

ON THE SHAPE OF BRUHAT INTERVALS

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ABSTRACT. Let (W, S) be a crystallographic Coxeter group (this includes all finite and affine Weyl groups), and $J \subseteq S$. Let W^J denote the set of minimal coset representatives modulo the parabolic subgroup W_J . For $w \in W^J$, let $f_i^{w,J}$ denote the number of elements of length i below w in Bruhat order on W^J (notation simplified to f_i^w in the case when $W^J = W$). We show that

$$0 \leq i < j \leq \ell(w) - i \implies f_i^{w,J} \leq f_j^{w,J}.$$

Furthermore, the case of equalities $f_i^w = f_{\ell(w)-i}^w$ for $i = 1, \dots, k$ is characterized in terms of vanishing of coefficients in the Kazhdan-Lusztig polynomial $P_{e,w}(q)$.

It is also shown that if W is finite then the number sequence $f_0^w, f_1^w, \dots, f_{\ell(w)}^w$ cannot grow too rapidly. Further, in the finite case, for any given $k \geq 1$ and any $w \in W$ of sufficiently great length (with respect to k):

$$f_{\ell(w)-k}^w \geq f_{\ell(w)-k+1}^w \geq \dots \geq f_{\ell(w)}^w.$$

The proofs rely for the most part on properties of the cohomology of Kac-Moody Schubert varieties.

1 Introduction

Let (W, S) be a Coxeter system with S finite. Fix a subset $J \subseteq S$ and let W^J denote the set of minimal coset representatives modulo the parabolic subgroup $W_J = \langle J \rangle$. For $w \in W^J$, let

$$f_i^{w,J} := \text{card}\{u \in W^J \mid u \leq w \text{ and } \ell(u) = i\}.$$

In words, $f_i^{w,J}$ is the number of length i elements in the Bruhat interval $[e, w]^J = [e, w] \cap W^J$. For terminology and basic facts concerning Coxeter groups, Weyl groups, and Bruhat order, we refer to [BB05, Hu90].

For a certain class of Coxeter groups we can apply geometric methods to study the $f_i^{w,J}$. The groups to which our methods are applicable are those for which the order of a product of two generators is 2, 3, 4, 6, or ∞ ; these are usually called the *crystallographic* Coxeter groups. They are precisely the groups that appear as Weyl groups of Kac-Moody algebras (cf. [Kac83, Prop. 3.13]). The main purpose of this paper is to prove the following relations and some of their ramifications.

Theorem A *Let (W, S) be a crystallographic Coxeter group, $J \subseteq S$ and $w \in W^J$. We have that*

$$0 \leq i < j \leq \ell(w) - i \implies f_i^{w,J} \leq f_j^{w,J}.$$

Combining the cohomological arguments used for proving Theorem A with a linear-algebraic argument of Stanley [St80] we can sharpen Theorem A to a combinatorial statement giving structural reasons for these inequalities.

Theorem B *Let (W, S) be a crystallographic Coxeter group, $J \subseteq S$. Fix $w \in W^J$ and i such that $0 \leq i < \ell(w)/2$. Then, in $[e, w]^J$ there exist $f_i^{w,J}$ pairwise disjoint chains $u_i < u_{i+1} < \dots < u_{\ell(w)-i}$ such that $\ell(u_j) = j$.*

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The inequalities of Theorem A are equivalent to the following two sets of inequalities combined:

$$f_i^{w,J} \leq f_{\ell(w)-i}^{w,J}, \quad \text{for all } i < \ell(w)/2, \quad (1)$$

and

$$f_0^{w,J} \leq f_1^{w,J} \leq \cdots \leq f_{\lfloor \ell(w)/2 \rfloor}^{w,J}. \quad (2)$$

For the rest of this section we treat only the $J = \emptyset$ case. Then $W^J = W$, so for simplicity we drop “ J ” from the notation. In this case the relations (1) sharpen the inequalities $\sum_{i \leq k} f_i^w \leq \sum_{i \leq k} f_{\ell(w)-i}^w$, for all $0 \leq k < \ell(w)/2$, due to Brion [Br00, Cor. 2].

The case of equality in some of the relations (1) is interesting. Fix $w \in W$, and let $m := \lfloor (\ell(w) - 1)/2 \rfloor$. Let

$$P_{e,w}(q) = 1 + \beta_0 + \beta_1 q + \cdots + \beta_m q^m$$

be the Kazhdan-Lusztig polynomial of the interval $[e, w]$. It is known [Ku02, Thm. 12.2.9] that all Kazhdan-Lusztig polynomials have non-negative coefficients if W is crystallographic, and that $\beta_0 = 0$.

Theorem C *Suppose that (W, S) is crystallographic. Let $w \in W$ and $0 \leq k \leq m$. Then the following conditions are equivalent:*

- (a) $f_i^w = f_{\ell(w)-i}^w$, for $i = 0, \dots, k$,
- (b) $\beta_i = 0$, for $i = 0, \dots, k$.

Furthermore, if $k < m$ then (a) and (b) imply

$$(c) \beta_{k+1} = f_{\ell(w)-k-1}^w - f_{k+1}^w.$$

In the case $k = m$ the equivalence of (a) and (b) specializes to a criterion for rational smoothness of the Schubert variety X_w due to Carrell and Peterson [Ca94].

The next result shows, among other things, that for finite groups the increasing sequence (2) cannot grow too fast. The condition of being an M -sequence is recalled in Section 6.

Theorem D *Let (W, S) be a finite Weyl group and $w \in W$. Then the vectors $(f_0^w, f_1^w, \dots, f_{\ell(w)}^w)$ and $(f_0^w, f_1^w - f_0^w, f_2^w - f_1^w, \dots, f_{\lfloor \ell(w)/2 \rfloor}^w - f_{\lfloor \ell(w)/2 \rfloor - 1}^w)$ are M -sequences.*

The increasing inequalities (2) have decreasing counterparts at the upper end of the Bruhat interval, but the information we are able to give about this is much weaker.

Theorem E *For all $k \geq 1$ there exists a number N_k , such that for every finite Coxeter group (W, S) and every $w \in W$ such that $\ell(w) \geq N_k$ we have that*

$$f_{\ell(w)-k}^w \geq f_{\ell(w)-k+1}^w \geq \cdots \geq f_{\ell(w)}^w.$$

The paper is organized as follows. Sections 2 and 3 contain preliminary material on the algebraic geometry underlying the proofs of Theorem A and C in Section 4. The proofs of Theorems B, D and E can be found in Sections 5, 6 and 7, respectively. Section 8 expands on some algebraic geometry needed for the proof of Theorem C.

2 The pure cohomology

Let F be an endomorphism of a graded \mathbf{Q}_ℓ -vector space V which is finite dimensional in each degree. It will be said to be *of weight $\leq w$* (resp. *of pure weight w*), with respect to a positive integer q , if the eigenvalues of F on V^i are algebraic numbers all of whose conjugates have the same absolute value $q^{j/2}$ for some $j \leq w + i$ (resp. $j = w + i$). A theorem of Deligne provides a large number of such vector spaces in the following way.

Consider a proper variety X_0 over a finite field \mathbf{F}_q and étale cohomology $H^*(X, \mathbf{Q}_\ell)$, considered as a graded vector space. (We shall follow the usual convention of using 0 as a subscript to denote objects over a finite field and drop the subscript when we extend scalars to an algebraic closure of that field). The Frobenius map on X_0 induces an endomorphism F of $H^*(X, \mathbf{Q}_\ell)$ and Deligne's theorem [BBD82, 5.1.14] says that the action of F on $H^*(X, \mathbf{Q}_\ell)$ is of weight ≤ 0 . We shall only be interested in the *pure part*, $H_p^*(X, \mathbf{Q}_\ell)$, of $H^*(X, \mathbf{Q}_\ell)$, which by definition is obtained from $H^*(X, \mathbf{Q}_\ell)$ by factoring out by the F -generalised eigenspaces of weight < 0 . Our first result will identify $H_p^*(X, \mathbf{Q}_\ell)$ with the image of $H^*(X, \mathbf{Q}_\ell)$ in the (middle perversity) *intersection cohomology* $\mathcal{I}H^*(X, \mathbf{Q}_\ell)$ of X .

