

# SEGRE AND REES PRODUCTS OF POSETS, WITH RING-THEORETIC APPLICATIONS

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**ABSTRACT.** We introduce (weighted) Segre and Rees products for posets and show that these constructions preserve the Cohen-Macaulay property over a field  $k$  and homotopically. As an application we show that the weighted Segre product of two affine semigroup rings that are Koszul is again Koszul. This result generalizes previous results by Crona on weighted Segre products of polynomial rings.

We also give a new proof of the fact that the Rees ring of a Koszul affine semigroup ring is again Koszul.

The paper ends with a list of some open problems in the area.

## 1. INTRODUCTION

We describe constructions of finite partially ordered sets (posets for short) that generalize situations arising in commutative algebra to a combinatorial setting. For all constructions the principal question asked is: “*Does this construction preserve the Cohen-Macaulay property ?*” The posets that are relevant for the commutative algebra situation are those that occur as intervals in affine semigroup posets. A result of Peeva, Reiner and Sturmfels [16] shows that an answer to our principal question for the class of intervals in affine semigroup posets will give a corresponding answer to the question whether certain ring theoretic constructions preserve the Koszul property.

The ring-theoretic constructions that motivate this study are weighted Segre products (see [11]) and Rees algebras. We define poset-theoretic analogues of these constructions and prove that the Cohen-Macaulay property is preserved. As corollaries we obtain that weighted Segre products of affine semigroup rings preserve the Koszul property.

The following theorem and its corollaries are our main results. Further definitions and background is given in Section 2.

**Segre products of posets:** Let  $f : P \rightarrow S$  and  $g : Q \rightarrow S$  be poset maps. Let  $P \circ_{f,g} Q$  be the induced subposet of the product poset  $P \times Q$  consisting of the pairs  $(p, q) \in P \times Q$  such that  $f(p) = g(q)$ . Recall that the product poset  $P \times Q$  is ordered by  $(p, q) \leq (p', q')$  if  $p \leq p'$  and  $q \leq q'$ . In the language of category theory the poset  $P \circ_{f,g} Q$  is the pullback of  $f : P \rightarrow S \leftarrow Q : g$ .

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We will be concerned only with the case when  $S = \mathbb{N} := \{0, 1, 2, \dots\}$  is the set of natural numbers equipped with its natural order. For a pure poset  $P$  the rank function serves as an example of a poset map from  $P$  to  $\mathbb{N}$ . In case  $P$  is a pure poset with rank function  $f = \text{rk}$  we write  $P \circ_g Q$  for  $P \circ_{\text{rk}, g} Q$  and call  $P \circ_g Q$  the *Segre product* of  $P$  and  $Q$  with respect to  $g$  (or, the  *$g$ -weighted Segre product* of  $P$  and  $Q$ ).

**Theorem 1.** *Let  $P$  and  $Q$  be pure posets. Let  $\text{rk} : P \rightarrow \mathbb{N}$  be the rank function of  $P$  and  $g : Q \rightarrow \mathbb{N}$  be a strict poset map such that  $g(Q) \subseteq \text{rk}(P)$ . If  $P$  and  $Q$  are Cohen-Macaulay over the field  $k$ , then the Segre product  $P \circ_g Q$  is Cohen-Macaulay over  $k$ . If  $P$  and  $Q$  are homotopically Cohen-Macaulay, then so is  $P \circ_g Q$ .*

Theorem 1 is a special case of a more general result for simplicial complexes, see Theorem 8 below.

**Rees products of posets:** Let  $P$  and  $Q$  be pure posets with rank functions  $\text{rk}$ . Let  $P * Q$  be the poset on the ground set  $\{(p, q) \in P \times Q \mid \text{rk}(p) \geq \text{rk}(q)\}$  with order relation

$$(p, q) \leq (p', q') \stackrel{\text{def}}{\iff} p \leq p', q \leq q' \text{ and } \text{rk}(p') - \text{rk}(p) \geq \text{rk}(q') - \text{rk}(q).$$

We call  $P * Q$  the *Rees product* of  $P$  and  $Q$ . Note that it is not in general an induced subposet of the product  $P \times Q$ . However, as will be shown, it is nevertheless a special case of the Segre product. Thus, using Theorem 1 we prove

**Corollary 2.** *Let  $P$  and  $Q$  be pure posets. If  $P$  and  $Q$  are Cohen-Macaulay over the field  $k$  and  $Q$  is acyclic over  $k$ , then the Rees product  $P * Q$  is Cohen-Macaulay over  $k$ . If  $P$  and  $Q$  are homotopically Cohen-Macaulay and  $Q$  is contractible, then  $P * Q$  is homotopically Cohen-Macaulay.*

**Affine semigroup rings:** Theorem 1 has the following ring-theoretic consequence, explained and further discussed in Section 4.

