COMPLEXES OF DIRECTED GRAPHS

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ABSTRACT. Let P be a monotone property of directed graphs on n vertices, and let Δ_n^P denote the abstract simplicial complex whose simplices are the edge sets of graphs having property P. We prove that:

1. If "P = acyclic" then Δ_n^P is homotopy equivalent to the $(n-2)$-sphere.
2. If "P = not strongly connected" then Δ_n^P has the homotopy type of a wedge of $(n-1)!$ spheres of dimension $2n-4$.

The lattice of all posets on $\{1, 2, \ldots, n\}$ plays an important role in the analysis. We also discuss a few other properties of directed graphs from this point of view.

1. Introduction

A property of graphs is called *monotone* if it is preserved under deletion of edges. Thus, a monotone graph property can be interpreted as a simplicial complex, and one can study its topological properties. This has been done for undirected graphs in a number of recent papers. See [1, 13] and the further references cited there.

Here we look at some monotone properties of directed graphs from this point of view. Let $[n] = \{1, 2, \ldots, n\}$. We identify a digraph G on the node set $[n]$ with the set $E(G)$ of its edges, which is a subset of the set $\Omega = [n] \times [n] \setminus \{(i,i) \mid 1 \leq i \leq n\}$. In particular, in this paper digraphs have no multiple edges and no loops.

A digraph is said to be *acyclic* if it contains no directed cycle of edges. Our first result is:

Theorem 1. *The complex Δ_n^{ACY} of acyclic digraphs on n vertices is homotopy equivalent to the $(n-2)$-dimensional sphere.*

A digraph is *strongly connected* if for every pair $(i,j) \in \Omega$ there is a sequence (v_k, v_{k+1}), $k = 0, \ldots, f$ of edges in $E(G)$ such that $v_0 = i$ and $v_{f+1} = j$. Thus, the property of being *not* strongly connected is monotone.

Theorem 2. *The complex Δ_n^{NSC} of not strongly connected digraphs on n vertices is homotopy equivalent to a wedge of $(n-1)!$ spheres of dimension $2n-4$.*

The proofs of both theorems, to be given in Sections 2 and 3 respectively, rely on an analysis of certain properties of the lattice of all posets on the ground set $[n]$. For two such posets P and P', say that P is less than P' if $i < j$ in P implies that $i < j$ in P' for all $(i,j) \in \Omega$. This partial order, augmented by a top element, is a lattice whose proper part we denote by Pos_n. We show that Pos_n is homotopy

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equivalent to the $(n-2)$-sphere by relating it to the covering of the $(n-2)$-sphere given by open hemispheres of the braid arrangement.

In Sections 4 – 6 we comment on complexes of some other classes of digraphs, *viz.* directed matchings, nonspanning digraphs and arborescences. We also present some computations and conjectures and remark on the action of the symmetric group on the homology of the graph properties considered in Theorems 1 and 2.

The computer calculations presented in Sections 4 and 6 were done using a program written by Frank Heckenbach.

2. Acyclic digraphs and the lattice of all posets

We will use a number of tools that have become fairly standard in topological combinatorics. For convenience these are summarized in an appendix (Section 7), to which we also refer for explanation of notation and terminology.

For any digraph G on the node set $[n]$, let $\widetilde{G}$ denote its *transitive closure*. Thus, (i,j) is an edge of $\widetilde{G}$ if and only if there is a directed path in G from i to j. The mapping $G \to \widetilde{G}$ is a closure operator on the Boolean lattice of all subsets of Ω.

If G is acyclic then $\widetilde{G}$ is the comparability graph of a poset on $[n]$, and conversely the comparability graphs of posets are precisely the digraphs of the form $\widetilde{G}$ for acyclic digraphs G. Thus the mapping $G \to \widetilde{G}$ restricts to a closure operator on $\overline{L}(\Delta_n^{ACY})$ with image Pos_n.

Lemma 3. *Δ_n^{ACY} and $\Delta(\mathsf{Pos}_n)$ are homotopy equivalent.*

Proof. Since $\Delta_n^{ACY} \cong \Delta(\overline{L}(\Delta_n^{ACY}))$ (barycentric subdivision) this follows directly from the Closure Lemma 20. □

The poset Pos_n consists of all partially ordered sets on the set $[n]$, except for the antichain. As was mentioned, Pos_n is the proper part of a lattice that we denote $\mathsf{Pos}_n^{\#}$. Indeed, in $\mathsf{Pos}_n^{\#}$ the meet of two posets $(P, \leq_P)$ and $(Q, \leq_Q)$ is given by the poset $(R, \leq_R)$ defined by $x \leq_R y$ if and only if $x \leq_P y$ and $x \leq_Q y$. The lattice $\mathsf{Pos}_n^{\#}$ is graded with rank function $\mathrm{rank}(P)$ being the number of strict order relations in P, i.e. the number of P's edges as an acyclic digraph. The bottom element of $\mathsf{Pos}_n^{\#}$ is the antichain (i.e., the poset with empty order relation). The atoms of $\mathsf{Pos}_n^{\#}$ are the posets with exactly one comparability relation, and the coatoms are the $n!$ total orders on $[n]$. In particular, the length of $\mathsf{Pos}_n^{\#}$ is $\binom{n}{2}+1$. Moreover, every poset $P \in \mathsf{Pos}_n^{\#}$ is the join of the atoms below it (whose number equals $\mathrm{rank}(P)$) and the meet of the coatoms above it. The latter is due to the well known fact that P is the intersection of its linear extensions (these are precisely the coatoms above P). Figure 1 shows the poset of posets for $n=3$.