We start by recalling how the map from ordinary cohomology to intersection cohomology is induced from a map of complexes of sheaves. Let X be an algebraic variety (we assume varieties to be reduced but not necessarily irreducible) and $j: U \rightarrow X$ the inclusion of its non-singular locus. The natural map $j_! \mathbf{Q}_\ell \rightarrow Rj_* \mathbf{Q}_\ell$ factors as $j_! \mathbf{Q}_\ell \rightarrow \mathbf{Q}_\ell \rightarrow Rj_* \mathbf{Q}_\ell$. This gives maps ${}^pH^0(j_! \mathbf{Q}_\ell) \rightarrow {}^pH^0(\mathbf{Q}_\ell) \rightarrow {}^pH^0(Rj_* \mathbf{Q}_\ell)$. Considering the distinguished triangle $\rightarrow j_! \mathbf{Q}_\ell \rightarrow \mathbf{Q}_\ell \rightarrow i_* \mathbf{Q}_\ell \rightarrow$, where $i: Y \rightarrow X$ is the inclusion of the singular locus, and as $i_* \mathbf{Q}_\ell \in {}^pD_c^{\leq -1}(X)$, we get that ${}^pH^0(j_! \mathbf{Q}_\ell) \rightarrow {}^pH^0(\mathbf{Q}_\ell)$ is surjective. As $j_! \mathbf{Q}_\ell$ (by definition) is the image of ${}^pH^0(j_! \mathbf{Q}_\ell) \rightarrow {}^pH^0(Rj_* \mathbf{Q}_\ell)$ we get a surjective map ${}^pH^0(\mathbf{Q}_\ell) \rightarrow j_{!*} \mathbf{Q}_\ell$, and as $\mathbf{Q}_\ell \in {}^pD_c^{\leq 0}(X)$ we have a map $\mathbf{Q}_\ell \rightarrow {}^pH^0(\mathbf{Q}_\ell)$. So, by composition we get a map $\mathbf{Q}_\ell \rightarrow j_{!*} \mathbf{Q}_\ell$. This map induces the desired map $H^*(X, \mathbf{Q}_\ell) \rightarrow \mathcal{I}H^*(X, \mathbf{Q}_\ell)$ from cohomology to intersection cohomology.

Note now that, by definition, if X_0 is an algebraic variety over a finite field \mathbf{F}_q , a mixed complex \mathcal{E}_0 on X_0 (cf. [De80, 1.1.2]) is of weight $\leq w$ precisely when the graded vector space $H^*(\mathcal{E}_{\bar{s}})$ is of weight $\leq w$ (with respect to q) for every geometric point \bar{s} of X_0 with image a closed point of X_0 whose residue field has cardinality q .

Theorem 2.1 *Let X_0 be a proper variety over a finite field. Then the kernel of the map from ordinary cohomology to intersection cohomology $H^*(X, \mathbf{Q}_\ell) \rightarrow \mathcal{I}H^*(X, \mathbf{Q}_\ell)$ consists exactly of the part of $H^*(X, \mathbf{Q}_\ell)$ of weight < 0 . In particular, $H_p^*(X, \mathbf{Q}_\ell)$ is the image of $H^*(X, \mathbf{Q}_\ell)$ in $\mathcal{I}H^*(X, \mathbf{Q}_\ell)$.*

PROOF: Let $j: U_0 \rightarrow X_0$ be the inclusion of the smooth locus. We fit the map $\mathbf{Q}_\ell \rightarrow j_{!*} \mathbf{Q}_\ell$ into a distinguished triangle

$$\rightarrow \mathcal{F} \rightarrow \mathbf{Q}_\ell \rightarrow j_{!*} \mathbf{Q}_\ell \rightarrow$$

and we start by showing that \mathcal{F} is of weight < 0 . We know, [BBD82, Cor 5.4.3], that $j_{!*} \mathbf{Q}_\ell$ is pure of weight 0. Let now \bar{s} be a geometric point of X_0 with image a closed point (whose residue field then is finite). We have an exact sequence

$$H^{i-1}((j_{!*} \mathbf{Q}_\ell)_{\bar{s}}) \rightarrow H^i(\mathcal{F}_{\bar{s}}) \rightarrow H^i((\mathbf{Q}_\ell)_{\bar{s}}) \rightarrow H^i((j_{!*} \mathbf{Q}_\ell)_{\bar{s}})$$

and as $H^{i-1}((j_{!*} \mathbf{Q}_\ell)_{\bar{s}})$ is of weight $\leq i-1$ it is enough to show that $H^i((\mathbf{Q}_\ell)_{\bar{s}}) \rightarrow H^i((j_{!*} \mathbf{Q}_\ell)_{\bar{s}})$ is injective, which is a non-trivial condition only for $i=0$. In that case it is indeed injective as the composite $\mathbf{Q}_\ell \rightarrow j_{!*} \mathbf{Q}_\ell \rightarrow Rj_* \mathbf{Q}_\ell$ induces an isomorphism on $H^0(-)$. By Deligne's theorem [BBD82, 5.1.14] we get that $H^*(X, \mathcal{F})$ is of weight < 0 , and as the sequence

$$H^*(X, \mathcal{F}) \rightarrow H^*(X, \mathbf{Q}_\ell) \rightarrow H^*(X, j_{!*} \mathbf{Q}_\ell)$$

is exact, we see that the kernel of $H^*(X, \mathbf{Q}_\ell) \rightarrow H^*(X, j_{!*} \mathbf{Q}_\ell)$ is of weight < 0 .

On the other hand, again by Deligne's theorem and duality, $H^*(X, j_{!*} \mathbf{Q}_\ell)$ is pure of weight 0, and hence everything in $H^*(X, \mathbf{Q}_\ell)$ of weight < 0 lies in the kernel. \square

Remark: i) Over the complex numbers this result is proved in [We04] (using instead Deligne's Hodge-theoretically defined weight filtration). For the applications of this paper that result could also be used. In any case, the filtration by weights of ℓ -adic cohomology is defined for a variety over any field, commutes appropriately under specialization of the base field, and coincides with Deligne's Hodge-theoretic weight filtration over the complex numbers. Thus our results are compatible with those of [We04].

ii) Nothing changes in the argument if one replaces U_0 by a smaller open dense subset and \mathbf{Q}_ℓ by $j_*\mathcal{E}$, where \mathcal{E} is a local system of pure weight 0.

It seems reasonable to introduce the *pure Betti numbers*, $b_i^p := \dim_{\mathbf{Q}_\ell} H_p^i(X, \mathbf{Q}_\ell)$, and we shall indeed do so. Our next result is a weakening of some well-known numeric consequences for the Betti numbers of a smooth and proper variety that arise as a consequence of the hard Lefschetz theorem. The stated restriction to varieties defined over finite fields is easily dispensed with, but we keep it to avoid too many details.

