty}^{(n)}$ as well. The standard norm estimate for convolutions is $\|f * g\|_q \leq \|f\|_1 \|g\|_q$ which holds provided that $1 \leq q \leq +\infty$. For our small exponents $0 < p < 1$ this is no longer true. However, there is a substitute, provided f is a finite sum of point masses:

$$\|f * g\|_p^p \leq \|f\|_{\ell^p}^p \|g\|_p^p, \quad \text{where } \|f\|_{\ell^p}^p = \sum_j |b_j|^p \text{ if } f = \sum_j b_j \delta_{x_j}, \tag{3.4}$$

for some finite collection of reals x_j. This follows immediately from the p-triangle inequality and the translation invariance of the L^p-norm. In our present context we see that

$$\begin{aligned}\|\Phi_{j,k}^{(n)}\|_p^p &= \|\Phi_{j,j+n-1}^{(n)} * \Phi_{j+n,k}\|_p^p \leq \|\Phi_{j,j+n-1}^{(n)}\|_{\ell^p}^p \|\Phi_{j+n,k}\|_p^p \\ &\leq \frac{2^n}{(a_j \cdots a_{j+n})^p} \sum_{l=j+n}^{k} a_l,\end{aligned} \tag{3.5}$$

where $n + j \leq k + 1$. Correspondingly for $k = +\infty$ we have that

$$\begin{aligned}\|\Phi_{j,\infty}^{(n)}\|_p^p &= \|\Phi_{j,j+n-1}^{(n)} * \Phi_{j+n,\infty}\|_p^p \leq \|\Phi_{j,j+n-1}^{(n)}\|_{\ell^p}^p \|\Phi_{j+n,\infty}\|_p^p \\ &\leq \frac{2^n}{(a_j \cdots a_{j+n})^p} \sum_{l=j+n}^{+\infty} a_l.\end{aligned} \tag{3.6}$$

3.2 Existence of invisible mollifiers in $W^{k,p}$

We now employ the repeated convolution procedure of Sect. 3.1, to exhibit mollifiers with L^p-vanishing properties.

Lemma 3.1 (Invisibility lemma) *Let $k \geq 0$ be an integer. Then for any given $\epsilon > 0$, there exists a non-negative function $\Phi \in C^{k-1,1}(\mathbb{R})$ such that*

$$\int_{\mathbb{R}} \Phi \, dx = 1, \quad \operatorname{supp}(\Phi) \subset [0, \epsilon], \quad \|\Phi\|_{k,p} < \epsilon.$$

Proof For positive decreasing numbers $\{a_j\}_{j=1}^{k+1}$ (to be determined), let $\Phi = \Phi_{1,k+1}$ be given by (3.1). Then clearly $\Phi \in C^{k-1,1}(\mathbb{R})$ with $\Phi \geq 0$ and $\int_{\mathbb{R}} \Phi \, dx = 1$. Moreover, the support of Φ equals $[0, a_1 + \cdots + a_{k+1}]$. By the estimate (3.5), it follows that

$$\|\Phi^{(n)}\|_p^p \leq \frac{2^n}{(a_1 \cdots a_{n+1})^p} \sum_{l=n+1}^{k+1} a_l. \tag{3.7}$$

We need to show that the finite sequence $\{a_l\}_{l=1}^{k+1}$ may be chosen such that the sum of the right-hand side in (3.7) over $0 \leq n \leq k$ is bounded by ϵ^p, while at the same time $\sum_{l=1}^{k+1} a_l \leq \epsilon$. As a first step, *we assume that $a_{j+1} \leq \frac{1}{2} a_j$ for integers $n \geq 0$*, and observe that it then follows that $\sum_{l=j}^{k+1} a_l \leq 2a_j$. Consequently, we get that $\operatorname{supp} \Phi \subset [0, 2a_0]$ and

$$\|\Phi^{(n)}\|_p^p \leq 2^{n+1} \frac{a_{n+1}}{(a_1 \cdots a_{n+1})^p}, \quad n = 0, \ldots, k.$$

We put

$$a_1 := \min \left\{ \left(\frac{\epsilon^p}{2(k+1)} \right)^{1/(1-p)}, \frac{\epsilon}{2} \right\}$$

and successively declare that

$$a_l := \min \left\{ \left(\frac{\epsilon^p (a_1 \cdots a_{l-1})^p}{2^l (k+1)} \right)^{1/(1-p)}, \frac{a_{l-1}}{2} \right\}, \quad l = 2, \ldots k+1.$$

It then follows that

$$\|\Phi^{(n)}\|_p^p \leq 2^{n+1} \frac{a_{n+1}}{a_1^p \cdots a_{n+1}^p} = a_{n+1}^{1-p} \frac{2^{n+1}}{(a_0 \cdots a_n)^p} \leq \frac{\epsilon^p}{k+1}, \quad n = 0, \ldots, k.$$

whence $\|\Phi\|_{p,k} \leq \epsilon$ and since also $\operatorname{supp} \Phi \subset [0, 2a_0] \subset [0, \epsilon]$, the constructed function Φ meets all the specifications. □

3.3 The definition of the lift β for $0 < p < 1$

The lift β maps boundedly $L^p \to W^{k,p}$, and we need to explain how it gets to be defined. Let $\mathscr{F}$ denote the collection of *step functions*, which we take to be the finite linear combination of characteristic functions of bounded intervals, and when also equip it with the quasinorm of L^p, we denote it by $\mathscr{F}_p := (\mathscr{F}, \|\cdot\|_p)$. We note that $\mathscr{F}_p$ is quasinorm dense in L^p. We first define $\boldsymbol{\beta} g$ for $g \in \mathscr{F}_p$. For $g \in \mathscr{F}_p$, we will write down a $W^{k,p}$-Cauchy sequence $\{g_j\}_j$ of test functions $g_j \in C_0^k(\mathbb{R})$, and declare $\boldsymbol{\beta} g \in W^{k,p}$ to be the abstract limit of the Cauchy sequence g_j as $j \to +\infty$.

We will require the following properties of the test functions g_j:

$$\lim_{j\to+\infty} \|g - g_j\|_p = 0 \quad \text{and} \quad \lim_{j\to+\infty} \|g_j'\|_{k-1,p} = 0. \tag{3.8}$$

If (3.8) can be achieved, then $\boldsymbol{\beta}\colon \mathscr{F}_p \to W^{k,p}$ becomes an isometry by the following calculation:

$$\|\boldsymbol{\beta} g\|_{k,p}^p = \lim_{j\to+\infty} \|g_j\|_{k,p}^p = \lim_{j\to+\infty} \left(\|g_j\|_p^p + \|g_j'\|_{k-1,p}^p\right) = \|g\|_p^p, \quad g \in \mathscr{F}.$$

These properties uniquely determine $\boldsymbol{\beta} g$ for $g \in \mathscr{F}$. Indeed, if $\tilde{g}_j$ were another Cauchy sequence satisfying (3.3), then $\{\tilde{g}_j\}_j$ and $\{g_j\}_j$ are equivalent as Cauchy sequences, in light of

$$\begin{aligned}\|\tilde{g}_j - g_j\|_{k,p}^p &= \|\tilde{g}_j - g_j\|_p^p + \|\tilde{g}_j' - g_j'\|_{k-1,p}^p \\ &\le \|\tilde{g}_j - g_j\|_p^p + \|\tilde{g}_j'\|_{k-1,p}^p + \|g_j'\|_{k-1,p}^p.\end{aligned}$$

In particular, $\boldsymbol{\beta}\colon \mathscr{F}_p \to W^{k,p}$ is then a well-defined bounded operator, and since $\mathscr{F}_p$ is dense in L^p it extends uniquely to a bounded operator $\boldsymbol{\beta}\colon L^p \to W^{k,p}$ which is actually an isometry.

In view of the above, it will be enough to define $\boldsymbol{\beta} g$ when g is the characteristic function of an interval $g = 1_{[a,b]}$ and to check (3.8) for it, since general step functions in $\mathscr{F}$ are obtained using finite linear combinations.

Let $\{\epsilon_j\}_{j=0}^{+\infty}$ be a sequence of numbers tending to zero with $0 < \epsilon_j < \frac{1}{2}(b-a)$. By Lemma 3.1 applied to $W^{k-1,p}$, there exists non-negative functions $\Phi_{\epsilon_j} \in C^{k-1}$, such that

$$\int_{\mathbb{R}} \Phi_{\epsilon_j} \mathrm{d}x = 1, \quad \operatorname{supp} \Phi_{\epsilon_j} \subset [0, \epsilon_j], \quad \left\|\Phi_{\epsilon_j}\right\|_{k-1,p} < \epsilon_j.$$

We define g_j by convolution: $g_j := \Phi_{\epsilon_j} * 1_{[a,b]}$. It is then clear that $g_j - g$ has support on $[a, a+\epsilon_j] \cup [b, b+\epsilon_j]$, and there, it is bounded by 1 in modulus. As a consequence,

$$\|g_j - g\|_p^p \le 2\epsilon_j,$$

so $g_j \to g$ in L^p. Next, we consider the derivative g_j', which we may express as $g_j' = (\tau_a - \tau_b)\Phi_{\epsilon_j}$, where we recall that τ with subscript is a translation operator. It is clear that

$$\|g_j'\|_{k-1,p}^p \le 2\|\Phi_{\epsilon_j}\|_{k-1,p}^p \le 2\epsilon_j.$$

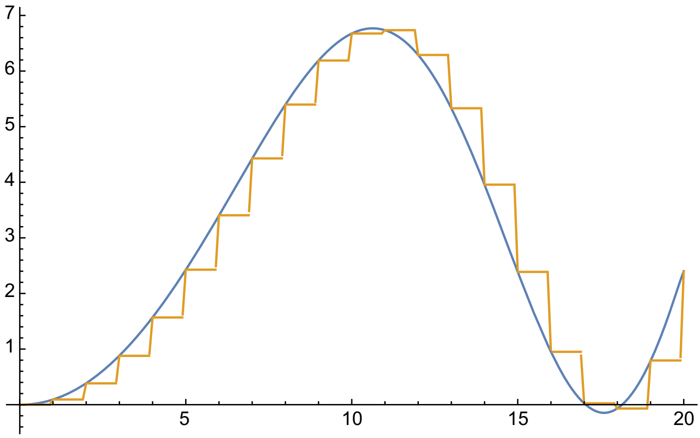

Fig. 1 The construction of $\boldsymbol{\beta}$ for $W^{1,p}$

Consequently, $\|g_j'\|_{k-1,p}^p$ also tends to zero, as needed. This establishes (3.8) (Fig. 1).