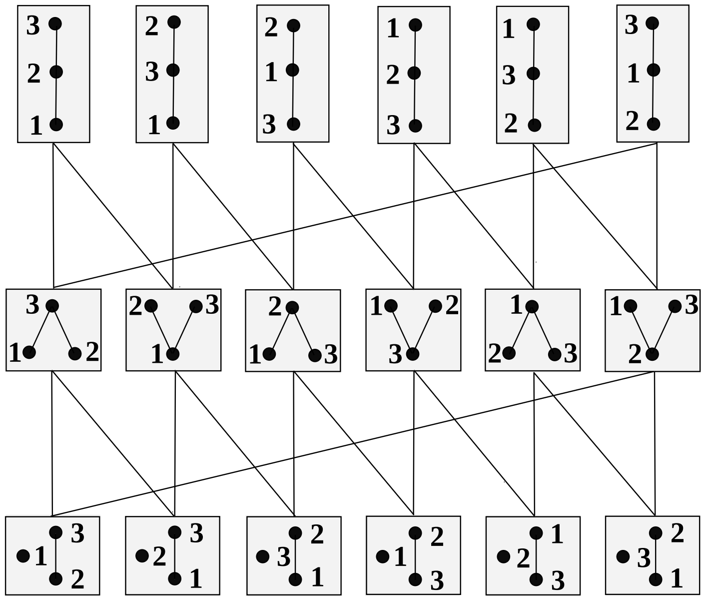

Figure 1. Pos_3.

Theorem 1 will follow from Lemma 3 and the following.

Theorem 4. $\Delta(\mathsf{Pos}_n)$ *is homotopy equivalent to the* $(n-2)$*-dimensional sphere.*

Proof. Let $X_n = \{\underline{\mathbf{x}} \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0 \text{ and } ||\underline{\mathbf{x}}|| = 1\}$. This is the unit sphere in an $(n-1)$-dimensional subspace of $\mathbb{R}^n$, hence X_n is an $(n-2)$-dimensional sphere.

With each atom $i < j$ of Pos_n we associate the open hemisphere $H_{i,j} = \{\underline{\mathbf{x}} \in X_n \mid x_i < x_j\}$. Let N denote the nerve of the covering of X_n by these hemispheres. Thus, N is the simplicial complex whose vertex set is the set A of atoms of Pos_n and whose simplices are the subsets of atoms corresponding to collections of hemispheres with nonempty intersection. By the Nerve Lemma (see e.g. [3, Th. 10.7]) $N \simeq X_n = S^{n-2}$.

It is easy to see that a collection of hemispheres $H_{i,j}$ has nonempty intersection if and only if the corresponding edges $i \to j$ determine an acyclic digraph. The transitive closure of this acyclic digraph is a poset. Therefore the nerve N equals the crosscut complex $\Gamma(\mathsf{Pos}_n^{\#}, A)$, and using the Crosscut Theorem 22 we get: $\Delta(\mathsf{Pos}_n) \simeq \Gamma(\mathsf{Pos}_n^{\#}, A) = N \simeq S^{n-2}$. □

In the remainder of this section we will study the structure of intervals in the lattice $\mathsf{Pos}_n^{\#}$. This has no immediate relevance for the topology of graph properties, so the rest of this section can be skipped without loss of continuity. Let us mention, however, that a different proof of Theorem 4 is given as a special case of the proof of Theorem 8 below.

We could not find any reference in the literature to the full lattice $\mathsf{Pos}_n^{\#}$ of all posets on $[n]$, although such a fundamental object must surely have been considered before. However, its lower intervals $[\hat{0}, P]$ for $P \neq \hat{1}$ have been studied by Edelman and Klingsberg [7] and others before them, see the references in [7]. A lattice of the form $[\hat{0}, P]$ is the lattice of all subposets of a given poset P. It is known that such lattices are *meet-distributive*, meaning that if x is the meet of all elements covered by y then the interval $[x, y]$ is Boolean. From this can be deduced that the order complex of each open interval (Q, P), $P \neq \hat{1}$, is either contractible or a sphere, and the Möbius function takes values $\mu(Q, P) \in \{0, \pm 1\}$ for $P \neq \hat{1}$.

We will extend this topological description of open intervals to *all* intervals of $\mathsf{Pos}_n^{\#}$, and make it more explicit in the known cases.

We call a poset $(P, \leq_P) \in \mathsf{Pos}_n^{\#}$ a *Coxeter poset* if it is the antichain (i.e., the bottom element $\hat{0}$) or if there is a point $(x_1, \ldots, x_n) \in X_n = \{\underline{\mathbf{x}} \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0 \text{ and } ||\underline{\mathbf{x}}|| = 1\}$ such that $i <_P j$ if and only if $x_i < x_j$. It follows that i and j are incomparable in P if and only if $x_i = x_j$. We say that the point $(x_1, \ldots, x_n)$ *realizes* P.

Lemma 5. *For a poset $(P, \leq_P) \in \mathsf{Pos}_n \cup \{\hat{0}\}$ the following are equivalent:*

(i) *P is a Coxeter poset.*
(ii) *No element in P is incomparable to both elements of some 2-element chain.*
(iii) *P is an ordinal sum of antichains.*

Proof. Assume i is incomparable to $j <_P k$. Then P is not a Coxeter poset, since otherwise $x_i = x_j < x_k = x_i$ for the coordinates of every point that realizes P.

We have shown that (i) $\Rightarrow$ (ii), and (ii) $\Rightarrow$ (iii) follows from an easy combinatorial argument.