Theorem 2.2 *Let X_0 be a proper variety over a finite field, of pure dimension n . We have that $b_i^p \leq b_{i+2j}^p$ for all $0 \leq j \leq n - i$. In particular, for $i \leq n$ we have that $b_{n-i}^p \leq b_{n+i}^p$.*

PROOF: It follows from Theorem 2.1 that the map $H^*(X, \mathbf{Q}_\ell) \rightarrow \mathcal{I}H^*(X, \mathbf{Q}_\ell)$ has $H_p^*(X, \mathbf{Q}_\ell)$ as its image. As it is induced by a map of \mathbf{Q}_ℓ -complexes of sheaves, this map is an $H^*(X, \mathbf{Q}_\ell)$ -module map. In particular, for $0 \leq j \leq n - i$ it commutes with multiplication by $c_1(\mathcal{L})^j$ (the first Chern class of a line bundle), giving a commutative diagram

$$\begin{array}{ccc} H_p^i(X, \mathbf{Q}_\ell) & \longrightarrow & \mathcal{I}H^i(X, \mathbf{Q}_\ell) \\ \downarrow \cap_{c_1(\mathcal{L})^j} & & \downarrow \cap_{c_1(\mathcal{L})^j} \\ H_p^{i+2j}(X, \mathbf{Q}_\ell) & \longrightarrow & \mathcal{I}H^{i+2j}(X, \mathbf{Q}_\ell). \end{array}$$

By what we have proven the horizontal maps are injective, and the right vertical map is an injection by the hard Lefschetz theorem (cf. [BBD82, Thm 5.4.10], note that there they have made a shift by n of the cohomology sheaves, hence the difference in indexing). This implies that the left vertical map is injective, giving

$$b_i^p = \dim_{\mathbf{Q}_\ell} H_p^i(X, \mathbf{Q}_\ell) \leq \dim_{\mathbf{Q}_\ell} H_p^{i+2j}(X, \mathbf{Q}_\ell) = b_{i+2j}^p.$$

□

Using these theorems for motivation and consistency we define, for any proper variety X over an algebraically closed field, the pure cohomology $H_p^*(X, \mathbf{Q}_\ell)$ as the image of $H^*(X, \mathbf{Q}_\ell)$ in $\mathcal{I}H^*(X, \mathbf{Q}_\ell)$, and similarly for rational coefficients when the base field is the field of complex numbers.

3 The number of cells

We define a *stratification* of a proper variety X as a (necessarily finite) collection $\{V_\alpha\}_{\alpha \in I}$ of subvarieties of X , called *strata*, such that X is the disjoint union of them and the closure of each stratum is a union of strata. We get a partial order on the index set I of the strata by saying that $\alpha \leq \beta$ when $V_\alpha \subseteq \overline{V_\beta}$. It then follows that if $J \subseteq I$ is *downwards closed* (i.e., if $\alpha \leq \beta$ and $\beta \in J$ then $\alpha \in J$) then $X_J := \cup_{\alpha \in J} V_\alpha$ is a closed subset, as it is the union of the closed subsets $\overline{V_\alpha} = \cup_{\beta \leq \alpha} V_\beta$ for all $\alpha \in J$.

An *algebraic cell decomposition* of X is a stratification for which each stratum (which in this case will also be called a *cell*) is isomorphic to the n -dimensional affine space \mathbf{A}^n for some n .

Theorem 3.1 *Let X be a proper variety over an algebraically closed field, having an algebraic cell decomposition with f_i cells of dimension i . Then:*

- i) $H^{2i+1}(X, \mathbf{Q}_\ell) = 0$ for all integers i . In particular, $b_{2i+1} = b_{2i+1}^p = 0$.
- ii) $H^{2i}(X, \mathbf{Q}_\ell) = H_p^{2i}(X, \mathbf{Q}_\ell)$ for all i , and this space has a basis in bijection with the set of cells of dimension i . In particular, $b_{2i} = b_{2i}^p = f_i$ for all i .
- iii) Assuming that X is of pure dimension n we have that $f_i \leq f_j$ for all $i \leq j \leq n - i$.

PROOF: For the first part, by standard specialization arguments we may assume that X is defined over the algebraic closure of a finite field \mathbf{F}_q , and after possibly extending the finite field

we may assume that X as well as the strata are defined over \mathbf{F}_q . We now prove the statements by induction over the number of cells of the stratification, whose index set we denote by I . Let $\alpha \in I$ be maximal with respect to the partial order on I . This means that the cell of index α is an open subset U of X with complement F , which is a proper algebraic variety with a cell decomposition with fewer cells than X . By definition, $U \cong \mathbf{A}^n$ for some n .

We have a long exact sequence of cohomology

$$\cdots \rightarrow H_c^i(U, \mathbf{Q}_\ell) \rightarrow H^i(X, \mathbf{Q}_\ell) \rightarrow H^i(F, \mathbf{Q}_\ell) \rightarrow \cdots$$

and as X , U , and F are defined over \mathbf{F}_q the Frobenius map acts on all the vector spaces involved compatibly with all the maps. (Note that this long exact sequence is for cohomology with compact supports, but as X and F are compact, cohomology with compact supports is equal to ordinary cohomology.) Furthermore, we have that $H_c^i(\mathbf{A}^n) = 0$ for $i \neq 2n$ and $H_c^{2n}(\mathbf{A}^n) = \mathbf{Q}_\ell$ is pure of weight $2n$. For i odd this shows that $H^i(X, \mathbf{Q}_\ell) = 0$, while for i even with $i \neq 2n$ we get that $H^i(X, \mathbf{Q}_\ell) = H^i(F, \mathbf{Q}_\ell)$. Finally, for $i = 2n$ we get a short exact sequence $0 \rightarrow H_c^{2n}(U, \mathbf{Q}_\ell) \rightarrow H^{2n}(X, \mathbf{Q}_\ell) \rightarrow H^{2n}(F, \mathbf{Q}_\ell) \rightarrow 0$, which together with the induction hypothesis and the quoted result on the cohomology of \mathbf{A}^n finishes the proof of part ii).

Part iii) now follows from Theorem 2.2. \square

4 Proofs of Theorems A and C

As was mentioned in the introduction, the crystallographic groups are precisely the ones that appear as Weyl groups of Kac-Moody algebras. We are now going to apply the results of previous sections to such Weyl groups. General references for background to this material are the books by Kac [Kac83] and Kumar [Ku02], for the algebraic-geometric aspects see [Ku02] and [Sl86].

We start by recalling some properties of the Schubert varieties for a Kac-Moody algebra (group). (It seems that no attempt has been made to extend the construction of Kac-Moody groups to positive characteristic, à la Chevalley, so we restrict ourselves to characteristic zero from now on.) Let (W, S) be the Weyl group of the Kac-Moody algebra (which is a Coxeter group on the generating set S), pick $J \subseteq S$ and let W_J be the subgroup of W generated by the elements of J . As is well-known, there is a unique element w of minimal length in any W_J -coset \bar{w} . The set of such elements will be denoted W^J . For each $w \in W^J$ there exists (cf. [Ku02, Ch. 7], [Sl86, §2.2]) a complex projective variety \bar{X}_w containing locally closed subvarieties X_u for all $u \in [e, w]^J$ whose closures are disjoint unions

$$\bar{X}_u = \bigsqcup_{z \leq u} X_z,$$

where z is assumed to be in W^J . The partial order \leq is the Bruhat order, and X_u is a subvariety of \bar{X}_w isomorphic to $\mathbf{A}^{\ell(u)}$ [Sl86, Thm. 2.4].

Remark: The variety \bar{X}_w depends, at least a priori, on the choice of a dominant weight. However, we are just going to use its existence and not any uniqueness. (It is in any case true that any two choices give varieties that are related by algebraic maps which are homeomorphisms [Sl86], and hence have the same cohomology.)

Going further, the Kac-Moody group has a Borel subgroup B which acts on each \bar{X}_w such that the X_u , $u \leq w$, are the orbits. Note that B is not an algebraic group but only a group scheme (i.e., not of finite type). However, the action on any \bar{X}_w factors through a quotient which is an algebraic group, and we shall therefore allow ourselves to act as if B itself was an algebraic group. Using this action we get the next result.

