A COMPARISON THEOREM FOR f-VECTORS OF SIMPLICIAL POLYTOPES

ANDERS BJÖRNER

DEDICATED TO ROBERT MACPHERSON ON THE OCCASION OF HIS 60TH BIRTHDAY

Abstract. Let $f_i(P)$ denote the number of i-dimensional faces of a convex polytope P. Furthermore, let $S(n,d)$ and $C(n,d)$ denote, respectively, the stacked and the cyclic d-dimensional polytopes on n vertices. Our main result is that for every simplicial d-polytope P, if

$$f_r(S(n_1,d)) \leq f_r(P) \leq f_r(C(n_2,d))$$

for some integers n_1, n_2 and r, then

$$f_s(S(n_1,d)) \leq f_s(P) \leq f_s(C(n_2,d))$$

for all s such that $r < s$.

For $r = 0$ these inequalities are the well-known lower and upper bound theorems for simplicial polytopes.

The result is implied by a certain "comparison theorem" for f-vectors, formulated in Section 4. Among its other consequences is a similar lower bound theorem for centrally-symmetric simplicial polytopes.

1. Introduction

The following extremal problem and its ramifications have a long tradition in the theory of convex polytopes: among all d-dimensional polytopes P with n vertices determine the maximum (or, minimum) of $f_i(P)$. The answers were given around 1970 by McMullen [5] and Barnette [1], who proved that (as had been conjectured) the upper bound is attained in all dimensions by the cyclic polytope $C(n,d)$ and the lower bound is attained in all dimensions by the stacked polytope $S(n,d)$.

What if we specify the number of r-dimensional faces of P, for some $r > 0$, and pose the analogous extremal problem? The following can be said in general.

Theorem 1. *Let P be a d-dimensional simplicial polytope.*
Suppose that

$$f_r(S(n_1,d)) \leq f_r(P) \leq f_r(C(n_2,d))$$

for some integers n_1, n_2 and $0 \leq r \leq d-2$. Then,

$$f_s(S(n_1,d)) \leq f_s(P) \leq f_s(C(n_2,d))$$

for all s such that $r < s < d$.

For $r = 0$ these inequalities are the lower and upper bound theorems of Barnette and McMullen [1], [5], [9, Ch. 8]. The $s = d-1$ case of the upper bound part is also

---

Date: November 17, 2006.

known; it is covered by the "generalized upper bound theorem" of Kalai [4, Theorem 2].

The proof of Theorem 1 relies on a comparison theorem for f-vectors of simplicial homology spheres (Theorem 4 in Section 4) together with Stanley's proof of necessity for the g-theorem [7]. By the same technique we obtain the following. Here $CS(2n,d)$ denotes the centrally-symmetric stacked d-dimensional polytopes on $2n$ vertices.

Theorem 2. *Let P be a d-dimensional centrally-symmetric simplicial polytope. Suppose that*

$$f_r(CS(2n,d)) \leq f_r(P)$$

for some integers n and $0 \leq r \leq d-2$. Then,

$$f_s(CS(2n,d)) \leq f_s(P)$$

for all s such that $r < s < d$.

Acknowledgment. This paper was written in response to a question of C. Smyth, who asked whether the upper bound part of Theorem 1 might be true (personal communication). Partial results in this direction have also been achieved by A. Werner and G. M. Ziegler (personal communication). I am grateful to G. M. Ziegler and to an anonymous referee for helpful comments on a preliminary version of the paper, and to S. Linusson who spotted and helped correct an error in the proof of Lemma 3.

2. Preliminaries

For the standard notions concerning convex polytopes and simplicial complexes we refer to the literature, see e.g. [9]. In this section we gather some basic definitions and recall some core results.

The *cyclic polytope* $C(n,d)$ is defined and extensively discussed in [9]. The *stacked polytope* $S(n,d)$, $n > d$, is obtained from the d-simplex by performing an arbitrary sequence of $n-d-1$ stellar subdivisions of facets. Similarly, the *centrally-symmetric stacked polytope* $CS(2n,d)$, $2n \geq 2d$, is obtained from the d-dimensional cross-polytope by performing an arbitrary sequence of $n-d$ pairs of centrally-symmetric stellar subdivisions of facets. For $n > d+1 > 3$ the combinatorial types of the resulting polytopes depend on choices made during the construction, but their f-vectors are well-defined.

