Integration Theory / Mathematical Analysis (5B1452/MA429)

Solutions to homework assignment # 3

Final exam will be on **December 12th, 10:00-14:00,** in classrooms **K51** and **K53** It will be written, and will contain a few problems similar to those in the homework assignments. You may use your class notes and Friedman's book (no other books, please, since they may contain some of the exam's problems).

1. a. Since $\{f_n\}$ converges to f almost uniformly, it also converges almost everywhere pointwise, so except for a null-set F one can write $\lim_n f_n(x) = f(x)$. Similarly, except for a null-set G one has $\lim_n f_n(x) = g(x)$. Thus, except for a null set $F \cup G$ and so almost everywhere, $f(x) = \lim_n f_n(x) = g(x)$.

b. Since $\{f_n\}$ converges to f in measure, there is a subsequence $\{f_{n'}\}$ converging to f almost uniformly. Since $\{f_n\}$ converges to g in measure, so does $\{f_{n'}\}$, and hence there is a subsubsequence $\{f_{n''}\}$ converging to g almost uniformly. It also converges almost uniformly to f (since $\{f_{n'}\}$ did), thus we can employ part a and deduce that $f = g$ a.e.

c. We can write

$$\int |f - g| d\mu \leq \int |f - f_n| d\mu + \int |f_n - g| d\mu \ ,$$

and two latter integrals tend to zero as $n \to \infty$. Hence $\int |f - g| d\mu = 0$ and $|f - g| = 0$ a.e. implying $f = g$ a.e.

Also convergence in the mean implies convergence in measure, so we could use part b.

2. Define $f_n(x) := e^{-a_n|x-x_n|}$, which is clearly measurable (being continuous). Since it is also non-negative one can write

$$\begin{aligned} \int_{\mathbb{R}} f_n(x)dx &= \lim_{t\to\infty} \left(\int_{x_n - t}^{x_n} e^{a_n(x-x_n)} + \int_{x_n}^{x_n+t} e^{-a_n(x-x_n)} \right) dx \\ &= \lim_{t\to\infty} \frac{1}{a_n} \left(e^{-a_n(x-x_n)}|_{x_n-t}^{x_n} - e^{-a_n(x-x_n)}|_{x_n}^{x_n+t} \right) = \frac{2}{a_n} \, . \end{aligned}$$

Thus by theorem on integration of series (all functions involved are non-negative)

$$\sum_{n=1}^{\infty} e^{-a_n|x-x_n|} = \int_{\mathbb{R}} \sum_n f_n(x)dx = \sum_n \int_{\mathbb{R}} f_n(x)dx = \sum_n \frac{1}{a_n} < \infty \, .$$

Therefore $\sum_n e^{-a_n|x-x_n|}$ is an integrable function and hence a.e. finite.

3. Take $\mu := \sum_{j=1}^{\infty} \frac{1}{j}\delta_j$ and $\nu := \sum_{j=1}^{\infty} 2\delta_j$ for the space $\mathbb{N}$ of positive integers with σ-algebra of all subsets. Clearly $\mu(E) = 0 \iff E = \emptyset \iff \nu(E) = 0$, so $\nu \ll \mu$. On the other hand for every postive integer j one has $\nu\{j\} = 2 > 1$ whereas $\mu\{j\} = \frac{1}{j}$ is arbitrarily small as $j \to \infty$.

4. Set $g_t(x) := e^{itx} f(x)$ for $t \in \mathbb{R}$. Clearly g_t is majorated by $h(x) := |f(x)| = |g_t(x)|$, which is integrable, since f is. Thus

$$|\hat{f}(t)| = |\int_{\mathbb{R}} g_t(x)dx| \leq \int_{\mathbb{R}} |g_t(x)|dx = \int_{\mathbb{R}} h(x)dx < \infty \,,$$

so $\hat{f}$ is bounded. Fix real t. Then as $s \to t$ functions g_s converge to g_t pointwise: indeed, $\lim_{s\to t} e^{isx} f(x) = e^{itx} f(x)$. By the Lebesgue bounded convergence theorem (applied with majorant h) for every real t one has

$$\lim_{s\to t} \hat{f}(s) = \lim_{s\to t} \int_{\mathbb{R}} g_s(x)dx \int_{\mathbb{R}} g_t(x)dx = \hat{f}(t) \,,$$

so $\hat{f}$ is continuous.

5. Consider intervals $[a_j, b_j] := [1/(\frac{3\pi}{2} + 2\pi j), 1/(\frac{\pi}{2} + 2\pi j)]$. Then they are disjoint and as $\jmath \to \infty$ get closer and closer to 0, so if we take all inetrvals from n-th to m-th, the sum of their lengths is arbitrarily small for big n:

$$\sum_{j=n}^{m} |a_j - b_j| \leq b_n = 1/(\frac{\pi}{2} + 2\pi n) < 1/n \,.$$

On the other hand, if we fix n, and increase m, the oscillation of f is arbitrarily big:

$$\begin{aligned}\sum_{j=n}^{m} |f(a_j) - f(b_j)| &= \sum_{j=n}^{m} \left| \sin(\frac{3\pi}{2} + 2\pi j)/(\frac{3\pi}{2} + 2\pi j) - \sin(\frac{\pi}{2} + 2\pi j)/(\frac{\pi}{2} + 2\pi j)] \right| \\ &= \sum_{j=n}^{m} \left| -1/(\frac{3\pi}{2} + 2\pi j) - 1/(\frac{\pi}{2} + 2\pi j)] \right| > \frac{1}{\pi} \sum_{j=n}^{m} \frac{1}{j+1} \overset{m\to\infty}{\to} \infty \,,\end{aligned}$$

So $f(x)$ *is not absolutely continuous.*