Proof of Lemma 2.1, part 1 We show that $\boldsymbol{\alpha\beta} = \mathrm{id}_{L^p}$ and $\boldsymbol{\delta\beta} = 0$. Since $\boldsymbol{\beta} g \in W^{k,p}$ is the abstract limit of the Cauchy sequence g_j with (3.8), and by definition $\boldsymbol{\alpha} g_j = g_j$ and $\boldsymbol{\delta} g_j = g_j'$, it follows from (3.8) that $\boldsymbol{\alpha\beta} g = g$ and $\boldsymbol{\delta\beta} g = 0$ for every $g \in L^p$. The assertion follows. □

3.4 The lift $\boldsymbol{\gamma}$

To construct $\boldsymbol{\gamma}$, we let $g \in W^{k-1,p}$ be an arbitrary element, which is by definition the abstract limit of some Cauchy sequence $\{g_j\}_j$, where $g_j \in C_0^{k-1}(\mathbb{R})$. For any given $\epsilon > 0$, Lemma 3.1 provides a function $\Phi_\epsilon \in C_0^{k-1}(\mathbb{R})$ with $\Phi_\epsilon \geq 0$, $\langle \Phi_\epsilon \rangle_{\mathbb{R}} := \int_{\mathbb{R}} \Phi_\epsilon(t)\mathrm{d}t = 1$, supported in $[0, \epsilon]$, while at the same time, $\|\Phi_\epsilon\|_{k-1,p} < \epsilon$. We use the functions Φ_{ϵ_j} to modify each $g_j(x)$ to have vanishing zeroth moment, by defining

$$\tilde{g}_j(x) := g_j(x) - \langle g_j \rangle_{\mathbb{R}} \Phi_{\epsilon_j}(x), \quad \langle g_j \rangle_{\mathbb{R}} := \int_{\mathbb{R}} g_j(t)\mathrm{d}t,$$

where the ϵ_j are chosen to tend to zero so quickly that

$$\lim_{j \to +\infty} \|g_j - \tilde{g}_j\|_{k-1,p} = \lim_{j \to +\infty} \|\Phi_{\epsilon_j}\|_{k-1,p} |\langle g_j \rangle_{\mathbb{R}}| = 0.$$

Next, we define the functions u_j as primitives:

$$u_j(x) = \int_{-\infty}^{x} \tilde{g}_j(t)\mathrm{d}t, \quad x \in \mathbb{R}.$$

Then as $\tilde{g}_j$ has integral 0, we see that $u_j \in C_0^k(\mathbb{R})$. We put $f_j := u_j - \boldsymbol{\beta} u_j \in W^{k,p}$, and observe that from the known properties of $\boldsymbol{\beta}$, it follows that

$$\boldsymbol{\alpha} f_j = \boldsymbol{\alpha} u_j - \boldsymbol{\alpha}\boldsymbol{\beta} u_j = u_j - u_j = 0 \quad \text{and} \quad \boldsymbol{\delta} f_j = \boldsymbol{\delta} u_j - \boldsymbol{\delta}\boldsymbol{\beta} u_j = \boldsymbol{\delta} u_j = u_j' = \tilde{g}_j. \tag{3.9}$$

Then, from the isometry of $\mathbf{A}\colon W^{k,p} \to L^p \oplus W^{k-1,p}$ (see (2.2)), we have that

$$\begin{aligned} \|f_j\|_{k,p}^p &= \|\boldsymbol{\alpha} f_j\|_p^p + \|\boldsymbol{\delta} f_j\|_{k-1,p}^p \\ &= \|\tilde{g}_j\|_{k-1,p}^p \le \|g_j\|_{k-1,p}^p + \|\Phi_{\epsilon_j}\|_{k-1,p}^p |\langle g_j \rangle_{\mathbb{R}}|^p, \end{aligned} \tag{3.10}$$

where in the last step, we applied the p-triangle inequality. A similar verification shows that $\{f_j\}_j$ is a Cauchy sequence, so that it has a limit $\boldsymbol{\gamma} g := \lim_{j\to+\infty} f_j$ in $W^{k,p}$. Moreover, in view of (3.9), it follows that

$$\boldsymbol{\alpha}\boldsymbol{\gamma} g = \lim_{j\to+\infty} \boldsymbol{\alpha} f_j = 0 \quad \text{and} \quad \boldsymbol{\delta}\boldsymbol{\gamma} g = \lim_{j\to+\infty} \boldsymbol{\delta} f_j = \lim_{j\to+\infty} \tilde{g}_j = g, \tag{3.11}$$

in L^p and $W^{k-1,p}$, respectively. In the construction of the sequence of functions f_j, there is some arbitrariness e.g. in the choice of the sequence of the ϵ_j (they were just asked to tend to 0 sufficiently quickly). To investigate whether this matters, we suppose another Cauchy sequence $\{F_j\}_j$ in $W^{k,p}$ is given, with properties that mimic (3.9): that $\boldsymbol{\alpha} F_j = 0$ in L^p, and that for some Cauchy sequence $\{G_j\}_j$ in $W^{k-1,p}$ converging to $g \in W^{k-1,p}$, we know that $\boldsymbol{\delta} F_j = G_j$, then

$$\begin{aligned} \|F_j - f_j\|_{k,p}^p &= \|\mathbf{A}(F_j - f_j)\|_{L^p \oplus W^{k-1,p}}^p = \|\boldsymbol{\alpha}(F_j - f_j)\|_p^p + \|\boldsymbol{\delta}(F_j - f_j)\|_{k-1,p}^p \\ &= \|G_j - \tilde{g}_j\|_{k-1,p}^p \le \|G_j - g_j\|_{k-1,p}^p + \|\Phi_{\epsilon_j}\|_{k-1,p}^p |\langle g_j \rangle_{\mathbb{R}}|^p \to 0, \\ &\quad \text{as } j \to +\infty, \end{aligned}$$

by the isometric properties of $\mathbf{A}$. If we let F denote the abstract limit of the Cauchy sequence F_j in $W^{k,p}$, we conclude that $F = f$ in $W^{k,p}$. In conclusion, it did not matter whether we used the prescribed Cauchy sequence or a competitor, and hence $\boldsymbol{\gamma} g \in W^{k,p}$ is well-defined for $g \in W^{k-1,p}$. Finally, we observe from (3.10) that

$$\|\boldsymbol{\gamma} g\|_{k,p} \le \|g\|_{k-1,p}, \quad g \in W^{k-1,p},$$

which makes $\boldsymbol{\gamma}\colon W^{k-1,p} \to W^{k,p}$ a linear contraction (Fig. 2).

Proof of Lemma 2.1, part 2 We show that the remaining properties, those involving $\boldsymbol{\gamma}$: $\boldsymbol{\alpha}\boldsymbol{\gamma} = 0$ and $\boldsymbol{\delta}\boldsymbol{\gamma} = \mathrm{id}_{W^{k-1,p}}$. We read off from (3.9) that $\boldsymbol{\alpha}\boldsymbol{\gamma} g = 0$ and $\boldsymbol{\delta}\boldsymbol{\gamma} g = g$ for $g \in W^{k-1,p}$, which does it. □

Remark 3.1 A classical theorem by Day [11] states that the L^p-spaces for $0 < p < 1$ have trivial dual. It follows from Douady–Peetre's isomorphism $W^{k,p} \cong L^p \oplus \cdots \oplus L^p$ that the same is true for the Sobolev spaces $W^{k,p}$. We note here that any space that

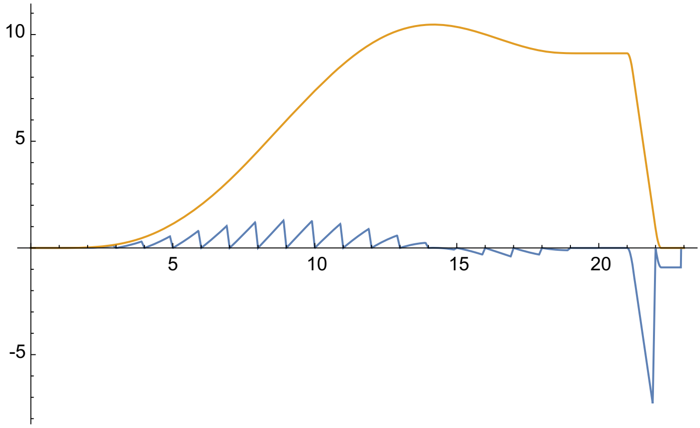

Fig. 2 The functions f_j and u_j, for $W^{1,p}$

could be realised as a space of distributions would necessarily admit nontrivial bounded functionals (the test functions, for instance). Hence it follows that the elements of $W^{k,p}$ cannot even be interpreted as distributions.

4 The smooth regime

4.1 Classes of test functions

Although we mainly focus on the classes $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$, we also mention the Hermite class $\mathscr{S}^{\mathrm{Her}}$ of weighted polynomials

$$\mathscr{S}^{\mathrm{Her}} = \left\{ f : \; f(x) = q(x)\mathrm{e}^{-x^2}, \;\; \text{where } q \in \mathrm{Pol}(\mathbb{R}) \right\},$$

where $\mathrm{Pol}(\mathbb{R})$ denotes the linear space of all polynomials of a real variable. We show that $\mathscr{S}^{\mathrm{Her}}$ consists of $(p, 0)$-tame functions. To get a class which fits into the associated L^p-Carleman class, we also intersect the Hermite class with the space $\{f \in C^\infty(\mathbb{R}) : \|f\|_{p,\mathcal{M}} < +\infty\}$ to obtain the $\mathscr{S}^{\mathrm{Her}}_{p,\mathcal{M}}$.