Let Δ be a $(d-1)$-dimensional simplicial complex, and let f_i be the number of i-dimensional faces of Δ. The sequence $\mathbf{f} = (f_0, \ldots, f_{d-1})$ is called the *f-vector* of Δ. We put $f_{-1} = 1$. The *h-vector* $\mathbf{h} = (h_0, \ldots, h_d)$ of Δ is defined by the equation

$$\sum_{i=0}^{d} f_{i-1} x^{d-i} = \sum_{i=0}^{d} h_i (x+1)^{d-i}.$$

From now on we fix the integer $d \geq 3$, and let $\delta = \lfloor \frac{d}{2} \rfloor$. The *$g$-vector* of Δ is the integer sequence $\mathbf{g} = (g_0, g_1, \ldots, g_\delta)$ defined by $g_0 = 1$ and

$$g_i = h_i - h_{i-1}, \qquad i = 1, \ldots, \delta.$$

The f-vector, h-vector and g-vector of a simplicial d-polytope are those of its boundary complex.

In the case when Δ is a homology sphere (or, more generally, a psedomanifold such that the complex itself as well as the link of every face has the Euler characteristic of a sphere of the same dimension) we have the *Dehn-Sommerville equations* $h_i = h_{d-i}$, which show that the f-vector of Δ is completely determined by its g-vector. The linear relation can be expressed as a matrix product (see e.g. [2] or [9, p. 269])

$$\mathbf{f} = \mathbf{g} \cdot M_d,$$

where the $(\delta+1) \times d$-matrix $M_d = (m_{ij})$ is defined by

$$m_{i,j} = \binom{d+1-i}{d-j} - \binom{i}{d-j}, \qquad \text{for } 0 \le i \le \delta,\ 0 \le j \le d-1.$$

Thus, the set of f-vectors of homology $(d-1)$-spheres coincides with the g-vector weighted linear span of the row vectors of M_d.

For instance, we have that

$$M_{10} = \begin{pmatrix} 11 & 55 & 165 & 330 & 462 & 462 & 330 & 165 & 55 & 11 \\ 1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 9 \\ 0 & 1 & 9 & 36 & 84 & 126 & 126 & 84 & 35 & 7 \\ 0 & 0 & 1 & 8 & 28 & 56 & 70 & 55 & 25 & 5 \\ 0 & 0 & 0 & 1 & 7 & 21 & 34 & 31 & 15 & 3 \\ 0 & 0 & 0 & 0 & 1 & 5 & 10 & 10 & 5 & 1 \end{pmatrix}$$

3. NONNEGATIVITY OF THE M_d MATRIX

We need the following technical property of the matrix M_d.

Lemma 3. *All 2×2 minors of the matrix M_d are nonnegative.*

Proof. For $0 \le a < b \le \delta$ and $0 \le r < s \le d-1$, let

$$\Phi^{a,b}_{r,s} \overset{\text{def}}{=} m_{a,r} m_{b,s} - m_{a,s} m_{b,r}.$$

We want to show that $\Phi^{a,b}_{r,s} \ge 0$.

Let $\overline{r} \overset{\text{def}}{=} d - r$, $\overline{s} \overset{\text{def}}{=} d - s$, $\tilde{a} \overset{\text{def}}{=} d + 1 - a$ and $\tilde{b} \overset{\text{def}}{=} d + 1 - b$. Then, by definition

$$\Phi^{a,b}_{r,s} = \left[\binom{\tilde{a}}{\overline{r}} - \binom{a}{\overline{r}}\right]\left[\binom{\tilde{b}}{\overline{s}} - \binom{b}{\overline{s}}\right] - \left[\binom{\tilde{a}}{\overline{s}} - \binom{a}{\overline{s}}\right]\left[\binom{\tilde{b}}{\overline{r}} - \binom{b}{\overline{r}}\right]$$

