Integration Theory / Mathematical Analysis (5B1452/MA429)

Solutions to homework assignment # 2

1. a. *Not integrable.* In fact, for any n one has $\int_{2\pi n}^{2\pi n+\pi} f(x)dx = 2$, so

$$\int_{\mathbf{R}} |f(x)|dx \geq \sum_{n=1}^{\infty} \int_{2\pi n}^{2\pi n+\pi} f(x)dx = \sum_{n=1}^{\infty} 2 = +\infty \ ,$$

and f cannot be integrable.
b. *Not integrable.* By the Lebesgue monotone convergence Theorem,

$$\int_{[0,1]} f(x)dx = \lim_{n\to\infty} \int_{1/n}^{1} f(x)dx = \lim_{n\to\infty} (n-1) = \infty \ ,$$

so the integral exists and is equal to $+\infty$, but f is not integrable.
c. *Integrable.* Same reasoning shows, that

$$\int_{[1,\infty]} f(x)dx = \lim_{n\to\infty} \int_{1}^{\infty} f(x)dx = \lim_{n\to\infty} (1 - \frac{1}{n}) = 1 < +\infty \ ,$$

so $f(x)$ is integrable since it is measurable and non-negative.
2. a. By a classical inequality, $1 + nx \leq (1+x)^n$, so we deduce that sunctions $f_n(x) := \frac{1+nx}{(1+x)^n}$ have an integrable majorant on $[0,1]$, namely $0 \leq f_n(x) \leq 1$. Also f_n converges pointwise to $f(x)$, which is 0 for non-zero x and 1 otherwise: $f(0) = 1$. Therefore by the Lebesgue bounded convergence Theorem,

$$\lim_{n\to\infty} \int_0^1 \frac{1+nx}{(1+x)^n} dx = \int_{[0,1]} f(x)dx = 0 \ .$$

b. Define functions $f_n(x)$ on $[0,\infty)$ by $f_n(x) := (1+x/n)^n e^{-2x}$, if $x \in [0,n]$, and $f_n(x) := 0$ otherwise. By a classical ineqaulity $(1+\frac{1}{y})^y < e$. Plugging in $y = n/x$, we obtain $0 \leq f_n(x) = (1+x/n)^n e^{-2x} = (1+x/n)^{(n/x)x} e^{-2x} \leq e^x e^{-2x} = e^{-x}$, the latter clearly integrable. Moreover, by an equally classical limit, $\lim_{n\to\infty}(1+x/n)^n = e^x$, and hence $f_n(x)$ conveges pointwise to e^{-x}.

Thus we can apply the Lebesgue bounded convergence Theorem, and write

$$\lim_{n\to\infty} \int_0^n (1+x/n)^n e^{-2x} dx = \lim_{n\to\infty} \int_{[0,\infty]} f_n(x)dx = \int_{[0,\infty]} e^{-x}dx = -e^{-x}|_{x=0}^{\infty} = 1 \ .$$

3. a. If g_1 is integrable, then we can apply the Lebsgue bounded convergence Theorem with majorant g_1 (since $0 \leq g_n \leq g_1$), and obtain integrability of g_n, g, and desired convergence of integrals.
b. On the real line with Lebesgue measure, take $g_n := \chi_{[n,\infty)}$, they clearly converge to $g(x) = 0$, without convergence of integrals.

4. If we prove this statements for (non-negative) functions f^+ and f^-, statements for f clearly follow, so we can assume from the beginning that we deal with a non-negative function f.

Consider (biinfinite) sequence of functions $\{g_n\}_{n=-\infty}^{+\infty}$, defined by $g_n(x) := f(x+n)$. They are clearly measurable, and denoting $g := \sum_{n=-\infty}^{+\infty} g_n$ we can apply the Theorem about integrals of series (for the interval $[0,1]$):

$$\begin{aligned}\int_0^1 g(x)dx &= \sum_{n=-\infty}^{+\infty} \int_0^1 g_n(x)dx = \sum_{n=-\infty}^{+\infty} \int_0^1 f(x+n)dx \\ &= \sum_{n=-\infty}^{+\infty} \int_n^{n+1} f(x)dx = \int_{-\infty}^{+\infty} f(x)dx < +\infty .\end{aligned}$$

Thus, function g is integrable on $[0,1]$, and part **b.** follows. Also it implies, that g is finite a.e.. Therefore the series in question converges for a.e. x in $[0,1]$ and hence for a.e. x on the real line (the sum of this series is periodic, $g(x) = g(x+1)$).

5. a. Function f is equal to the function

$$g(x) := \sum_{n=1}^{\infty} n\chi_{(1/(n+1),1/n]}(x) ,$$

changed on the set of rational numbers to be equal to zero there. Since g is a some of measurable functions (in fact, chracteristic functions of measurable sets), it is measurable. But $f = g$ almost everywhere, thus (Lebesgue measure is complete, so all nullsets are measurable) f is also measurable.

b. Functions f and g have the same integrals (since $f = g$ a.e.), so we can write

$$\begin{aligned}\int_0^1 f(x)dx &= \sum_{n=1}^{\infty} \int_{(1/(n+1),1/n]} f(x)dx = \sum_{n=1}^{\infty} \int_{(1/(n+1),1/n]} ndx \\ &= \sum_{n=1}^{\infty} (1/n - 1/(n+1))n = \sum_{n=1}^{\infty} 1/(n+1) = +\infty .\end{aligned}$$