APPENDIX TO “HYPERPLANE ARRANGEMENTS AND BOX–SPLINES” BY C. DE CONCINI AND C. PROCESI

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In this note we review some facts about face rings of matroid complexes that are relevant for the paper of De Concini and Procesi.

1. Shellability and face rings. An abstract simplicial complex Δ is *pure* if all its *facets* (maximal faces) are of the same dimension. A linear order $F_1, F_2, \dots, F_t$ of its facets is called a *shelling* if each facet F_i, $i > 1$, intersects the complex $\Delta_{i-1} = 2^{F_1} \cup \cdots 2^{F_{i-1}}$ generated by the preceding facets in a pure $(\dim \Delta - 1)$-dimensional subcomplex. Equivalently, each facet F_i has a subface $\widehat{F_i}$, called its *restriction*, such that $\Delta_i \setminus \Delta_{i-1} = \{G : \widehat{F_i} \subseteq G \subseteq F_i\}$ for all $i > 1$. We put $\widehat{F_1} = \emptyset$. A pure complex Δ is said to be *shellable* if it admits a shelling order of its facets. See [**?**, pp. 228–232] or [**?**, pp. 78–83] for motivation and more details about the notion of shellability.

Let $V = \{x_1, \dots, x_n\}$ be the set of vertices of a simplicial complex Δ. To each subset $F \subseteq V$ we associate a squarefree monomial $x(F) = \prod_{x_i \in F} x_i$. The commutative ring

$$\mathbf{k}[\Delta] = \mathbf{k}[x_1, \dots, x_n]/(x(F) | F \notin \Delta)$$

($\mathbf{k}$ a field), is called the *face ring* (or Stanley-Reisner ring, or discrete Hodge algebra) of Δ.

The relevance of the concept of shellability for commutative algebra is the following theorem, showing that it induces a combinatorial decomposition of face rings from which Hilbert series can be read.

Theorem 1. *Let $F_1, F_2, \dots, F_t$ be a shelling of the simplicial complex Δ, and let $(\theta) = (\theta_1, \theta_2, \dots, \theta_d)$ be a linear system of parameters for the ring $\mathbf{k}[\Delta]$. Then $\mathbf{k}[\Delta]$ is Cohen–Macaulay and*

The realization that shellability implies Cohen-Macaulayness can be traced back to the seminal work of Hochster [**?**] and Stanley [**?**]. The more detailed form of the theorem is due to Garsia [**?**], Kind and Kleinschmidt [**?**] and Stanley [**?**, Thm 2.5, p. 82].

2. Matroids. A *matroid* $M = (E, IN)$ consists of a family IN of subsets of a finite set E such that

M1. (Closure) $A \subseteq B \in IN$ implies $A \in IN$, and

M2. (Exchange) $A, B \in IN$ and $|A| > |B|$ implies that $B \cup \{x\} \in IN$ for some $x \in A \setminus B$.

The sets in IN are called *independent*, and maximal independent sets are *bases*. All bases have the same cardinality, called the *rank* of M. The minimal dependent sets are called *circuits*. A premier example of a matroid is given by linear independence among a finite set of vectors. See e.g. the book series [**?**] for an exposition on matroids.

Matroid theory contains a pleasant duality operation, defined as follows. If M is a matroid on the ground set E, there is a *dual matroid* M^* (having the same ground set) whose bases are given by the set complements $E \setminus B$ of bases B of M. The circuits of M^* are called *cocircuits* of M, and $rank(M^*) = |E| - rank(M)$.

If B is a basis of M and $p \notin B$ then there is a unique circuit $cir(B, p)$ contained in $B \cup p$ (we here dispense with set brackets for singletons). Dually, if $q \in B$ then there is a unique cocircuit $cocir(B, q)$ contained in $(E \setminus B) \cup q$. These *basic circuits and cocircuits* are related in the following way:

$$q \in cir(B, p) \;\Leftrightarrow\; (B \setminus q) \cup p \text{ is a basis } \;\Leftrightarrow\; p \in cocir(B, q).$$

From now on, assume that the ground set E is linearly ordered. Given a basis B and an element $p \notin B$ we say that p is *externally active* in B if p is the least element of $cir(B, p)$. Dually, an element $q \in B$ is said to be *internally active* in B if q is the least element of $cocir(B, p)$. Note that these concepts are dual: p is externally active in the basis B of M if and only if p is internally active in the basis $E \setminus B$ of M^*.

Let $ext(B)$ denote the number of elements that are externally active

It follows from the axioms that the independent sets $IN(M)$ of a matroid M form a pure simplicial complex whose facets are the bases of M. The following result appears in [**?**, pp. 233–236].

Theorem 2. *The lexicographic order of the bases of M, induced by the linear order of the ground set E, is a shelling of $IN(M)$. The corresponding restriction operator sends a basis B to the subset $\widehat{B} = \{p \in B \mid p \text{ is not internally active in } B\}$.*

3. Face ring of a matroid. Combining the information from sections 1 and 2 we draw the following conclusions about the face ring of a matroid complex. Let M be a matroid of rank r on set $E = \{x_1, \dots, x_n\}$. Consider the polynomial ring $\mathbf{k}[M] := \mathbf{k}[x_1, \dots, x_n]/J$, where J is the ideal generated by the squarefree monomials $x(C)$ corresponding to the cocircuits C of M. Then $\mathbf{k}[M]$ is the face ring of the simplicial complex $IN(M^*)$ of independent sets of the dual matroid M^*, a complex which is pure $(n-r-1)$-dimensional.

According to Theorem 2 the complex $IN(M^*)$ is shellable and

$$\sum_B z^{\|\widehat{E \setminus B}\|} = \sum_B z^{n-r-ext(B)} = z^{n-r} T_M(1, 1/z),$$

with summation running over all bases B of M. Then we have from Theorem 1 that the face ring $\mathbf{k}[M]$ is Cohen-Macaulay with Hilbert series

$$Hilb_{\mathbf{k}[M]}(z) = \frac{z^{n-r} T_M(1, 1/z)}{(1-z)^{n-r}}.$$

Hence, modding out by a linear system of parameters we reach the following conclusion.

Theorem 3. *Let (θ) be an l.s.o.p for $\mathbf{k}[M]$. Then $\mathbf{k}[M]/(\theta)$ is a finite-dimensional algebra with Hilbert series*

$$Hilb_{\mathbf{k}[M]/(\theta)}(z) = z^{n-r} T_M(1, 1/z).$$

Furthermore, for each basis B of M, let $Ext(B)$ be the set of its externally active elements. Then the system of squarefree monomials

$$\{x((E \setminus B) \setminus Ext(B)) \mid B \text{ matroid basis of } M\}$$

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