Suppose that $P = A_1 \oplus A_2 \oplus \cdots \oplus A_k$, where the A_i's denote antichains on the respective blocks A_i of a partition of the set $[n]$. If $k = 1$ assertion (i) follows trivially. Assume $k \geq 2$. Define a point $\underline{\mathbf{y}} = \{y_1, \ldots, y_n\} \in \mathbb{R}^n$ by setting $y_i = j$ if $i \in A_j$, and let $N = \sum_1^n y_i$. Then let $\underline{\mathbf{z}}$ be the point obtained by subtracting N/n from each coordinate of $\underline{\mathbf{y}}$, and finally let $\underline{\mathbf{x}}$ be the vector $\underline{\mathbf{z}}$ normalized to length one. One easily checks that $\underline{\mathbf{x}} \in X_n$ and that $\underline{\mathbf{x}}$ realizes P. Hence, P is a Coxeter poset. □

For example, the Coxeter posets in Pos_3 are the twelve elements on the two top rank levels, see Figure 1.

In the sequel we denote by Cox_n the *Coxeter complex* of the symmetric group Σ_n. The Coxeter complex is constructed as follows. For each $p \in X_n$ let $\overline{p}$ be the intersections of all closed hemispheres of the form $\{\underline{\mathbf{x}} \in X_n \mid x_i \leq x_j\}$ that contain p. The sets of the form $\overline{p}$ are the simplices of a triangulation of the $(n-2)$-sphere X_n called the *Coxeter complex* Cox_n, see e.g. [5, Chapter 2.3]. We describe an inclusion map from the face poset of Cox_n to Pos_n.

Lemma 6. *The subposet of Pos_n consisting of all Coxeter posets is isomorphic to the proper part of the face lattice of the Coxeter complex Cox_n.*

Proof. This follows immediately from Lemma 5 and its proof. □

A Coxeter poset can easily be identified with a chain in the Boolean lattice B_n. Namely, if the poset is the ordinal sum $A_1 \oplus A_2 \oplus \cdots \oplus A_k$, where the A_i's denote antichains, the corresponding chain in B_n is $A_1, A_1 \cup A_2, A_1 \cup A_2 \cup A_3, \ldots$. Thus, the face lattice of Cox_n is isomorphic to the order complex of the proper part $\overline{B_n}$ of the Boolean lattice, or, which is the same, to the barycentric subdivision of the boundary of an $(n-1)$-simplex. This is a well known combinatorial description of Cox_n.

We will call a poset $(P, \leq_P)$ a *chainbreaker* for a poset $(Q, \leq_Q)$ if $P < Q$ and for at least one chain $i <_Q j <_Q k$ in Q the element i is incomparable to k in P.

The second part of the following lemma expresses the meet-distributivity of lower intervals, cf. [7].

Lemma 7. *Let $[P,Q]$ be an interval in $\mathsf{Pos}_n^{\#}$, with $Q \neq \hat{1}$. If P is a chainbreaker for Q then the open interval (P,Q) is contractible. Otherwise $[P,Q]$ is isomorphic to the Boolean lattice on a* $\mathrm{rank}(Q) - \mathrm{rank}(P)$ *element set.*

Proof. Let $i <_Q j <_Q k$ be a chain in Q such that i is incomparable to k in P. The order relation $i < k$ will be present in any poset covered by Q. In particular, in the meet of all coatoms of $[P,Q]$ we will have the order relation $i < k$. Thus P is not the meet of these coatoms. But this shows that the crosscut complex $\Gamma([P,Q]^*, A)$ of the dual lattice (i.e., all order relations reversed) of $[P,Q]$ is the full simplex over the set A of coatoms of $[P,Q]$. In particular, it is contractible.

Now assume that P is not a chainbreaker for Q. Let $i <_Q k$ be an order relation in Q such that i and k are incomparable in P. Let $(R, \leq_R)$ be the poset which inherits all order relations from Q except $i <_Q k$. We have to show that R is indeed a poset. Only the transitivity of the order relation is not completely trivial. But it follows from the observation that there are no chains $i <_Q j <_Q k$ in Q, since otherwise P would be a chainbreaker for Q. Thus we can remove an arbitrary order relation from $\mathrm{Rel}(Q) \setminus \mathrm{Rel}(P)$ and obtain a poset R covered by Q. Since P is not a chainbreaker for R the assertion follows by induction. □

We are now ready to formulate a theorem describing all intervals of $\mathsf{Pos}_n^{\#}$ up to homotopy type. The *height* of a poset P is by definition the maximal number k such that P contains some chain $y_0 < y_1 < \cdots < y_k$.

Theorem 8. *Let (P,Q) be an open interval in $\mathsf{Pos}_n^{\#}$.*

- (i) *If either $Q \neq \hat{1}$ and P is a chainbreaker for Q or $Q = \hat{1}$ and P is not a Coxeter poset, then the interval (P,Q) is contractible.*
- (ii) *If $Q \neq \hat{1}$ and P is not a chainbreaker for Q, then (P,Q) is homeomorphic to a sphere of dimension* $\mathrm{rank}(Q) - \mathrm{rank}(P) - 2$.
- (iii) *If $Q = \hat{1}$ and P is a Coxeter poset, then (P,Q) is homotopy equivalent to a sphere of dimension* $n - 2 - \mathrm{height}(P)$.

It follows as a special case of part (iii) that $\mathsf{Pos}_n = (\hat{0}, \hat{1})$ is homotopy equivalent to the $(n-2)$-dimensional sphere, so we obtain below a new proof for Theorem 4.

Proof. Lemma 7 settles the case $Q \neq \hat{1}$. It therefore remains to show the following for the case $Q = \hat{1}$:

- (a) If P is not a Coxeter poset, then the interval $(P, \hat{1})$ is contractible.
- (b) If P is a Coxeter poset, then $(P, \hat{1})$ is homotopy equivalent to a sphere of dimension $n - 2 - \mathrm{height}(P)$.

All assertions are trivial for $n = 1$, so we assume $n \geq 2$.