**Corollary 3.** *Let  $\Lambda \subseteq \mathbb{N}^d$  and  $\Gamma \subseteq \mathbb{N}^e$  be two homogeneous affine semigroups. Assume that the semigroup rings  $k[\Lambda]$  and  $k[\Gamma]$  are Koszul. Let  $g$  be a grading of  $k[\Gamma]$ . Then the weighted Segre product  $k[\Lambda] \circ_g k[\Gamma]$  with respect to  $g$  is Koszul.*

Similarly, Theorem 2 has the following consequence, which can also be deduced by ring-theoretic arguments from a result of Backelin and Fröberg [2, Proposition 3].

**Corollary 4.** *Let  $k[\Lambda]$  and  $k[\Gamma]$  be Koszul affine semigroup rings. Let  $k[\Lambda]_i$  and  $k[\Gamma]_i$  denote their  $i$ -th graded components and set  $\mathfrak{m}_\Lambda = \bigoplus_{i \geq 1} k[\Lambda]_i$ . Then the  $k$ -algebra*

$$k[\Lambda] * k[\Gamma] = \bigoplus_{i \geq 0} \mathfrak{m}_\Lambda^i \otimes_k k[\Gamma]_i$$

*is Koszul.*

Note that for  $k[\Gamma] = k[t]$  the Rees product  $k[\Lambda] * k[\Gamma]$  is the Rees ring  $\mathcal{R}[k[\Lambda], \mathfrak{m}_\Lambda]$  of  $k[\Lambda]$  with respect to its maximal ideal  $\mathfrak{m}_\Lambda$ .

2. TOOLS FROM TOPOLOGICAL COMBINATORICS

We begin with a review of some basic definitions.

A *chain*  $C$  in a poset  $P$  is a linearly ordered subset, its *length*  $\ell(C)$  is one less than its number of elements. A poset  $P$  is *pure* if all maximal chains have the same length. For each element  $p \in P$  of a pure poset  $P$  the length of a maximal chain in  $P_{\leq p} := \{q \in P \mid q \leq p\}$  is called the *rank*  $\text{rk}(p)$  of  $p$  in  $P$ .

A poset  $P$  is called *bounded* if there is a unique minimal element  $\hat{0}$  and a unique maximal element  $\hat{1}$  in  $P$ . For two elements  $x \leq y$  in  $P$  we write  $[x, y]$  for the *closed interval*  $\{z \mid x \leq z \leq y\}$  in  $P$ , and similarly  $(x, y)$  for the *open interval*  $\{z \mid x < z < y\}$ . Clearly,  $[x, y]$  is a bounded poset. A poset  $P$  is called *graded* if it is both bounded and pure. Let  $\hat{P} := P \cup \{\hat{0}, \hat{1}\}$  denote  $P$  augmented by new bottom and top elements  $\hat{0}$  and  $\hat{1}$ . Thus,  $\hat{P}$  is graded iff  $P$  is pure.

A map  $f : P \rightarrow Q$  is a *poset map* [resp. a *strict poset map*] if  $x < y$  implies  $f(x) \leq f(y)$  [resp.  $f(x) < f(y)$ ] for all  $x, y \in P$ .

We write  $\Delta(P)$  for the simplicial complex of all chains of  $P$ . By  $\tilde{H}_i(P; k)$  we denote the  $i$ -th reduced simplicial homology group of  $\Delta(P)$  with coefficients in  $k$ . Also, if convenient we identify  $P$  with the geometric realization of  $\Delta(P)$ .

A poset  $P$  is called *Cohen-Macaulay* over the field  $k$  if for all  $x < y$  in  $\hat{P}$  the reduced simplicial homology  $\tilde{H}_i((x, y); k)$  vanishes for  $i \neq \text{rk}(y) - \text{rk}(x) - 2$ . A poset  $P$  is called *homotopically Cohen-Macaulay* if for all  $x < y$  in  $\hat{P}$  the interval  $(x, y)$  is homotopy equivalent to a wedge of spheres of dimension  $\text{rk}(y) - \text{rk}(x) - 2$ . Cohen-Macaulay posets are pure.

