

## A Note on Blockers in Posets

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**Abstract.** The blocker  $A^*$  of an antichain  $A$  in a finite poset  $P$  is the set of elements minimal with the property of having with each member of  $A$  a common predecessor. The following is done:

- (1) The posets  $P$  for which  $A^{**} = A$  for all antichains are characterized.
- (2) The blocker  $A^*$  of a symmetric antichain in the partition lattice is characterized.
- (3) Connections with the question of finding minimal size blocking sets for certain set families are discussed.

*Keywords:* antichain, blocker, partition, dominance, refinement, blocking set, Turán

### 1. Introduction

The blocker  $A^*$  of a set family  $A$  is a well-known construction in combinatorics and combinatorial optimization. Among the early references are [6] and [4], and the concept is discussed in several elementary textbooks. A crucial property in this setting is that if  $A$  is an antichain (no set contains another), then  $A^{**} = A$ .

The construction of blockers can be directly generalized to antichains in any finite bounded poset. In this paper we work in this generality. The generalized blocker construction has previously been considered by Matveev [9] and by Björner, Peeva and Sidman [3].

For general posets all that remains of blocker duality is the relation  $A^{***} = A^*$ , valid for every antichain  $A$ . The first question we deal with is: *What posets have the property that  $A^{**} = A$  for all antichains  $A$ ?* Such “strong blocker duality” is characterized in Section 2.

In [3] symmetric antichains and their blockers in the partition lattice  $\Pi_n$  play an important role due to their relevance for the theory of subspace arrangements. The second question we address is: *How does one compute the blocker of a symmetric antichain in  $\Pi_n$ ?* The answer, presented in Section 3, involves both the dominance and the refinement orderings of number partitions.

In the final section we discuss an algebraic approach to finding minimal size blocking sets to set families that can be realized as families of flats in a geometric lattice realizable over a field.

## 2. Posets with Strong Blocker Duality

We begin by agreeing on some notation. A poset is *bounded* if it contains unique bottom and top elements, denoted by  $\hat{0}$  and  $\hat{1}$ , respectively. Let  $P$  be a bounded poset. We denote by  $\Lambda$  its set of *atoms*, i.e., elements that cover  $\hat{0}$ , and given  $x \in P$  we let  $\Lambda(x) \subseteq \Lambda$  be the set of atoms below  $x$ .

If  $P$  is a lattice, then  $x \vee y$  and  $x \wedge y$  denotes the *join* (supremum) and *meet* (infimum), respectively, of two elements  $x, y \in P$ .

We say that a set  $A \subseteq P$  is an *antichain* if  $\hat{0} \notin A \neq \emptyset$ , and the elements of  $A$  are pairwise incomparable with respect to the partial ordering in  $P$ .

**Definition 2.1.** *Let  $A$  be an antichain in a finite bounded poset  $P$ . The blocker of  $A$  is the antichain*

$$A^* = \min\{x \in P \mid \Lambda(x) \cap \Lambda(a) \neq \emptyset \text{ for every } a \in A\},$$

where  $\min E$  denotes the set of minimal elements of a subset  $E \subseteq P$ .

*Remark 2.2.* The requirement in this paper that  $P$  is bounded is for convenience only. The bottom element  $\hat{0}$  plays no role whatsoever, and the top element  $\hat{1}$  has as only function to make sure that  $A^* \neq \emptyset$  for all antichains  $A$ . Everything can be reformulated for general (non-bounded) posets having at least one element  $x$  (not necessarily unique) above all its atoms. We have chosen the formulation for bounded posets since this is notationally simpler, and since the examples we have in mind are bounded.

A partial order on the antichains in  $P$  is defined as follows: we say that  $A \leq B$  for two antichains if for each  $b \in B$  there exists an  $a \in A$  such that  $a \leq b$ .