Lemma 4.1 *The restriction of a B -complex of \bar{X}_w to some X_u , for $u \leq w$, is constant.*

PROOF: Recall (cf. [Sl86, 1.8]) that there is a subvariety U_u of B and a point x on X_u such that the map $g \mapsto gx$ gives an isomorphism $U_u \rightarrow X_u$. Now, if C is the B -complex, then by

assumption we have an isomorphism between p_2^*C and m^*C on $B \times \overline{X_w}$. Taking the restriction of this isomorphism to $U_u \times \{x\}$ we obtain an isomorphism between C and the constant extension of C_x to X_u . \square

Remark: By Proposition 8.5 this result can be applied to the intersection complex. In the setup of [BM01], rather than assuming a B -action, an equisingularity condition along X_u is assumed (which should follow from the fact that X_u is a B -orbit). This implies that the restriction to X_u is locally constant and then the fact that X_u is isomorphic to an affine space and hence contractible implies that locally constant complexes are constant.

Using the Kac-Moody Schubert varieties we can now undertake the proofs of the main results.

PROOF OF THEOREM A: This follows immediately from Proposition 3.1 applied to $\overline{X_w}$. \square

Remark: In connection with the inequalities of Theorem A it might be tempting to speculate that the f -vectors $(f_0^{w,J}, f_1^{w,J}, \dots, f_{\ell(w)}^{w,J})$ are *unimodal*, meaning that they increase up to some maximum and then decrease. However, this is false. See Stanton [Sta90] for non-unimodal examples in groups of type A_n modulo maximal parabolic subgroups.

For the proof of Theorem C we need to extend the monotonicity theorem of Braden and MacPherson (see [BM01, Cor. 3.7]) to the case of a general Kac-Moody Schubert variety.

Theorem 4.2 *Let $x < y < z$ in a crystallographic Coxeter group. Then $P_{x,z}^i \geq P_{y,z}^i$, where $P_{y,z}^i$ denotes the coefficients of q^i of the respective Kazhdan-Lusztig polynomials.*

Remark: It is no doubt true that the Whitney stratification condition, which is one of the standing hypotheses of [BM01], is indeed fulfilled also in this case. But rather than trying to verify that, we note that the fact that the Schubert cells are the orbits of a group action can be used more directly to prove the necessary conditions. As the proofs of [BM01] are also sometimes somewhat sketchy we have therefore chosen to go through the needed steps rather than leaving to the reader the task of checking that the proofs of Braden and MacPherson go through. At the same time this allows us to give the results in our context of ℓ -adic cohomology rather than in the de Rham-cohomology context of [BM01]. We have deferred to Section 8 the part of the argument that does not directly pertain to Kac-Moody Schubert varieties. We will here take that material for granted.

PROOF: Here, as in the more familiar case of finite and affine Weyl groups, these Kazhdan-Lusztig coefficients can be interpreted as the dimension of the fibre of the cohomology of the intersection complex for $\overline{X_z}$ at a point of X_x . Hence the monotonicity theorem 4.2 follows from our analogue of [BM01, Thm. 3.6], which we now have the appropriate tools for proving. We state it as a separate proposition. \square

Proposition 4.3 *For $x \leq y \leq z \in W^J$ we have a surjective map of \mathbf{Q}_ℓ -vector spaces $\mathcal{IH}^*(\overline{X_z})_x \rightarrow \mathcal{IH}^*(\overline{X_z})_y$, where the $\mathcal{IH}^*(\overline{X_z})$ are the cohomology sheaves of the intersection complex of $\overline{X_z}$ and $(-)_t$ denotes a fibre at any point of X_t .*

PROOF: This is [BM01, Thm. 3.6] in our context. Its proof, as well as proofs of the supporting Lemma 3.1, Lemma 3.3, Proposition 3.4, and Theorem 3.5 of [BM01], can now be carried through:

- Lemma 3.1 follows from Proposition 8.2 and Proposition 8.5.
- Lemma 3.3, Proposition 3.4, and Theorem 3.5 can be proved with the same proofs, using [De80] instead of [Sa89] for the weight results (note that in sheaf theory the relative cohomology $H^*(X, U, \mathcal{F})$ is defined as the cohomology with support $H_Y^*(X, \mathcal{F})$, where $Y := X \setminus U$).
- In the proof of (the analogue of) Theorem 3.6 we use Lemma 4.1 to conclude that the intersection complex is constant on each X_t and hence the cohomology of it on X_t is equal to its fibre for any point on X_t .

- To get a contracting action on X_z with the same fixed points as for the torus $T \subset B$ we use Corollary 8.4 applied to Z consisting of a single fixed point (note that $\overline{X_z}$ is irreducible). To verify the needed conditions we use that $\overline{X_z}$ lies an ind-variety which set-theoretically is equal to G/P_J , where G is the Kac-Moody group and P_J is the parabolic subgroup corresponding to J . The fixed points of T on G/P_J are the cosets wP_J , where w runs over W^J (where W is identified with the normaliser N of T in G divided by T itself). What we want to show is that the cone spanned by the weights of the cotangent space of $x = xP_J \in \overline{X_z}$ does not contain a line. For this it is enough to show the same thing for the cotangent space of x in G/P_J . By multiplying by $x^{-1} \in N$ we reduce to $x = e$ and by (vector space) duality to the case of the tangent space of $e \in G/P_J$. However, that tangent space is equal to $\mathfrak{g}/\mathfrak{p}_J$, where \mathfrak{g} is the corresponding Kac-Moody Lie algebra and \mathfrak{p}_J the corresponding parabolic subalgebra. That means that the weights form a subset of the weights of $\mathfrak{g}/\mathfrak{b}$, where \mathfrak{b} is the Lie algebra of B . These weights are exactly the negative roots of the root system of the Lie algebra and they lie in the cone generated by the negatives of the simple roots, a cone that indeed does not contain a line.

□

We are now ready to prove our third main theorem.

PROOF OF THEOREM C: Assume condition (b), and let

$$F_w(q) = \sum_{i=0}^{\ell(w)} a_i q^i := \sum_{x \leq w} q^{\ell(x)} P_{x,w}(q).$$

Theorem 4.2 implies that the q^i -coefficient of $P_{x,w}(q)$ is 0 for all $i = 1, \dots, k$ and all $x \leq w$. Hence, since also $\deg P_{x,w}(q) \leq \lfloor (\ell(w) - \ell(x) - 1)/2 \rfloor$, we get

$$\begin{array}{ll} a_0 = 1 & a_{\ell(w)} = 1 \\ a_1 = f_1^w & a_{\ell(w)-1} = f_{\ell(w)-1}^w \\ \vdots & \vdots \\ a_k = f_k^w & a_{\ell(w)-k} = f_{\ell(w)-k}^w \\ a_{k+1} = f_{k+1}^w + \beta_{k+1} & a_{\ell(w)-k-1} = f_{\ell(w)-k-1}^w \end{array}$$

Here the last row requires that $k < m$.

Now use that $a_i = a_{\ell(w)-i}$ for all i . This is valid in all Coxeter groups by [KL79, Lemma 2.6 (v)] (in this case it is also implied by Poincaré duality of middle intersection cohomology of $\overline{X_w}$). From this we conclude condition (a), as well as (c).

