Integration Theory / Mathematical Analysis (5B1452/MA429)

Homework assignment # 4

This assignment is due Thursday, December 5th. **No late homework will be accepted.**

Final exam will be on **December 12th, 10:00-14:00,** in classrooms **K51** and **K53**. It will be written, and will contain a few problems similar to those in the homework assignments.

1.a. Show that $f(x,y) = 1/\sqrt{x^2+y^2}$ is Lebesgue-integrable in the square $[0,1] \times [0,1]$.
b. Evaluate $\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dxdy$ and $\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dydx$. Why Fubini theorem does not apply?
2. Let $\mathcal{B}_j$ and $\mathcal{L}_j$ denote Borel (i.e. generated by open sets) and Lebesgue (i.e. of Lebesgue-measurable sets) σ-algebras in $\mathbb{R}^j$ correspondingly.
a. Show that $\mathcal{B}_1 \times \mathcal{B}_1 = \mathcal{B}_2$
b. Show that $\mathcal{L}_1 \times \mathcal{L}_1 \neq \mathcal{L}_2$.
Hint: Show that every subset of any line in the plane belongs to $\mathcal{L}_2$, but not every – to $\mathcal{L}_1 \times \mathcal{L}_1$.
3.a. Consider some measure space $(X, \mathcal{A}, \mu)$ with finite measure μ. Show that if $1 \leq p \leq r \leq \infty$, then $L^p(X,\mu) \supset L^r(X,\mu)$.
b. Show that the latter is no longer true for infinite measures μ.
4. Show that translation is continuous in $L^1(\mathbb{R})$, i.e. that for $f \in L^1(\mathbb{R})$ one has

$$\lim_{\epsilon \to 0} \|f(x) - f(x+\epsilon)\|_1 = 0$$

Hint: Prove first for continuous functions, which are zero outside interval $[a,b]$. Then prove for L^1 functions, which are zero outside $[a,b]$ (approximating them by continuous – Lusin's theorem), and finally for general L^1 functions.
5. Convolutions. Convolution $h = f * g$ of two Lebesgue measurable on the real line functions f, g is defined by

$$h(x) \ := \ \int_{\mathbb{R}} f(x-y)g(y)dy \ ,$$

provided this integral exist.
a. Show that if $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, then $f * g \in L^1(\mathbb{R})$ (particularly, $f * g$ is well-defined and a.e. finite).
b. Show that $f * g = g * f$
c. Show that if $f \in L^1(\mathbb{R})$ and g is continuous, zero outside some finite interval $[a,b]$, then $f * g$ is continuous.