Lemma 4.1 *The Hermite class $\mathscr{S}^{\mathrm{Her}}$ consists of $(p, 0)$-tame functions.*

Proof Let $f \in \mathscr{S}^{\mathrm{Her}}$. Then $f(x) = q(x)\mathrm{e}^{-x^2}$ for some polynomial $q \in \mathrm{Pol}(\mathbb{R})$. Let d be the degree of q, and assume that all coefficients of the polynomial $q(x)$ are bounded by some number $m = m_q$. Let $\mathbf{L}$ be the operator on $\mathrm{Pol}(\mathbb{R})$ defined by the relation

$$\mathbf{L}q(x) = \mathrm{e}^{x^2} \frac{\mathrm{d}}{\mathrm{d}x}\left(q(x)\mathrm{e}^{-x^2}\right) = q'(x) - 2xq(x), \quad q \in \mathrm{Pol}(\mathbb{R}).$$

It follows that the Taylor coefficients $\widehat{\mathbf{L}q}(j)$ of $\mathbf{L}q$ can be estimated rather crudely in terms of d and m_q:

$$|\widehat{\mathbf{L}q}(j)| \le (d+2)m_q, \quad j = 0, 1, \ldots, d+1, \tag{4.1}$$

and the coefficients vanish for $j > d+1$, so that the degree of $\mathbf{L}q$ is at most $d+1$. By repeating the same argument, with $\mathbf{L}q$ in place of q, we obtain

$$|\widehat{\mathbf{L}^2 q}(j)| \le (d+3)(d+2)m_q,$$

and, by iteration of (4.1), it follows more generally that for $j = 0, 1, 2, \ldots,$

$$\begin{aligned} |\widehat{\mathbf{L}^n q}(j)| &\le (d+2+n-1)(d+n-2)\cdots(d+2)m_q \\ &= (d+2)_n m_q, \quad n = 1, 2, \ldots, \end{aligned} \tag{4.2}$$

where we use the standard Pochhammer notation $(x)_k = x(x+1)\cdots(x+k-1)$ for $x \in \mathbb{R}$.

We proceed to estimate the L^∞-norm of $f^{(n)}$. By the way $\mathbf{L}$ was defined, we may estimate

$$\begin{aligned} \|f^{(n)}\|_\infty &= \sup_{x\in\mathbb{R}} |\mathrm{e}^{-x^2}\mathbf{L}^n q(x)| = \sup_{x\in\mathbb{R}} \left| \mathrm{e}^{-x^2} \sum_{j=0}^{d+n} \widehat{\mathbf{L}^n q}(j) x^j \right| \\ &\le \sum_{j=0}^{d+n} |\widehat{\mathbf{L}^n q}(j)| \sup_{x\in\mathbb{R}} |x|^j \mathrm{e}^{-x^2}. \end{aligned}$$

By trivial calculus, the supremum on the right-hand side is attained at $|x| = \sqrt{j/2}$, and this we may implement in the above estimate while we recall the coefficient estimate (4.2):

$$\begin{aligned} \|f^{(n)}\|_\infty &\le m_q (d+2)_n \sum_{j=0}^{d+n} \left(\frac{j}{2}\right)^{j/2} \mathrm{e}^{-j/2} \\ &\le m_q (d+2)_{n+1} \left(\frac{d+n}{2}\right)^{(d+n)/2} \mathrm{e}^{-(d+n)/2}, \end{aligned} \tag{4.3}$$

where in the last step we just estimated by the number of terms multiplied by the largest term. Next, we take logarithms and apply elementary estimates to arrive at

$$\begin{aligned} &(1-p)^n \log \|f^{(n)}\|_\infty \le (1-p)^n \\ &\quad \times \left\{ \log m_q + (n+1)\log(d+n+2) + \frac{d+n}{2}\left(\log\frac{d+n}{2} - 1\right) \right\}. \end{aligned}$$

As the expression in brackets is of growth order $\mathrm{O}\left((n+d)\log(n+d)\right)$, it follows that the right-hand side expression tends to 0 as $n \to +\infty$. It is then immediate that f is $(p, 0)$-tame, and since f was an arbitrary element $f \in \mathscr{S}^{\mathrm{Her}}$, the claim follows. □

Next, we turn to the question of what is required of the weight sequence $\mathcal{M}$ in order for $\mathscr{S}^{\mathrm{Her}}_{p,\mathcal{M}}$ to contain nontrivial functions. We thus need to estimate the L^p-norms of $f^{(n)}$ for $f \in \mathscr{S}^{\mathrm{Her}}$. By performing the same estimates as in the above proof and by appealing to the p-triangle inequality (which says that $(a+b)^p \le a^p + b^p$ for $0 < p \le 1$ and positive a and b), we see that

$$\|f^{(n)}\|_p^p = \int_{\mathbb{R}} |f^{(n)}(x)|^p \mathrm{d}x \le m_q^p (d+2)_n^p \sum_{j=0}^{d+n} \int_{\mathbb{R}} |x|^{pj} \mathrm{e}^{-px^2} \mathrm{d}x.$$

The integrals on the right-hand side are easily evaluated:

$$\int_{\mathbb{R}} |x|^{pj} \mathrm{e}^{-px^2} \mathrm{d}x = p^{-(pj+1)/2} \Gamma((pj+1)/2),$$

so that the above gives the estimate

$$\begin{aligned} \|f^{(n)}\|_p^p &\le m_q^p (d+2)_n^p \sum_{j=0}^{d+n} p^{-(pj+1)/2} \Gamma((pj+1)/2) \\ &\le m_q^p (d+n+1)(d+2)_n^p p^{-(pd+pn+1)/2} \Gamma((pd+pn+1)/2). \end{aligned}$$

Next, using Stirling's formula yields the estimate that

$$\Gamma((pd+pn+1)/2) = \mathrm{O}\left((n!)^{p/2} n^{p(d-1)/2} \left(\frac{p}{2}\right)^{pn/2} \right),$$

whence

$$\begin{aligned} & m_q^p (d+n+1)(d+2)_n^p p^{-(pd+pn+1)/2} \Gamma((pd+pn+1)/2) \\ & = \mathrm{O}\left(\frac{n^{1+\frac{3p(d-1)}{2}+2p}}{2^{pn}} (n!)^{p+\frac{p}{2}} \right) \end{aligned}$$

Ignoring the specific constants, we find that

$$\|f^{(n)}\|_p^p = \mathrm{O}\left((n!)^{3p/2} \frac{n^\alpha}{\beta^n} \right),$$

where $\alpha > 0$ and $\beta > 1$ are some constants, and hence it follows that $\|f^{(n)}\|_p = \mathrm{O}\left((n!)^{3/2}\right)$. We do not proceed to analyse in full detail what $\mathcal{M} = \{M_n\}$ should have to fulfill in order for $\mathscr{S}^{\mathrm{Her}}_{p,M}$ to be a meaningful test class, but we note that the above implies that $\mathscr{S}^{\mathrm{Her}}_{p,M} = \mathscr{S}^{\mathrm{Her}}$ if e.g. $\mathcal{M} = \{M_n\}_n$ meets $M_n \geq (n!)^\sigma$ for any $\sigma \geq 3/2$, which we understand as a *Gevrey class* condition.

4.2 The smooth regime: Theorem 1.2

We have already presented the bootstrap argument, which is the main ingredient in the proof of Theorem 1.2, in the introduction. We proceed to fill in the remaining details.

Proof of Theorem 1.2 We consider the canonical mapping $\boldsymbol{\pi}: f \mapsto (f, f', f'', \ldots)$, defined initially from $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$ into $\ell^\infty(L^p, \mathcal{M})$. The quasinorm on the sequence space $\ell^\infty(L^p, \mathcal{M})$ is supplied in Eq. (1.15). Then, as a matter of definition,

$$\|\boldsymbol{\pi} f\|_{\ell^\infty(L^p,\mathcal{M})} = \|f\|_{p,\mathcal{M}}, \quad f \in \mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$$

and passing to the completion the mapping extends to an isometry $\boldsymbol{\pi}: W^{p,\theta}_{\mathcal{M}} \to \ell^\infty(L^p, \mathcal{M})$. In particular, the mapping $\boldsymbol{\pi}$ is injective. We recall that we think of an element $f \in W^{p,\theta}_{\mathcal{M}}$ as an abstract limit of a Cauchy sequence $\{f_j\}_j$ in the norm $\|\cdot\|_{p,\mathcal{M}}$, with $f_j \in \mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$. For such an element, the mapping is obtained by taking the L^p-limit in all coordinates (that is, the sequence of higher order derivatives). This is well-defined, since if we were to take two Cauchy sequences $\{f_j\}$ and $\{\tilde{f}_j\}$ with the same abstract limit $f \in W^{p,\theta}_{\mathcal{M}}$, the images under the mapping would agree in each coordinate since

$$\left\| f_j^{(n)} - \tilde{f}_j^{(n)} \right\|_p \leq M_n \|f_j - \tilde{f}_j\|_{p,\mathcal{M}}.$$

Next, we show that the image of $W^{p,\theta}_{\mathcal{M}}$ under the mapping is actually inside $C^\infty \times C^\infty \times \cdots$. Indeed, since all the differences $f_j - f_k$ are in $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$, it follows from (1.13) that

$$\left\| f_j^{(n)} - f_k^{(n)} \right\|_\infty \leq \left\{ \mathrm{e}^{\theta(1-p)^{-n} + p(1-p)^{-n-1}\kappa(p,\mathcal{M})} \prod_{l=1}^{n} M_l^{-(1-p)^{l-n-1}p} \right\} \|f_j - f_k\|_{p,\mathcal{M}},$$

where the right hand side tends to zero as $j, k \to +\infty$. It is now pretty obvious that under the finite p-characteristic condition $\kappa(p, \mathcal{M}) < +\infty$, each function $f_j^{(n)}$ has a limit $g_n \in C(\mathbb{R})$ as $j \to +\infty$. Moreover, as all the derivatives converge uniformly, we conclude that $g_n = g'_{n-1} = \cdots = g_0^{(n)}$ for each $n = 0, 1, 2, \ldots$, and hence that

$$\partial \boldsymbol{\pi}_n f = \boldsymbol{\pi}_{n+1} f, \quad n = 0, 1, 2, \ldots.$$

From this relation it follows that the first coordinate map $\boldsymbol{\pi}_0$ is injective. Indeed, if $\boldsymbol{\pi}_0 f = 0$, then by iteration $\boldsymbol{\pi}_n f = 0$ for $n = 0, 1, 2, \ldots$. Hence $\boldsymbol{\pi} f = 0$ and thus $f = 0$ by the injectivity of $\boldsymbol{\pi}$.