Finally, assume that condition (b) fails, say $d \leq k$ is minimal such that $\beta_d \neq 0$. Applying the implication (b) \Rightarrow (c) we get that $f_{\ell(w)-d}^w - f_d^w = \beta_d \neq 0$, so also condition (a) fails. □

5 Proof of Theorem B

We proceed with the proof of Theorem B. As mentioned in the introduction we shall use an argument of Stanley. When adapting it, it is both from a geometric and linear algebra point of view more natural to consider homology than cohomology (it will be clear that formally this is not required). Even though there is a sheaf theoretic definition, for our purposes it is enough to define the homology $H_i(X, \mathbf{Q}_\ell)$, for a projective variety X , as the dual vector space to $H^i(X, \mathbf{Q}_\ell)$. Then the homology becomes a covariant instead of contravariant functor. We denote by f_* the map induced by a map $f: X \rightarrow Y$. Furthermore, if X is n -dimensional, then there is the *trace map* $H^{2n}(X, \mathbf{Q}_\ell) \rightarrow \mathbf{Q}_\ell$ which is surjective and hence gives an element, the *fundamental class*, $[X] \in H_{2n}(X)$. If X is a closed subvariety of Y , then we get a fundamental class $[Y] = i_*[X]$, the *class* of Y , where $i: X \rightarrow Y$ is the inclusion. Now, if $U \subseteq X$ is an open n -dimensional

subset, then the composite $H_c^{2n}(U) \rightarrow H^{2n}(X) \rightarrow \mathbf{Q}_\ell$ is surjective and depends only on U . When $U = \mathbf{A}^n$, the map $H_c^*(\mathbf{A}^n) \rightarrow \mathbf{Q}_\ell$ is an isomorphism. Assuming now that X has a cell decomposition, combining this with Proposition 3.1 (or rather its proof) gives that $H_{2i}(X)$ has the classes of the closures of the i -dimensional cells as basis.

Remark: This result may seem to contradict Theorem 2.1 of [St80], as together they would imply that if X is n -dimensional, then the number of i -cells is equal to the number of $(n-i)$ -cells, which is not true in general (e.g., for a Schubert variety this is true, by the Carrell-Peterson criterion, if and only if it is rationally smooth). When X is smooth the Theorem 2.1 is true however, and that is the only case considered in [St80].

Now, as $H^*(X)$ is a \mathbf{Q}_ℓ -algebra, $H_*(X)$ becomes a module over $H^*(X)$. Furthermore, if $f: X \rightarrow Y$ is a map, then $H_*(X)$ becomes a $H^*(Y)$ -module through $y \cdot x = f^*y \cdot x$ and then, purely formally, we have the *projection formula* $y \cdot f_*x = f_*(f^*y \cdot x)$ for $y \in H^*(Y)$ and $x \in H_*(X)$. We are now ready to prove the analogue of [St80, Lemma 2.2] (which as it stands is true only in the smooth case).

Lemma 5.1 *Let X be a variety with a cell decomposition and \mathcal{L} a line bundle on X . Then for any cell C the expansion $c_1(\mathcal{L}) \cdot [C] = \sum_D d_{C,D} [\overline{D}]$, where D runs over the cells of X , has the property that $d_{C,D} = 0$ unless $D \subset \overline{C}$.*

PROOF: Let $i: C \rightarrow X$ be the inclusion and consider $[C] \in H_*(C)$. As \overline{C} has a cell decomposition, the cells of which are the D for which $D \subseteq \overline{C}$, we get that $i^*c_1(\mathcal{L}) \cdot [C] = \sum_D d_{C,D} [\overline{D}] \in H_*(\overline{C})$, where D runs over the cells of \overline{C} . Applying i_* to this formula gives $c_1(\mathcal{L}) \cdot [C] = \sum_{D \subset \overline{C}} d_{C,D} i_*[\overline{D}]$. However, $[\overline{D}] \in H_*(\overline{C})$ is equal to $j_*[\overline{D}]$, where $j: \overline{D} \rightarrow \overline{C}$ is the inclusion. Hence $i_*[\overline{D}] = i_*j_*[\overline{D}] = (ij)_*[\overline{D}]$, but the right hand side is by definition the class of \overline{D} in X . \square

We are now almost ready to adapt the proof of Stanley. However, in [St80] the hard Lefschetz theorem is combined with Lemma 1.1 for the desired conclusion and in our situation the hard Lefschetz theorem does not quite give a bijective map. Luckily the proof of [St80, Lemma 1.1] needs only a slight modification to be applicable to our situation (where we also, contrary to [St80], do not turn the geometric poset “upside down”).

Lemma 5.2 *Let P be a finite graded poset of rank n . Let P_j denote the set of its elements of rank j and let V_j be the vector space with basis P_j over some given field. For each $i \leq j < n-i$ assume given a linear transformation $\varphi_j: V_{j+1} \rightarrow V_j$ such that the following two conditions are satisfied:*

(a) *The composite transformation $\varphi_i \circ \varphi_{i+1} \circ \cdots \circ \varphi_{n-i-2} \circ \varphi_{n-i-1}$ is surjective.*

(b) *If $x \in P_{j+1}$ and $\varphi_j(x) = \sum_{y \in P_j} c_{x,y}^j y$, then $c_{x,y}^j = 0$ unless $y < x$.*

Then, in P there exist $\text{card}(P_i)$ pairwise disjoint chains $x_i < x_{i+1} < \cdots < x_{n-i}$ such that $\text{rank}(x_j) = j$ for all j .

PROOF: The proof of [St80, Lemma 1.1] goes through in this situation with a slight modification, for the reader's convenience we repeat the argument. Let $m := \text{card}(P_i)$ and put $\varphi := \varphi_i \circ \cdots \circ \varphi_{n-i-1}$. As φ is surjective so is $\Lambda^m \varphi: \Lambda^m V_{n-i} \rightarrow \Lambda^m V_i$. By the definition of φ we get that $\Lambda^m \varphi = \Lambda^m(\varphi_i) \circ \cdots \circ \Lambda^m(\varphi_{n-i-1})$. Using the bases P_j we get bases for $\Lambda^m V_j$ and hence a matrix for each $\Lambda^m(\varphi_j)$. An entry of the product matrix of the composite $\Lambda^m(\varphi_i) \circ \cdots \circ \Lambda^m(\varphi_{n-i-1})$ has the form

$$\sum \det \varphi_i[Q_{i+1}, Q_i] \det \varphi_{i+1}[Q_{i+2}, Q_{i+1}] \cdots \det \varphi_{n-i-1}[Q_{n-i}, Q_{n-i-1}],$$

where $Q_j \subseteq P_j$ with $\text{card}(Q_j) = m = \text{card}(P_i)$, Q_{n-i} specifies the entry of the matrix of the composite, $\varphi_j[Q_{j+1}, Q_j]$ is the submatrix of φ_j corresponding to the sets of basis elements Q_j and Q_{j+1} , and the sum runs over all the choices of Q_j , $i \leq j < n-i$. By assumption there is a

Q_{n-i} such that this sum is non-zero, and hence there is a summand that is non-zero. This gives us a set of Q_j such that all the $\det \varphi_j[Q_{j+1}, Q_j]$ are non-zero. In particular, one term of the expansion of this matrix must be non-zero, which gives us a bijection $\sigma_j: Q_{j+1} \rightarrow Q_j$ such that $c_{x, \sigma_j(x)}^j \neq 0$ for all $x \in Q_{j+1}$. By assumption (b) this implies that $\sigma_j(x) < x$ for all $x \in Q_{j+1}$. \square

We can now prove Theorem B.

