Integration Theory / Mathematical Analysis (5B1452/MA429)

Homework assignment # 1, due Thursday, October 17th.

1. Let μ be a measure. Prove, that for any two measurable sets A and B

$$\mu(A) + \mu(B) = \mu(A \cap B) + \mu(A \cup B) \ .$$

2. Let X be the set of all real numbers, and $\mathcal{K}$ be collection consisting of X, an empty set, and all one-point sets. Set

$$\lambda(\emptyset) := 0, \ \lambda\{a\} := 0, \ \lambda(X) := 1 \ .$$

a. Describe outer measure μ^*, which is obtained from λ.
b. Describe measure μ and σ-algebra $\mathcal{A}$ (of all μ^*-measurable sets) which is obtained from an outer measure μ^*.
3. Suppose that an outer measure μ^* on a space X is additive, i.e.

$$\mu^*(A) + \mu^*(B) = \mu^*(A \cup B) \ ,$$

for any two disjoint sets A and B.
Prove, that μ^* is actually a measure (whose domain contains all subsets of X).
4. Represent the text of the complete works of W. Shakespeare as a binary string $S = s_1 s_2 \ldots s_n$ of digits $s_j \in \{0, 1\}$ (as it would be stored in the computer memory). Prove, that Lebesgue-almost every number in the interval $[0, 1]$ contains S in its binary expansion.
Remark. Of course, important thing is that S is just some sequence of digits of finite length n. The statement above says that the set E of numbers from $[0, 1]$, which do not have sequence S in their binary expansion, has Lebesgue measure zero.
Another remark. One of the most important applications of Measury Theory is Probability. The statement above has a common translation in the probabilistic language: if a monkey sits at the typewriter and hits keys at random, then almost surely it will type complete works of W. Shakespeare sooner or later (and lots and lots of random sequences).
5. Let f be a real-valued, increasing function on the real line (usually it is assumed to be right-continuous, i.e. for any real a one has $\lim_{x \to a+} f(x) = f(a)$) Let $\mathcal{K}$ consist of an empty set and all open intervals (a, b), and define λ by $\lambda(\emptyset) := 0$ and

$$\lambda\{(a, b)\} := f(b) - f(a) \ .$$

The outer measure μ_f^* determined by $\mathcal{K}$ and λ is called the *Lebesgue-Stieltjes outer measure induced by f*. The corresponding measure μ_f is called the *Lebesgue-Stieltjes measure induced by f*.
a. Prove, that all Borel sets are Lebesgue-Stieltjes measurable.
b. Prove, that if f is continuous, μ_f satisfies $\mu_f[a, b] = f(b) - f(a)$.
c. Define $f(x) := 0, \ x < 0$ and $f(x) := 1, \ x \geq 0$. Find μ_f.