**Lemma 2.3.** (cf. [3] and [9]) *Let  $A$  and  $B$  be antichains in a finite bounded poset  $P$ .*

- (1) *If  $A \leq B$ , then  $B^* \leq A^*$ .*
- (2)  *$A^{**} \leq A$ .*
- (3)  *$A^{***} = A^*$ .*

*Proof.* The first two parts are straightforward from the definitions. By part (2) we get that  $A^{***} \leq A^*$ . On the other hand, part (1) applied to  $A^{**} \leq A$  yields  $A^{***} \geq A^*$ . ■

*Remark 2.4.* The poset of antichains in  $P$  is in fact a distributive lattice with meet operation  $A \wedge B = \min(A \cup B)$ , on which the mapping  $A \mapsto A^*$  is a Galois connection (see e.g. [1] for the definitions). These properties are not used in what follows.

As was mentioned in the Introduction, the blocker construction is well-known for the special case when  $P = 2^V$  is the Boolean lattice of all subsets of a finite set  $V$ . In this case,  $A^{**} = A$  for all antichains  $A$ . This can be seen, for instance, by applying the next lemma to the case  $P = 2^V$ .

**Lemma 2.5.** *Let  $V$  be a finite set, and suppose  $P$  is an induced subposet of the Boolean lattice  $2^V$  such that  $\emptyset, V$  and all singletons belong to  $P$ . Then, for two antichains  $A, B \subset P$ , we have  $B = A^*$  if and only if the following property is satisfied:*

*Property C: For all  $U \in P$ ,  $V \setminus U$  contains no member of  $A$  if and only if  $U$  contains a member of  $B$ .*

*Proof.* Note that Property C is equivalent to the assertion “for all  $U \in P$ ,  $U \cap a \neq \emptyset$  for all  $a \in A$  if and only if  $U \supseteq b$  for some  $b \in B$ .” Thus, Property C is satisfied if and only if  $B$  is the antichain of minimal elements in the set  $\{x \in P \mid x \cap a \neq \emptyset \text{ for all } a \in A\}$ . This antichain is precisely  $A^*$ . ■

**Definition 2.6.** Let  $V$  be a finite set. A subposet of the Boolean lattice  $2^V$  induced by a family  $S \subseteq 2^V$  is called well-complemented if (i) the empty set and all singletons belong to  $S$ , and (ii)  $S$  is closed under taking complements in  $V$ .

By the symmetry of Property C, it is immediate that  $A^{**} = A$  for all antichains  $A$  in a well-complemented poset. In fact, well-complemented posets are characterized by this property, as we now show.

**Theorem 2.7.** Let  $P$  be a finite bounded poset. Then the following are equivalent:

- (1)  $A^{**} = A$  for all antichains  $A$  in  $P$ .
- (2)  $P$  is isomorphic to a well-complemented subposet of a Boolean lattice.
- (3)  $P$  satisfies
  - (i) if  $\Lambda(x) \subseteq \Lambda(y)$  then  $x \leq y$ , for all  $x, y \in P$ ,
  - (ii) for all  $x \in P$  there exists  $y \in P$  such that  $\Lambda \setminus \Lambda(x) = \Lambda(y)$ .

*Proof.* The implication (2)  $\Rightarrow$  (1) follows from Lemma 2.5.

We show that (1)  $\Rightarrow$  (3). Assume that  $A^{**} = A$  for all antichains  $A \subset P$ . Note that, in particular, this implies that the map  $A \mapsto A^*$  is injective on antichains in  $P$ . Suppose  $\Lambda(x) \subseteq \Lambda(y)$  for some  $x, y \in P$ . If  $x$  and  $y$  are incomparable, then  $\{x, y\}^* = \{x\}^* = \Lambda(x)$ , contradicting injectivity of  $A \mapsto A^*$ . If, instead,  $x > y$ , we must have  $\Lambda(x) = \Lambda(y)$ . A similar contradiction is then obtained from  $\{x\}^* = \{y\}^* = \Lambda(x)$ . We conclude that  $x \leq y$ , proving part (i).