PROOF OF THEOREM B: We apply Lemma 5.2 to the interval $[e, w]^J$, a graded poset of rank $\ell(w)$. We identify the vector space V_j with $H_{2j}(X)$, where X is the Schubert variety of w and let $\varphi_j: H_{2j+2}(X) \rightarrow H_{2j}(X)$ be multiplication by $c_1(\mathcal{L})$. Condition (a) of the lemma follows from the proof of Proposition 2.2, which shows that multiplication by $c_1(\mathcal{L})^{n-2i}$ gives an injective map from $H^{2i}(X)$ to $H^{2n-2i}(X)$ and hence by duality it gives a surjective map from $H_{2n-2i}(X)$ to $H_{2i}(X)$. Condition (b) follows from Lemma 5.1. \square

6 Proof of Theorem D

We begin by recalling the definition of an M -sequence. For $n, k \geq 1$ there is a unique expansion

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_i}{i},$$

with $a_k > a_{k-1} > \cdots > a_i \geq i \geq 1$. This given, let

$$\begin{aligned} \partial^k(n) &:= \binom{a_k-1}{k-1} + \binom{a_{k-1}-1}{k-2} + \cdots + \binom{a_i-1}{i-1}, \\ \partial^k(0) &:= 0. \end{aligned}$$

Theorem (Macaulay-Stanley [St78, Thm. 2.2])

For an integer sequence $(1, m_1, m_2, \dots)$ the following conditions are equivalent (and this defines an M -sequence):

- (1) $\partial^k(m_k) \leq m_{k-1}$, for all $k \geq 1$,
- (2) some family of monomials, closed under divisibility, contains exactly m_k monomials of degree k ,
- (3) $\dim(A_k) = m_k$ for some graded commutative algebra $A = \bigoplus_{k \geq 0} A_k$ (over some field), such that A is generated by A_1 .

PROOF OF THEOREM D: The f -vector $f^w = \{f_0, f_1, \dots, f_{\ell(w)}\}$ satisfies $f_k = \dim H^{2k}(X_w)$. Hence, to prove that f^w is an M -sequence we need to show that $H^*(X_w)$, the cohomology algebra of the Schubert variety X_w (over \mathbf{C}), is generated in degree one (or, equivalently, in $\dim = 2$). For $w = w_0$ this is classical — it can be seen either from the description of $H^*(X_{w_0})$ in terms of special Schubert classes, or from the isomorphism of $H^*(X_{w_0})$ with the coinvariant algebra of W .

For $w \neq w_0$ we use that the inclusion $X_w \hookrightarrow X_{w_0}$ induces an injective map on homology $H_*(X_w) \rightarrow H_*(X_{w_0})$, as is apparent from the cell decomposition. Hence, dually there is algebra surjection $H^*(X_{w_0}) \rightarrow H^*(X_w)$. Since $H^*(X_{w_0})$ is generated in degree one, so is $H^*(X_w)$.

Finally, to prove that $(f_0^w, f_1^w - f_0^w, f_2^w - f_1^w, \dots, f_{\lfloor \ell(w)/2 \rfloor}^w - f_{\lfloor \ell(w)/2 \rfloor - 1}^w)$ is an M -sequence we apply the Macaulay-Stanley theorem to the algebra $H^*(X, \mathbf{Q}_\ell)/c_1(\mathcal{L})H^*(X, \mathbf{Q}_\ell) + H^{>\ell(w)}(X, \mathbf{Q}_\ell)$, which by Theorem 2.2 corresponds to the desired vector. \square

Remark: i) The M -sequence property for f -vectors of lower intervals $[e, w]$ does not seem to extend beyond the case of finite groups. E.g., it fails for the affine Weyl group \tilde{C}_2 , whose Poincaré series gives

$$\sum q^{\ell(w)} = 1 + 3q + 5q^2 + 8q^3 + \cdots$$

Let $u \in \widetilde{C}_2$ be above all elements of length 3. Then $f^u = (1, 3, 5, 8, \dots)$, which is not an M -sequence (since $\partial^3(8) = 6 \not\leq 5$).

An algebraic consequence of this is that $H^*(X_u)$ is not generated in degree one.

ii) The M -sequence property fails also for general intervals $[x, w]$ in finite groups. For instance, for a particular $x \in C_4$:

$$\sum_{x \leq y \leq w_0} q^{\ell(y) - \ell(x)} = 1 + 4q + 11q^2 + \dots$$

This information can be read off from Goresky's tables [Go81] by letting $x = zw_0$, where z is element number 377 of C_4 .

We are grateful to E. Nevo for help with finding these examples, and to B. Shapiro for help with the proof of Theorem D.

7 Proof of Theorem E

Let $[u, v]$ be a Bruhat interval. The elements of $[u, v]$ of length $\ell(u) + 1$ are its *atoms*. We let $f_{\ell(u)+1}^{u,v}$ denote the number of atoms in $[u, v]$.

It has been shown by Dyer [Dy91] that (up to isomorphism) only finitely many posets of each given length r occur as intervals in the Bruhat order on finite Coxeter groups. Therefore, the following function is well-defined,

$$M(r) := \max_{[u,v]} \{ f_{\ell(u)+1}^{u,v} \mid \ell(v) - \ell(u) = r \}.$$

That is, $M(r)$ denotes the maximum number of atoms of a Bruhat interval of length r occurring in any finite Coxeter group. The initial values $M(2) = 2$, $M(3) = 4$, $M(4) = 8$ are known, see [Hul03]. Let $\widetilde{M}(r) := \max_{t \leq r} M(t)$. (Actually, $\widetilde{M}(r) = M(r)$, but we don't need this.)

As usual, denote by w_0 the element of maximal length in any given finite group W . It follows from the classification of irreducible finite Coxeter groups that the following number-theoretic function is well-defined,

$$Q(s) := \max_{(W,S)} \{ \ell(w_0) \mid \text{card}(W) < \infty, \text{card}(S) = s \},$$

where the maximum is taken over all finite Coxeter groups of rank s , of which there are only finitely many. The classification shows that $Q(s) = s^2$ for $s \geq 9$ (the maximum occurring in type B), whereas there are irregularities occurring for $s \leq 8$ due to the exceptional groups.

Lemma 7.1 *Let (W, S) be a finite crystallographic Coxeter group, and $w \in W$. We have that*

$$\ell(w) > Q(j) \quad \Rightarrow \quad f_{\ell(w)-1}^w > j.$$

PROOF: Suppose that $f_{\ell(w)-1}^w \leq j$. By Theorem A we have that $f_1^w \leq f_{\ell(w)-1}^w$. Hence, $f_1^w \leq j$, which means that there is a set $J \subseteq S$ of cardinality $|J| \leq j$ (the set of atoms of $[e, w]$) such that every reduced expression for w uses letters only from the set J (by the subword property of Bruhat order [Hu90, Thm. 5.10]). In particular, $w \in W_J$, so $\ell(w) \leq \ell(w_0(J)) \leq Q(j)$, where $w_0(J)$ denotes the element of maximal length in the parabolic subgroup W_J . \square

PROOF OF THEOREM E: Assume that W is crystallographic. The easy extension of the proof to the general case (when W has components of type H_3 and H_4) is left to the reader.

Put

$$N_k := Q(\widetilde{M}(k) - 1) + k,$$

and $n := \ell(w) \geq N_k$. For r such that $1 \leq r \leq k$ let $L_{n-r} := \{x \in [e, w] \mid \ell(x) = n - r\}$. Consider the bipartite graph with vertices $L_{n-r} \cup L_{n-r+1}$ and edges $E_r = \{(x, y) \in L_{n-r} \times L_{n-r+1} \mid x < y\}$. If $(x, y) \in E_r$ then

$$\deg(x) \leq \widetilde{M}(r) \leq \widetilde{M}(k),$$

where $\deg(x)$ denotes the number of edges adjacent to x in E_r . Similarly, by Lemma 7.1,

$$\deg(y) \geq \widetilde{M}(k).$$

Thus,

$$|L_{n-r}| \cdot \widetilde{M}(k) \geq |E_r| \geq |L_{n-r+1}| \cdot \widetilde{M}(k)$$

and hence

$$f_{\ell(w)-r}^w = |L_{n-r}| \geq |L_{n-r+1}| = f_{\ell(w)-r+1}^w.$$

□

In closing we would like to raise the question: Does there exist $\alpha < 1$ such that

$$f_{\lfloor \alpha \cdot \ell(w) \rfloor}^w \geq \cdots \geq f_{\ell(w)}^w$$

for all $w \in W$?