Rearranging terms, and letting $B^{p,q}_{t,u}$ denote the binomial determinant

$$B^{p,q}_{t,u} \overset{\text{def}}{=} \det \begin{pmatrix} \binom{p}{t} & \binom{p}{u} \\ \binom{q}{t} & \binom{q}{u} \end{pmatrix}$$

we can write

$$\Phi^{a,b}_{r,s} = B^{a,\tilde{b}}_{\overline{s},\overline{r}} + B^{\tilde{b},\tilde{a}}_{\overline{s},\overline{r}} - B^{a,b}_{\overline{s},\overline{r}} - B^{b,\tilde{a}}_{\overline{s},\overline{r}} \tag{1}$$

Step 1. Note that

$$\det \begin{pmatrix} m_{i,t} & m_{i,u} \\ m_{j,t} & m_{j,u} \end{pmatrix} \geq 0 \quad \Leftrightarrow \quad \frac{m_{i,t}}{m_{i,u}} \geq \frac{m_{j,t}}{m_{j,u}}, \tag{2}$$

if $i < j$, $t < u$ and $m_{j,u} > 0$.

An elementary argument based on this observation shows that it suffices to prove nonnegativity of $\Phi^{a,b}_{r,s}$ for the special case when $b = a + 1$.

(*Remark:* We could also reduce to the case $s = r + 1$; however, this leads to no simplification in what follows.)

Step 2.

In order to show that $\Phi^{a,a+1}_{r,s} \geq 0$ we put to use the lattice-path interpretation of binomial determinants, due to Gessel and Viennot [3].

Let $L^{p,q}_{t,u}$ denote the set of pairs (P, Q) of vertex-disjoint NE-lattice paths in $\mathbb{Z}^2$, such that P leads from $(0, -p)$ to $(t, -t)$ and Q from $(0, -q)$ to $(u, -u)$. By a NE-*lattice path* we mean a path taking steps N=$(0, 1)$ to the *north* and steps E=$(1, 0)$ to the *east.*

The formula of Gessel and Viennot [3, Theorem 1] states that

$$B^{p,q}_{t,u} = \# L^{p,q}_{t,u}$$

Thus, from equation (1) we have

$$\Phi^{a,a+1}_{r,s} = \# L^{a,\tilde{a}-1}_{\overline{s},\overline{r}} + \# L^{\tilde{a}-1,\tilde{a}}_{\overline{s},\overline{r}} - \# L^{a,a+1}_{\overline{s},\overline{r}} - \# L^{a+1,\tilde{a}}_{\overline{s},\overline{r}}$$

For ease of notation we from now let $L^{p,q} \overset{\text{def}}{=} L^{p,q}_{\overline{s},\overline{r}}$. The proof will be concluded by producing an injective mapping

$$\varphi : L^{a,a+1} \cup L^{a+1,\tilde{a}} \to L^{a,\tilde{a}-1} \cup L^{\tilde{a}-1,\tilde{a}}$$

The construction of the mapping φ proceeds by cases.

Case 1: $(P, Q) \in L^{a,a+1}$. Then $\varphi(P, Q) \in L^{a,\tilde{a}-1}$ is constructed by keeping the path P and extending the path Q by an intitial vertical segment (a sequence of North steps) so that it begins at the point $(0, -(\tilde{a} - 1))$.

Case 2: $(P, Q) \in L^{a+1,\tilde{a}}$.

Subcase 2a: Both Q and P begin with N steps. Then $\varphi(P, Q) \in L^{a,\tilde{a}-1}$ is constructed by removing the first step from both paths.

Subcase 2b: Q begins with an E step. Then $\varphi(P, Q) \in L^{\tilde{a}-1,\tilde{a}}$ is constructed by keeping the path Q and extending the path P by an intitial vertical segment so that it originates in $(0, -(\tilde{a} - 1))$.

Subcase 2c: Q begins with an N step, and P begins with an E step. Then $\varphi(P, Q) \in L^{\tilde{a}-1,\tilde{a}}$ is constructed as follows. We may assume that $a \geq \overline{s}$, since otherwise some binomial coefficients are zero and the situation simplifies. Thus, the path P begins with a sequence of E steps, say k of them, followed by a N step. Denoting the rest of P by P' we can write: $P = E^k N P'$. Similarly, Q has the factorization $Q = NRENᵛEQ'$, where the two E:s designate the k-th and $(k+1)$-st occurrences of the letter "E" in Q. See Figure 1 for the geometric idea.