Finally, we show that the given test class is stable under the completion. We already saw that the limiting functions are of class C^∞, and naturally $\|f\|_{p,\mathcal{M}} < +\infty$ for each $f \in W^{p,\theta}_{\mathcal{M}}$. What remains is to show that

$$\limsup_{n\to+\infty}(1-p)^n \log \|f^{(n)}\|_\infty \le \theta.$$

However, the bound (1.13) tells us that

$$\|f^{(n)}\|_\infty \le \|f\|_{p,\mathcal{M}}\, \mathrm{e}^{\theta(1-p)^{-n}+p(1-p)^{-n-1}\kappa(p,\mathcal{M})} \prod_{j=1}^{n} M_j^{-(1-p)^{j-n-1}p},$$

so that

$$(1-p)^n \log \|f^{(n)}\|_\infty \le \theta + \frac{p}{(1-p)}\left\{\kappa(p,\mathcal{M}) - \sum_{j=1}^{n}(1-p)^j \log M_j\right\} + (1-p)^n \log \|f\|_{p,\mathcal{M}}.$$

By Remark 1.3(b), it follows that

$$\kappa(p,\mathcal{M}) - \sum_{j=1}^{n}(1-p)^j \log M_j \to 0, \quad n \to +\infty,$$

and hence

$$\limsup_{n\to+\infty}(1-p)^n \log \|f^{(n)}\|_\infty \le \theta,$$

as claimed. $\square$

5 Independence of derivatives

5.1 Some notation and preparatory material

We previously considered the mapping $\boldsymbol{\pi}: f \mapsto (f, f', f'', \ldots)$. To conform with notation introduced earlier in this paper as well as in the work of Peetre, we will use the letter $\boldsymbol{\alpha}$ instead of $\boldsymbol{\pi}_0$ to refer to the first coordinate map $\boldsymbol{\alpha}: W^{p,\theta}_{\mathcal{M}} \to L^p$. Likewise, we write $\boldsymbol{\delta}$ for the mapping $W^{p,\theta}_{\mathcal{M}} \to W^{p,\theta(1-p)^{-1}}_{\mathcal{M}_1}$ of taking the derivative (on the test functions). Here, we recall that $\mathcal{M}_1$ is the shifted sequence $\mathcal{M}_1 = \{M_{n+1}\}_n$. More precisely, in terms of a Cauchy sequence $\{f_j\}_j$ of test functions in $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$ which converges in quasinorm to an abstract limit $f \in W^{p,\theta}_{\mathcal{M}}$, we write

$$\boldsymbol{\alpha} f = \lim_j \boldsymbol{\alpha} f_j = \lim_j f_j \in L^p \quad \text{and} \quad \boldsymbol{\delta} f = \lim_j \boldsymbol{\delta} f_j = \lim_j f_j' \in W^{p,\theta(1-p)^{-1}}_{\mathcal{M}_1}.$$

As we saw in connection with the Sobolev $W^{k,p}$-spaces for $0 < p < 1$, the key step was the construction of the two lifts $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. If we may find the analogous lifts in the present setting, the rest of the proof will carry over almost word for word from the Sobolev space case. It turns out that the lifts $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are built in the same manner as before, using the existence of invisible mollifiers Φ_ϵ analogous to the case of $W^{k,p}$ (compare with Lemma 3.1), with slight technical obstacles in the construction of $\boldsymbol{\gamma}$, related to the unbounded support of test functions.

Before turning to the mollifiers, we make an observation regarding the weight sequence. We will make a dichotomy between the cases

$$\lim_{n\to+\infty}(1-p)^n \log M_n = 0 \quad \text{and} \quad \liminf_{n\to+\infty} \log(1-p)^n \log M_n > 0$$

in the definition of p-regularity (Definition 1.4).

Lemma 5.1 *Let $\mathcal{M}$ be a log-convex sequence such that $\kappa(p,\mathcal{M}) = +\infty$ and assume that*

$$\liminf_{n\to\infty}(1-p)^n \log M_n > 0.$$

Then there exists a p-regular minorant sequence $\mathcal{N}$ with $\kappa(p,\mathcal{N}) = +\infty$, such that

$$\lim_{n\to\infty}(1-p)^n \log N_n = 0.$$

For more complete details of this proof, see [4], as well as Behm's thesis [3], where the former paper is discussed.

Proof sketch Let $c_0 = \liminf_{n\to+\infty}(1-p)^n \log M_n > 0$. Then

$$c_0(1-p)^{-n} \le (1+\mathrm{o}(1)) \log M_n,$$

so that

$$\frac{c_0}{n+1}(1-p)^{-n} \le \frac{1}{n+1}(1+\mathrm{o}(1)) \log M_n \le \log M_n + \mathrm{O}(1).$$

We now put

$$N_n = C_1 \mathrm{e}^{c_0(n+1)^{-1}(1-p)^{-n}},$$

where the positive constant C_1 is chosen such that $N_n \le M_n$ for all integers $n \ge 0$. It follows that $\mathcal{N} = \{N_n\}_n$ is p-regular with $\kappa(p,\mathcal{N}) = +\infty$, and that $(1-p)^n \log N_n = \mathrm{o}(1)$, as needed. □

5.2 The construction of mollifiers

The invisibility lemma runs as follows.

Lemma 5.2 (Invisibility lemma for L^p-Carleman spaces) *Assume that $\mathcal{M}$ is p-regular and has infinite p-characteristic $\kappa(p,\mathcal{M}) = +\infty$. For any $\epsilon > 0$ there exists a non-negative function $\Phi \in \mathscr{S}^{\circledast}_{p,0,\mathcal{M}}$ (with $\theta = 0$) such that*

$$\int_{\mathbb{R}} \Phi\, dx = 1, \quad \operatorname{supp} \Phi \subset [0,\epsilon], \quad \|\Phi\|_{p,\mathcal{M}} < \epsilon.$$

The tools needed to prove the lemma were developed back in Sect. 3.1, in connection with the Invisibility Lemma for $W^{k,p}$. Given a sequence $\{a_l\}_{l=1}^{\infty}$, we form the associated convolution product $\Phi_{1,\infty}$ (see (3.1)). We recall that $\Phi_{1,\infty}$ has support $[0, \sum_{l\geq 1} a_l]$, and enjoys the estimates (3.3) and (3.6), provided that $\{a_j\}_j$ is a decreasing ℓ^1-sequence.

In order to bound the support of $\Phi_{1,\infty}$, will need to estimate the sums $\sum_{j\geq n+1} a_j$. If we write $a_j = c_j \alpha_j$ and ask for $\{\alpha_j\}_j$ to be a decreasing sequence of positive numbers, and that the c_j are positive with $\{c_j\}_j \in \ell^1$, we obtain the bound

$$\sum_{j=n+1}^{+\infty} a_j \leq c\alpha_{n+1}, \quad n \geq 0, \tag{5.1}$$

where $c = \|\{c_j\}_j\|_{\ell^1}$. Then the right-hand side of (3.6) may be estimated further:

$$\|\Phi_{1,\infty}^{(n)}\|_p^p \leq 2^n \frac{c\,\alpha_{n+1}}{(a_1 \cdots a_{n+1})^p} = \frac{2^n c\,\alpha_{n+1}}{(c_1 \cdots c_{n+1})^p (\alpha_1 \cdots \alpha_{n+1})^p}. \tag{5.2}$$

We now briefly describe what we ask the sequence $a = \{a_l\}_l = \{c_l \alpha_l\}_l$ to satisfy to guarantee that $\Phi_{1,\infty}$ meets the conclusion of the lemma. The $(p,0)$-tameness is, in view of (3.3), ensured by

$$\limsup_{n\to\infty} (1-p)^n \log \frac{1}{a_1 \cdots a_n} \leq 0. \tag{5.3}$$

Next, to get the norm control $\|\Phi_{1,\infty}\|_{p,\mathcal{M}} < \epsilon$, we require in view of (5.2) that

$$\frac{2^n c\,\alpha_{n+1}}{(c_1 \cdots c_{n+1})^p (\alpha_1 \cdots \alpha_{n+1})^p} \leq \epsilon^p M_n^p, \quad n = 0,1,2,\ldots. \tag{5.4}$$

Moreover, the assertion that $\operatorname{supp} \Phi_{1,\infty} \subset [0,\epsilon]$ is equivalent to having $\|a\|_{\ell^1} \leq \epsilon$, which follows if we assume that $\alpha_1 \leq \epsilon/c$.

Remark 5.1 We now digress on the structural consequences for positive sequences $\{\alpha_l\}_l$ satisfying a requirement of the form

$$\frac{\alpha_{l+1}}{(\alpha_1 \cdots \alpha_{l+1})^p} \le \delta M_l^r,$$

where r and δ are positive constants. While r is fixed, we want to keep some flexibility in the choice of δ. Given the product structure, we make a quotient ansatz $\alpha_l = b_l/b_{l+1}$ with $b_1 = B^{1/p}$, where $B > 1$ is large. In terms of $\{b_l\}_{l\ge1}$, the above relations read

$$\frac{b_{l+1}}{b_{l+2}^{1-p}} \le \delta B M_l^r, \quad l \ge 0.$$

We assume now that $\delta B = 1$, and study the instance when all the inequalities are equalities. Then it follows that

$$\frac{b_{l+1}}{b_{l+2}^{1-p}} = M_l^r$$

which gives iteratively the sequence $\{b_l\}_l$ as

$$b_{l+2} = (b_{l+1} M_l^{-r})^{1/(1-p)},$$

for $l = 0, 1, 2, \ldots$. Proceeding iteratively, we would obtain

$$b_{l+2} = \big(B\, M_0^{-r} M_1^{-r(1-p)} \cdots M_l^{-r(1-p)^l}\big)^{(1-p)^{-l-1}}, \quad l = 0, 1, 2, \ldots.$$

Now, if $\kappa(p, \mathcal{M})$ were finite, we would put $B = \prod_{j\ge0} M_j^{r(1-p)^j}$ and obtain the formula

$$b_l = \prod_{j\ge0} M_{l+j-1}^{r(1-p)^j}, \quad l = 1, 2, 3, \ldots. \tag{5.5}$$

In the case at hand, we have $\kappa(p, \mathcal{M}) = +\infty$, so such a choice will not work directly. However, it does give a hint as to how to select these sequences.

With these preliminaries, we begin our construction.

Proof Without loss of generality we may assume that

$$\lim_{n\to+\infty} (1-p)^n \log M_n = 0.$$

Indeed, if $\mathcal{M}$ does not meet this requirement, then we would apply Lemma 5.1 and replace $\mathcal{M}$ by a minorant $\mathcal{N}$, and proceed as below. The result would then follow, since $\mathscr{S}^{\circledast}_{p,0,\mathcal{N}} \subset \mathscr{S}^{\circledast}_{p,0,\mathcal{M}}$, where the inclusion contracts the quasinorm.

