Integration Theory / Mathematical Analysis (5B1452/MA429)

Solutions to homework assignment # 1

1. Sets $A \cap B$, $A - B$, and $B - A$ are disjoint. Hence, since μ is additive,

$$\begin{aligned} \mu(A) + \mu(B) &= (\mu(A - B) + \mu(A \cap B)) + (\mu(A \cap B) + \mu(B - A)) \\ &= \mu(A \cap B) + (\mu(A - B) + \mu(A \cap B) + \mu(B - A)) = \mu(A \cap B) + \mu(A \cup B) \ . \end{aligned}$$

2. a. If set E is countable or finite, then we can cover it by one-point sets $\{a_j\}$ and obtain $0 \leq \mu^*(E) \leq \sum_j \lambda\{a_j\} = 0$. On the other hand, if E is uncountable, $\mu^*(E) \leq \lambda(X) = 1$ and for any cover A_j one of the sets, say A_p, has to coincide with X (if all A_j are one-point or empty, then E is countable), hence $\sum_j \lambda(A_j) \geq \lambda(A_p) = \lambda(X) = 1$.

Thus we conclude, that $\mu^*(E) = 0$ if E is countable and $\mu^*(E) = 1$ if E is uncountable.

b. If set E is countable, then for countable A both $A \cap E$ and $A - E$ are countable and

$$\mu^*(A) = 0 = 0 + 0 = \mu^*(A \cap E) + \mu^*(A - E) \ .$$

For uncountable A set $A \cap E$ is countable while $A - E$ is uncountable and

$$\mu^*(A) = 1 = 0 + 1 = \mu^*(A \cap E) + \mu^*(A - E) \ .$$

We conclude that countable sets are μ^* measurable, and hence their complements also are.

On the other hand, if both sets E and E^c are uncountable, taking $A = X$ we obtain

$$\mu^*(A) = \mu^*(X) = 1 \neq 1 + 1 = \mu^*(E) + \mu^*(E^c) = \mu^*(A \cap E) + \mu^*(A - E) \ ,$$

so E is not μ^*-measurable.

Therefore $\mathcal{A} = \{E : \ E \ or \ E^c \ is \ countable\}$, and μ is zero for countable sets and one for uncountable ones.

3. Take arbitrary set E, then for any A one has

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A - E) \ ,$$

so E is μ^*-measurable. Thus all sets are μ^*-measurable, and μ^* is a measure.

4. Denote $\sigma := s_1 \dots s_n$ (the binary number). Numbers, where the string S occurs as the digits number $(kn + 1), \dots, (kn + n)$, are exactly those in the intervals $[\sigma/2^{kn+n}, (\sigma + 1)\sigma/2^{kn+n})$. Simlarly to construction in class, one sees by induction in l that the set E_l of numbers, where the string S does not occur as the digits number $(kn + 1), \dots, (kn + n)$ for $1 \leq k \leq l$, can be covered by $(2^n - 1)^l$ intervals of length 2^{-ln}. Therefore, the Lebesgue measure of E_l satisfies $\mu(E_l) \leq (2^n - 1)^l \cdot 2^{-ln} = [(2^n - 1)/2^n]^l$. As l tends to infinity, the latter tends to zero, because $2^n - 1)/2^n < 1$. Thus, intersection $\cap_l E_l$ has measure zero. But it contains all the numbers which do not have S in their binary expansion, so latter is a nullset.

5.a. It is sufficient to show that μ^* is a *metric outer measure*, i.e. that for any sets A and B, separated by an interval of positive length, $\mu_f^*(A) + \mu_f^*(B) = \mu_f^*(A \cup B)$, or, equivalently, "$\leq$", since the opposite inequality always holds. This definition was given in class, and it is weaker, than one in the book.