Pick  $x \in P$ . We must show that  $\Lambda \setminus \Lambda(x) = \Lambda(y)$  for some  $y \in P$ , so suppose that this is not the case. Note that  $(\Lambda \setminus \Lambda(x))^* = \min\{z \in P \mid \Lambda(z) \supset \Lambda \setminus \Lambda(x)\}$ . Hence, we have  $\Lambda(z) \cap \Lambda(x) \neq \emptyset$  for all  $z \in (\Lambda \setminus \Lambda(x))^*$ , implying that  $x \geq t$  for some  $t \in (\Lambda \setminus \Lambda(x))^{**}$ . This, however, implies  $(\Lambda \setminus \Lambda(x))^{**} \neq (\Lambda \setminus \Lambda(x))$ , a contradiction.

It remains to show that (3)  $\Rightarrow$  (2). Let  $2^\Lambda$  denote the Boolean lattice of all subsets of the set  $\Lambda$  of atoms in  $P$ . Define a map  $\psi : P \rightarrow 2^\Lambda$  by  $x \mapsto \Lambda(x)$ . Clearly,  $\psi$  is order-preserving, and property (i) implies both that  $\psi$  is injective and that the inverse mapping  $\psi(P) \rightarrow P$  given by  $\Lambda(x) \mapsto x$  is order-preserving. Thus,  $P$  is isomorphic to  $\psi(P)$ . By construction,  $\psi(P)$  contains all singletons and the empty set, and property (ii) shows that it is closed under taking complements. ■

The theorem has the somewhat unexpected consequence that strong blocker duality forces  $P$  to be isomorphic to its order dual.

**Corollary 2.8.** Suppose that  $A^{**} = A$  for all antichains  $A$  in  $P$ . Then  $P$  admits a fixed-point-free, order-reversing bijection of order 2 onto itself.

*Proof.* This is a direct consequence of the implication (1)  $\Rightarrow$  (2). ■

The equivalence (1)  $\Leftrightarrow$  (2) shows that the posets with strong blocker duality and  $n$  labeled atoms are precisely the ones obtained from the full Boolean lattice  $\mathbf{2}^{\{1, \dots, n\}}$  by deleting an arbitrary family of complementary pairs of subsets, avoiding cardinalities  $0, 1, n-1, n$ . Thus, there are

$$2^{2^{n-1}-n-1}$$

such posets, and they are pairwise distinct. Dividing by the possible symmetries we obtain the following estimate for the number  $N_n$  of nonisomorphic  $n$ -atom posets with strong blocker duality:

$$N_n \geq \frac{2^{2^{n-1}-n-1}}{n!} \geq \frac{2^{2^n/n}}{2^{n^2}} = 2^{(2^n/n)-n^2}.$$

Out of this doubly-exponential number of posets there is, however, only one that is a lattice.

**Corollary 2.9.** *Let  $L$  be a finite lattice. Then the following are equivalent:*

- (1)  $A^{**} = A$  for all antichains  $A$  in  $L$ .
- (2)  $L$  is Boolean.

*Proof.* We already know that (2)  $\Rightarrow$  (1). To prove (1)  $\Rightarrow$  (2), it suffices to show that, for finite  $V$ , the only well-complemented subposet of  $\mathbf{2}^V$  which is a lattice is  $\mathbf{2}^V$  itself. Let  $P \subset \mathbf{2}^V$  be another well-complemented subposet, and suppose  $S \subset V$  is maximal with the property  $S \notin P$ . All coatoms (elements covered by  $\hat{1} = V$ ) and  $V$  belong to  $P$ . Hence,  $S$  is covered by more than one element in  $\mathbf{2}^V$ . This means that  $\Lambda(S)$  has multiple minimal upper bounds in  $P$ , so that  $P$  cannot be a lattice. ■

As a small example, Figure 1 shows one of the three 4-atom posets with strong blocker duality that are not lattices.