By the assumption of p-regularity, the number

$$\beta := \liminf_{n\to\infty} \frac{\log M_n}{n \log n} \tag{5.6}$$

satisfies $1 < \beta \le +\infty$. Let r and ρ be numbers with $0 < r \le p$ and $\rho > 0$, such that

$$(p-r)\beta > (1+\rho)p,$$

which is possible since $\beta > 1$. Guided by the considerations in Remark 5.1, we consider instead of (5.5) the truncated products

$$b_{l,k} = \prod_{j=0}^{k} M_{l-1+j}^{r(1-p)^j}, \quad l = 1, 2, 3, \ldots,$$

where the parameter k remains to be chosen. Next, we write

$$\alpha_{l,k} = \frac{b_{l,k}}{b_{l+1,k}}, \quad c_l = \frac{1}{l^{1+\rho}}, \qquad l = 1, 2, 3, \ldots,$$

and we put $a_{l,k} = c_l \alpha_{l,k}$ as before. Since $\mathcal{M}$ is logarithmically convex by assumption, we have that

$$\alpha_{l,k} = \prod_{j=0}^{k} \left(\frac{M_{l-1+j}}{M_{l+j}} \right)^{r(1-p)^j} \le \prod_{j=0}^{k} \left(\frac{M_{l-2+j}}{M_{l+j-1}} \right)^{r(1-p)^j} = \alpha_{l-1,k}, \quad l = 2, 3, 4, \ldots,$$

so that the sequence $\{\alpha_{l,k}\}_l$ is decreasing. Moreover, the fact that $\{c_l\}_l \in \ell^1$ implies that the above discussion is applicable. In particular, the support of $\Phi_{1,\infty}$ is contained in the interval $[0, c\alpha_{1,k}]$, where $c = \|\{c_j\}_j\|_{\ell^1}$ and

$$c\alpha_{1,k} = c \prod_{j=0}^{k} \left(\frac{M_j}{M_{j+1}} \right)^{r(1-p)^j} = \frac{c\, M_0^r}{M_{k+1}^{r(1-p)^k} \prod_{j=1}^{k} M_j^{rp(1-p)^{j-1}}}.$$

Since $\kappa(p, \mathcal{M}) = +\infty$ is assumed and as $M_{k+1} \ge 1$ holds for large enough k, this expression tends to 0 as $k \to +\infty$.

To check that $\Phi_{1,\infty}$ is $(p, 0)$-tame, we observe that

$$\begin{aligned}(1-p)^n \log\|\Phi^{(n)}_{1,\infty}\|_\infty &\le (1-p)^n \log \frac{2^n}{a_{1,k}\cdots a_{n+1,k}}\\ &= n(1-p)^n \log 2 + (1-p)^n \log \frac{1}{\alpha_{1,k}\cdots\alpha_{n+1,k}} + (1-p)^n \log \frac{1}{c_1\cdots c_{n+1}}\\ &= n(1-p)^n \log 2 + (1-p)^n \log \frac{b_{n+2,k}}{b_{1,k}} + (1+\rho)(1-p)^n \sum_{j=1}^{n+1} \log j\\ &= n(1-p)^n \log 2 - (1-p)^n \log b_{1,k} + (1+\rho)(1-p)^n \log((n+1)!)\\ &\quad + r\sum_{j=0}^{k}(1-p)^{j+n} \log M_{n+j+1}.\end{aligned}$$

Consequently, if k is fixed but large enough for $b_{1,k} \ge 1$ to hold, we have that

$$(1-p)^n \log\|\Phi^{(n)}\|_\infty \le r(1-p)^{-1}\sum_{j=0}^{k}(1-p)^{j+n+1}\log M_{n+j+1} + \mathrm{o}(1) = \mathrm{o}(1),$$

as $n \to +\infty$, since it is given that $\lim_n (1-p)^n \log M_n = 0$.

We turn to the property (5.4). We first observe that

$$\frac{2^n c}{(c_1\cdots c_{n+1})^p} = \mathrm{e}^{f(n)},$$

where $f(n) = \log c + n\log 2 + p(1+\rho)\log((n+1)!)$. In addition, we see that

$$\frac{\alpha_{n+1,k}}{(\alpha_{1,k}\cdots\alpha_{n+1,k})^p} = \frac{\frac{b_{n+1,k}}{b_{n+2,k}}}{\left(\frac{b_{1,k}}{b_{n+2,k}}\right)^p} = \frac{b_{n+1,k}}{b^p_{1,k} b^{1-p}_{n+2,k}} = \frac{M_n^r}{b^p_{1,k} M^{r(1-p)^{k+1}}_{n+k+1}}.$$

By the p-regularity, for large enough k, we have that $M_{n+k+1} \ge 1$, and hence the above calculation combined with (3.6) gives the estimate

$$\|\Phi^{(n)}_{1,\infty}\|^p_p \le \frac{1}{b^p_{1,k}}\mathrm{e}^{f(n)-(p-r)\log M_n} M_n^p.$$

In particular, observe that

$$\begin{aligned}f(n) - (p-r)\log M_n &\le \log c + n\log 2\\ &\quad + p(1+\rho)\left\{\log(n+1)! - \frac{p-r}{p(1+\rho)}\log M_n\right\} \le \Lambda\end{aligned}$$

for some positive constant Λ independent of k, if we use Stirling's formula and recall that $\frac{p-r}{(1+\rho)p} > \frac{1}{\beta}$, while taking into account the definition of the parameter β in (5.6).

In terms of quasinorm control, this means that

$$\|\Phi_{1,\infty}^{(n)}\|_p^p \le \frac{\mathrm{e}^{\Lambda}}{b_{1,k}^p} M_n^p.$$

Since we know that $b_{1,k} \to +\infty$ as $k \to +\infty$, while Λ is independent of k, the desired estimate on $\|\Phi_{1,\infty}\|_{p,\mathcal{M}}$ follows by choosing k large enough. The proof is complete. □

5.3 Construction of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$

The existence of invisible mollifiers finally allows us to define the lifts $\boldsymbol{\beta}\colon L^p \to W^{p,\theta}_{\mathcal{M},0}$ and $\boldsymbol{\gamma}\colon W^{p,\theta_1}_{\mathcal{M}_1,0} \to W^{p,\theta}_{\mathcal{M},0}$.

Proposition 5.1 *Suppose $\mathcal{M}$ is p-regular with $\kappa(p,\mathcal{M}) = +\infty$. Then there exists a continuous linear mapping $\boldsymbol{\beta}\colon L^p \to W^{p,\theta}_{\mathcal{M},0}$ such that*

$$\boldsymbol{\alpha} \circ \boldsymbol{\beta} = \mathrm{id}_{L^p} \quad \textit{and} \quad \boldsymbol{\delta} \circ \boldsymbol{\beta} = 0.$$

Proof It is enough to demonstrate the result for $\theta = 0$, since the general case then follows by inclusion. The definition procedure remains the same as that in Sect. 3.3. We denote by $\mathscr{F}_p$ the space of step functions, and to each $g \in \mathscr{F}_p$ we aim to associate a Cauchy sequence $\{g_j\}_j \subset \mathscr{S}^{\circledast}_{p,0,\mathcal{M}} \cap C_0^\infty$ such that

$$\lim_{j\to+\infty} \|g - g_j\|_p = 0 \qquad \lim_{j\to+\infty} \|g_j'\|_{p,\mathcal{M}_1} = 0, \tag{5.7}$$

where $\mathcal{M}_1$ is the shifted sequence $\mathcal{M}_1 = \{M_{j+1}\}_{j\ge 0}$. We then declare $\boldsymbol{\beta} g$ to be the abstract limit $\lim_j g_j$ in $W^{p,0}_{\mathcal{M},0}$. If this can be achieved, these properties show that $\boldsymbol{\beta}$ is a rescaled isometry on $\mathscr{F}_p$: Indeed, the norm splits as

$$\|f\|_{p,\mathcal{M}} = \max\left\{\frac{\|f\|_p}{M_0},\ \|f\|_{p,\mathcal{M}_1}\right\}, \tag{5.8}$$

so we may write

$$\|\boldsymbol{\beta} g\|_{p,\mathcal{M}}^p = \lim_{j\to+\infty} \|g_j\|_{p,\mathcal{M}}^p = \lim_{j\to+\infty} \max\left\{\frac{\|g_j\|_p^p}{M_0^p},\ \|g_j'\|_{p,\mathcal{M}_1}^p\right\} = \frac{\|g_j\|_p^p}{M_0^p}$$

and they uniquely determine $\boldsymbol{\beta} g$. Indeed, any other Cauchy sequence $\{\tilde{g}_j\}_j$ for which (5.7) holds will be equivalent to $\{g_j\}$ in $W^{p,0}_{\mathcal{M},0}$:

$$\|g_j - \tilde{g}_j\|_{p,\mathcal{M}}^p \le \frac{\|g - g_j\|_p^p}{M_0} + \frac{\|g - \tilde{g}_j\|_p^p}{M_0} + \|g_j'\|_{p,\mathcal{M}_1}^p + \|\tilde{g}_j'\|_{p,\mathcal{M}_1}^p = \mathrm{o}(1),$$

as $j \to +\infty$. Moreover, (5.7) clearly shows that

$$(\boldsymbol{\alpha} \circ \boldsymbol{\beta})g = g \quad \text{and} \quad (\boldsymbol{\delta} \circ \boldsymbol{\beta})g = 0, \qquad g \in \mathscr{F}_p,$$

and since these properties are stable under extension by continuity to L^p, the proof is complete once the existence of a Cauchy sequence $\{g_j\}_j$ satisfying (5.7) is demonstrated.