By Lemma 5 we know that if $(P, \leq_P)$ is not a Coxeter poset then there is an element j that is incomparable to a 2-element chain $i <_P k$. Now consider the poset $(R, \leq_R)$ which is the meet of all linear extensions of P such that $i <_P j <_P k$ (such extensions exist!). Clearly, $i <_R j <_R k$ in R and hence $(R, \leq_R) \neq (P, \leq_P)$. We claim that there is no poset $(S, \leq_S)$ in $(P, \hat{1})$ that complements R (i.e., such that the meet of S and R is P and their join is $\hat{1}$). Let us distinguish three cases:

- "$j <_S k$": Then j is smaller than k in the meet of S and R. In particular, the meet is not P.

- "$i <_S j$": Then j is larger than i in the meet of S and R. In particular, the meet is not P.
- "j is incomparable to $i <_S k$ in S": Then there is a linear extension $(T, \leq_T)$ of S such that $i \leq_T j \leq_T k$. But then $T \geq R$ and the join of S and R is not $\hat{1}$.

By Proposition 21 it follows that $(P, \hat{1})$ is contractible.

For part (b) we consider the inclusion map $f : \overline{L}(\mathsf{Cox}_n) \hookrightarrow \mathsf{Pos}_n$ from the face poset of Cox_n to Pos_n described in Lemma 6. We will use Quillen's Fiber Lemma 19. Let $(P, \leq_P)$ be a poset in Pos_n and set $Q := f^{-1}((\mathsf{Pos}_n)_{\geq P})$.

Claim. Q is contractible.

Let $R := \overline{L}(\mathsf{Cox}_n) \setminus Q$, and let

$$X_P = \Big\{(x_1, \ldots, x_n) \in X_n \,\Big|\, i <_P j \Rightarrow x_i < x_j\Big\}.$$

We have that X_P is an intersection of open hemispheres. From the fact that P is a poset it follows that X_P is non-empty. Thus X_P is contractible. On the other hand, $X \setminus X_P$ is triangulated by $\Delta(R)$, so via a retraction argument (see e.g. [5, Lemma 4.7.27]) $\Delta(Q)$ and X_P are homotopy equivalent.

Thus the Quillen Fiber Lemma 19 implies that for all P in the image of f the interval $(P, \hat{1})$ in $\mathsf{Pos}_n^{\#}$ is homotopy equivalent to the interval $(P, \hat{1})$ in $L(\mathsf{Cox}_n)$. An upper interval $(\sigma, \hat{1})$ in the face lattice of a simplicial complex is homeomorphic to the link of σ in the complex. Now, Cox_n is a PL triangulation of the $(n-2)$-sphere, so the link of a simplex of dimension c is homeomorphic to a sphere of dimension $n - 3 - c$. It is easily seen that the dimension of P as a simplex of Cox_n is $\mathrm{height}(P) - 1$. This completes the proof. □

Corollary 9. *The Möbius function of* $\mathsf{Pos}_n^{\#}$ *is for* $P < Q$ *given by:*

$$\mu(P,Q) = \begin{cases} 0 & \textit{if } Q \neq \hat{1} \textit{ and } P \textit{ is a chainbreaker for } Q, \\ (-1)^{\mathrm{rank}(Q)-\mathrm{rank}(P)} & \textit{if } Q \neq \hat{1} \textit{ and } P \textit{ is not a chainbreaker for } Q, \\ (-1)^{n-\mathrm{height}(P)} & \textit{if } Q = \hat{1} \textit{ and } P \textit{ is a Coxeter poset}, \\ 0 & \textit{if } Q = \hat{1} \textit{ and } P \textit{ is not a Coxeter poset}. \end{cases}$$

3. Not strongly connected digraphs

Let G be a digraph on the node set $[n]$. Say that two nodes u and v are G-equivalent if there is a directed path from u to v and a directed path from v to u. This is clearly an equivalence relation, and hence determines a partition of the set of nodes into equivalence classes B_i. We will denote this partition by $\pi(G) = |B_1| \cdots |B_k|$. The induced subgraphs G_{B_i} on the node sets B_i are called the *strongly connected components* of G.

We have defined an order-preserving map π from the set of all digraphs on [n] (ordered by inclusion of edge sets) to the partition lattice Π_n. Note that $\pi(G) = |1| \cdots |n|$ (the bottom element of Π_n) if and only if G is acyclic. Also, $\pi(G) = |1 \cdots n|$ (the top element of Π_n) if and only if G is strongly connected.

Let $\mathsf{Pos}_n \oplus \overline{\Pi_n}$ be the ordinal sum of Pos_n and the proper part $\overline{\Pi_n}$ of the partition lattice Π_n, and define a map $\varphi : \Delta_n^{NSC} \setminus \{\emptyset\} \to \mathsf{Pos}_n \oplus \overline{\Pi_n}$ by

$$\varphi : G \mapsto \begin{cases} \widetilde{G} \in \mathsf{Pos}_n & \text{if } G \text{ is acyclic}, \\ \pi(G) \in \overline{\Pi_n} & \text{otherwise}. \end{cases}$$

Thus, φ sends a not strongly connected digraph G to its transitive closure $\widetilde{G}$ in case $\pi(G)$ is the bottom element of Π_n and to the partition $\pi(G)$ otherwise. This mapping is clearly order-preserving, and from now on we think of it as a poset map defined on the face poset of Δ_n^{NSC}.

Lemma 10. *The poset mapping* $\varphi : \overline{L}(\Delta_n^{NSC}) \to \mathsf{Pos}_n \oplus \overline{\Pi_n}$ *induces homotopy equivalence of order complexes.*

Proof. We will use Quillen's Fiber Lemma 19. To simplify notation, let $Q := \mathsf{Pos}_n \oplus \overline{\Pi_n}$ for the duration of this proof. Let $q \in Q$. We have to show that the fiber $\varphi^{-1}(Q_{\leq q})$ is contractible. There are two cases to consider.