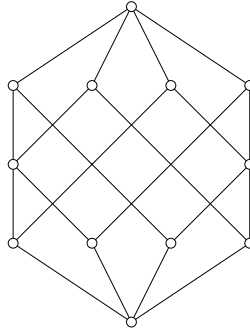


Figure 1.

### 3. Symmetric Blockers in Partition Lattices

Recall that the *partition lattice*  $\Pi_n$  consists of all set partitions of  $[n] = \{1, \dots, n\}$  ordered by refinement. In other words,  $\sigma \leq \tau \in \Pi_n$  if the equivalence relation corresponding to  $\tau$  contains the one corresponding to  $\sigma$ .

We are interested in antichains in  $\Pi_n$  that are invariant with respect to the natural action of the symmetric group  $\mathfrak{S}_n$  on  $\Pi_n$ . Since, clearly, the blocker of any  $\mathfrak{S}_n$ -invariant antichain is itself  $\mathfrak{S}_n$ -invariant, the subject can be formulated solely in terms of orbits, i.e., in terms of number partitions.

We need some notation. Let  $\mathfrak{P}_n$  be the set of partitions of the number  $n$ ,  $\text{Ref}(n)$  the refinement order on  $\mathfrak{P}_n$ , and  $\text{Dom}(n)$  the dominance order. These partial orderings are defined as follows. Let  $\lambda = (\lambda_1, \lambda_2, \dots), \mu = (\mu_1, \mu_2, \dots) \in \mathfrak{P}_n$ , with the parts  $\lambda_i$  and  $\mu_j$  decreasingly arranged and  $\sum \lambda_i = \sum \mu_j = n$ . Then

- (1)  $\lambda \leq \mu$  in  $\text{Dom}(n)$  if  $\sum_{i \leq k} \lambda_i \leq \sum_{j \leq k} \mu_j$  for all  $k$ ,
- (2)  $\lambda \leq \mu$  in  $\text{Ref}(n)$  if  $\lambda$  can be obtained from  $\mu$  by partitioning the parts  $\mu_j$ .

Note that the identity mapping  $\text{Ref}(n) \rightarrow \text{Dom}(n)$  is order-preserving.

Let  $\text{SH}: \Pi_n \rightarrow \mathfrak{P}_n$  be the shape map  $\{\tau_1, \dots, \tau_r\} \mapsto \{|\tau_1|, \dots, |\tau_r|\}$  (multiset). In the other direction, given  $\lambda \in \mathfrak{P}_n$  we let  $\text{FI}(\lambda)$  denote the fiber (inverse image) in  $\Pi_n$ , i.e.,  $\tau \in \text{FI}(\lambda)$  iff  $\text{SH}(\tau) = \lambda$ . Similarly, for  $S \subseteq \mathfrak{P}_n$  we define  $\text{FI}(S) = \text{SH}^{-1}(S)$ .

The following theorem, characterizing blocker duality of symmetric antichains in the partition lattice  $\Pi_n$ , is based on the fact that every symmetric (i.e.,  $\mathfrak{S}_n$ -invariant) antichain in  $\Pi_n$  is of the form  $\text{FI}(A)$  for some antichain  $A$  in  $\text{Ref}(n)$ .

For a poset  $P$  and  $x \in P$ , we write  $P_{\leq x} = \{y \in P \mid y \leq x\}$ , and  $P_{< x}$  is defined similarly. The transpose of a number partition  $\lambda$  is denoted by  $\lambda'$ .

**Theorem 3.1.** *Let  $A \subseteq \text{Ref}(n)$  be an antichain. We have  $\text{FI}(A)^* = \text{FI}(B)$  in  $\Pi_n$ , where*

$$B = \min_{\text{Ref}(n)} \mathfrak{P}_n \setminus \left( \cup_{\lambda \in A} \text{Dom}(n)_{\leq \lambda'} \right).$$

*In other words, to construct  $B$  we take the refinement-minimal number partitions among those that are not dominated by any  $\lambda', \lambda \in A$ .*

*Proof.* It suffices to show that, given  $\lambda, \mu \in \mathfrak{P}_n$ , there exist set partitions  $\sigma \in \text{FI}(\lambda)$  and  $\tau \in \text{FI}(\mu)$  with  $\sigma \wedge \tau = \hat{0}$  if and only if  $\lambda'$  dominates  $\mu$ .