The properties of $\mathcal{M}$ needed to apply Lemma 5.2 are inherited by $\mathcal{M}_1$, so for each j we may apply the lemma to $W^{p,0}_{\mathcal{M}_1,0}$ in order to produce functions $\Phi_j \in \mathscr{S}^{\circledast}_{p,0,\mathcal{M}_1}$ with support in $[0, \epsilon_j]$ such that $\|\Phi_j\|_{\mathcal{M}_1} \le \epsilon_j$, where ϵ_j is a decreasing sequence tending to 0. Let g be an arbitrary step function in $\mathscr{F}_p$, and put $g_j = g * \Phi_j$, so that g_j is smooth with uniformly bounded support. We observe that for $n \ge 1$,

$$(g * \Phi_j)^{(n)} = g' * \Phi_j^{(n-1)},$$

where g' is thought of as a finite sum of point masses. By (3.4), it follows that

$$\|g_j^{(n)}\|_p = \left\|(g * \Phi_j)^{(n)}\right\|_p \le \|g'\|_{\ell^p} \|\Phi_j^{(n-1)}\|_p \le \epsilon_j \|g'\|_{\ell^p} M_n, \quad n = 1, 2, 3, \ldots. \tag{5.9}$$

Clearly we have the convergence $g_j \to g$ in L^p, that is, $\lim_j \|g_j - g\|_p = 0$. It also follows from (5.9) that

$$\lim_{j\to+\infty} \|g_j'\|^p_{p,\mathcal{M}_1} = \lim_{j\to+\infty} \sup_{n\ge1} \frac{\|g_j^{(n)}\|_p}{M_n} \le \lim_{j\to+\infty} \epsilon_j \|g'\|_{\ell^p} = 0.$$

We conclude that for each j, $g_j \in \mathscr{S}^{\circledast}_{p,0,\mathcal{M}} \cap C_0^\infty$, and moreover, since $g_j \to g$ in L^p and $g_j' \to 0$ in $W^{p,0}_{\mathcal{M}_1,0}$, the sequence $\{g_j\}_j$ is Cauchy in the quasinorm of $W^{p,0}_{\mathcal{M}}$. We declare the abstract limit of this Cauchy sequence to be $\boldsymbol{\beta} g \in W^{p,0}_{\mathcal{M},0}$ for the given step function g. In view of the above calculations,

$$\|\boldsymbol{\beta} g\|_{p,\mathcal{M}} = \lim_j \|g_j\|_{p,\mathcal{M}} = \lim_j \sup_{n\ge0} \frac{\|g_j^{(n)}\|_p}{M_n} = \frac{\|g\|_p}{M_0}.$$

As of right now, $\boldsymbol{\beta}$ is a densely defined bounded operator $\mathscr{F}_p \to W^{p,0}_{\mathcal{M},0}$. By extending $\boldsymbol{\beta}$ to the entire space, we obtain a well-defined linear operator $\boldsymbol{\beta}\colon L^p \to W^{p,0}_{\mathcal{M},0}$ such that $\boldsymbol{\alpha}\boldsymbol{\beta} = \mathrm{id}$ and $\boldsymbol{\delta}\boldsymbol{\beta} = 0$. □

Recall the notation $\mathcal{M}_1 = \{M_{k+1}\}_k$ and $\theta_1 = \theta/(1-p)$.

Proposition 5.2 *Assume that $\mathcal{M}$ is p-regular with $\kappa(p, \mathcal{M}) = +\infty$. Then there exists a continuous linear mapping $\boldsymbol{\gamma}\colon W^{p,\theta_1}_{\mathcal{M}_1,0} \to W^{p,\theta}_{\mathcal{M},0}$ such that*

$$\boldsymbol{\delta} \circ \boldsymbol{\gamma} = \mathrm{id}_{W^{p,\theta_1}_{\mathcal{M}_1,0}} \quad \textit{and} \quad \boldsymbol{\alpha} \circ \boldsymbol{\gamma} = 0.$$

Proof To construct $\boldsymbol{\gamma}$, we let $g \in W^{p,\theta_1}_{\mathcal{M}_1,0}$ be an arbitrary element, which is by definition the abstract limit of some Cauchy sequence $\{g_j\}_j$ in $\mathscr{S}^{\circledast}_{p,\theta_1,\mathcal{M}_1} \cap C_0^\infty$. For any given $\epsilon > 0$, Lemma 5.2 provides a function $\Phi_\epsilon \in \mathscr{S}^{\circledast}_{p,\theta_1,\mathcal{M}_1} \cap C_0^\infty$ with the following properties: $\Phi_\epsilon \geq 0$, $\langle \Phi_\epsilon \rangle_{\mathbb{R}} := \int_{\mathbb{R}} \Phi_\epsilon(t)\mathrm{d}t = 1$, Φ_ϵ supported in $[0, \epsilon]$, while at the same time, $\|\Phi_\epsilon\|_{p,\mathcal{M}_1} \leq \epsilon$. For a sequence ϵ_j, we use the functions Φ_{ϵ_j} to modify each $g_j(x)$ to have vanishing zeroth moment, by defining

$$\tilde{g}_j(x) := g_j(x) - \langle g_j \rangle_{\mathbb{R}} \Phi_{\epsilon_j}(x), \quad \langle g_j \rangle_{\mathbb{R}} := \int_{\mathbb{R}} g_j(t)\mathrm{d}t,$$

where the ϵ_j are chosen to tend to zero so quickly that

$$\lim_{j\to+\infty} \|g_j - \tilde{g}_j\|_{p,\mathcal{M}_1} = \lim_{j\to+\infty} \|\Phi_{\epsilon_j}\|_{p,\mathcal{M}_1} |\langle g_j \rangle_{\mathbb{R}}| = 0.$$

Next, we define the functions u_j as primitives:

$$u_j(x) = \int_{-\infty}^{x} \tilde{g}_j(t)\mathrm{d}t, \quad x \in \mathbb{R}.$$

Then as $\tilde{g}_j$ has integral 0, we see that $u_j \in C_0^\infty(\mathbb{R})$, and it is easy to see that $u_j \in \mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$. We put $f_j := u_j - \boldsymbol{\beta} u_j \in W^{p,\theta}_{\mathcal{M},0}$, and define $\boldsymbol{\gamma} g$ as the limit of $\{f_j\}_j$ in $W^{p,\theta}_{\mathcal{M},0}$. That this makes the mapping $\boldsymbol{\gamma}$ well-defined with the asserted properties follows in a fashion analogous to that of the Sobolev case (Sect. 3.4). □

Sketch of proof of Theorem 1.3 In principle, we just follow the algebraic approach from the proof of Theorem 1.1 supplied in Sect. 2.3. So, we define a linear operator $\mathbf{A}\colon W^{p,\theta}_{\mathcal{M},0} \to L^p \oplus W^{p,\theta_1}_{\mathcal{M}_1,0}$ given by $\mathbf{A} f := (\boldsymbol{\alpha} f, \boldsymbol{\delta} f)$ and analyze it using the properties of the four maps $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\gamma}$. □

5.4 A remark on the size of $W^{p,\theta}_{\mathcal{M}}$

We conclude this section with a proposition which illustrates the size of $W^{p,\theta}_{\mathcal{M}}$ when $\kappa(p, \mathcal{M}) = +\infty$.

Proposition 5.3 *In the setting of Theorem 1.3, we have the inclusions*

$$c_0(L^p, \mathcal{M}) \subset \boldsymbol{\pi} W^{p,\theta}_{\mathcal{M},0} \subset \boldsymbol{\pi} W^{p,\theta}_{\mathcal{M}} \subset \ell^\infty(L^p, \mathcal{M})$$

Proof The second and third inclusions are clear. To obtain the first, let $(f_0, f_1, f_2, \ldots)$ be an arbitrary element of $c_0(L^p, \mathcal{M})$. By iterating Theorem 1.3 as in Remark 1.4, for each fixed k there exists an element $F_k \in W^{p,\theta}_{\mathcal{M},0}$ such that $\boldsymbol{\pi}(F_k) = (f_0, \cdots, f_k, 0, \ldots)$. We claim that $\lim_j F_j$ exists as an element of $W^{p,\theta}_{\mathcal{M},0}$, and that $\boldsymbol{\pi}(\lim_j F_j) = (f_0, f_1, f_2, \ldots)$. Since $(f_0, f_1, f_2, \ldots) \in c_0(L^p, \mathcal{M})$, the cut-off sequences $(f_0, \ldots, f_k, 0, \ldots)$ converge in the quasinorm of $l^\infty(L^p, \mathcal{M})$ to

$(f_0, f_1, f_2, \ldots)$ as $k \to +\infty$. Moreover, since $\boldsymbol{\pi}$ is an isometry $W^{p,\theta}_{\mathcal{M}} \to l^\infty(L^p, \mathcal{M})$, this means that the elements $F_k \in W^{p,\theta}_{\mathcal{M},0}$ converge in quasinorm as $k \to +\infty$. Finally, $\boldsymbol{\pi}$ applied to the limit coincides with the sequence $(f_0, f_1, f_2, \ldots)$. □

6 The quasianalyticity transition and the parameter θ

6.1 A remark on the case $1 \le p < +\infty$

It is of course a basic question is if the Denjoy–Carleman theorem remains valid in the setting of the L^p-Carleman Classes in the parameter range $1 \le p < +\infty$ (without any tameness requirement of course). This is of course true and should be well-known, but we have unfortunately not been able to find a suitable references for this fact. For this reason, we supply a short self-contained presentation. We begin with the following lemma.

Lemma 6.1 *If $\mathcal{M}$ is logarithmically convex, the Carleman class $\mathscr{C}_{\mathcal{M}}$ is an algebra.*

Proof By the logarithmic convexity of $\mathcal{M}$, it follows that

$$M_j M_{n-j} \le M_0 M_n, \quad 0 \le j \le n.$$

Indeed, if $j \le n - j$, since $M_0 = 1$,

$$M_j M_{n-j} = M_0 \left(\frac{M_1}{M_0} \cdots \frac{M_j}{M_{j-1}}\right)\left(\frac{M_{n-j}}{M_{n-j+1}} \cdots \frac{M_{n-1}}{M_n}\right) M_n.$$

We next observe that

$$\frac{M_k}{M_{k-1}} \le \frac{M_{k+1}}{M_k}, \quad k \ge 1,$$

so we may estimate the product

$$\begin{aligned}&\left(\frac{M_1}{M_0} \cdots \frac{M_j}{M_{j-1}}\right)\left(\frac{M_{n-j}}{M_{n-j+1}} \cdots \frac{M_{n-1}}{M_n}\right)\\ &\quad \le \left(\frac{M_{n-j+1}}{M_{n-j}} \cdots \frac{M_n}{M_{n-1}}\right)\left(\frac{M_{n-j}}{M_{n-j+1}} \cdots \frac{M_{n-1}}{M_n}\right) = 1.\end{aligned}$$

The claim now follows.