The Cohen-Macaulay-over-a-field- $k$  property for posets is a special case of a property defined for all finite simplicial complexes. This general notion of Cohen-Macaulayness is in turn equivalent to a particular instance of the ring-theoretic Cohen-Macaulay concept. For this connection with Commutative Algebra, see Stanley [20].

The construction of Segre products of posets generalizes a well known concept; namely, Segre products of posets subsume rank selection of posets. For a pure poset  $P$  with rank function  $\text{rk}$ , let  $S \subseteq \text{rk}(P)$  be a set of ranks of the poset  $P$ . The *rank-selected subposet* of  $P$  determined by  $S$  is the induced subposet  $P_S$  of all elements  $x$  of  $P$  such that  $\text{rk}(x) \in S$ .

In the late 1970's, rank selection was shown to preserve Cohen-Macaulayness over  $k$  by Baclawski, Munkres, Stanley and Walker. See e.g. [3]. Complete references are given in [4, p. 1858], where also a proof of the homotopy version is sketched.

**Proposition 5.** [3, Thm. 6.4] [4, Thm. 11.13] *Let  $P$  be a Cohen-Macaulay poset over the field  $k$  and let  $S \subseteq \text{rk}(P)$ . Then the rank-selected subposet  $P_S$  is Cohen-Macaulay over  $k$ . If  $P$  is homotopically Cohen-Macaulay, then so is  $P_S$ .*

In order to realize rank selection as a Segre product, let  $Q$  be the chain on  $|S|$  elements  $\{1, \dots, r\}$ . Let  $S = \{s_1 < \dots < s_r\}$  and let  $g$  be the map that sends  $i \in Q$  to  $s_i$ . Then it is easily seen that  $P \circ_g Q \cong P_S$ . Unfortunately, Theorem 1 does not give a new proof of Proposition 5, but rather uses this fact as an essential point in the argumentation.

As a second tool we need another result, which is due to Baclawski [3] for Cohen-Macaulayness over  $k$  and to Quillen [17] for homotopical Cohen-Macaulayness. Both versions are proved in slightly greater generality in [6].

**Proposition 6.** [3, Thm. 5.2], [17, Cor. 9.7] *Let  $P$  and  $Q$  be pure posets and  $f : P \rightarrow Q$  a rank-preserving and surjective poset map. Assume that for all  $q \in Q$  the fiber  $f^{-1}(Q_{\leq q})$  is Cohen-Macaulay over  $k$ . If  $Q$  is Cohen-Macaulay over  $k$ , then so is also  $P$ . The same is true with “Cohen-Macaulay over  $k$ ” everywhere replaced by “homotopically Cohen-Macaulay”.*

We also need the following result on barycentric subdivisions, which can be obtained from the fact that Cohen-Macaulayness is invariant under homeomorphisms.

First recall a few definitions. If  $\Gamma$  is a simplicial complex then we can consider  $\Gamma$  as a poset, namely as the partially ordered set of its faces ordered by inclusion. Denote by  $\mathcal{F}(\Gamma)$  this *face poset*. The simplicial complex  $\Delta(\mathcal{F}(\Gamma))$  is called the *barycentric subdivision* of  $\Gamma$  and is well known to be homeomorphic to  $\Gamma$ .

**Proposition 7.** *A poset  $P$  is Cohen-Macaulay over  $k$  (resp. homotopically Cohen-Macaulay) if and only if the poset  $\mathcal{F}(\Delta(P))$  has the same property.*

### 3. PROOFS AND COMMENTS

In this section we prove the main poset theoretic theorems and discuss some related questions.

The  $g$ -weighted Segre product  $P \circ_g Q$  of two pure posets  $P$  and  $Q$  was defined in Section 1. Note that  $P \circ_g Q$  is also pure, and that  $\text{rk}_{P \circ_g Q}(p, q) = \text{rk}_Q(q)$ , for all  $(p, q) \in P \circ_g Q$ . In particular,  $\text{rk}(P \circ_g Q) = \text{rk}(Q)$ .

Proof of Theorem 1. Let

$$f : \mathcal{F}(\Delta(P \circ_g Q)) \rightarrow \mathcal{F}(\Delta(Q))$$

be the poset map that sends each chain  $(p_0, q_0) < \cdots < (p_\ell, q_\ell)$  to its projection  $q_0 < \cdots < q_\ell$ . This map is surjective and rank-preserving. For an element  $c = (q_0 < \cdots < q_\ell)$  in  $\mathcal{F}(\Delta(Q))$ , the fiber  $f^{-1}(\mathcal{F}(\Delta(Q))_{\leq c})$  consists of all subchains of chains  $(p_0, q_0) < \cdots < (p_\ell, q_\ell)$  for which  $\text{rk}(p_i) = g(q_i)$  for all  $i$ . Setting  $S = \{g(q_0), \dots, g(q_\ell)\}$ , then clearly  $f^{-1}(\mathcal{F}(\Delta(Q))_{\leq c})$  is isomorphic to  $\mathcal{F}(\Delta(P_S))$ .