We have a 1-1 correspondence between pairs of set partitions  $\sigma, \tau \in \Pi_n$  and bipartite graphs with  $n$  labeled edges and no isolated vertices as follows. Given  $\sigma, \tau \in \Pi_n$ , the vertex set of the graph can be thought of as the set of blocks in  $\sigma$  and  $\tau$ . The graph is constructed by letting the  $i$ -th edge connect the block containing  $i$  in  $\sigma$  and the block containing  $i$  in  $\tau$ . The crucial observation is that this graph contains multiple edges if and only if  $\tau \wedge \sigma \neq \hat{0}$ . By the Gale-Ryser Theorem, there is a bipartite graph with degree sequences  $\lambda, \mu \in \mathfrak{P}_n$  without multiple edges if and only if  $\lambda'$  dominates  $\mu$ . Hence the theorem.  $\blacksquare$

*Example 3.2.* As a special case of the theorem we observe that the antichain  $\text{FI}((p, 1^{n-p}))$  of all set partitions of hook type  $(p, 1^{n-p})$  is the blocker in  $\Pi_n$  of the antichain  $A_p$  consisting of all partitions with  $p-1$  blocks. Conversely,  $A_p$  is the blocker of  $\text{FI}((p, 1^{n-p}))$ .

**Corollary 3.3.** *The blocker of every  $\mathfrak{S}_n$ -invariant antichain in  $\Pi_n$  contains a hook shape antichain  $\text{FI}((p, 1^{n-p}))$  for some  $p$ . In particular,  $\text{FI}(\lambda)$  is itself a blocker if and only if  $\lambda$  is a hook shape.*

*Proof.* Let  $A \subseteq \text{Ref}(n)$  be an antichain. In  $\text{Ref}(n)$  as well as in  $\text{Dom}(n)$ , the hook shapes form a chain from the bottom element to the top element. Thus,  $\mathfrak{P}_n \setminus (\cup_{\lambda \in A} \text{Dom}(n)_{\leq \lambda'})$  contains a unique  $\mu = (p, 1^{n-p})$  which is both refinement-minimal and dominance-minimal among the hook shapes. Now,  $\text{Ref}(n)_{< \mu}$  has a unique dominance-maximal element, namely  $(p-1, 1^{n-p+1})$ . Thus,  $\mu$  is refinement-minimal in  $\mathfrak{P}_n \setminus (\cup_{\lambda \in A} \text{Dom}(n)_{\leq \lambda'})$ .

For the last assertion, see Example 3.2.  $\blacksquare$

**Corollary 3.4.** *The map  $A \mapsto \text{FI}(A)^*$  determines a bijection between antichains in  $\text{Dom}(n)$  and  $\mathfrak{S}_n$ -invariant blockers in  $\Pi_n$ .*

*Proof.* In this proof, let  $X = \{A \subseteq \text{Ref}(n) \mid A \text{ is an antichain}\}$  and  $Y = \{A \subseteq \text{Dom}(n) \mid A \text{ is an antichain}\}$ . Define  $\phi : X \rightarrow Y$  by letting  $\phi(A)$  be the set of dominance-minimal elements in  $A$ . From the fact that the identity mapping  $\text{Ref}(n) \rightarrow \text{Dom}(n)$  is order-preserving follows that  $\phi$  is surjective.

