Topological Combinatorics

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Anders Björner
Dept. of Mathematics
Kungl. Tekniska Högskolan, Stockholm

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Influence of R. MacPherson on topological combinatorics:

- Intersection homology
  Convex polytopes (via toric varieties), toric g-vector
  Bruhat order (via Schubert varieties)

- Subspace arrangements
  Goresky-MacPherson formula
  Application to complexity

- Oriented matroids
  CD (combinatorial differential) manifolds
  MacPhersonians (discrete Grassmannians)

- And more . . .

Two topics for this talk:

- Goresky-MacPherson formula,
  — with an application to complexity

- Bruhat order
  — with an application of intersection cohomology

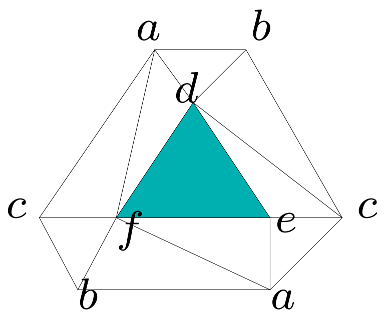
Connections Topology $\leftrightarrow$ Combinatorics

Simplest case: Space $\leftrightarrow$ Triangulation

Example: The real projective plane

$$\boxed{\mathbb{R}P^2}$$

$$\updownarrow$$



$$\leftrightarrow \quad \{abd, acf, adf, ace, abf, aef, bcd, bcf, cde, def\}$$

Topic 1: Goresky-MacPherson formula for subspace arrangements

$\mathcal{A} \stackrel{\text{def}}{=}$ collection of affine subspaces of $\mathbb{R}^d$ – an arrangement

$M_{\mathcal{A}} \stackrel{\text{def}}{=} \mathbb{R}^d \setminus \cup\mathcal{A}$ – its complement

$L_{\mathcal{A}} \stackrel{\text{def}}{=}$ family of nonempty intersections of members of $\mathcal{A}$, ordered by reverse containment – its intersection semi-lattice.

THM ("Goresky-MacPherson formula"):

$$\widetilde{H}^i(M_{\mathcal{A}}) \cong \bigoplus_{x \in L_{\mathcal{A}},\, x > \hat{0}} \widetilde{H}_{\mathrm{codim}(x)-2-i}(\Delta(\hat{0}, x))$$

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Proof: Stratified Morse Theory (1988)

Other proofs by several authors

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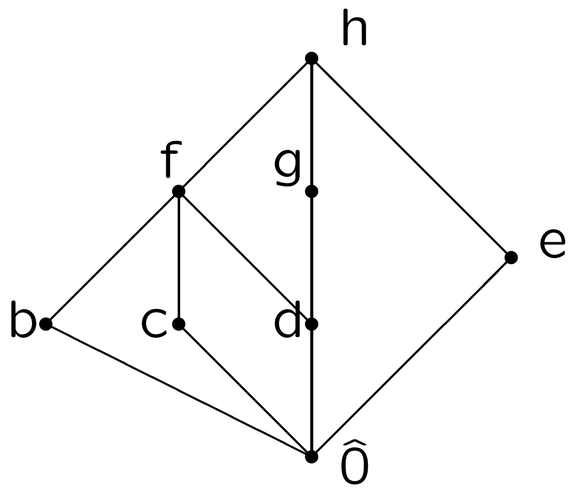
Here $\Delta(\hat{0}, x)$ is the simplicial complex of

$$\{z_1, z_2, \ldots, z_k\}$$

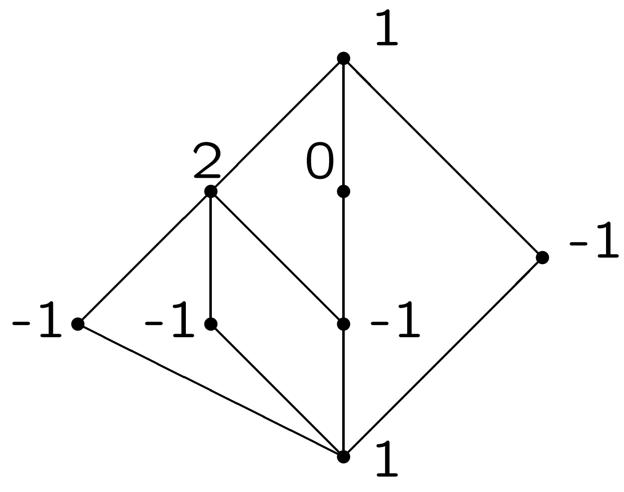
such that

$$\hat{0} < z_1 < z_2 < \cdots z_k < x$$

called the *order complex* of the open interval $(\hat{0}, x)$ in $L_{\mathcal{A}}$.



A small poset



Values of its Möbius function $\mu(\hat{0}, x)$

$\mu(\hat{0}, x) \stackrel{\text{def}}{=} \sum_{\hat{0} \le y < x} \mu(\hat{0}, y)$ $\mu(\hat{0}, x) = \text{Euler char}(\Delta(\hat{0}, x)) - 1$

Special cases of G-M formula:

Hyperplane arr'ts over $\mathbb{R} \rightarrow$ Zaslavsky's formula for number of connected components of $M_{\mathcal{A}}$

Hyperplane arr'ts over $\mathbb{C} \rightarrow$ Brieskorn-Orlik-Solomon formula for cohomology groups of $M_{\mathcal{A}}$

Application of G-M formula
to complexity of algorithms

Given: a string of real numbers

$$x_1, x_2, \ldots, x_n$$

Sought: Efficient algorithms to decide some property of the sequence or to restructure it using only pairwise comparisons.

The question: *How many such comparisons must be made in the worst case when using the best algorithm?* This number, $c(n)$, is called the *complexity* of the problem.

Note: $c(n) \leq n^2$ is immediate

Well-known examples:

1. **Sorting.** *To rearrange the* n *numbers increasingly* $x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_n}$ *requires* $\Theta(n \log n)$ *comparisons.*

2. **Median.** *To find* j *such that* x_j *is "in the middle" requires* $\Theta(n)$ *comparisons, where* $2n \leq \Theta(n) \leq 3n$*.*

3. **Distinctness.** *To decide whether all entries* x_i *are distinct (i.e., if* $x_i \neq x_j$ *when* $i \neq j$*) requires* $\Theta(n \log n)$ *comparisons.*

A generalization of the distinctness problem (the $k = 2$ case).