Case 1: $q \in \mathsf{Pos}_n$. Here the fiber $\varphi^{-1}(Q_{\leq q})$ has a unique greatest element, namely the comparability graph of the poset q. So its order complex is a cone and hence contractible.

Case 2: $q \in \overline{\Pi_n}$. We will use the $k = 2$ case of Lemma 25. Now q is a nontrivial partition of the node set $[n]$. Assume that B is a non-singleton block in this partition and choose two elements in B. Without loss of generality we may assume that the two elements are 1 and 2.

The fiber $\Delta := \varphi^{-1}(Q_{\leq q})$ is the subcomplex of Δ_n^{NSC} consisting of those digraphs G such that $\pi(G) \leq q$. Let $\Delta_1 \subseteq \Delta$ be the subcomplex consisting of those graphs in Δ that contain no directed path from 1 to 2 except possibly for the edge $(1, 2)$. The map $G \mapsto G\pm(2, 1)$ (defined in connection with Lemma 25) maps Δ_1 into itself. If $(2, 1) \in G$ this is clear. If $(2, 1) \notin G$ then $\pi(G\pm(2, 1))$ might differ from $\pi(G)$ by the merging of the blocks B_1 and B_2 containing 1 and 2. But then we still have that $\pi(G\pm(2, 1)) \leq q$, since B_1 and B_2 are both subsets of B. Similarly, $G \mapsto G\pm(1, 2)$ maps $\Delta \setminus \Delta_1$ into itself, because $\pi(G\pm(1, 2)) = \pi(G)$. Hence, by Lemma 25, Δ is contractible. □

The following result will together with Lemma 10 imply the truth of Theorem 2.

Lemma 11. $\Delta(\mathsf{Pos}_n \oplus \overline{\Pi_n})$ *has the homotopy type of a wedge of* $(n-1)!$ *spheres of dimension* $2n - 4$.

Proof. From the definition of ordinal sum follows that $\Delta(\mathsf{Pos}_n \oplus \overline{\Pi_n}) = \Delta(\mathsf{Pos}_n) * \Delta(\overline{\Pi_n})$, where "$*$" denotes join of simplicial complexes. It is a known fact that if Δ_1 is homotopy equivalent to a wedge of k_1 spheres of dimension d_1 and Δ_2 is homotopy equivalent to a wedge of k_2 spheres of dimension d_2, then their join $\Delta_1 * \Delta_2$ is homotopy equivalent to a wedge of $k_1 \cdot k_2$ spheres of dimension $d_1 + d_2 + 1$ (see for example [6, Lemma 2.5]). It is also well known that the proper part $\overline{\Pi_n}$ of the partition lattice has the homotopy type of a wedge of $(n-1)!$ spheres of dimension $n-3$, see e.g. [6]. Also by Theorem 4 we know that Pos_n is homotopy equivalent to an $(n-2)$-sphere. Hence $\Delta(\mathsf{Pos}_n \oplus \overline{\Pi_n})$ is homotopy equivalent to a wedge of $(n-1)!$ spheres of dimension $(n-2) + (n-3) + 1 = 2n - 4$. □

4. Directed matchings

An undirected graph is called a matching if the degree (number of incident edges) at every node is at most 1. Similarly, we define a *directed matching* to be a digraph for which both the in-degree and out-degree at every vertex is at most one. So, the components of a directed matching are either directed paths or directed cycles.

Let Δ_n^{DM} be the simplicial complex of directed matchings on the node set $[n]$. This complex, whose set of vertices is Ω, can be described differently as follows.

Let A be a finite set of squares on a large enough chessboard. The *chessboard complex* $\mathcal{C}_A$ is the simplicial complex whose vertex set is A and whose simplices are the subsets of A corresponding to nontaking rook positions (i.e., no two squares in the same row or in the same column). Such complexes have been studied in several papers, see [1, 14] and the further references given there.

The following is immediately clear by identifying the edges (i,j) of Ω with the corresponding squares (i,j) of the $n \times n$ chessboard.

Lemma 12. *Let A be the $n \times n$ chessboard minus one diagonal. Then $\mathcal{C}_A \cong \Delta_n^{DM}$.*

Using this lemma and a result of Ziegler [14] we can deduce this connectivity lower bound.

Theorem 13. *The complex Δ_n^{DM} is $(\lfloor \frac{2n+1}{3} \rfloor - 2)-$connected.*

Proof. This is a consequence of [14, Theorem 3.3]. That theorem says that if a chessboard A contains a certain "admissible k-shape" $\Sigma(m,n,k)$ then the $(k-1)$-skeleton of its complex $\mathcal{C}_A$ is vertex-decomposable of dimension $k-1$, and hence in particular is $(k-2)$-connected. Now, the $n \times n$ chessboard minus one diagonal contains an isomorphic copy of the admissible k-shape $\Sigma(n,n,k)$ for $k = \lfloor \frac{2n+1}{3} \rfloor$. $\square$

It is easy to see that the $(n-1)$-dimensional complex Δ_n^{DM} collapses to its $(n-2)$-skeleton. Hence we may deduce the following vanishing result:

$$\widetilde{H}_i(\Delta_n^{DM}) \neq 0 \quad \Longrightarrow \quad (\lfloor \frac{2n+1}{3} \rfloor - 1) \leq i \leq n-2.$$

These bounds are sharp for $n \leq 7$, as the following table shows:

$n\backslash i$	0	1	2	3	4	5	6
2	0	0	0	0	0	0	0
3	0	$\mathbb{Z}^2$	0	0	0	0	0
4	0	0	$\mathbb{Z}^4$	0	0	0	0
5	0	0	$\mathbb{Z}$	$\mathbb{Z}^{13}$	0	0	0
6	0	0	0	$\mathbb{Z}^{24} \oplus \mathbb{Z}_3^5$	$\mathbb{Z}^{32}$	0	0
7	0	0	0	0	$\mathbb{Z}^{415} \oplus \mathbb{Z}_3^{15}$	$\mathbb{Z}^{95}$	0

Table: Homology groups $\widetilde{H}_i(\Delta_n^{DM})$.

