Integration Theory / Mathematical Analysis (5B1479/MA429)

Final Exam

You may use Friedman's book and class notes (no other books, please). Explain everything, mentioning appropriate theorems (from class or Friedman's book). **Good luck!**

1. Find the limit

$$\lim_{n\to\infty}\int_0^\infty \left(1+\frac{x}{n}\right)^{-n}\cdot\left(1-\sin\frac{x}{n}\right)\,dx\ .$$

2. Let $(X,\mathcal{A},\mu)$ be a measure space, and let f be a measurable real valued function on X. Show that

$$\int_X \exp(f(x))d\mu(x)\cdot\int_X \exp(-f(x))d\mu(x)\ \geq\ \mu(X)^2\ .$$

3. Suppose that function f is Lebesgue integrable on the real line. Define function f_h by $f_h(x)=\frac{1}{h}\int_x^{x+h} f(t)dt$. Show that for positive h function f_h is Lebesgue integrable on the real line, and $\|f_h\|_1\leq\|f\|_1$.

4. Let $(X,\mathcal{A},\mu)$ be a measure space, and let $\{f_n\}_{n=1}^\infty$ be a sequence of measurable functions. Show that the set

$$E\ :=\ \{x\in X\ :\ \text{there is } \textbf{exactly one} \text{ index } n \text{ such that } f_n(x)=1\}\ ,$$

is measurable.

5. a. Suppose that function $F(x)$ is absolutely continuous on the interval $[0,1]$. Prove that function $G(x):=F(x)^2$ is also absolutely continuous on $[0,1]$.

b. Suppose that function f is Lebesgue-integrable on the interval $[0,1]$. Show that there exists a Lebesgue-integrable on $[0,1]$ function g such that for every $x\in[0,1]$ one has

$$\int_0^x g(t)dt\ =\ \left(\int_0^x f(t)dt\right)^2\ .$$

Notation. In problems 3 and 5 $\int\ldots dt$ denotes integral with respect to Lebesgue measure on the real line. In problem 3 $\|\ldots\|_1$ denotes the usual norm in the space $L^1(\mathbb{R})$ of Lebesgue integrable functions on the real line.