The k-equal problem: *for* $k \geq 2$*, decide whether some* k *entries are equal, that is, can we find* $i_1 < i_2 < \cdots < i_k$ *such that* $x_{i_1} = x_{i_2} = \cdots = x_{i_k}$*?*

For example, are there nine equal entries in the following list of numbers?

24791374685848713955196742346159463
31486772955924362854117836972581932

Question repeated: are there nine equal entries in the following list of numbers?

247913746858487139551967423461594633
148677295592436285411783697258193 2

Answer: Yes, there are nine copies of the number “4”. Are there ten equal entries? Answer: No.

THM (Bj-Lovász-Yao '92) *The complexity of the k-equal problem is*

$$\Theta(n \log \frac{2n}{k}).$$

More precisely,

$$C_1 \cdot n \log \frac{2n}{k} \leq c_k(n) \leq C_2 \cdot n \log \frac{2n}{k},$$

where

$$\frac{C_2}{C_1} \leq 16.$$

Upper bound: Sorting algorithms
Lower bound: Topological method
(involving G-M formula)

Before sketch of lower bound argument, need more tools

Examples of interesting subspace arr'ts in codimension $k-1$:

- $\mathcal{A}_{n,k} \stackrel{\text{def}}{=} \{x_{i_1} = \cdots = x_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$

- $\mathcal{D}_{n,k} \stackrel{\text{def}}{=} \{\varepsilon_1 x_{i_1} = \cdots = \varepsilon_k x_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n, \varepsilon_i \in \{\pm 1\}\}$

- $\mathcal{B}_{n,k} \stackrel{\text{def}}{=} \mathcal{D}_{n,k} \cup \{x_{j_1} = \cdots = x_{j_{k-1}} = 0 \mid 1 \leq j_1 < \cdots < j_{k-1} \leq n\}$

Note: for $k = 2$ get Coxeter reflection arrangements

Computing cohomology of complement of $\mathcal{A}$ reduces (via G-M formula) to computing homology of order complex of $L_{\mathcal{A}}$.

How compute homology of poset $L_{\mathcal{A}}$?

$\exists$ combinatorial method that works surprisingly often:
lexicographic shellability

P — a poset with $\hat{0}$ and $\hat{1}$

$\mathcal{E}(P) = \{(x, y) \in P \times P \mid x \lessdot y\}$ — its covering relation

Def: An *EL-labeling* of P is a map $\lambda : \mathcal{E}(P) \to \mathbb{Z}$, such that for every interval $[x, y]$:

1. there is a unique maximal chain $\mathbf{m}_{[x,y]}$ whose associated label $\lambda(\mathbf{m}_{[x,y]}) = (a_1, \ldots, a_p)$ is increasing $a_1 < a_2 < \cdots < a_p$,

2. if $\mathbf{m}'$ is any other maximal chain in $[x, y]$ then $\lambda(\mathbf{m}') > \lambda(\mathbf{m}_{[x,y]})$ in the lexicographic order on strings with elements from $\mathbb{Z}$.

The poset P is said to be *lexicographically shellable* (or for short: *EL-shellable*) if it admits an EL-labeling.

EL-shellability, when applicable, reduces homology computations for posets to a combinatorial labeling game. Call a maximal chain $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = x$, *falling* if

$$\lambda(x_0 \lessdot x_1) \;\geq\; \lambda(x_1 \lessdot x_2) \;\geq\; \ldots \;\geq\; \lambda(x_{k-1} \lessdot x_k).$$

THM (Bj-Wachs '96) EL-shellable $\Rightarrow$ $\Delta(\hat{0}, x)$ has the homotopy type of a wedge of spheres, for $\forall x > \hat{0}$. Furthermore, for any fixed EL-labeling:

- $\widetilde{H}_i(\Delta(\hat{0}, x); \mathbb{Z}) \cong \mathbb{Z}^{\#\text{ falling chains of length } (i+2)}$

- a basis for i-dimensional (co)homology is induced by the falling chains of length $i + 2$.

Combining Goresky-MacPherson formula for $M_{\mathcal{A}}$ with lexicographic shellability of $L_{\mathcal{A}}$ we get:

THM For arrangement $\mathcal{A}$, suppose $L_{\mathcal{A}}$ is EL-shellable. Then $\widetilde{H}^i(M_{\mathcal{A}})$ is torsion-free, and the Betti number $\widetilde{\beta}^i(M_{\mathcal{A}})$ is equal to the number of falling chains $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_g$ such that $\mathrm{codim}(x_g) - g = i$.

THM EL-shellability works for
(1) hyperplane arr'ts (over *any* field)
(2) $\mathcal{A}_{n,k}$ and $\mathcal{B}_{n,k}$,
(3) some other cases . . .

Conjecture: Works for $\mathcal{D}_{n,k}$.

Incidentally,

THM (Khovanov '96) Complements of $\mathcal{A}_{n,3}$ and $\mathcal{B}_{n,3}$ are $K(\pi, 1)$ spaces.

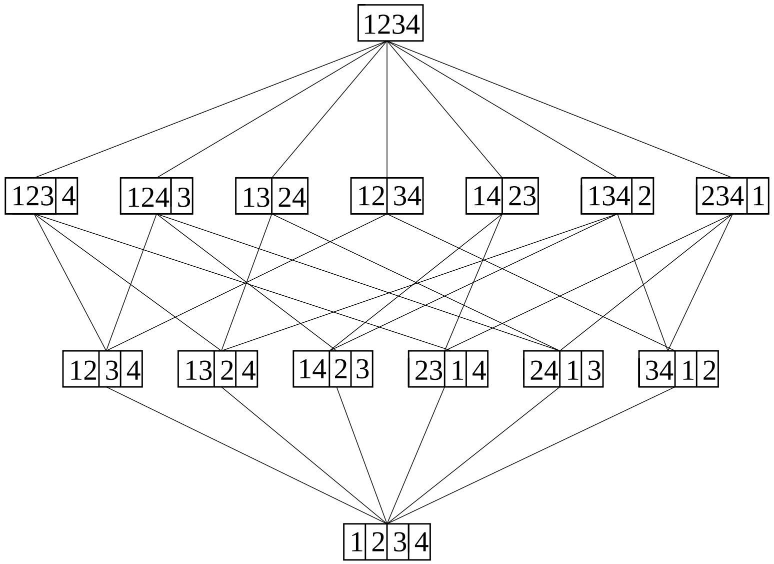
Conjecture: Complement of $\mathcal{D}_{n,3}$ is a $K(\pi, 1)$ space.