Note that, for  $A, B \in X$ , we have  $\cup_{\lambda \in A} \text{Dom}(n)_{\leq \lambda'} = \cup_{\lambda \in B} \text{Dom}(n)_{\leq \lambda'}$  if and only if  $\phi(A) = \phi(B)$ . By Theorem 3.1, this implies  $\phi(A) = \phi(B) \Leftrightarrow \text{FI}(A)^* = \text{FI}(B)^*$ . Thus,  $\phi(A) \mapsto \text{FI}(\phi(A))^* = \text{FI}(A)^*$  is a bijection from  $Y$  to the set of  $\mathfrak{S}_n$ -invariant blockers in  $\Pi_n$ .  $\blacksquare$

#### 4. Subspace Arrangements and Blocker Ideals

Here we review some necessary background for the following section. This concerns subspace arrangements, which provided the motivation for the blocker construction in [3]. For background and details concerning subspace arrangements, see [2].

Let  $\mathbf{k}$  be a field, and consider an arrangement  $\mathcal{A}$  of subspaces of  $\mathbf{k}^n$ . The *vanishing ideal*  $I_{\mathcal{A}} \subseteq \mathbf{k}[x_1, \dots, x_n]$  is the ideal of polynomials that are identically zero on all subspaces in  $\mathcal{A}$ . It is an intriguing problem to determine generators for  $I_{\mathcal{A}}$ .

Now, the arrangement  $\mathcal{A}$  can always be embedded in a hyperplane arrangement  $\mathcal{H}$ . In particular,  $\mathcal{A}$  can be considered an antichain in the intersection lattice  $L_{\mathcal{H}}$  (a geometric lattice). In this setting, we may define the *blocker ideal*

$$\mathcal{B}_{\mathcal{A}, \mathcal{H}} = \langle \{ \prod_{H \in \Lambda(B)} \ell_H \mid B \in \mathcal{A}^* \} \rangle,$$

where  $\ell_H$  is the defining linear form of the hyperplane  $H$ .

It is easy to see that  $\mathcal{B}_{\mathcal{A}, \mathcal{H}} \subseteq I_{\mathcal{A}}$ , and this inclusion is in general strict. However, it turns out that in several of the cases where generators for  $I_{\mathcal{A}}$  are known, we actually have  $\mathcal{B}_{\mathcal{A}, \mathcal{H}} = I_{\mathcal{A}}$ .

One particularly interesting and rich class of subspace arrangements is the class of orbit arrangements, which we now define. The *braid arrangement*  $\mathcal{A}_n$  is the arrangement of hyperplanes defined by the equations  $x_i = x_j$  for  $1 \leq i < j \leq n$ . Its intersection lattice  $L_{\mathcal{A}_n}$  is naturally isomorphic to the partition lattice  $\Pi_n$ . The symmetric group  $\mathfrak{S}_n$  acts on the braid arrangement by permuting the indices, and the subspace arrangements that correspond to  $\mathfrak{S}_n$ -invariant antichains in  $L_{\mathcal{A}_n}$  we call *orbit arrangements*. As in the previous section, there is a 1-1 correspondence between orbit arrangements and antichains in  $\text{Ref}(n)$ . We let  $\mathcal{A}_{\lambda}$  denote the arrangement corresponding to the partition  $\lambda$ .

Two interesting cases where it is known that  $\mathcal{B}_{\mathcal{A}, \mathcal{H}} = I_{\mathcal{A}}$  are when  $\mathcal{A} = \mathcal{A}_{(p, 1^{n-p})}$  and  $\mathcal{A} = \cup \mathcal{A}_{\lambda}$  (union over all  $\lambda$  with  $p-1$  parts). These results are due to Li and Li [7] and to Kleitman and Lovász [8], respectively. In view of Example 3.2, note that (given  $p$ ) either of the two arrangements is the blocker of the other. Actually, only blockers can be expected to have the property  $\mathcal{B}_{\mathcal{A}, \mathcal{H}} = I_{\mathcal{A}}$ . This is so because of the following consequence of [3, Theorem 3.3.4], if  $\mathbf{k}$  is algebraically closed:

$$\mathcal{B}_{\mathcal{A}, \mathcal{H}} = I_{\mathcal{A}} \implies \mathcal{A}^{**} = \mathcal{A}. \quad (4.1)$$

## 5. Minimal Blocking Sets

Again, suppose  $A$  is an antichain in a finite bounded poset  $P$ . We say that a subset  $S \subseteq \Lambda$  of the atoms is  $A$ -*intersecting* if  $S \cap \Lambda(a) \neq \emptyset$  for all  $a \in A$ . Clearly,  $\Lambda(b)$  is  $A$ -intersecting for every  $b \in A^*$ .