The integers k and v are determined by the definition of the paths P and Q. Let h be the number of occurrences of the letter "N" in R. Let $\overline{P}$ and $\overline{Q}$ be the paths

$$\overline{P} = N^{\tilde{a}-a-h-3}ERN^2P' \quad \text{and} \quad \overline{Q} = E^kN^vEN^{h+1}Q',$$

originating in the points $(0, -\tilde{a}+1)$ and $(0, -\tilde{a})$, respectively. A straightforward inspection of the construction shows that these paths are disjoint. Namely, the lowest point on $\overline{P}$ and the highest point on $\overline{Q}$ with first coordinate k are, respectively, $(k, -a-h-2)$ and $(k, -\tilde{a}+v)$. Their distance is $\tilde{a}-a-h-v-2 > 0$. Let $\varphi(P,Q) = (\overline{P}, \overline{Q}) \in L^{\tilde{a}-1,\tilde{a}}$.

This defines the mapping φ in all cases. Each case separately is clearly injective. That there is no interference among the four cases, and hence that φ is injective globally, is most easily seen from following properties of the construction:

- $\varphi(P,Q) \in L^{a,\tilde{a}-1}$ in cases 1 and 2a
- $\varphi(P,Q) \in L^{\tilde{a}-1,\tilde{a}}$ in cases 2b and 2c
- $(0,-a-1) \in \varphi(Q)$ in cases 1 and 2b
- $(0,-a-1) \notin \varphi(Q)$ in cases 2a and 2c

This completes the proof. □

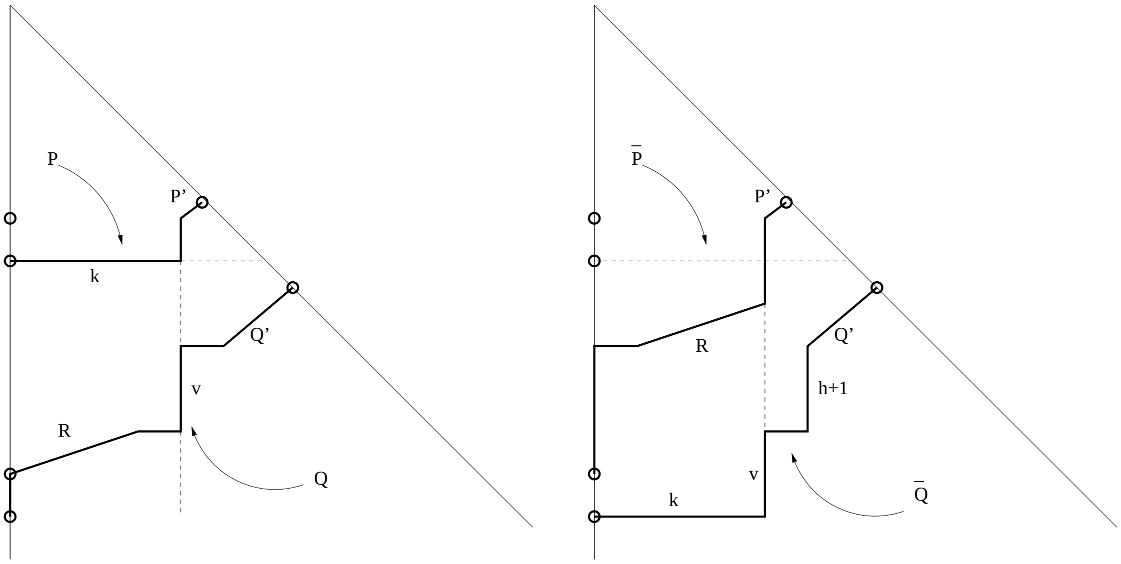

Figure 1: A sketch of subcase 2c.

Remark: We conjecture that the matrix M_d is *totally nonnegative*, meaning that all minors of all orders are nonnegative. This has been verified for all $d \leq 13$ by A. Hultman.