Example: EL-shellability-based computation for $\mathcal{A}_{n,k}$

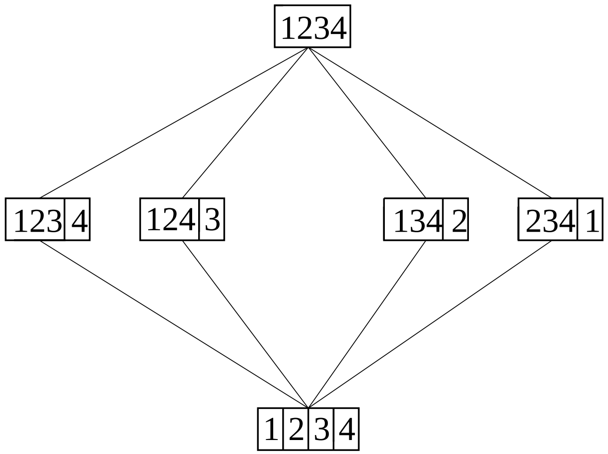
(Following 4 slides are based on joint work with
M. Wachs '95 and V. Welker '95.)

Let $\Pi_{n,k}$ be family of all partitions of $\{1, 2, \ldots, n\}$ that have no parts of sizes $2, 3, \ldots, k-1$. Order them by refinement.

Fact: *The intersection lattice of $\mathcal{A}_{n,k}$ is (isomorphic to) $\Pi_{n,k}$.*



$\Pi_{4,2}$



$\Pi_{4,3}$

Labeling of $L_{\mathcal{A}_{n,k}} \cong \Pi_{n,k}$ with elements from the totally ordered set

$$\bar{1} < \bar{2} < \cdots < \bar{n} < 1 < 2 < \cdots < n$$

Covering	Label
New k-block B created from singletons	$\max(B)$
Non-singleton block B merged with singleton $\{a\}$	a
Two non-singleton blocks B_1 and B_2 merged	$\overline{\max(B_1 \cup B_2)}$

For instance, the following maximal chain in $\Pi_{8,3}$ (only non-singleton blocks are shown)

$$\hat{0} \lessdot 25\textcolor{red}{8} \lessdot 2\textcolor{red}{4}58 \lessdot 13\textcolor{red}{7} \mid 2458 \lessdot 123457\textcolor{red}{8} \lessdot \hat{1}$$

receives the label $(8, 4, 7, \overline{8}, 6)$.

This is an EL-labeling of $\Pi_{n,k}$.

Hence,

* $\Delta_{n,k} = \Delta(\hat{0}, \hat{1})$ has homotopy type of a wedge of spheres

* the Betti numbers $\widetilde{\beta}^d_{n,k} = \text{rank } \widetilde{H}_d(\Delta_{n,k}; \mathbb{Z})$ satisfy

$$\widetilde{\beta}^d_{n,k} \neq 0 \text{ iff } d = n - 3 - t(k-2) \text{ for some } 1 \leq t \leq \left\lfloor \frac{n}{k} \right\rfloor,$$

and

$$\begin{aligned} \widetilde{\beta}^{n-3-t(k-2)}_{n,k} &= (t-1)! \sum_{\substack{0=i_0 \leq \cdots \leq i_t \\ = n-tk}} \prod_{j=0}^{t-1} \binom{n - jk - i_j - 1}{k-1} (j+1)^{i_{j+1} - i_j} \\ &= \sum_{\substack{j_1 + \cdots + j_t = n \\ j_i \geq k}} \binom{n-1}{j_1 - 1, j_2, \ldots, j_t} \prod_{i=1}^{t} \binom{j_i - 1}{k-1}. \end{aligned}$$

Also,

* the Betti numbers of the complement $M_{n,k}$ of $\mathcal{A}_{n,k}$:

$$\beta^i(M_{n,k}) \neq 0 \quad \Leftrightarrow \quad i = t(k-2), \text{ for } 0 \leq t \leq \lfloor \frac{n}{k} \rfloor.$$

Back to algorithmic k-equal problem: geometric point of view (following Bj-Lovász '94)

The k-equal problem is to determine whether a given point $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ lies in the union of all the subspaces

$$x_{i_1} = x_{i_2} = \cdots = x_{i_k}.$$

Equivalently, does it belong to $M_{n,k}$, the complement of the k-equal arrangement $\mathcal{A}_{n,k}$.

Let

$$\beta(M_{n,k}) = \sum_{i=0}^{n} \operatorname{rank} H^i(M_{n,k}).$$

Fact 1. *The complexity of the k-equal problem is at least* $\log_3 \beta(M_{n,k})$.

Note: Recall computation of $\beta(M_{n,k})$ via "GM+EL method", messy sums/products of binomial coeff's . . .

Let $\mu_{n,k}$ be the *Möbius function* computed over the poset $\Pi_{n,k}$.

Fact 2. $\beta(M_{n,k}) \geq |\mu_{n,k}|$. (Again based on G-M formula)

We turn to generating functions and prove:

$$\exp\left(\sum_{n\geq 1} \mu_{n,k} \frac{x^n}{n!}\right) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{k-1}}{(k-1)!}.$$

This implies

Fact 3. *Let* $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ *be the complex roots of the polynomial* $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{k-1}}{(k-1)!}$. *Then*

$$\mu_{n,k} = -(n-1)! \left(\alpha_1^{-n} + \alpha_2^{-n} + \cdots + \alpha_{k-1}^{-n}\right).$$

Collecting the facts, we have:

$$c_k(n) \geq \log_3 \beta(M_{n,k}) \geq \log_3 |\mu_{n,k}|$$

Now either estimate $\beta(M_{n,k})$ via expressions given by EL-shelling,
or estimate $|\mu_{n,k}|$ via Fact 3.
This gives:

$$c_k(n) \geq \ldots\ldots \geq C \cdot n \log \frac{2n}{k}$$

Q.E.D.