**Definition 5.1.** *The antichain  $A$  has the Turán property if the smallest cardinality of any  $A$ -intersecting atom set is  $\min\{|\Lambda(b)| \mid b \in A^*\}$ .*

To motivate this definition, again consider the antichain  $\text{FI}((p, 1^{n-p}))$  in  $\Pi_n$  consisting of all set partitions of shape  $(p, 1^{n-p})$  for some fixed  $p$ . We may think of  $\Pi_n$  as the lattice of all clique graphs (i.e., graphs such that every connected component is a clique) on vertex set  $[n]$ , the atoms of  $\Pi_n$  corresponding to the set of edges. Then, the assertion that  $\text{FI}((p, 1^{n-p}))$  has the Turán property is equivalent to the assertion that the smallest number of edges in any graph that intersects every  $p$ -clique is attained in a clique graph on  $p-1$  cliques. By passing to complements, one sees that this is precisely the famous Turán theorem of graph theory.

It seems reasonable to inquire which antichains have the Turán property. In particular, if antichains in  $\Pi_n$  corresponding to  $\mathfrak{S}_n$ -orbits have the Turán property, this gives rise to Turán type graph theorems.

In their paper, Li and Li [7] point out that their theorem implies the original Turán theorem. Their argument can be generalized to obtain the following.

**Theorem 5.2.** *Let  $\mathcal{A}$  be a subspace arrangement embedded in a hyperplane arrangement  $\mathcal{H}$ . If  $\mathcal{B}_{\mathcal{A}, \mathcal{H}} = I_{\mathcal{A}}$ , then  $\mathcal{A}$  has the Turán property (viewed as an antichain in  $L_{\mathcal{H}}$ ).*

*Proof.* Suppose  $\mathcal{A}$  does not have the Turán property. Then there exists a set of hyperplanes  $S \subseteq \mathcal{H}$  whose union contains all subspaces in  $\mathcal{A}$ , and  $|S| < |\Lambda(B)|$  for all  $B \in \mathcal{A}^*$ . Thus, by definition, we have  $\deg(p) > |S|$  for all  $p \in \mathcal{B}_{\mathcal{A}, \mathcal{H}}$ . However, it is easy to see that

$$\prod_{H \in S} \ell_H \in I_{\mathcal{A}},$$

where, again,  $\ell_H$  is the defining linear form of a hyperplane  $H$ . This polynomial has degree  $|S|$ , and therefore  $\mathcal{B}_{\mathcal{A}, \mathcal{H}} \neq I_{\mathcal{A}}$ .  $\blacksquare$

*Example 5.3.* We illustrate what this says with a small example, where  $\mathcal{H}$  is taken to be the braid arrangement  $\mathcal{A}_6$  and hence  $L_{\mathcal{H}} \cong \Pi_6$ .

Let  $A = \{222, 3111\}$  and  $B = \{42, 51\}$  be two antichains in  $\text{Ref}(6)$ . One sees from Theorem 3.1 that  $\text{FI}(A)^* = \text{FI}(B)$  and  $\text{FI}(B)^* = \text{FI}(A)$  in  $\Pi_6$ . It was checked in Example 3.4.3 of [3] that the blocker ideal equals the vanishing ideal for the corresponding orbit arrangements in both cases. Thus, Theorem 5.2 applies.