The only torsion appearing in the table is modulo 3. Thus the directed matching complexes Δ_n^{DM} seem to share the mysterious "torsion mod 3" phenomenon empirically observed for undirected matching complexes, see the discussion in [1, Section 9.1].

5. Nonspanning digraphs and arborescences

In this section we will first consider a digraph property that is defined with respect to a root node. Since only out-going edges from the root play a role, this introduces a small asymmetry into the ground sets of nodes and of edges. Thus we enlarge our standard ground sets $[n]$ to $[n]_0 = [n] \cup \{0\}$ and Ω (defined in Section 1) to $\Omega_0 = \Omega \cup \{(0,j) \mid 1 \leq j \leq n\}$. This means simply that we introduce 0 as the root node and add edges from 0 to all other nodes j.

A rooted digraph is called *spanning* (or, *initially connected*) if it contains a directed path from the root 0 to every other node i. The nonspanning digraphs form a simplicial complex on the vertex set Ω_0 that we denote Δ_n^{NS}.

Theorem 14. *The complex Δ_n^{NS} is contractible.*

Proof. Let $f : \overline{L}(\Delta_n^{NS}) \to B_n$ be the map that assigns to each digraph in $\overline{L}(\Delta_n^{NS})$ the subset of $[n]$ consisting of all nodes that can be reached from 0 on a directed path. Clearly, this map is order preserving and its image consists of all subsets of $[n]$ of cardinality $\leq n-1$. In particular, the order complex of $f(\overline{L}(\Delta_n^{NS}))$ is contractible, since it is a cone with the empty set as apex. Thus we are done once we show that f induces a homotopy equivalence. For this we apply the Quillen Fiber Lemma 19.

For $j = 0, 1, \ldots, n-1$, let $\Delta_j := \{G \in \Delta_n^{NS} | f(G) \subseteq [j]\}$. Thus, Δ_j consists of all digraphs such that only a subset of $[j] = \{1, \ldots, j\}$ can be reached along a directed path from 0. We have that $\Delta_0 \subseteq \Delta_1 \subseteq \cdots \subseteq \Delta_{n-1}$ and plan to use Lemma 25.

First observe that Δ_0 is the full simplex on the vertex set Ω, and hence is a cone. Then, notice that $G \mapsto G \pm (j+1, j)$ maps $\Delta_j \setminus \Delta_{j-1}$ into itself for all $1 \leq j \leq n-1$. Hence, by Lemma 25, every complex Δ_j is contractible.

The argument is now finished, since by symmetry the Quillen fiber $f^{-1}((B_n)_{\leq A})$ is isomorphic to $\Delta_{|A|}$ for every subset A of $[n]$ such that $|A| \leq n-1$. □

Another digraph property we will consider here is that of being an *arborescence* (or, *directed forest*). This means that each connected component is a directed tree, i.e., a tree having a node r such that every other node of the tree can be reached from r along a directed path. In other words, every component of an arborescence is a tree in which all edges are directed "away" from some particular node. Being an arborescence is clearly a hereditary property. We denote by Δ_n^{AR} the complex of arborescences. It is a simplicial complex on the vertex set Ω_0.

Relying on some results on greedoids by Björner, Korte and Lovász [4] we can deduce the following.

Theorem 15. *The complex Δ_n^{AR} is shellable and contractible.*

Proof. We will here freely use greedoid terminology. See [4] for all definitions.

Consider the branching greedoid G of the complete digraph Ω_0 rooted at 0. The feasible sets of G are the directed trees rooted at 0, and the bases are the maximal arborescences (by necessity rooted at 0). Hence, Δ_n^{AR} is the primal complex of G, which by [4, Theorem 5.8] is shellable and by [4, Theorem 5.9] is contractible. □

For $n \geq 4$ Theorem 14 can also be proved using methods from greedoid theory.

Greedoid Proof of Theorem 14. Let once more G be the the branching greedoid of the complete digraph Ω_0 rooted at 0. A digraph D is spanning if and only if D contains a basis of G, which comes to the same as saying that the complement $\Omega_0 \setminus D$ belongs to the dual complex $G^\perp$. Hence, Δ_n^{NS} is Alexander dual to $G^\perp$ in the sense of Proposition 23, which therefore gives:

$$\widetilde{H}_i(\Delta_n^{NS}) \cong \widetilde{H}^{n^2-i-3}(G^\perp).$$

By [4, Corollary 5.12] $G^\perp$ is homeomorphic to a ball. Hence, $\widetilde{H}_i(\Delta_n^{NS}) = 0$ for all i.

If $n \geq 4$ then Δ_n^{NS} has a complete 2-skeleton, so Δ_n^{NS} is simply-connected in this case and therefore contractible. □

Remark 16. The complex of arborescences has been studied also by Kozlov [8]. He proves shellability both for Ω_0 (as in Theorem 15) and for the smaller ground set Ω.

Remark 17. The argument by Alexander duality in the last proof can be extended to all greedoids. Let G be a rank r greedoid on a ground set of cardinality m. By [4, Theorem 5.1] the dual complex $G^\perp$ is shellable. Since it is also pure and of dimension $m-r-1$ it follows that $\widetilde{H}^j(G^\perp) = 0$ for all $j \neq m-r-1$ and $\widetilde{H}^{m-r-1}(G^\perp)$ is free. The complex NS of nonspanning sets of G, which is Alexander dual to $G^\perp$, therefore satisfies $\widetilde{H}_i(NS) = 0$ for all $i \neq m-(m-r-1)-3 = r-2$, and $\widetilde{H}_{r-2}(NS)$ is free. If $r \geq 4$ then NS has a complete 2-skeleton, and is therefore simply-connected. From these facts we may conclude (cf. [3, (9.18)]) that NS has the homotopy type of a wedge of $(r-2)$-spheres.