Take such sets A and B, then we can chose a point x so that $A \subset (-\infty, x)$ and $B \subset (x, +\infty)$ (or vice versa). Take any covering of $A \cup B$ by a collection of open intervals $\{I_j\}$. For a given I_j denote by I'_j and I''_j the intervals $I_j \cap (-\infty, x)$ and $I_j \cap (x, +\infty)$ correspondingly. Clearly, $\lambda(I_j) = \lambda(I'_j) + \lambda(I''_j)$, and collections $\{I'_j\}$ and $\{I'_j\}$ cover A and B. Therefore

$$\begin{aligned} \mu_f^*(A) + \mu_f^*(B) &\leq \sum_j \lambda(I'_j) + \sum_j \lambda(I''_j) \\ &= \sum_j (\lambda(I'_j) + \lambda(I''_j)) = \sum_j \lambda(I_j) , \end{aligned}$$

and taking infimum over all covers $\{I'_j\}$ we arrive at desired inequality $\mu_f^*(A) + \mu_f^*(B) \leq \mu_f^*(A \cup B)$.

b. We will not assume the continuity of f. Take any cover of $[a, b]$ by open intervals. By the Heine-Borel lemma the closed interval is compact, hence we can chose a finite subcover $\{I_j\}_{j=1\ldots n}$. We can through away any interval non-intersecting $[a, b]$, and among the remaining ones there will be two intervals I_j and I_i which intersect. Replacing them by their union I. If $I = (x, y)$, we can assume that $I_i = (x, x')$ and $I_j = (y', y)$, with $y' < x'$, and hence

$$\lambda(I) = f(y) - f(x) \leq f(y) - f(y') + f(x') - f(x) = \lambda(I_j) + \lambda(I_i) .$$

Therefore joining two intervals only decreases $\sum_j \lambda(I_j)$. Repeat this operation untill only one interval $J = (c, d)$ remains. Clearly, $c < a$, $b < d$, thus (function f is increasing!)

$$\lim_{x \to b+} f(x) - \lim_{x \to a-} f(x) \leq f(d) - f(c) = \lambda(J) \leq \sum_j \lambda(I_j) .$$

Since we have started with an arbitrary cover, this reasoning shows, that $\lim_{x \to b+} f(x) - \lim_{x \to a-} f(x) \leq \mu_f[a, b]$. On the other hand, covering by an interval $(a - \varepsilon, b + \varepsilon)$, and then tending ε to zero, shows that the reverse inequality also holds. Thus,

$$\mu_f[a, b] = \lim_{x \to b+} f(x) - \lim_{x \to a-} f(x) .$$

Particularly, applying this to one-point closed inetrval $[a, a]$ we get

$$\mu_f\{a\} = \lim_{x \to a+} f(x) - \lim_{x \to a-} f(x) .$$

Since all Borel sets are Lebesgue-Stieltjes measurable,

$$\begin{aligned} \mu_f(a, b) &= \mu_f[a, b] - \mu_f\{a\} - \mu_f\{b\} \\ &= (\lim_{x \to b+} f(x) - \lim_{x \to a-} f(x)) - (\lim_{x \to a+} f(x) - \lim_{x \to a-} f(x)) - (\lim_{x \to b+} f(x) - \lim_{x \to b-} f(x)) \\ &= \lim_{x \to b-} f(x) - \lim_{x \to a+} f(x) . \end{aligned}$$

c. By the definition, $\lambda(I) = 1$ if $0 \in I$ and $\lambda(I) = 0$ if $0 \notin I$. Any set E, not containing 0 can be covered by intervals $(-\infty, 0)$ and $(0, +\infty)$ (which do not contain 0), hence $\mu_f^*(E) = 0$. If E contains 0, then any covering contains some interval I_p, containing 0, and hence $\mu_f^*(E) \geq 1$. Covering E by the interval $(-\infty, +\infty)$ we obtain the reverse inequality. So $\mu_f^*(E) = 1$ if $0 \in E$ and $\mu_f^*(E) = 0$ if $0 \notin E$. It is easy to check that all sets are measurable.