Remark : Goresky-MacPherson formula over finite fields

ℓ-adic étale cohomology $H^i(X;\mathbb{Q}_\ell)$ versions of G-M:
Bj-Ekedahl '97, Yan '00, Deligne-Goresky-MacPherson '00, ...

Briefly: Let $\mathcal{A}$ be a d-dim'l subspace arr't in $\mathbb{F}_q^n$, $q = p^r$. Let $\alpha_{i,j}$ be the eigenvalues of Frobenius acting on $H_c^i(M_\mathcal{A};\mathbb{Q}_\ell)$, and $P_i(t) \stackrel{\text{def}}{=} \prod_j (1-\alpha_{i,j}t)$. Then,

$$P_i(t) = \prod_{j=0}^{d} (1-q^j t)^{\beta^{\oplus j}_{i-2j-2}},$$

where $\beta_i^{\oplus j} \stackrel{\text{def}}{=}$ sum over $\forall x \in L_\mathcal{A}$ such that $\dim(x) = j$ of i-th Betti number of $\Delta(\hat{0}, x)$ (i.e., order complex homology).

Suggestion: Étale cohomology could be a secret tool for complexity theory.

Observation: Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is simply a subset of affine n-space over $GF(2)$.

Program:

1. Find good description of some NP-complete f as a variety,
2. Compute the étale Betti numbers of f,
3. Show that big Betti numbers force big Boolean circuits,
4. Conclude that $NP \neq P$.

Topic 2: Bruhat order

Figures on following slides are mostly taken from the book "Combinatorics of Coxeter Groups" by Björner–Brenti, Springer 2005.

Help with figures by F. Incitti and F. Lutz is gratefully acknowledged.

The pair (W, S) is a *Coxeter group* (Coxeter system) if W is a group with presentation

$$\begin{cases} \text{Generators:} & S \\ \text{Relations:} & (ss')^{m(s,s')} = e,\ (s,s') \in S^2, \end{cases}$$

where $m : S \times S \to \{1, 2, \ldots, \infty\}$ satisfies

$$m(s,s') = m(s',s)\,;$$
$$m(s,s') = 1 \ \Leftrightarrow \ s = s'.$$

In particular,

$$s^2 = e, \quad \text{for all } s \in S,$$

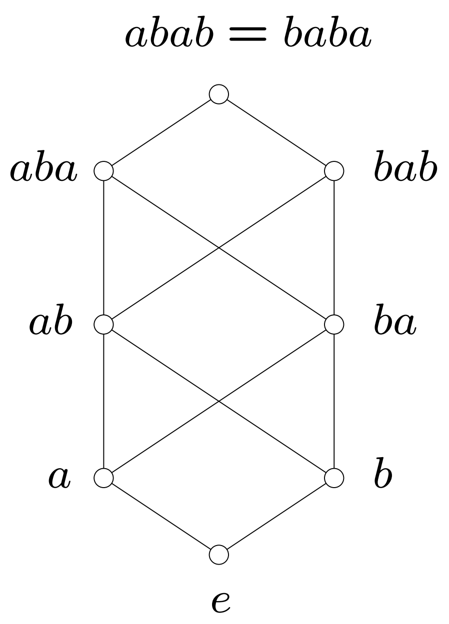
and

$$\underbrace{s\, s'\, s\, s'\, s \ldots}_{m(s,s')} = \underbrace{s'\, s\, s'\, s\, s' \ldots}_{m(s,s')}$$

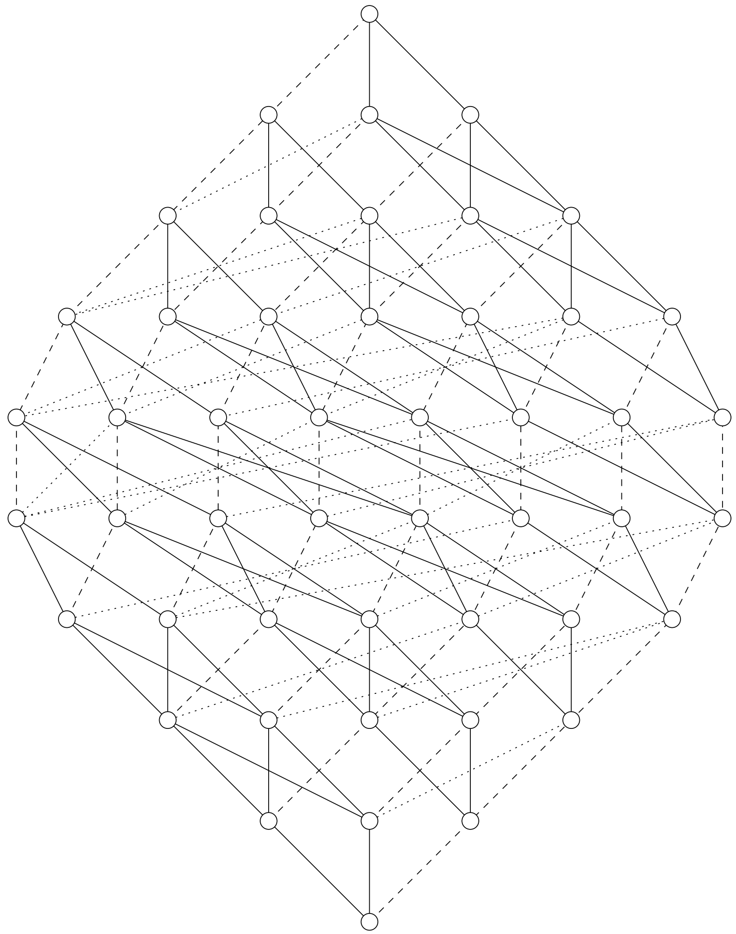
∃ classification of finite (affine, hyperbolic) Coxeter groups: type A_n, B_n, ... etc.