A particular case of this is given by the complex of disconnected undirected graphs on the vertex set $[n]$, denoted by Δ_n^1 in [1]. This is the complex of nonspanning sets in the circuit matroid of the complete graph K_n, a greedoid of rank $n-1$. It is well known that Δ_n^1 has nonvanishing homology only in dimension $n-3$, see e.g. [1, Proposition 2.1].

6. Final remarks

6.1. **Strongly i-connected digraphs.** Following the program pursued in [1] one can extend the considerations of Section 3 to the complex $\Delta_n^{NSC,i}$ of not strongly i-connected digraphs on the node set $[n]$. We will say that a digraph $(V(G), E(G))$ is *strongly i-connected*, for a number $0 < i < |V(G)|$, if for any j vertices $v_1, \ldots, v_j$, $j < i$, the digraph that is obtained from G by removing $v_1, \ldots, v_j$ and all incident edges is strongly connected. In particular, $\Delta_n^{NSC} = \Delta_n^{NSC,1}$. Computer calculations for $i = 2$ yield the following homology groups (no non-vanishing homology occurs for j not in the table).

$n\backslash j$	0	1	2	3	4	5	6	7	8	9	10
3	0	0	0	0	$\mathbb{Z}$	0	0	0	0	0	0
4	0	0	0	0	0	0	0	$\mathbb{Z}^4$	0	0	0
5	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}^{18}$

Table: Homology groups $\widetilde{H}_j(\Delta_n^{NSC,2})$

Based on this table one is led to suspect the following.:

Conjecture 18. *$\Delta_n^{NSC,2}$ is homotopy equivalent to a wedge of $(n-2)!(n-2)$ spheres of dimension $3n-5$.*

Now we consider situations when i is close to n. A digraph is not strongly $(n-1)$-connected if there is a pair of vertices such that the subgraph induced on these vertices is not strongly connected. But this is equivalent to the condition that the graph is not the complete digraph on $[n]$ vertices. Thus, $\Delta_n^{NSC,n-1}$ is the full boundary of an (n^2-n-1)-simplex, and hence $\Delta_n^{NSC,n-1} \cong S^{n^2-n-2}$.

In order to study $\Delta_n^{NSC,n-2}$ we consider the dual complex in the sense of Proposition 23. It is easily derived from the definitions that a digraph lies in $(\Delta_n^{NSC,n-2})^*$ if and only if the out-degree and in-degree at each node is at most 1. Hence, by Proposition 23

$$\widetilde{H}_i(\Delta_n^{NSC,n-2}) \cong \widetilde{H}^{n^2-n-i-3}(\Delta_n^{DM}),$$

and the results of Section 4 apply.

6.2. **Action of the symmetric group.** For each monotone property P of digraphs on n nodes, the complex Δ_n^P is invariant under the action of the symmetric group Σ_n by permutations of the node set n. This action induces a Σ_n-representation on each homology group $\widetilde{H}_i(\Delta_n^P;\mathbb{C})$. For Δ_n^{ACY} and Δ_n^{NSC} we can describe these representations.

Essentially, what has to be done is to keep track that all the homotopy equivalences established in Sections 2 and 3 are compatible with the group action. This can be done using an equivariant version of the Quillen Fiber Lemma due to Thévenaz and Webb [12] and the equivariant versions of Crosscut Theorem and Homotopy Complementation that can be derived from it. Using this strategy the following results can be obtained:

- The complex Δ_n^{ACY} is Σ_n-homotopy equivalent to the Coxeter complex Cox_n.
- The complex Δ_n^{NSC} is Σ_n-homotopy equivalent to the join of Cox_n and $\Delta(\overline{\Pi_n})$.

It is easily seen and well known that on the single non-vanishing homology group $\widetilde{H}_{n-2}(\mathsf{Cox}_n;\mathbb{C})$ of Cox_n the group Σ_n acts by the sign-character $\mathbf{sign}_n$. By results of Stanley [11] the character of Σ_n on the only non-vanishing homology group $\widetilde{H}_{n-3}(\Delta(\overline{\Pi_n});\mathbb{C})$ of $\Delta(\overline{\Pi_n})$ is given by $\mathbf{sign}_n \cdot \mathbf{lie}_n$. Here $\mathbf{lie}_n := \mathbf{e}^{\frac{2\pi i}{n}} \uparrow_{\mathbf{C_n}}^{\Sigma_n}$, where C_n denotes the cyclic group of order n generated by an n-cycle and $\mathbf{e}^{\frac{2\pi i}{n}}$ the character of C_n which assumes the value $e^{\frac{2\pi i}{n}}$ on a generator of the group. Using these two facts and the Σ_n-homotopy equivalences stated above we immediately obtain:

- The character of Σ_n on $\widetilde{H}_{n-2}(\Delta_n^{ACY};\mathbb{C})$ is the sign-character.
- The character of Σ_n on $\widetilde{H}_{2n-4}(\Delta_n^{NSC};\mathbb{C})$ is the character $\mathbf{lie}_n$.

We remark that $\mathbf{lie}_n$ is also the character of Σ_n on the multigraded part of the free Lie algebra on n generators (see e.g. [10]).

7. Appendix: Notation and Tools

In this section we will summarize the main tools used. We refer the reader to the survey paper [3] for more details and references.

Let P be a finite partially ordered set – *poset* for short. If P has a unique minimum element $\hat{0}$ and a unique maximum element $\hat{1}$, we denote by $\overline{P}$ the *proper part* of P, that is the poset obtained by removing from P the elements $\hat{0}$ and $\hat{1}$. By $\Delta(P)$ we denote the simplicial complex of all chains in P, called the *order complex* of P. Via the functor $\Delta(\cdot)$ one can speak of the homology and homotopy type of a poset P.