8 Appendix: Contracting \mathbf{G}_m -actions

As was mentioned in Section 4, we have deferred to here some general algebraic-geometric material needed for the proof of Theorem 4.2.

If (X, \mathcal{F}) and (Y, \mathcal{G}) are pairs consisting of an algebraic variety and a complex of ℓ -adic sheaves on it, then a *morphism* $(X, \mathcal{F}) \rightarrow (Y, \mathcal{G})$ consists of a map $f: X \rightarrow Y$ together with a map of complexes (or more precisely a map in the derived category) $\mathcal{G} \rightarrow Rf_*\mathcal{F}$, or equivalently by adjunction, a map $f^*\mathcal{G} \rightarrow \mathcal{F}$. Such maps form a category with composition of $\mathcal{H} \rightarrow Rg_*\mathcal{G}$ and $\mathcal{G} \rightarrow Rf_*\mathcal{F}$ over maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ being obtained by first applying Rg_* to $\mathcal{G} \rightarrow Rf_*\mathcal{F}$ and then taking the composite $\mathcal{H} \rightarrow Rg_*\mathcal{G} \rightarrow Rg_*Rf_*\mathcal{F} = R(gf)_*\mathcal{F}$. A morphism $(X, \mathcal{F}) \rightarrow (Y, \mathcal{G})$ induces a map on cohomology $H^*(Y, \mathcal{G}) \rightarrow H^*(X, \mathcal{F})$ by applying $H^*(X, -)$ to the map $\mathcal{G} \rightarrow Rf_*\mathcal{F}$ and then using $H^*(Y, Rf_*\mathcal{F}) = H^*(X, \mathcal{F})$. We shall say that two maps $f, g: (X, \mathcal{F}) \rightarrow (Y, \mathcal{G})$ are *homotopic* if there is a connected variety I , a map $F: (I \times X, p_2^*\mathcal{F}) \rightarrow (Y, \mathcal{G})$, where $p_2: I \times X \rightarrow X$ is the projection, and two points $0, 1 \in I$ such that the composites of F and the maps $0, 1: (X, \mathcal{F}) \rightarrow (I \times X, p_2^*\mathcal{F})$ equal f resp. g .

Proposition 8.1 *If $f, g: (X, \mathcal{F}) \rightarrow (Y, \mathcal{G})$ are two homotopic maps, then they induce the same map in cohomology (over an algebraically closed field \mathbf{k}).*

PROOF: Using the notation in the definition of homotopicity, f^* and g^* factor as 0^*F^* and 1^*F^* , respectively. Hence it is enough to show that $0^* = 1^*: H^*(I \times X, p_2^*\mathcal{F}) \rightarrow H^*(X, \mathcal{F})$. However, $H^*(I \times X, p_2^*\mathcal{F}) = H^*(X, Rp_{1*}p_2^*\mathcal{F})$ and by the projection formula $Rp_{1*}p_2^*\mathcal{F} = R\Gamma(I, \mathbf{Z}_\ell) \otimes \mathcal{F}$. In the latter description 0^* and 1^* are induced by tensoring \mathcal{F} by the maps $R\Gamma(I, \mathbf{Z}_\ell) \rightarrow R\Gamma(\mathbf{Spec} \mathbf{k}, \mathbf{Z}_\ell) = \mathbf{Z}_\ell$, the equality being true as the spectrum of an algebraically closed field is a point from the cohomological perspective, induced by the inclusions $0, 1: \mathbf{Spec} \mathbf{k} \rightarrow I$. However, these maps are completely determined by the maps induced on zero-th cohomology $H^0(I, \mathbf{Z}_\ell) \rightarrow \mathbf{Z}_\ell$. As I is connected, the structure map $I \rightarrow \mathbf{Spec} \mathbf{k}$ induces an isomorphism $\mathbf{Z}_\ell = H^0(\mathbf{Spec} \mathbf{k}, \mathbf{Z}_\ell) \rightarrow H^0(I, \mathbf{Z}_\ell)$. As the two composites $\mathbf{Spec} \mathbf{k} \xrightarrow{0,1} I \rightarrow \mathbf{Spec} \mathbf{k}$ both are the identity, this implies that $0, 1: \mathbf{Spec} \mathbf{k} \rightarrow I$ induce the same map $H^0(I, \mathbf{Z}_\ell) \rightarrow H^0(\mathbf{Spec} \mathbf{k}, \mathbf{Z}_\ell)$. □

We now want to apply this result to the case of an equivariant complex of sheaves for a contracting action. Thus we are dealing with an algebraic action of a 1-dimensional torus $\mathbf{G}_m := \mathbf{Spec} \mathbf{k}[t, t^{-1}]$ on an algebraic variety X , i.e., a map $m: \mathbf{G}_m \times X \rightarrow X$ fulfilling the required conditions for an action. It is said to be *contracting* if it can be extended to a map $\mathbf{A}^1 \times X \rightarrow X$. Our next step says, roughly, that the cohomology of a \mathbf{G}_m -equivariant complex of sheaves on X (assuming that the action is contracting) is equal to the cohomology of the restriction of the complex to the fixed point set. There are (at least) two possible definitions of what we should

mean by an equivariant complex of sheaves (the difference being whether we demand an action on an actual complex or only on an object of the derived category). It turns out that the complexes to which we want to apply our results have an action of \mathbf{G}_m in the strongest possible sense, but on the other hand to prove our results we need a very weak notion of action which is very easy to verify. We therefore say that a complex C of ℓ -adic sheaves is a \mathbf{G}_m -complex if there exists an isomorphism $\phi: p_2^*C \rightarrow m^*C$ in the derived category, where $p_2: \mathbf{G}_m \times X \rightarrow X$ is the projection on the second component and m is the given action as above. Note that one has to be very careful when using this notion; we make for instance no claim that \mathbf{G}_m -sheaves form a triangulated category.

Proposition 8.2 *Let X be a variety with a contracting \mathbf{G}_m -action and C a \mathbf{G}_m -complex of ℓ -adic sheaves on X . If Z is the fixed point variety of the action, then the restriction map $H^*(X, C) \rightarrow H^*(Z, C)$ is an isomorphism.*

PROOF: Let $i: Z \rightarrow X$ and $j: U \rightarrow X$ be the inclusions, where $U := X \setminus Z$. We then have a distinguished triangle $\rightarrow j_!j^*C \rightarrow C \rightarrow i_*C \rightarrow$ which pulls back by m and p_2 to distinguished triangles, and the assumed isomorphism $\phi: p_2^*C \rightarrow m^*C$ induces a morphism of distinguished triangles. By the 5-lemma we are thus reduced to the cases where either C is supported on Z or zero on it. The first case is clear, so we may assume that C is zero on Z . We now have two maps $(X, C) \rightarrow (X, C)$, one being the identity map and the other the identity map on X and zero on the complexes. If we can show that they are homotopic, then we conclude that $H^*(X, C)$ is zero and hence the map to the restriction is an isomorphism.

Now, we consider the map, also denoted m , $m: \mathbf{A}^1 \times X \rightarrow X$, which extends the \mathbf{G}_m -action. We want to define a map $(\mathbf{A}^1 \times X, p_2^*C) \rightarrow (X, C)$ over m , i.e., a map of complexes $m^*C \rightarrow p_2^*C$. However, m^*C and p_2^*C are by assumption isomorphic over $\mathbf{G}_m \times X$, and as m^*C is zero in its complement in $\mathbf{A}^1 \times X$ (as $m(\{0\} \times X) \subseteq Z$) this isomorphism extends to a map $m^*C \rightarrow p_2^*C$. Composing with $1: \mathbf{Spec} \mathbf{k} \times X \rightarrow \mathbf{A}^1 \times X$ given by the point 1 of \mathbf{A}^1 clearly gives the identity map, and composing with $0: \mathbf{Spec} \mathbf{k} \times X \rightarrow \mathbf{A}^1 \times X$ given by the point 0 of \mathbf{A}^1 gives the zero map, as the image of $\{0\} \times X$, as was just observed, is contained in Z and C is zero on Z . \square

In order to be able to apply these results we shall need to on the one hand give a method for deciding when a \mathbf{G}_m -action is contracting, and on the other show that the intersection complex of a variety with a \mathbf{G}_m -action is a \mathbf{G}_m -complex in our sense. We start with the first problem.