Bruhat order: For $u, w \in W$:

$$u \leq w \overset{\text{def}}{\Longleftrightarrow} \text{for } \forall \text{ reduced expression} w = s_1 s_2 \ldots s_q$$
$$\exists \text{ a reduced subexpression} u = s_{i_1} s_{i_2} \ldots s_{i_k},$$
$$1 \leq i_1 < \ldots < i_k \leq q.$$



Bruhat order of B_2



Bruhat order of B_3.

$\exists$ analogy

Intervals $[e, w]$ in Bruhat order	$\leftrightarrow$	Face lattices of convex polytopes
Weyl group	$\leftrightarrow$	rational polytope
Schubert variety	$\leftrightarrow$	toric variety
Kazhdan-Lusztig polynomial	$\leftrightarrow$	g-polynomial

Also: Both determine regular CW decompositions of a sphere
Intersection cohomology lurks in the background

Remark:

For all polytopes: $\exists$ combinatorial intersection cohomology theory satisfying hard Lefschetz (recent work of K. Karu and others)

Question: $\exists$??? combinatorial intersection cohomology theory for all Coxeter groups ("virtual Schubert varieties")?

THM (Bj-Wachs'82, Bj'84) Let $[u, w]$ be a Bruhat interval. Then $\exists$ regular CW decomposition $\Gamma_{u,w}$ of the $(\ell(w) - \ell(u) - 2)$-dimensional sphere with cells σ_x, $u < x < w$, such that

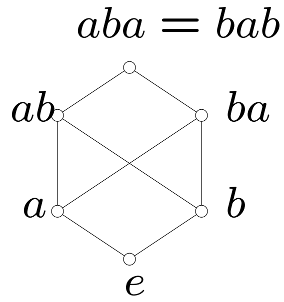
$$\dim(\sigma_x) = \ell(x) - \ell(u) - 1$$

and

$$\sigma_x \subseteq \overline{\sigma_z} \quad \Leftrightarrow \quad x \leq z.$$

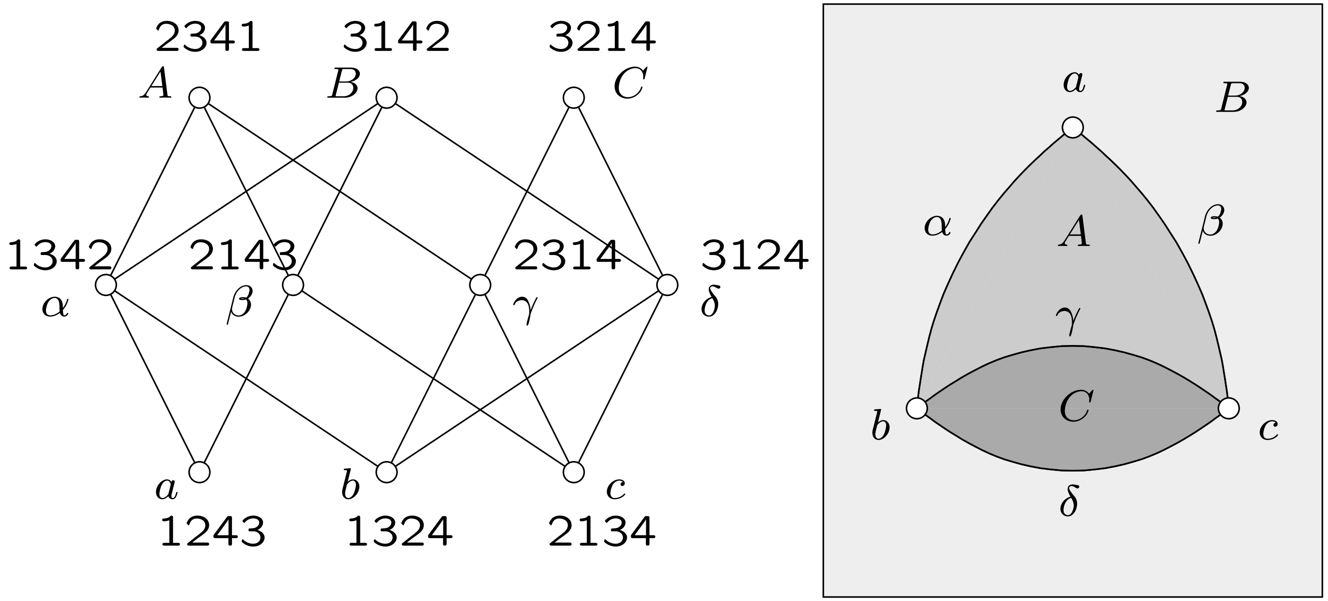
Proof idea: via lexicographic shellability of Bruhat order

Example: Lex. shelling of $W = S_3$, generators $S = \{a, b\}$

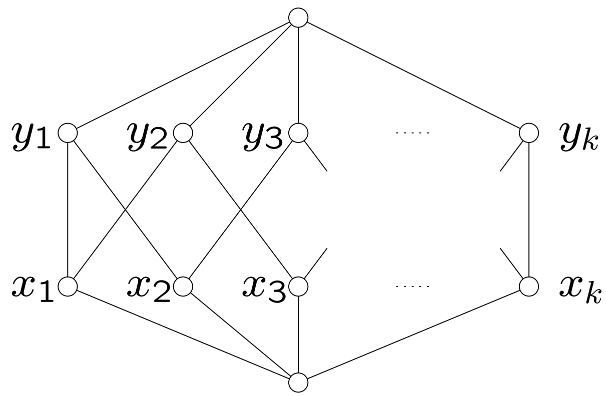


Choosing "aba" as reduced expression for the top element the induced labels of the four maximal chains are

$$\lambda(aba \;\triangleright\; *ba \;\triangleright\; **a \;\triangleright\; ***) = (1, 2, 3),$$
$$\lambda(aba \;\triangleright\; *ba \;\triangleright\; *b* \;\triangleright\; ***) = (1, 3, 2),$$
$$\lambda(aba \;\triangleright\; ab* \;\triangleright\; *b* \;\triangleright\; ***) = (3, 1, 2),$$
$$\lambda(aba \;\triangleright\; ab* \;\triangleright\; a** \;\triangleright\; ***) = (3, 2, 1).$$



Regular CW interpretation of a Bruhat interval.



