Integration Theory / Mathematical Analysis (5B1479/MA429)
Solutions to homework assignment # 4

1.a. Use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and note that mentioned square is contained in the region $\{0 \le r\sqrt{2}, 0 \le \theta\frac{\pi}{2}\}$, then $f(x,y) = \frac{1}{r}$ and

$$\int_{[0,1]\times[0,1]} f(x,y)dxdy \le \int_{[0,\sqrt{2}]\times[0,\frac{\pi}{2}]} f\cdot rdrd\theta = \int_{[0,\sqrt{2}]\times[0,\frac{\pi}{2}]} 1drd\theta = \frac{\sqrt{2}\pi}{2}.$$

b.

$$\begin{aligned}\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}dxdy &= \int_0^1\int_0^1 \left(\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3}\right)dxdy = \int_0^1 \left(-\frac{1}{(x+y)^1} + \frac{y}{(x+y)^2}\right)\Bigg|_{x=0}^1 dy \\ &= \int_0^1 \left(-\frac{1}{1+y} + \frac{1}{y} + \frac{y}{(1+y)^2} - \frac{y}{y^2}\right)dy = \int_0^1 -\frac{1}{(1+y)^2}dy = \frac{1}{1+y}\Bigg|_{y=0}^1 = -\frac{1}{2}\ .\end{aligned}$$

Likewise one shows that $\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}dydx = \frac{1}{2}$. Fubini theorem cannot be applied, since $\frac{x-y}{(x+y)^3}$ is not Lebesgue integrable in the square $[0,1]\times[0,1]$. One can see that by evaluating he iterated integral of its absolute value:

$$\begin{aligned}\int_0^1\int_0^1 \left|\frac{x-y}{(x+y)^3}\right|dxdy &= \int_0^1 \left(\int_0^y -\frac{x-y}{(x+y)^3}dx + \int_y^1 \frac{x-y}{(x+y)^3}dx\right)dy \\ &= \int_0^1 \left(-\left(-\frac{1}{(x+y)^1} + \frac{y}{(x+y)^2}\right)\Bigg|_{x=0}^y + \left(-\frac{1}{(x+y)^1} + \frac{y}{(x+y)^2}\right)\Bigg|_{x=y}^1\right)dy \\ &= \int_0^1 \left(\frac{1}{2y} - \frac{1}{(1+y)^2}\right)dy = +\infty\ .\end{aligned}$$

2.a. $\mathcal{B}_1$ is generated by open intervals, so $\mathcal{B}_1\times\mathcal{B}_1$ is generated by open rectangles, while $\mathcal{B}_2$ is generated by open discs. Any open rectangle is a countable union of open circles, and vice versa, therefore $\mathcal{B}_1\times\mathcal{B}_1 = \mathcal{B}_2$.

b. Consider a set $E := A\times\{0\}$ for some Lebesgue non-measurable subset A of the real line. Then $E\subset\mathbb{R}\times\{0\}$, so it is a null set for the Lebesgue measure in $\mathbb{R}^2$, and hence E belongs to $\mathcal{L}_2$. On the other hand section $E^0 = \{x : (x,0)\in E\} = A \notin \mathcal{L}_1$, so by the Lemma 2.15.1 set E does not belong to $\mathcal{L}_1\times\mathcal{L}_1$.

One can also easily show, that $\mathcal{L}_1\times\mathcal{L}_1\subset\mathcal{L}_2$.

3.a. Suppose that function $h\in L^r(X,\mu)$. Then by Hölder inequality for functions $f = |h|^p$ and $g = 1$ and powers $P = \frac{r}{p}$ and $Q = \frac{r}{r-p}$ (clearly $\frac{1}{P} + \frac{1}{Q} = 1$) one has

$$\begin{aligned}\int |h|^p d\mu &= \int fg\mu \le \left(\int f^P d\mu\right)^{\frac{1}{P}}\left(\int g^Q d\mu\right)^{\frac{1}{Q}} \\ &= \left(\int |h|^{p\frac{r}{p}}d\mu\right)^{\frac{1}{P}}\left(\int 1d\mu\right)^{\frac{1}{Q}} = \left(\int |h|^r d\mu\right)^{\frac{1}{P}}\mu(X)^{\frac{1}{Q}} \le +\infty\ ,\end{aligned}$$

since $h\in L^r$ and $\mu(X) < \infty$. Thus $\int |h|^p d\mu < \infty$ and $h\in L^p(X,\mu)$.

One can give a proof not employing the Hölder inequality, but using obvious $|f|^p \le 1 + |f|^r$ instead.

b. Set $X := [1,+\infty)$, and let μ be the usual Lebesgue measure. It is easy to check that function $h(x) = x^{-1/r}$ belongs to $L^p[1,+\infty)$ for any $p > r$, but not to $L^r[1,+\infty)$.