4. HOMOLOGY SPHERES

A key role for this paper is played by the following comparison theorem for f-vectors of homology spheres.

Theorem 4. *Let Δ and Γ be $(d-1)$-dimensional simplicial homology spheres whose g-vectors for some t $(0 \leq t \leq \delta)$ satisfy*

- $g_i(\Delta) \geq g_i(\Gamma)$ *for* $i = 1, \ldots, t$
- $g_i(\Delta) \leq g_i(\Gamma)$ *for* $i = t+1, \ldots, \delta$.

Suppose that

$$f_r(\Delta) \leq f_r(\Gamma)$$

for some $0 \leq r \leq d-2$. *Then*

$$f_s(\Delta) \leq f_s(\Gamma)$$

for all s such that $r < s < d$.

Proof. Let $v_i = g_i(\Delta) - g_i(\Gamma)$. Now,

$$0 \geq f_r(\Delta) - f_r(\Gamma) = \sum_{i=0}^{\delta} v_i m_{i,r} = \sum_{i=0}^{\delta} v_i m_{i,s} \frac{m_{i,r}}{m_{i,s}} \tag{3}$$

Lemma 3 implies, in view of equivalence (2), that

$$\frac{m_{0,r}}{m_{0,s}} \geq \frac{m_{1,r}}{m_{1,s}} \geq \cdots \geq \frac{m_{\delta,r}}{m_{\delta,s}} \geq 0$$

(*Remark:* It is possible that $m_{i,s} = 0$ for $i = k, \ldots, \delta$. Then also $m_{i,r} = 0$ for $i = k-1, \ldots, \delta$ while $m_{i,s} > 0$ for all $i < v$. This requires notational adjustments in our argument, but no new ideas.)

By assumption, the vector $v = (v_0, v_1, \ldots, v_\delta)$ satisfies

$$v_1, \ldots, v_t \geq 0 \quad \text{and} \quad v_{t+1}, \ldots, v_\delta \leq 0.$$

Thus,

$$\sum_{i=0}^{\delta} v_i m_{i,s} \frac{m_{i,r}}{m_{i,s}} \geq \left(\sum_{i=0}^{t} v_i m_{i,s} \right) \frac{m_{t,r}}{m_{t,s}} + \left(\sum_{i=t+1}^{\delta} v_i m_{i,s} \right) \frac{m_{t,r}}{m_{t,s}}$$

which implies that

$$0 \geq f_r(\Delta) - f_r(\Gamma) \geq \frac{m_{t,r}}{m_{t,s}} \left(\sum_{i=0}^{\delta} v_i m_{i,s} \right) = \frac{m_{t,r}}{m_{t,s}} \left(f_s(\Delta) - f_s(\Gamma) \right)$$

It follows that

$$0 \geq f_s(\Delta) - f_s(\Gamma),$$

as desired. ☐

We will say that an integer vector $(n_0, \ldots, n_\delta)$ is an m-sequence if $n_0 = 1$ and $n_j \geq \binom{m}{j}$ implies that $n_{j-1} \geq \binom{m-1}{j-1}$, for all $m \geq j > 1$. In particular, if some entry in an m-sequence is positive then so are all earlier entries. The notion of m-sequence is less restrictive than the well-established concept of M-sequence, recalled in Section 5.

Corollary 5. (Upper bounds) *Let* Δ *be a* $(d-1)$*-dimensional homology sphere whose* g*-vector is an* m*-sequence. Suppose that*

$$f_r(\Delta) \leq f_r(C(n,d))$$

for some integers n *and* $0 \leq r \leq d-2$. *Then*

$$f_s(\Delta) \leq f_s(C(n,d))$$

for all s *such that* $r < s < d$.

Proof. The g-vector of the cyclic polytope $C(n,d)$ is

$$g_i(C(n,d)) = \binom{n-d-2+i}{i}$$

Thus, since $g(\Delta)$ is an m-sequence the conditions of Theorem 4 are satisfied. ☐

Stanley's upper bound theorem for homology spheres [6] shows that in the special case when $r = 0$ Corollary 5 is valid also without the assumption that $g(\Delta)$ is an m-sequence.