A k-crown.

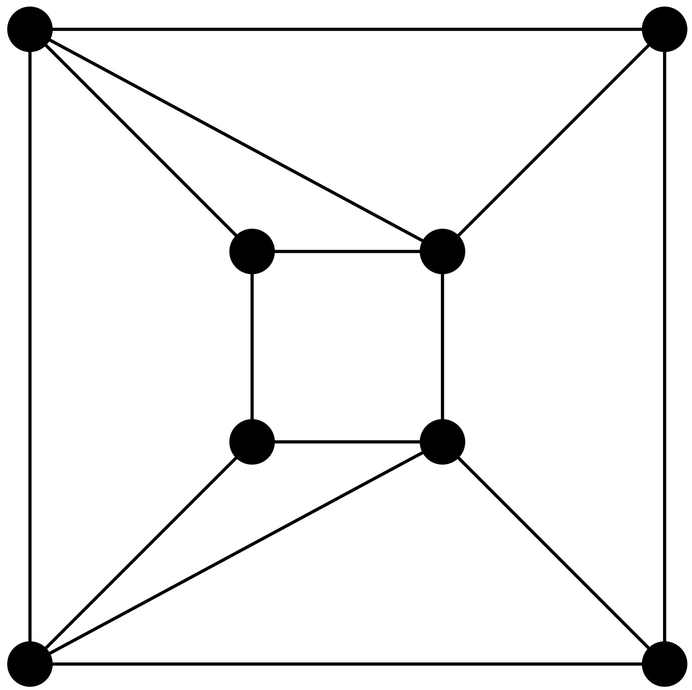
All Bruhat intervals of length 3 are k-crowns, $k \geq 2$.

Finite case $\Rightarrow$ only $k = 2, 3, 4$ possible.
(And for type H also $k = 5$.)

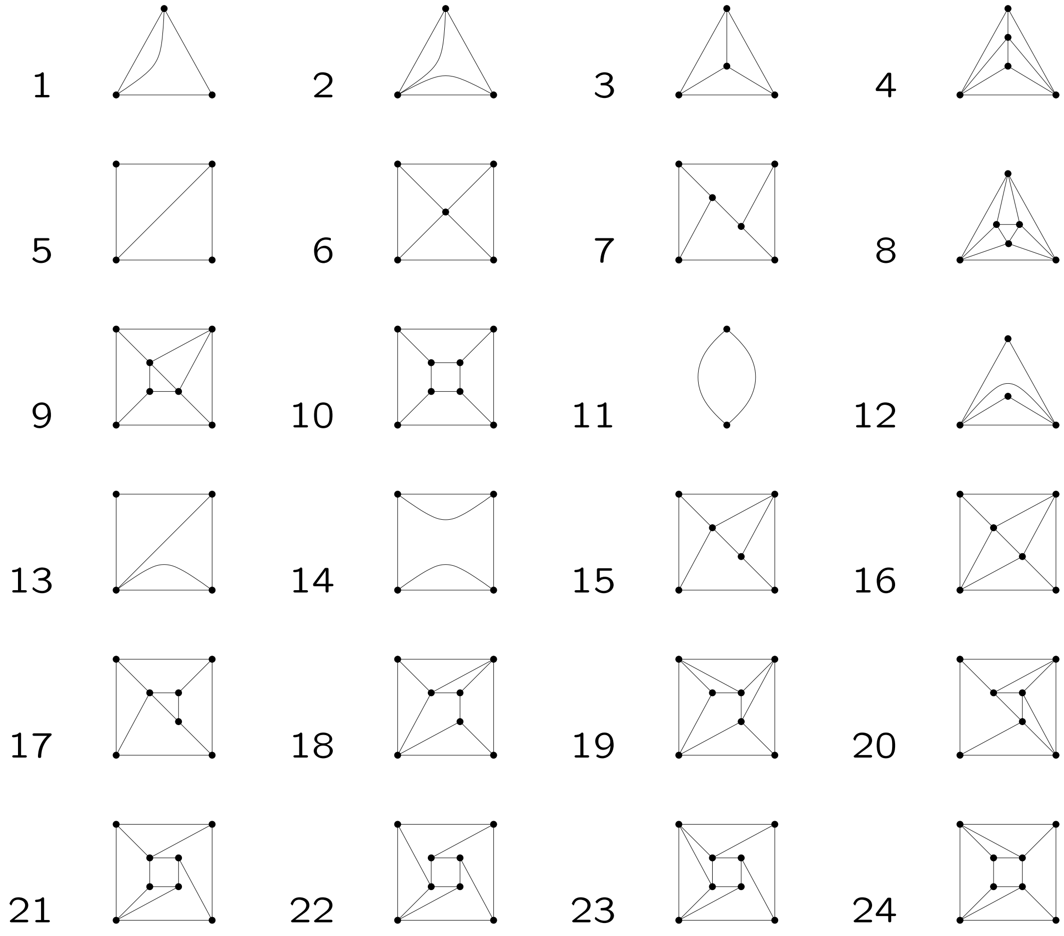
THM (Dyer'91). For each m, there exist only finitely many isomorphism classes of length m intervals in finite Coxeter groups.

THM (Hultman'03). There are 24 types of length 4 intervals in finite Weyl groups.

Only 7 of them occur in the symmetric groups.
All 24 show up in F_4.



A Bruhat interval of length 4
(rendered as a CW complex)



All length 4 intervals that appear in finite Weyl groups.

Shape of lower interval $[e, w]$:

$$f^w = \{f_0^w, f_1^w, \ldots, f_m^w\},$$

$m \stackrel{\text{def}}{=} \ell(w)$
$f_i^w \stackrel{\text{def}}{=}$ number of elements $x \leq w$ of length i.

