Integration Theory / Mathematical Analysis (5B1452/MA429)

Homework assignment # 2, due October 31

1. Which of the functions below are Lebesgue-integrable?
a. $f(x) = \sin(x) + \cos(x)$ on the whole real line.
b. $f(x) = 1/x^2$ on the interval $[0, 1]$.
c. $f(x) = 1/x^2$ on the interval $[1, +\infty]$.
2. Find the following limits:

a.
$$\lim_{n\to\infty} \int_0^1 \frac{1+nx}{(1+x)^n} dx \ .$$

b.
$$\lim_{n\to\infty} \int_0^n (1+x/n)^n e^{-2x} dx \ .$$

*Hint:*Use Lebesgue bounded or monotone convergence theorems.
3. Let $\{g_n\}$ be a monotone decreasing sequence of non-negative measurable functions, converging to a function g.
a. Prove, that if g_1 is integrable, then all g_n and g are integrable and

$$\lim_{n\to\infty} \int g_n d\mu \ = \ \int g d\mu \ . \qquad (\star)$$

b. Construct a counterexample to $(\star)$, with g_1 non-integrable (i.e. $\int g_1 = +\infty$).
4. Let $f(x)$ be a Lebesgue integrable function on the real line.
a. Prove that for Lebesgue almost every real x the (biinfinite) series $\sum_{n=-\infty}^{+\infty} f(x+n)$ converges absolutely.
b. Prove that for a function g defined by $g(x) := \sum_{n=-\infty}^{+\infty} f(x+n)$, one has

$$\int_0^1 g(x)dx = \int_{-\infty}^{+\infty} f(x)dx \ .$$

Hint: Show that it is sufficient to deal with non-negative function f, and then use theorem about summing series under the integral sign (and translation invariance of Lebesgue measure).
4. Define function f on the interval $[0, 1]$ by the following rule: $f(x) = 0$ for rational x and $f(x) = [1/x]$ for irrational x ($[\ldots]$ denotes the integral part).
a. Show, that f is measurable.
b. Show, that $\int_0^1 f(x)dx = +\infty$.