By convention we include the empty set $\emptyset$ in every simplicial complex. For any simplicial complex Δ, the *face lattice* $L(\Delta)$ is the poset of faces of Δ, ordered by inclusion and enlarged by an additional greatest element $\hat{1}$. The proper part $\overline{L}(\Delta) = L(\Delta) \setminus \{\emptyset, \hat{1}\}$ is called the *face poset*. The order complex $\Delta(\overline{L}(\Delta))$ is homeomorphic to Δ — indeed, $\Delta(\overline{L}(\Delta))$ is the barycentric subdivision of Δ.

The *ordinal sum* $P \oplus Q$ of two posets P and Q is the poset on their set union for which the order relation on pairs of elements of P (resp. Q) is inherited, and each element of P is defined to be less than each element of Q. This operation on posets is associative, so repeated ordinal sums $P_1 \oplus P_2 \oplus \cdots \oplus P_k$ are well defined.

For a poset P and $p \in P$ we denote by $P_{\leq p}$ the sub-poset $\{p' \mid p' \in P;\ p' \leq p\}$, and similarly for $P_{\geq p}$. For $p \leq p'$ in P we denote by $[p, p']$ the *closed interval*

$P_{\geq p} \cap P_{\leq p'}$, and by (p, p') the *open interval* $[p, p'] \setminus \{p, p'\}$. By a map $f : P \to Q$ of posets we always mean a poset homomorphism (i.e., $x \leq y$ implies $f(x) \leq f(y)$).

Proposition 19 (Quillen Fiber Lemma [9] [3, Th. 10.5])**.** *Let $f : P \to Q$ be a map of posets. Assume that $f^{-1}(Q_{\leq q})$ is contractible for all $q \in Q$. Then $\Delta(P)$ and $\Delta(Q)$ are homotopy equivalent.*

A map $f : P \to P$ from a poset to itself is called a *closure operator* if $f(x) \geq x$ and $f(f(x)) = f(x)$ for all $x \in P$. The Quillen Fiber Lemma immediately implies the following fact.

Corollary 20 (Closure Lemma)**.** *Let $f : P \to P$ be a closure operator on the partially ordered set P. Then $\Delta(P)$ and $\Delta(f(P))$ are homotopy equivalent.*

A poset L is called a *lattice* if suprema, denoted by "$\vee$", and infima, denoted by "$\wedge$", exist. Note that if L is a finite lattice then there is a least element $\hat{0}$ and a greatest element $\hat{1}$ in L. The elements covering $\hat{0}$ are called *atoms* and the elements covered by $\hat{1}$ are called *coatoms*. In a lattice an element x is called *complement* of the element y if $x \vee y = \hat{1}$ and $x \wedge y = \hat{0}$.

Proposition 21 (Homotopy Complementation [2] [3, Th. 10.15])**.** *If some element of a lattice L has no complement then $\Delta(\overline{L})$ is contractible.*

Let A be the set of atoms of a lattice L. Define the *crosscut complex* $\Gamma(L, A)$ to be the simplicial complex on the vertex set A whose simplices are the subsets $S \subseteq A$ such that $\vee S < \hat{1}$.

Proposition 22 (Crosscut Theorem [2] [3, Th. 10.8])**.** *The complexes $\Gamma(L, A)$ and $\Delta(\overline{L})$ are homotopy equivalent.*

Our next tool is the combinatorial version of a standard duality theorem from algebraic topology. See e.g. [1, Prop. 10.4] for a proof.

Proposition 23 (Combinatorial Alexander Duality)**.** *Let Δ be a finite simplicial complex on vertex set V and define*

$$\Delta^* = \{B \subseteq V \mid V \setminus B \notin \Delta\}.$$

Then

$$\widetilde{H}_i(\Delta) \cong \widetilde{H}^{|V|-i-3}(\Delta^*).$$

Finally, we make use of the following simple collapsing argument. Assume there is a fixed ground set V and for $F \subseteq V$ and $a \in V$ define the operation

$$F \pm a = \begin{cases} F \cup \{a\} & \text{if } a \notin F, \\ F \setminus \{a\} & \text{if } a \in F. \end{cases}$$

Lemma 24. *Let $\Delta_1 \subseteq \Delta_2$ be simplicial complexes. Assume there exists some vertex a such that $F \mapsto F \pm a$ maps $\Delta_2 \setminus \Delta_1$ into itself. Then Δ_2 collapses to Δ_1.*

Proof. Take a pair $(F, F \pm a)$, $a \notin F \in \Delta_2 \setminus \Delta_1$, of maximal dimension among all such pairs. Suppose that F is not a free face. Then $F \cup \{b\} \in \Delta_2 \setminus \Delta_1$ for some $b \notin F \cup \{a\}$. But since $a \notin F \cup \{b\}$ then also $F \cup \{b\} \cup \{a\} \in \Delta_2 \setminus \Delta_1$, contradicting the choice by maximal dimension. Hence, F is a free face, so the removal of $\{F, F \pm a\}$ is an elementary collapse step. Now continue by induction. □

The following generalization of the concept of a cone (the $k = 1$ case) is an immediate consequence.

Lemma 25. *Let* $\Delta_1 \subseteq \Delta_2 \cdots \subseteq \Delta_k = \Delta$ *be simplicial complexes, and put* $\Delta_0 = \emptyset$. *Assume there exist vertices* $a_1, a_2, \ldots, a_k$ *such that* $F \mapsto F \pm a_i$ *maps* $\Delta_i \setminus \Delta_{i-1}$ *into itself, for* $i = 1, \ldots, k$. *Then* Δ *is collapsible.*

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