Integration Theory / Mathematical Analysis (5B1452/MA429)

Homework assignment # 3, due November 14

1. Let $\{f_n\}$ be a sequence of measurable functions, which converges (**a.** almost uniformly, **b.** in measure, **c.** in the mean) to a function f, and also to a function g. Prove that $f = g$ almost everywhere.

2. Let $\{a_n\}_1^\infty$ be a sequence of positive numbers with $\sum_n 1/a_n < \infty$. Show that for any real numbers $\{x_n\}$ the series

$$\sum_{n=1}^{\infty} e^{-a_n|x-x_n|}$$

converges for Lebesgue almost every real x.

Hint: Use theorem on integration of series, or Lebesgue bounded convergence theorem.

3. On the definition of absolute continuity. Find an example, which shows that finiteness assumption is necessary to conclude in the Theorem 2.12.2 that "weak definition" 2.12.1 of absolute continuity implies "strong definition" 2.8.1 of absolute continuity. Namely, construct such measures ν and μ, that

- $\nu \ll \mu$, i.e. $\mu(E) = 0$ implies $\nu(E) = 0$,
- there are sets E with $\nu(E) > 1$ and arbitrarily small $\mu(E)$.

Hint: Try to look at various sums of Dirac δ-measures, like $\mu := \sum_{j=1}^{\infty} a_j \delta_{x_j}$, and carefully choose coefficients.

4. Fourier transform. Let f be a Lebesgue integrable function on the real line $\mathbf{R}$. *Fourier transform of* f is a function $\hat{f}$ on the real line, defined by

$$\hat{f}(t) \ := \ \int_{\mathbf{R}} e^{itx} f(x)dx \ = \ \int_{\mathbf{R}} \cos(tx) f(x)dx \ + \ i \int_{\mathbf{R}} \sin(tx) f(x)dx \ .$$

Show, that $\hat{f}$ is continuous and bounded.

Hint: Use Lebesgue bounded convergence theorem.

Remark: Integrals of complex-valued functions are easily defined as above: if $h(x)$ is complex-valued, we can write it as a sum $h(x) = h_1(x) + ih_2(x)$ of real and imaginary parts, which are real valued functions, and set $\int_X h d\mu := \int_X h_1 d\mu + \int_X h_2 d\mu$.

5. Define functon $f(x)$ on the real line by setting $f(x) := x\sin(1/x)$ for non-zero x and $f(0) := 0$. Is $f(x)$ absolutely continuous?

Hint: Look at oscillation of f on small intervals near 0.