Next, let $f, g \in \mathscr{C}_{\mathcal{M}}$, and suppose that the estimates

$$\|f^{(n)}\|_\infty \le A_f^n M_n \quad \text{and} \quad \|g^{(n)}\|_\infty \le A_g^n M_n,$$

hold for $n = 0, 1, 2, \ldots$, where A_f and A_g are appropriate positive constants. Then, by the Leibniz rule, we have that

$$\begin{aligned}\|(fg)^{(n)}\|_\infty &\le \sum_{j=0}^{n} \binom{n}{j} \|f^{(j)}\|_\infty \|g^{(n-j)}\|_\infty \le \sum_{j=0}^{n} \binom{n}{j} A_f^j A_g^{n-j} M_j M_{n-j} \\ &\le (A_f + A_g)^n M_0 M_n, \end{aligned} \tag{6.1}$$

which proves that $fg \in \mathscr{C}_{\mathcal{M}}$. By linearity, the conclusion extends to all $f, g \in \mathscr{C}_{\mathcal{M}}$. □

Theorem 6.1 *When* $1 \le p < +\infty$ *the p-Carleman class* $\mathscr{C}^p_{\mathcal{M}}$ *is quasianalytic if and only if* $\mathscr{C}_{\mathcal{M}}$ *is.*

Proof We obtain the contrapositive statements. Assume that $\mathscr{C}_{\mathcal{M}}$ is not quasianalytic. Then there exist functions in $\mathscr{C}_{\mathcal{M}}$ with compact support. Indeed, by the Denjoy–Carleman theorem it follows that

$$\sum_{n \ge 1} \frac{M_{n-1}}{M_n} < +\infty,$$

so $\Phi = \Phi_{1,\infty}$ constructed according to (3.1) with $a_n = \frac{M_{n-2}}{M_{n-1}}$ and $a_1 = \frac{1}{M_0}$, lies in the class $\mathscr{C}_{\mathcal{M}}$. As a consequence of the estimate

$$\|\Phi^{(n)}\|_p \le |\mathrm{supp}\,\Phi|^{1/p} \|\Phi^{(n)}\|_\infty \le |\mathrm{supp}\,\Phi|^{1/p} A_\Phi^n M_n,$$

where $|E|$ denotes the length of the set $E \subset \mathbb{R}$, we find that $\Phi \in \mathscr{C}^p_{\mathcal{M}}$ as well. In particular, the class $\mathscr{C}^p_{\mathcal{M}}$ cannot be quasianalytic.

As for the other direction, we assume that $\mathscr{C}^p_{\mathcal{M}}$ is not quasianalytic. Then there exists a function $f \in \mathscr{C}^p_{\mathcal{M}}$ which vanishes to infinite degree at a point a but is nontrivial at some other point b (so that $f(b) \ne 0$). In view of translation as well dilatation invariance, we may assume that $a = 0$ and $b = \frac{1}{2}$. We now estimate the supremum norm on the unit interval $I = [0, 1]$:

$$|f^{(n)}(x)| = |f^{(n)}(x) - f^{(n)}(a)| \le \|f^{(n+1)}|_I\|_1 \le \|f^{(n+1)}|_I\|_p \le C_f A_f^{n+1} M_{n+1}, \tag{6.2}$$

using Hölder's inequality, since we are considering $1 \le p < +\infty$. Next, we form $g(x) = f(x) f(1-x)$ for $0 \le x \le 1$ and put $g(x) = 0$ off this interval. It turns out that it follows from (6.1) that

$$\|g^{(n)}\|_\infty \le 2^n C_f^2 A_f^{n+2} M_1 M_{n+1}.$$

In particular, we have that $g \in \mathscr{C}_{\mathcal{M}_1}$, where $\mathcal{M}_1 = \{M_{n+1}\}_n$ is the shifted weight sequence, so that $C_{\mathcal{M}_1}$ is not quasianalytic. Since a simple weight shift does not affect whether or not a Carleman class is quasianalytic, it follows that $\mathscr{C}_{\mathcal{M}}$ is nonquasianalytic as well. □

6.2 Quasianalytic classes for $0 < p < 1$

For decay-regular sequences (see Definition 1.5), part (ii) of Theorem 1.4 asserts that the class $\mathscr{C}_{\mathcal{M}}^{p,0}$ is quasianalytic if and only if $\mathscr{C}_{\mathcal{N}}$ is. Here, the numbers N_n are the bounds that appeared in controlling $\|f^{(n)}\|_\infty$ by $\|f\|_{\mathcal{M}}$ where $f \in \mathscr{S}^{\circledast}_{p,0,\mathcal{M}}$, that is,

$$N_n = \prod_{j=0}^{+\infty} M_{n+j}^{p(1-p)^j}, \tag{6.3}$$

where the product converges because $\kappa(p, \mathcal{M}) < +\infty$. In addition, part (i) of Theorem 1.4 maintains that the class $\mathscr{C}_{\mathcal{M}}^{p,\theta}$ is never quasianalytic if $\theta > 0$. In the context of $\theta > 0$, we introduce the related sequence $\mathcal{N}_\theta = \{N_{n,\theta}\}_n$ given by

$$N_{n,\theta} = \mathrm{e}^{\theta(1-p)^{-n+1}} \prod_{j=0}^{+\infty} M_{n+j}^{(1-p)^j p}. \tag{6.4}$$

Proof of Theorem 1.4 We define the a sequence of positive numbers α_n by

$$\alpha_n = N_{n-1}/N_n, \quad n = 1, 2, 3, \ldots, \tag{6.5}$$

and observe that this sequence is decreasing. This follows from the logarithmic convexity of $\mathcal{M}$, since

$$\alpha_{n+1} = \prod_{j=0}^{+\infty} \left(\frac{M_{n+j}}{M_{n+1+j}}\right)^{(1-p)^j p} \le \prod_{j=0}^{+\infty} \left(\frac{M_{n-1+j}}{M_{n+j}}\right)^{(1-p)^j p} = \alpha_n, \quad n = 1, 2, 3, \ldots.$$

A calculation shows that the numbers α_j enjoy the property

$$\frac{\alpha_{n+1}^{1-p}}{(\alpha_1 \cdots \alpha_n)^p} = \frac{N_n}{N_0^p N_{n+1}^{1-p}} = \frac{M_n^p}{N_0^p}. \tag{6.6}$$

We begin with the assertion (i). We put

$$a_n = \mathrm{e}^{-p\theta(1-p)^{-n+1}} \alpha_n, \quad n = 1, 2, 3, \ldots, \tag{6.7}$$

and observe that the numbers a_n may be expressed in terms of the sequence $\mathcal{N}_\theta$ by

$$a_n = \frac{N_{n-1,\theta}}{N_{n,\theta}}, \quad n = 1, 2, 3, \ldots.$$

Since

$$a_{n+1} = \mathrm{e}^{-p\theta(1-p)^{-n}}\alpha_{n+1} \le \mathrm{e}^{-p\theta(1-p)^{-n}}\alpha_n = \mathrm{e}^{-p\theta(1-p)^{-n}+p\theta(1-p)^{-n+1}}a_n$$
$$= \mathrm{e}^{-p^2\theta(1-p)^{-n}}a_n, \tag{6.8}$$

and rather trivially,

$$0 < \mathrm{e}^{-p^2\theta(1-p)^{-n}} \le \rho < 1$$

holds for some appropriate constant ρ so that $a_{n+1} \le \rho a_n$, the sequence decays at least geometrically. We now consider the functions $\Phi_{1,\infty}$ defined as in Sect. 3.1 by the formula

$$\Phi_{1,\infty} = \lim_{j\to+\infty} a_1^{-1}1_{[0,a_1]} * \cdots a_j^{-1}1_{[0,a_j]}.$$

Then $\Phi_{1,\infty}$ is a nonnegative compactly supported C^∞-smooth function, since $\{a_j\}_j \in \ell^1$ and each convolution mollifies. In view of the estimate (3.6) and the geometric decay of the sequence $\{a_n\}_n$ as stated above, we find that

$$\|\Phi_{1,\infty}^{(n)}\|_p^p \le \frac{2^n a_{n+1}}{(1-\rho)(a_1\cdots a_{n+1})^p} = \frac{2^n a_{n+1}^{1-p}}{(1-\rho)(a_1\cdots a_n)^p}$$
$$= \mathrm{e}^{p^2\theta+p^2\theta(1-p)^{-1}+\cdots+p^2\theta(1-p)^{-n+1}-p\theta(1-p)^{-n+1}}\frac{2^n\alpha_{n+1}^{1-p}}{(1-\rho)(\alpha_1\cdots\alpha_n)^p}$$
$$= \mathrm{e}^{-p\theta(1-p)}\frac{2^n M_n^p}{(1-\rho)N_0^p}, \tag{6.9}$$

where we have summed the finite geometric series and applied the identity (6.6). This means that the norm estimate associated with the L^p-Carleman classes $\mathscr{C}_{\mathcal{M}}^{p,\theta}$ is fulfilled for the function $\Phi_{1,\infty}$. It remains to verify that $\Phi_{1,\infty}$ is (p,θ)-tame. However, by (3.3), we see that

$$(1-p)^n \log\|\Phi_{1,\infty}^{(n)}\|_\infty \le (1-p)^n\log\left(\frac{2^n}{a_1a_2\cdots a_{n+1}}\right)$$
$$= (1-p)^n n\log 2 + (1-p)^n\log N_{n+1,\theta} - (1-p)^n\log N_{0,\theta}$$
$$= \theta + (1-p)^n\sum_{j=0}^{+\infty} p(1-p)^j\log M_{n+j+1} + \mathrm{o}(1)$$
$$= \theta + p\sum_{j=n+1}^{+\infty}(1-p)^{j-1}\log M_j = \theta + \mathrm{o}(1), \tag{6.10}$$

as $n \to +\infty$, where we have used the factorization $N_{n+1,\theta} = \mathrm{e}^{\theta(1-p)^{-n}}N_{n+1,0}$ as well as the fact that the series on the right-hand side is the tail sum of the convergent series representing $\kappa(p, \mathcal{M}) < +\infty$. It now follows from (6.9) and (6.10) that $\Phi_{1,\infty} \in \mathscr{C}_{\mathcal{M}}^{p,\theta}$, and since $\Phi_{1,\infty}$ is a nontrivial compactly supported function, the class $\mathscr{C}_{\mathcal{M}}^{p,\theta}$ cannot be quasianalytic.