Suppose that $X = \mathbf{Spec} R$ is an affine variety with a \mathbf{G}_m -action. The action immediately translates to a grading $R = \bigoplus_{i \in \mathbf{Z}} R_i$ so that $\lambda \in \mathbf{k}^* = \mathbf{G}_m(\mathbf{k})$ acts as $\lambda \cdot r = \lambda^i r$ for $r \in R_i$. The condition that the action be contracting is then equivalent to $R_i = 0$ for $i < 0$. This implies that if $x \in X(\mathbf{k})$ is a \mathbf{k} -point, then the linear action of \mathbf{G}_m on the cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$ has the property that only non-negative weights occur. (Recall that a linear representation of \mathbf{G}_m is a direct sum of 1-dimensional representations of the form $\lambda \mapsto \lambda^n$, and that n is the weight of that subrepresentation.) It turns out that there is a converse to this result (where we for simplicity, and because it is the only case we shall use, only consider the irreducible case).

Proposition 8.3 *Let X be an irreducible variety with a \mathbf{G}_m -action, and let Z be a closed subvariety of fixed points such that for each closed point of Z the action of \mathbf{G}_m on its cotangent space has only non-negative weights. Then the action of \mathbf{G}_m is contracting on the union of the open \mathbf{G}_m -invariant affine subsets that meet Z .*

PROOF: An extension of the \mathbf{G}_m -action to an \mathbf{A}^1 -action is unique if it exists, as \mathbf{G}_m is dense in \mathbf{A}^1 . Hence it is enough to show that the action extends to any open \mathbf{G}_m -invariant affine subset that meets Z . We may therefore assume that $X = \mathbf{Spec} R$ and the \mathbf{G}_m -action then corresponds to a grading $R = \bigoplus_i R_i$. We may further assume that Z contains the closed point z . What we want to show is that $R_i = 0$ for $i < 0$.

Now, as Z is irreducible we have that R embeds in the local ring $R_{\mathfrak{m}_z}$, and by a theorem of Krull we have that $\bigcap_n \mathfrak{m}_z^n = 0$ in $R_{\mathfrak{m}_z}$. Hence it will be enough to show that $R_i \subseteq \mathfrak{m}_z^n$ for $i < 0$ and every n . We do this by induction on n , where the case $n = 1$ is true as z is a fixed point so that \mathbf{G}_m acts trivially on R/\mathfrak{m}_z . The case $n = 2$ is then true as the map $R_i \rightarrow \mathfrak{m}_z/\mathfrak{m}_z^2$ is a map

from a \mathbf{G}_m -representation consisting only of negative weight representations to a representation that by assumption contains only non-negative weight representations. Multiplication in R now gives a surjective \mathbf{G}_m -equivariant map $S^n(\mathfrak{m}_z/\mathfrak{m}_z^2) \rightarrow \mathfrak{m}_z^n/\mathfrak{m}_z^{n+1}$, which implies that $\mathfrak{m}_z^n/\mathfrak{m}_z^{n+1}$ consists only of non-negative weight representations. By the induction assumption we have an induced map $R_i \rightarrow \mathfrak{m}_z^n/\mathfrak{m}_z^{n+1}$, which therefore also is the zero map. \square

Remark: The case of the action of \mathbf{G}_m on \mathbf{P}^2 given by $(x : y : z) \mapsto (x : \lambda y : \lambda^2 z)$ is instructive. The point $(1 : 0 : 0)$ is then a fixed point whose cotangent space has weights 1 and 2. The only invariant affine open subset that contains $(1 : 0 : 0)$ is $\{x \neq 0\}$, and on it the action visibly extends to a \mathbf{A}^1 -action; $(1 : y : z) \mapsto (1 : \lambda y : \lambda^2 z)$ makes sense also for $\lambda = 0$.

This proposition has the following corollary which is the only consequence that we shall actually use.

Corollary 8.4 *Assume that T is an algebraic torus (i.e., isomorphic to \mathbf{G}_m^n for some $n > 0$) and let Γ be its group of characters (i.e., the group of algebraic homomorphisms $T \rightarrow \mathbf{G}_m$). Let X be a variety with an action of T such that there exists an ample line bundle on X with a compatible T -action. Suppose $z \in X$ is a T -fixed point such that the cone generated by the characters that appear in the cotangent space of z does not contain a line (or equivalently no non-zero element and its inverse). Then there is an algebraic group homomorphism $\mathbf{G}_m \rightarrow T$ such that $X^T = X^{\mathbf{G}_m}$, and such that there exists an affine T -invariant neighbourhood of z for which the action is contracting.*

PROOF: Let Γ^* be the group of cocharacters of T (i.e., algebraic group homomorphisms $\mathbf{G}_m \rightarrow T$) which is identified with the dual of Γ by pairing $\phi: \mathbf{G}_m \rightarrow T$ and $\varphi: T \rightarrow \mathbf{G}_m$ using the integer n for which the composite $\varphi \circ \phi$ is of the form $\lambda \mapsto \lambda^n$. We want to apply the proposition, so what we are looking for is an element $\phi \in \Gamma^*$ such that its fixed point set is equal to that of T and such that its weights on the cotangent space of z are all non-negative. For the first condition it is well-known that there is a finite set of hyperplanes in Γ^* such that if ϕ is not contained in any of them, then it has the same fixed point set as T . For the second condition it is clear that we are looking for a ϕ with non-negative values on each character of T that appears in the cotangent space of z . By the assumption on the cone generated by them (and the fact that Γ^* is the dual of Γ) there is such a ϕ outside of any finite number of hyperplanes, so that we may simultaneously fulfill both conditions. Now, by the assumption that X admits a T -linearised ample line bundle we may find a T -invariant affine neighbourhood of z , and then the proposition is applicable. \square

We are now left with establishing that the intersection complex is a \mathbf{G}_m -complex. However, we shall need the corresponding results for other actions so we put ourselves in a somewhat more general situation. Thus, if G is an algebraic group acting on a variety X , we say that a complex C of ℓ -adic sheaves is a G -complex if the two complexes p_2^*C and m^*C are isomorphic in the derived category, where $p_2, m: G \times X \rightarrow X$ is the projection and G -action map respectively.

Proposition 8.5 *Let X be a variety on which the algebraic group G acts. Then the intersection complex of X is a G -complex.*

PROOF: We start by noticing that m is the composite of p_2 and the automorphism

$$\begin{aligned} t: G \times X &\rightarrow G \times X \\ (g, x) &\mapsto (g, gx) \end{aligned}$$

so that it will be enough to show that t^*C is isomorphic to C , where C is the pullback by p_2 of the intersection complex on X . However, as G is smooth, pullback by p_2 takes perverse sheaves to perverse sheaves and pulls back $j_*\mathbf{Q}_\ell$ to $j'_*\mathbf{Q}_\ell$, resp. $j_!\mathbf{Q}_\ell$ to $j'_!\mathbf{Q}_\ell$. Here j is the inclusion of the smooth locus U of X into X and j' is the inclusion of $G \times U$, which is the smooth locus of $G \times X$. This implies that C is isomorphic to the middle extension $j'_{!*}\mathbf{Q}_\ell$. Now, it is characterised in a fashion which makes it isomorphic to its pullback by any automorphism of $G \times X$. \square

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