Note: Analogy with *f-vector* of convex polytope

Known for f-vector of simplicial $(d+1)$-dimensional polytope:

(1) $f_i \leq f_j$ if $i < j \leq d - i$. In particular,

- $f_0 \leq f_1 \leq \cdots \leq f_{d/2}$ and $f_i \leq f_{d-i}$

(2) $f_{3d/4} \geq f_{(3d/4)-1} \geq \cdots \geq f_d$

(3) The bounds $d/2$ and $3d/4$ are best possible.

Conjecture: (2) is true for all polytopes.

Does it make sense to ask such questions for f^w-vectors of Bruhat intervals $[u, w]$?

Perhaps . . . — consider this:

THM (Carrell-Peterson '94) A Shubert variety X_w is rationally smooth
$\Leftrightarrow$
$f_i^w = f_{m-i}^w$ for all i

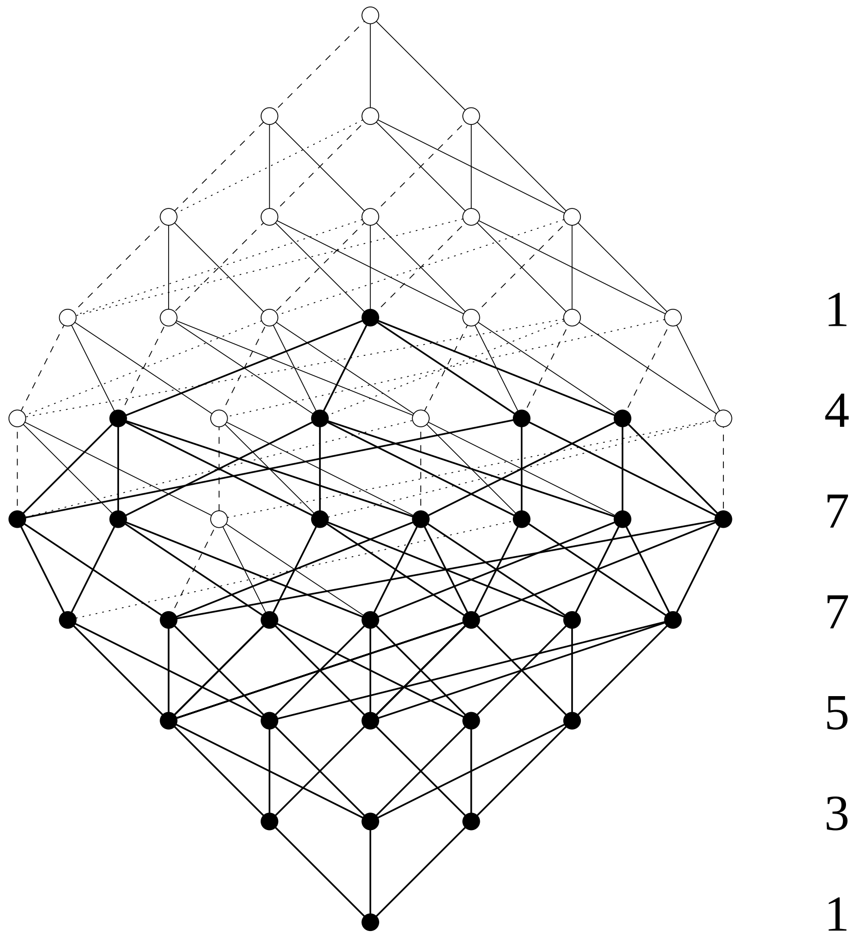
THM (Bj-Ekedahl '04) For the f^w-vector $f^w = \{f_0, f_1, \ldots, f_m\}$ of interval $[e, w]$ in a (Kac-Moody) Weyl group:

(1) $f_i \leq f_j$ if $i < j \leq m - i$. In particular,

- $f_0 \leq f_1 \leq \cdots \leq f_{m/2}$ and $f_i \leq f_{m-i}$

(2) If finite then also $f_* \geq f_{*+1} \geq \cdots \geq f_m$ (*= to be explained)

Conjecture: This is true for all Coxeter groups.



f^w-vector of Bruhat interval $[e, w]$

Idea of proof of (1):

For $X = X_w$, $H^*(X,\mathbb{Q}_\ell) \to IH^*(X,\mathbb{Q}_\ell)$ is an $H^*(X,\mathbb{Q}_\ell)$-module map
$\Rightarrow$ for $i \le j \le m-i$ it commutes with multiplication by $c_1(\mathcal{L})^{j-i}$
$\Rightarrow$ commutative diagram

$$\begin{array}{ccc} H^{2i}(X,\mathbb{Q}_\ell) & \longrightarrow & IH^{2i}(X,\mathbb{Q}_\ell) \\ \Big\downarrow {\scriptstyle \cap c_1(\mathcal{L})^{j-i}} & & \Big\downarrow {\scriptstyle \cap c_1(\mathcal{L})^{j-i}} \\ H^{2j}(X,\mathbb{Q}_\ell) & \longrightarrow & IH^{2j}(X,\mathbb{Q}_\ell). \end{array}$$

The horisontal maps are injective and the right vertical map is an injection by hard Lefschetz. Hence the left vertical map is injective, giving

$$f_i^w = \dim_{\mathbb{Q}_\ell} H^{2i}(X,\mathbb{Q}_\ell) \le \dim_{\mathbb{Q}_\ell} H^{2j}(X,\mathbb{Q}_\ell) = f_j^w.$$

For monotonicity at upper end (part (2)), all we can prove is

For $\forall k > 1 \, \exists N_k$ *such that for* $\forall$ *finite Weyl group and* $\forall w \in W$ *such that* $m = \ell(w) \geq N_k$*:*

$$f_{m-k} \geq f_{m-k+1} \geq \cdots \geq f_m$$

Question: Does there exist $\alpha < 1$ such that

$$f_{\lfloor \alpha m \rfloor} \geq f_{\lfloor \alpha m \rfloor + 1} \geq \cdots \geq f_m$$

for all w in all Coxeter groups?