We continue with the assertion (ii). We derive the contrapositive statement, and begin by assuming that $\mathscr{C}_{\mathcal{M}}^{p,0}$ is not quasianalytic. Let $f \in W_{\mathcal{M}}^{p,0}$. After some inspection, the inequality (1.13) asserts that

$$\|f^{(n)}\|_\infty \leq N_{n+1}\|f\|_{p,\mathcal{M}}.$$

By applying appropriate dilations, it now follows that we have the inclusion

$$\mathscr{C}_{\mathcal{M}}^{p,0} \subset \mathscr{C}_{\mathcal{N}^\star},$$

where $\mathcal{N}^\star = \{N_{j+1}\}_j$ is the shifted sequence, and $\mathscr{C}_{\mathcal{N}^\star}$ denotes the usual Carleman class associated to $\mathcal{N}^\star$. Since by assumption, $\mathscr{C}_{\mathcal{M}}^{p,0}$ is not quasianalytic, $\mathscr{C}_{\mathcal{N}^\star}$ cannot be a quasianalytic class either. Finally, by the Denjoy–Carleman theorem, the same holds true for $\mathscr{C}_{\mathcal{N}}$.

Finally, as for the remaining implication, we assume that $\mathscr{C}_{\mathcal{N}}$ is non-quasianalytic. We single out the two cases when $\sum_n M_{n-1}/M_n = +\infty$ and when $\sum_n M_{n-1}/M_n < +\infty$. In the latter case, there are nontrivial compactly supported functions in $\mathscr{C}_{\mathcal{M}}$. Take one such function f, and set $B := |\mathrm{supp}\, f|$. We now claim that $f \in \mathscr{C}_{\mathcal{M}}^{p,0}$. First note that since $\kappa(p, \mathcal{M})$ is finite, the terms $(1-p)^n \log M_n$ must tend to zero as $n \to +\infty$. The norm estimate required for a function f to be in the Carleman class $\mathscr{C}_{\mathcal{M}}$ then gives that

$$(1-p)^n \log\|f^{(n)}\|_\infty \leq (1-p)^n \log(C_f A_f^n M_n) = \mathrm{o}(1),$$

from which it follows that f is automatically $(p, 0)$-tame. Next, it is clear that

$$\|f^{(n)}\|_p \leq B^{1/p} C_f A_f^n M_n,$$

since $\|f^{(n)}\|_\infty \leq C_f A_f^n M_n$ by assumption and, in addition, the support of f is bounded. This shows that $f \in \mathscr{C}_{\mathcal{M}}^{p,0}$, as needed.

It remains to investigate the case when $\sum_n M_{n-1}/M_n = +\infty$. Then our regularity assertion tells us that

$$M_{n-1}/M_n \geq \epsilon^n, \tag{6.11}$$

holds for some $\epsilon > 0$. We recall the decreasing sequence $\{\alpha_n\}$ defined by (6.5), and consider the corresponding convolution product $\Phi_{1,\infty}$, this time defined in terms of the numbers $\{\alpha_n\}_n$. In view of the Denjoy–Carleman theorem and the assumption that $\mathscr{C}_{\mathcal{N}}$ is non-quasianalytic, we know that the sequence $\{\alpha_n\}_n$ is in ℓ^1, from which it follows that $\Phi_{1,\infty}$ is a non-trivial compactly supported function. It follows from

(6.10) applied with $\theta = 0$ that $\Phi_{1,\infty}$ is $(p, 0)$-tame. Finally, we study the p-norms of $\Phi^{(n)}_{1,\infty}$. By the estimate (6.11), we find that

$$\alpha_n = \frac{N_{n-1}}{N_n} = \prod_{j=0}^{+\infty} \left(\frac{M_{n+j-1}}{M_{n+j}}\right)^{p(1-p)^j} \geq \prod_{j=0}^{+\infty} \epsilon^{(j+n)p(1-p)^j} \geq \epsilon^n.$$

Then, if R is large enough, it follows that $R^n \alpha_n \geq \|\{\alpha_j\}\|_{\ell^1}$, and consequently that $\sum_{j \geq n} \alpha_j \leq R^n \alpha_n$. Using this observation together with the estimate (3.6), we obtain

$$\|\Phi^{(n)}_{1,\infty}\|_p^p \leq 2^n \frac{\sum_{j=n+1}^{+\infty} \alpha_j}{(\alpha_1 \cdots \alpha_{n+1})^p} \leq (2R)^n M_n^p,$$

from which it follows that $\Phi_{1,\infty} \in \mathscr{C}^{p,0}_{\mathcal{M}}$, as we already know that $\Phi_{1,\infty}$ is $(p, 0)$-tame. Since $\Phi_{1,\infty}$ is nontrivial and compactly supported, this means that the class $\mathscr{C}^{p,0}_{\mathcal{M}}$ cannot be quasianalytic. □

We now conclude with the proof of our last remaining theorem.

Proof of Theorem 1.5 We are given a logarithmically convex increasing sequence $\mathcal{M}$ such that $\kappa(p, \mathcal{M}) < +\infty$, and numbers $0 \leq \theta < \theta'$, and aim to prove that the inclusion

$$W^{p,\theta}_{\mathcal{M}} \subset W^{p,\theta'}_{\mathcal{M}}$$

is strict for $\theta < \theta'$. First, we note that by Theorem 1.2 the space $W^{p,\theta}_{\mathcal{M}}$ equals the class $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$ of test functions. Hence, in order to show that the containment $W^{p,\theta'}_{\mathcal{M}} \supset W^{p,\theta}_{\mathcal{M}}$ is strict, it is enough to exhibit a function $f \in W^{p,\theta'}_{\mathcal{M}}$ which is not (p, θ)-tame. To this end, we put $a_n = N_{n-1,\theta'}/N_{n,\theta'}$, where $\{N_{n,\theta'}\}_n$ is given by (6.4), and consider the associated convolution product $\Phi_{1,\infty} = \Phi_{1,\infty,\theta'} \in \mathscr{S}^{\circledast}_{p,\theta',\mathcal{M}}$ as in the proof of Theorem 1.4, part (i). It is a simple consequence of (6.8) that

$$\sum_{j>n} a_j = \mathrm{o}(a_n), \quad n \to +\infty.$$

It follows that there exists some n_0 such that

$$\|\Phi_{n,\infty}\|_\infty = \frac{1}{a_n}, \quad n \geq n_0.$$

Next, we write $\Phi^{(n)}_{1,\infty} = \Phi^{(n)}_{1,n} * \Phi_{n+1,\infty}$, and note that $\Phi^{(n)}_{1,n}$ is a sum of finitely many point masses, of mass $(a_1 \cdots a_n)^{-1}$, located at 2^n points $\{x_j\}_j$:

$$\Phi^{(n)}_{1,n} = \sum_{j=1}^{2^n} \epsilon(j) \frac{1}{a_1 \cdots a_n} \delta_{x_j},$$

where $\epsilon(j) \in \{1, -1\}$, and we have ordered the x_j in increasing order. Moreover, as each x_j is the sum $\sum_{k\in J_j} a_k$ over a subset $J_j \subset \{1, \ldots, n\}$, it follows that there exists at least one x_{j_0} such that $|x_j - x_{j_0}| \geq a_n$ for any $j \neq j_0$. For instance, we may take $x_1 = 0$ and note that $x_2 = a_n$ so that $|x_1 - x_2| = a_n$. As the support of $\Phi_{n+1,\infty}$ is contained in an interval of length $\mathrm{o}(a_n)$, no interference can occur on the interval $[x_{j_0}, x_{j_0+1})$, and we find that

$$\|\Phi^{(n)}_{1,\infty}\|_\infty \geq \frac{1}{a_1 \cdots a_n}\|\Phi_{n+1,\infty}\|_\infty = \frac{1}{a_1 \cdots a_{n+1}},$$

for n large enough. By a similar computation as carried out in (6.10), it follows from this estimate that $(1-p)^n \log\|\Phi^{(n)}_{1,\infty}\|_\infty \to \theta'$ as $n \to +\infty$, which completes the proof. □

7 Concluding remarks

A number of questions remain to be investigated. For instance, we would like to better understand the space $W^{p,\theta}_{\mathcal{M}}$ in the uncoupled regime. Note that when the p-characteristic $\kappa(p, \mathcal{M})$ is finite, we have a strict inclusion $W^{p,\theta}_{\mathcal{M}} \subset W^{p,\theta'}_{\mathcal{M}}$, $W^{p,\theta}_{\mathcal{M}} \neq W^{p,\theta'}_{\mathcal{M}}$ when $\theta < \theta'$ (see Proposition 1.5). The corresponding inclusions also hold for the spaces $W^{p,\theta}_{\mathcal{M},0}$. When $\kappa(p, \mathcal{M}) = +\infty$ we do not know whether this is the case. It is conceivable that these spaces are so large that the θ-dependence is lost. We note that Proposition 5.3, gives the inclusions

$$c_0(L^p, \mathcal{M}) \subset \boldsymbol{\pi} W^{p,\theta}_{\mathcal{M},0} \subset \boldsymbol{\pi} W^{p,\theta}_{\mathcal{M}} \subset \ell^\infty(L^p, \mathcal{M}).$$

Another issue is whether we can drop some of the regularity assumptions in Theorems 1.3 and 1.4.

In addition, it appears to be of interest to study the case when the maximal class $\mathscr{S}^{\circledast}_{p,\theta,\mathcal{M}}$ is replaced with some other smaller class of test functions, for which (1.4) remains valid. For instance, we might consider the Hermite class

$$\mathscr{S}^{\mathrm{Her}}_{p,\mathcal{M}} = \left\{ f(x) = \mathrm{e}^{-x^2} p(x) : \ p \text{ a polynomial such that } \|f\|_{p,\mathcal{M}} < +\infty \right\}$$

which was mentioned earlier, or, in the case of the unit circle, the class $\mathscr{P}_{\mathrm{trig}}$ of trigonometric polynomials, and ask whether the corresponding L^p-Carleman spaces undergo the same phase transitions.

Acknowledgements We wish to thank Gustav Behm for fruitful collaboration and valuable discussions in the early stages of this project. In addition, we would like to thank Misha Sodin and Avner Kiro for their interest in this work, and the referee for valuable comments.

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