4. Suppose that ϕ is continuous and zero outside interval $[-M, M]$. Then it is uniformly continuous on $[-M, M]$, meaning that $\lim_{\epsilon\to 0}\omega_\phi(\epsilon) = 0$, where $\omega_f(\epsilon) := \sup_{|x-y|\le\epsilon}|f(x)-f(y)|$ is its *modulus of continuity.* In this case

$$\|\phi(x)-\phi(x+\epsilon)\|_1 = \int_M^{M+\epsilon}\|\phi(x)-\phi(x+\epsilon)\|dx \le \int_M^{M+\epsilon}\omega_\phi(\epsilon)dx = (2M+\epsilon)\omega_\phi(\epsilon) \overset{\epsilon\to 0}{\longrightarrow} 0,$$

so $\lim_{\epsilon\to 0}\|\phi(x)-\phi(x+\epsilon)\|_1 = 0$.

Now take arbitrary function $f \in L^1(\mathbb{R})$. Fix some $\delta > 0$. Since $\lim_{M\to\infty}\int_{-\infty}^{-M}+\int_M^\infty |f|dx = 0$, we can pick such M that $\int_{-\infty}^{-M}+\int_M^\infty |f|dx < \delta/4$. Now by the corollary of Lusin's theorem one can find a continuous function ϕ, zero outside $[-M, M]$, such that $\int_{-M}^{M}|f-\phi|dx < \delta/4$. Then

$$\|f-\phi\|_1 = \int_{\mathbb{R}}|f-\phi|dx = \int_{-M}^{M}|f-\phi|dx + \int_{-\infty}^{-M}+\int_M^\infty |f|dx < \delta/2.$$

Hence we can write

$$\begin{aligned}\limsup_{\epsilon\to 0}\|f(x)-f(x+\epsilon)\|_1 &\le \limsup_{\epsilon\to 0}\left(\|f(x)-\phi(x)\|_1 + \|\phi(x)-\phi(x+\epsilon)\|_1 + \|\phi(x+\epsilon)-f(x+\epsilon)\|_1\right)\\ &\le \|f(x)-\phi(x)\|_1 + \limsup_{\epsilon\to 0}\|\phi(x)-\phi(x+\epsilon)\|_1 + \|f(x)-\phi(x)\|_1 \le \delta.\end{aligned}$$

Since δ was arbitrary, this implies that $\lim_{\epsilon\to 0}\|f(x)-f(x+\epsilon)\|_1 = 0$.

5.a. Using in $(*)$ Fubini theorem for non-negative functions, we get

$$\begin{aligned}\int_{\mathbb{R}}|f*g(x)|dx &= \int_{\mathbb{R}}\left|\int_{\mathbb{R}}f(x-y)g(y)dy\right|dx \le \int_{\mathbb{R}}\int_{\mathbb{R}}|f(x-y)g(y)|\,dydx \overset{*}{=} \int_{\mathbb{R}}\int_{\mathbb{R}}|f(x-y)g(y)|\,dydx\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}|f(x-y)||g(y)|dxdy = \int_{\mathbb{R}}\left(\int_{\mathbb{R}}|f(x-y)|dx\right)|g(y)|dy\\ &= \int_{\mathbb{R}}\|f\|_1|g(y)|dy = \|f\|_1\int_{\mathbb{R}}|g(y)|dy = \|f\|_1\|g\|_1 < \infty.\end{aligned}$$

Thus $f*g$ belongs to $L^1(\mathbb{R})$.

b. Changing the variable $y = x-u$ we get (whenever the first integral makes sense)

$$\begin{aligned}f*g(x) &= \int_{-\infty}^{+\infty}f(x-y)g(y)dy = \int_{+\infty}^{-\infty}f(x-(x-u))g(x-u)d(x-u)\\ &= -\int_{+\infty}^{-\infty}f(u)g(x-u)du = \int_{-\infty}^{+\infty}f(u)g(x-u)du = g*f(x).\end{aligned}$$

c. Since g is zero outside $[a, b]$ and continuous, it belongs to L^1 and thus convolution makes sense by part a. Denoting by $\omega_g(\epsilon)$ its modulus of continuity we can write

$$\begin{aligned}|g*f(x+\epsilon) - g*f(x)| &= \left|\int_{\mathbb{R}}g(x+\epsilon-y)f(y)+g(x-y)f(y)dy\right|\\ &\le \int_{\mathbb{R}}|g(x+\epsilon-y)f(y)+g(x-y)f(y)|\,dy = \int_{\mathbb{R}}|g(x+\epsilon-y)-g(x-y)|\cdot|f(y)|dy\\ &\le \int_{\mathbb{R}}\omega_g(\epsilon)\cdot|f(y)|dy = \omega_g(\epsilon)\int_{\mathbb{R}}|f(y)|dy = \omega_g(\epsilon)\|f\|_1 \overset{\epsilon\to 0}{\longrightarrow} 0,\end{aligned}$$

and $f*g = g*f$ is continuous.