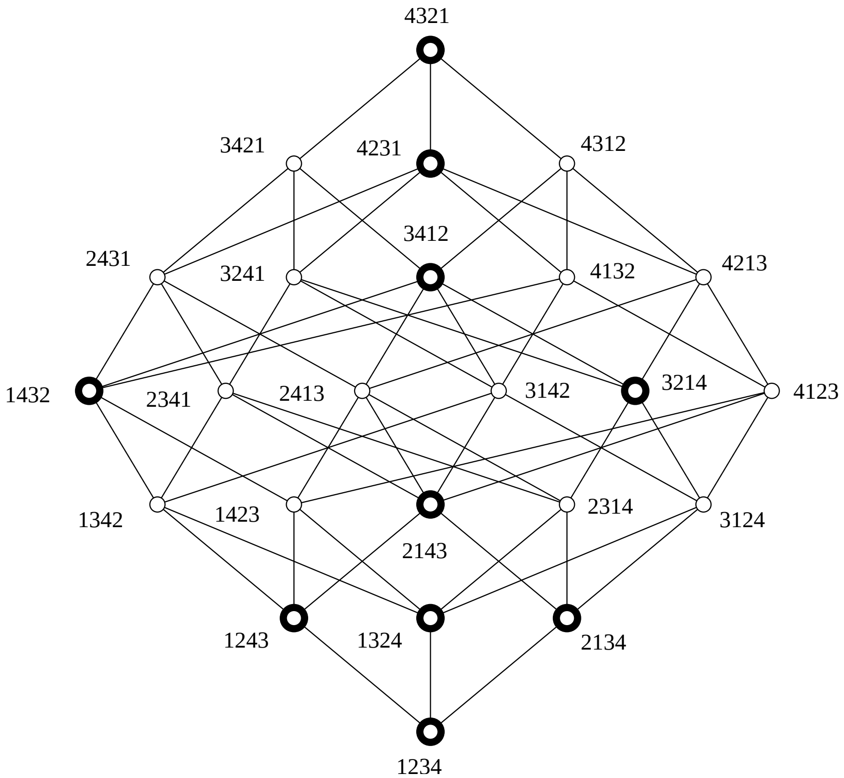
Conjecture: Yes, and $\alpha = 3/4$ will work

Let Invol(W) $\stackrel{\text{def}}{=}$ involutions of W with induced Bruhat order.

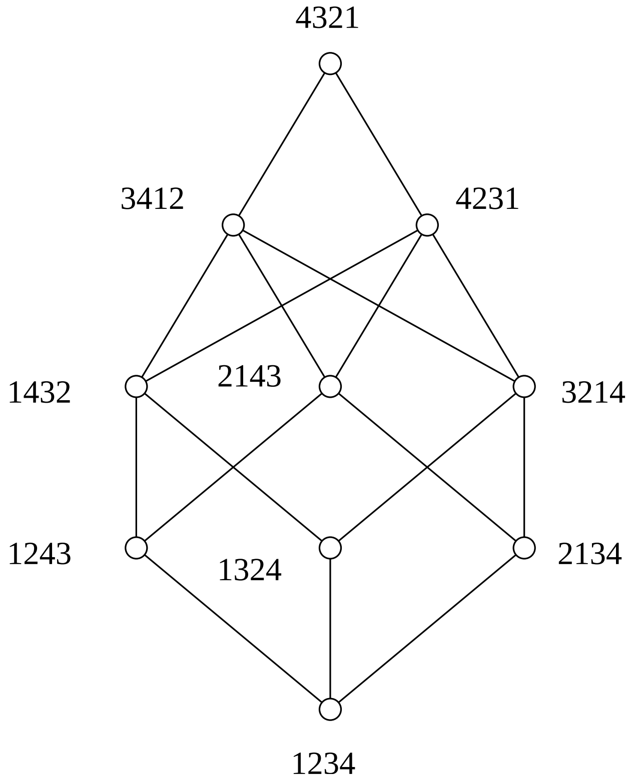
Studied by Richardson-Springer '94, Incitti '03, Hultman '04.

Has wonderful properties as poset, much as W itself:
pure, regular CW spheres for the classical Weyl groups (via EL-shellability), intervals= homology spheres in general, . . .

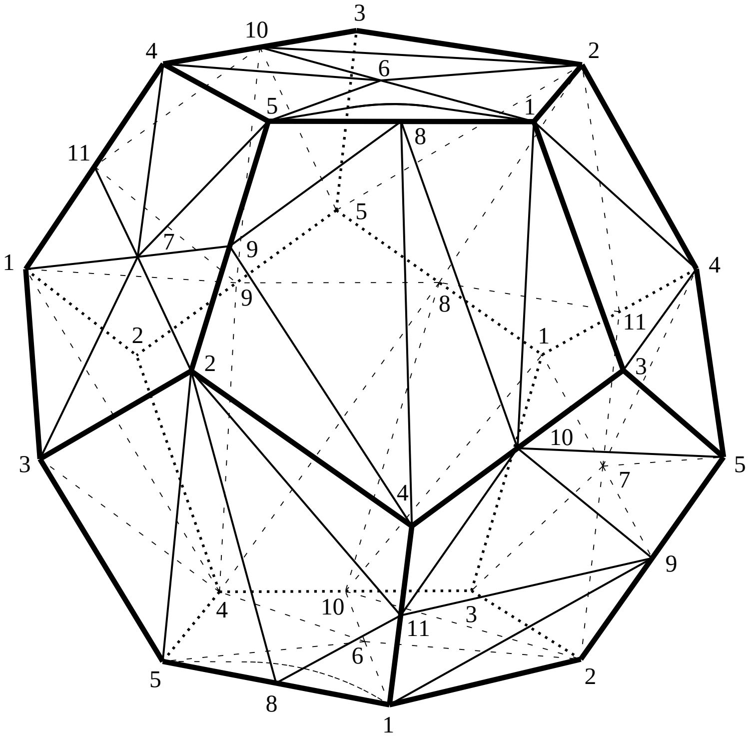
Poset rank function: $\text{rk}(w) = \frac{\ell(w)+a\ell(w)}{2}$,
where $a\ell(w)$ is absolute length



Involutions in S_4



Invol(S_4)



Happy Birthday, Bob !