Integration Theory / Mathematical Analysis (5B1479/MA429)

Solutions to the Final Exam

1. Using famous inequality $x\sin(1/x) \leq 1$ we can see that $n\sin(x/n) \leq x$ and $f_n(x) = n\sin(x/n)/x^3 \leq 1/x^2$. It is also well known that that $x\sin(1/x) \to 1$ as $x \to 0$, hence $n\sin(x/n) \to x$ as $n \to \infty$. So we can apply Lebesgue bounded convergence theorem (since $1/x^2$ is integrable on $[1,\infty)$):

$$\lim_{n\to\infty} \int_1^\infty \frac{n\sin(x/n)}{x^3}dx = \int_1^\infty \lim_{n\to\infty} \frac{n\sin(x/n)}{x^3}dx = \int_1^\infty \frac{x}{x^3}dx = 1.$$

2. We want to prove that iterated integral of $|g|$ is finite and then by Fubini theorem g is integrable.

$$\int_a^b \int_a^b |f(x+y)|dxdy = \int_a^b \int_{a+y}^{b+y} |f(t)|dtdy \leq \int_a^b \int_{\mathbb{R}} |f(t)|dtdy \leq |b-a| ||f||_1 < \infty$$

where the first equality holds since Lebesgue measure is translation invariant.

3. One can write

$$\begin{aligned} E &= \bigcup_{1\leq i<j<\infty} \{x : x \in E_i, x \in E_j, x \notin E_k \text{for} k \neq i,j\} = \\ & \bigcup_{1\leq i<j<\infty} \left((E_i \cup E_j) \setminus \bigcup_{k\neq i,j} E_k \right). \end{aligned}$$

Thus E can be written as countable union/difference of measurable sets, so it is also measurable.

4. Set $f_t(x) = f(x)/(x^2+t^2)$. It is obvious that $|f_t(x)| < |f(x)|$ for any x and $t \in [1,\infty)$. From this we can immediately see that

$$|g(x)| = \left| \int_{\mathbb{R}} \frac{f(x)}{x^2+t^2}dx \right| \leq \int_{\mathbb{R}} \frac{|f(x)|}{x^2+t^2}dx \leq \int_{\mathbb{R}} |f(x)|dx < \infty,$$

so $|g|$ is bounded. To prove continuity we first observe that $f_t(x) \to f_{t_0}(x)$ poitwise as $t \to t_0$ and that $|f_t| < |f|$ hence we can apply Lebesgue bounded convergence theorem and obtain

$$\lim_{t\to t_0} g(t) = \lim_{t\to t_0} \int_{\mathbb{R}} \frac{f(x)}{x^2+t^2}dx = \int_{\mathbb{R}} \lim_{t\to t_0} \frac{f(x)}{x^2+t^2}dx = \int_{\mathbb{R}} \frac{f(x)}{x^2+t_0^2}dx = g(t_0).$$

5. We want to show that for any $\epsilon > 0$ there is $\delta > 0$ such that $\sum |G(F(a_i)) - G((F(b_i))| < \epsilon$ if $\sum |a_i - b_i| < \delta$. Fix $\epsilon > 0$. Since F is absolutely continuous, there is $\delta > 0$ such that $\sum |F(a_i) - F(b_i)| < \epsilon/M$ if $\sum |a_i - b_i| < \delta$. For this choice of δ and $\sum |a_i - b_i| < \delta$ one can estimate $\sum |G(F(a_i)) - G((F(b_i))| < \sum M|F(a_i) - F(b_i)| < \epsilon$.