Corollary 6. (Lower bounds) *Let* Γ *be a* $(d-1)$*-dimensional homology sphere whose* g*-vector is nonnegative. Suppose that*

$$f_r(S(n,d)) \leq f_r(\Gamma)$$

for some integers n *and* $r \leq d-2$. *Then*

$$f_s(S(n,d)) \leq f_s(\Gamma)$$

for all s *such that* $r < s < d$.

Proof. The g-vector of the stacked polytope $S(n,d)$ is

$$g_i(S(n,d)) = \begin{cases} 1, & \text{for } i = 0 \\ n-d-1, & \text{for } i = 1 \\ 0, & \text{for } i > 1 \end{cases}$$

Thus, since $g(\Gamma)$ is nonnegative the conditions of Theorem 4 are satisfied. ☐

5. Polytopes

We recall the definition of an M-sequence. For any integers $k, n \geq 1$ there is a unique way of writing

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_i}{i},$$

so that $a_k > a_{k-1} > \ldots > a_i \geq i \geq 1$. Then define

$$\partial^k(n) = \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \ldots + \binom{a_i - 1}{i-1}.$$

Also let $\partial^k(0) = 0$.

A nonnegative integer sequence $(n_0, n_1, n_2, \ldots)$ such that $n_0 = 1$ and

$$\partial^k(n_k) \leq n_{k-1} \qquad \text{for all } k > 1$$

is called an *M-sequence.* Clearly, an M-sequence is an m-sequence (as defined in connection with Corollary 5), but not conversely.

Proof of Theorem 1. The g-vector of a simplicial polytope is an M-sequence, by the theorem of Stanley [7]. In particular, it is a nonnegative m-sequence, so both Corollaries 5 and 6 apply. $\square$

Proof of Theorem 2. The g-vector of the centrally-symmetric stacked polytope $CS(2n, d)$ is

$$g_i(CS(n,d)) = \begin{cases} 1, & \text{for } i = 0 \\ 2n - d - 1, & \text{for } i = 1 \\ \binom{d}{i} - \binom{d}{i-1}, & \text{for } i > 1 \end{cases}$$

Stanley [8] has shown that

$$g_i(P) \geq \binom{d}{i} - \binom{d}{i-1}, \text{ for } i \geq 1$$

holds for every centrally-symmetric simplicial polytope P. Hence, Theorem 4 applies. $\square$

References

[1] D. W. Barnette, *A proof of the lower bound conjecture for convex polytopes*, Pacific J. Math. **46** (1973), 349–354.

[2] A. Björner, *Face numbers of complexes and polytopes*, in "Proceedings of the International Congress of Mathematicians, Berkeley, 1986" , Amer. Math. Soc. (1987), pp. 1408–1418.

[3] I. Gessel and G. Viennot, *Binomial determinants, paths, and hook length formulae*, Advances in Math. **58** (1985), 300–321.

[4] G. Kalai, *The diameter of graphs of convex polytopes and f-vector theory*, in "Applied Geometry and Discrete Mathematics, The Victor Klee Festschrift", DIMACS Series in Discrete Math. and Theor. Computer Sci., Vol. **4**, Amer. Math. Soc., Providence, R.I., 1991, pp. 387–411.

[5] P. McMullen, *The maximum numbers of faces of a convex polytope*, Mathematica **17** (1970), 179–184.

[6] R. P. Stanley, *The upper bound conjecture and Cohen-Macaulay rings*, Studies in Applied Math. **54** (1975), 135 – 142.

[7] R. P. Stanley, *The number of faces of simplicial convex polytopes*, Advances in Math. **35** (1980), 236 – 238.

[8] R. P. Stanley, *On the number of faces of centrally-symmetric simplicial polytopes*, Graphs and Combinatorics **3** (1987), 55 – 66.

[9] G. M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics, Vol. **152**, Springer-Verlag, Berlin, 1995. Revised edition 1998.

E-mail address: `bjorner@math.kth.se`

DEPARTMENT OF MATHEMATICS, ROYAL INSTITUTE OF TECHNOLOGY, S-100 44 STOCKHOLM, SWEDEN