Integration Theory / Mathematical Analysis (5B1479/MA429)

Final Exam Solutions

1. Set $f_n(x) := (1+\frac{x}{n})^{-n}\cdot(1-\sin\frac{x}{n})$. By the classical facts, for $x>0$ sequence $\{(1+\frac{x}{n})^{-n}\}$ is positive, decreasing, and tends to e^{-x} as $n\to\infty$. It is easy to see, that $(1-\sin\frac{x}{n})$ is always between 0 and 2, and tends to 1 as $n\to\infty$. Hence, $0 < f_n(x) < (1+\frac{x}{2})^{-2}\cdot 2 =: g(x)$ and $\lim_{n\to\infty} f_n(x) = e^{-x}$. Since $\int_0^\infty g(x)dx = \int_0^\infty \frac{2dx}{(1+x/2)^2} < \infty$, we can apply Lebesgue bounded convergence theorem with majorating function $g(x)$ and obtain

$$\lim_{n\to\infty}\int_0^\infty \left(1+\frac{x}{n}\right)^{-n}\cdot\left(1-\sin\frac{x}{n}\right)dx = \int_0^\infty \lim_{n\to\infty} f_n(x)dx = \int_0^\infty e^{-x}dx = 1\,.$$

2. By Hölder's inequality for positive functions $e^{f(x)/2}$ and $e^{-f(x)/2}$ with $p=q=2$:

$$\sqrt{\int_X e^{f(x)}d\mu(x)\cdot\int_X e^{-f(x)}d\mu(x)} \ \geq\ \int_X e^{\frac{f(x)}{2}}e^{-\frac{f(x)}{2}}d\mu(x) \ =\ \int_X 1d\mu(x) \ =\ \mu(X)\,.$$

3. $f(x+t)$ is clearly measurable on $[0,h]\times\mathbb{R}$ so by Fubini's theorem for non-negative functions:

$$\begin{aligned}\int_{\mathbb{R}}|f_h(x)|dx &= \int_{\mathbb{R}}\left|\frac{1}{h}\int_x^{x+h}f(t)dt\right|dx \leq \int_{\mathbb{R}}\frac{1}{h}\int_x^{x+h}|f(t)|dtdx = \int_{\mathbb{R}}\frac{1}{h}\int_0^h|f(x+t)|dtdx\\ &= \frac{1}{h}\int_0^h\int_{\mathbb{R}}|f(x+t)|dxdt = \frac{1}{h}\int_0^h\|f\|_1dt = \frac{1}{h}h\|f\|_1 = \|f\|_1\,,\end{aligned}$$

it follows that $|f_h|$ is integrable and $\|f_h\|_1 = \|\,|f_h|\,\|_1 \leq \|f\|_1$. *One fine point:* we did not show yet that f_h itself is integrable (or measurable, which would be sufficient). To do it apply Fubini's without absolute value sign (which works since $\int_0^h\int_{\mathbb{R}}|f(x+t)|dxdt < \infty$). Another solution: set $F(x) := \int_{-\infty}^x f(t)dt$, then it is absolutely continuous and $f_h(x) = \frac{1}{h}(F(x+h)-F(x))$ also is, so it is measurable and since $\int|f_h| < \infty$ integrable.

4. $E = \cup_n\{x : f_n(x) = 1,\ f_j(x)\neq 1 \text{ for } j\neq n\} = \cup_n\{f_n^{-1}\{1\}\setminus\cup_{j\neq n}f_j^{-1}\{1\}\}$. Since $f_n^{-1}\{1\} = X\setminus f_n^{-1}(\mathbb{R}\setminus\{1\})$, it is measurable, thus E, which we wrote as a countable union/difference of measurable sets also is.

5. a. Since F is absolutely continuous, it is also continuous and $M := \max|F| < \infty$. Pick up any $\delta > 0$. By the absolute continuity of F there is such an ϵ that $\sum_j|b_j-a_j| < \epsilon$ implies $\sum_j|F(b_j)-F(a_j)| < \frac{\delta}{2M}$. But then $\sum_j|b_j-a_j| < \epsilon$ implies

$$\sum_j|G(b_j)-G(a_j)| = \sum_j|F(b_j)-F(a_j)|\cdot|F(b_j)+F(a_j)| < \sum_j|F(b_j)-F(a_j)|\cdot 2M < \frac{\delta}{2M}2M = \delta.$$

Since $\delta > 0$ was arbitrary, this shows that G is absolutely continuous.

b. Since f is Lebesgue-integrable, its indefinite integral $F(x) := \int_0^x f(t)dt$ is absolutely continuous. By part a function $G(x) := F^2(x)$ is also absolutely continuous. Hence it is a.e. differentiable, its derivative $g(x) := G'(x)$ is Lebesgue integrable, and $\int_0^x g(t)dt = G(x)-G(0) = G(x)$. Therefore, $g(x)$ is the desired function: $\int_0^x g(t)dt = G(x) = F^2(x) = \left(\int_0^x f(t)dt\right)^2$.