What the Turán property then means in the case  $A = \{222, 3111\}$  is the following: *The maximal number of edges of a graph on 6 vertices not containing three independent edges or a 3-clique equals*

$$\max\{\#K_{4,2}, \#K_{5,1}\} = \max\{8, 5\} = 8.$$

Here  $\#K_{n,m}$  denotes the number of edges in the complete bipartite graph  $K_{n,m}$ . Note that if we excluded only a 3-clique, the answer would be  $\max\{\#K_{3,3}, \#K_{4,2}, \#K_{5,1}\} = 9$ , which of course agrees with Turán's theorem.

Similarly, what the Turán property means in the case  $B = \{42, 51\}$  is: *The maximal number of edges of a graph on 6 vertices not containing either a 4-clique and an independent edge or a 5-clique, equals*

$$\max\{\#K_{2,2,2}, \#K_{3,1,1,1}\} = \max\{12, 12\} = 12.$$

*Remark 5.4.* The converse of Theorem 5.2 does not hold in general, not even for blockers. A construction of D. Kozlov (see [3, Example 4.2.2]) yields an arrangement  $\mathcal{A}$  of two subspaces embedded in an arrangement  $\mathcal{H}$  of four hyperplanes such that  $\mathcal{A}$  has the Turán property, but  $\mathcal{B}_{\mathcal{A}, \mathcal{H}} \neq I_{\mathcal{A}}$ . Moreover,  $\mathcal{A}$  is a blocker in  $L_{\mathcal{H}}$ .

*Example 5.5.* A graph-theoretic theorem by Simonovits [10, Theorem 2.2] implies (as a special case) that for a fixed number partition  $\lambda \in \mathfrak{P}_m$ , and for  $n$  large enough, the largest graph on vertex set  $[n]$  that does not contain the clique graph corresponding to  $\lambda$  is the complement of a clique graph. Phrased in our language, this means precisely that the antichain  $\text{FI}((\lambda, 1^{n-m}))$  in  $\Pi_n$  has the Turán property. By Corollary 3.3, this antichain is not a blocker (unless  $\lambda$  is a hook shape), so, by (4.1), its blocker ideal does not equal its vanishing ideal. Thus, we have another example showing that the converse of Proposition 5.2 is false.

*Example 5.6.* It is easy to find antichains in  $\Pi_n$  that do not satisfy the Turán property. One example is the antichain of any pair of atoms in  $\Pi_3$ . However, the only class of  $\mathfrak{S}_n$ -invariant counterexamples that we know of is the following.

Consider the partition  $\lambda = (2^r)$  for some  $r$ . Using Theorem 3.1, one readily verifies that  $\text{FI}(\lambda)^* = \text{FI}((r+1, 1^{r-1}))$  in  $\Pi_{2r}$ . Thus, the assertion that  $\text{FI}(\lambda)$  has the Turán property is equivalent to the assertion that the smallest number of edges in any graph on vertex set  $[2r]$  that intersects every complete matching is attained in an  $(r+1)$ -clique. However, a star (the graph containing every possible edge from a single vertex) also intersects every complete matching, and the star has fewer edges than the  $(r+1)$ -clique if  $r \geq 3$ .

We end by describing a class of symmetric antichains with the Turán property which is not produced by Theorem 5.2. Let  $\mathbb{F}_q$  be the finite field on  $q$  elements, and consider the geometric lattice  $L_q^n$  of all subspaces of  $\mathbb{F}_q^n$  ordered by inclusion. The analogue of orbit arrangements would in this case be antichains that are invariant under the action of  $\text{GL}(n, q)$ , i.e., antichains  $\mathcal{A}_k$  that contain every subspace of a given dimension  $k$ .

Clearly,  $\mathcal{A}_k^* = \mathcal{A}_{n-k+1}$ . The following proposition is therefore a reformulation of [5, p. 87, Theorem 3.5], which says that a set of points in  $PG(n, q)$  that intersects every  $k$ -dimensional subspace has cardinality at least  $1 + q + \dots + q^{n-k}$ .

**Proposition 5.7.** *The antichain  $\mathcal{A}_k \subset L_q^n$ , which consists of all  $k$ -dimensional subspaces of  $\mathbb{F}_q^n$ , has the Turán property.*

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