By Proposition 5 we know that  $P_S$  is Cohen-Macaulay over  $k$  (resp. homotopically Cohen-Macaulay), since  $P$  is. Also, Proposition 7 shows that since  $P_S$  is Cohen-Macaulay over  $k$  (resp. homotopically Cohen-Macaulay) then so is also  $\mathcal{F}(\Delta(P_S))$ . Hence, we get from Proposition 6 that  $\mathcal{F}(\Delta(P \circ_g Q))$  is Cohen-Macaulay. The assertion now follows via Proposition 7.  $\square$

We don't see any reasonable way to go beyond Theorem 1 in its poset version. Consider these obstacles:

- If  $g(Q) \subseteq \text{rk}(P)$  is not required, then if  $P$  is a chain we can realize arbitrary lower order ideals in  $Q$  as Segre products  $P \circ_g Q$ .
- If  $g$  is not *strict*, then a counterexample to the conclusion of the theorem can be constructed as follows. Let  $P$  be a two element antichain, let  $Q = \{x < y\}$  be a two element chain, and let  $g(x) = g(y) = 0$ . Then  $P \circ_g Q$  is the disjoint union of two chains of length 1, and hence the poset is not Cohen-Macaulay.

However, there is a rather straight-forward generalization of the Segre product to simplicial complexes.

Let  $\Gamma_1$  and  $\Gamma_2$  be simplicial complexes on vertex sets  $V_1$  resp.  $V_2$ , with  $\dim \Gamma_2 \leq \dim \Gamma_1 = d - 1$ . Assume that there are maps  $g_i : V_i \rightarrow \{1, \dots, d\}$  such that:

- (i)  $g_1$  restricts to a bijection on each maximal face of  $\Gamma_1$ ,
- (ii)  $g_2$  is injective on each maximal face of  $\Gamma_2$ .

Define a simplicial complex  $\Gamma_1 \circ_{g_1, g_2} \Gamma_2$  on the vertex set  $V_1 \times V_2$  as having faces

$$\{(x_1, y_1), \dots, (x_k, y_k)\}$$

for all  $\{x_1, \dots, x_k\} \in \Gamma_1$  and  $\{y_1, \dots, y_k\} \in \Gamma_2$  such that  $g_1(x_i) = g_2(y_i)$  for all  $i$ .

**Theorem 8.** *If  $\Gamma_1$  and  $\Gamma_2$  are Cohen-Macaulay over  $k$  (resp. homotopically Cohen-Macaulay), then so is  $\Gamma_1 \circ_{g_1, g_2} \Gamma_2$ .*

*Proof.* Essentially the same proof as for Theorem 1 goes through. Instead of “rank-selected subposets” one has here to use “type-selected subcomplexes”, for which the preservation of Cohen-Macaulayness is also known, see [4, p. 1858].  $\square$

In our opinion, even the following specialization of Theorem 1, to what might be called “unmixed Segre products”, is somewhat unexpected from the combinatorial point of view.

**Corollary 9.** *Let  $P$  and  $Q$  be pure posets and let  $\text{rk}$  denote the rank function for either poset. If  $P$  and  $Q$  are Cohen-Macaulay over  $k$  then the poset  $P \circ_{\text{rk}} Q = \{(p, q) \mid \text{rk}(p) = \text{rk}(q)\}$  is Cohen-Macaulay over  $k$ . If  $P$  and  $Q$  are homotopically Cohen-Macaulay, then so is  $P \circ_{\text{rk}} Q$ .*

**Example 10.** Let  $M_n$  denote the poset of all minors (square submatrices) of an  $n \times n$  matrix. As a special case of Corollary 9 one sees that this *poset of minors* is Cohen-Macaulay. Namely, if  $B_n$  denotes the Boolean lattice of all subsets of  $[n] := \{1, \dots, n\}$ , then  $M_n$  is clearly isomorphic to the *Segre square*  $B_n \circ_{\text{rk}} B_n = \{(A, B) \mid A, B \subseteq [n], |A| = |B|\} \subseteq B_n \times B_n$ . Such Segre powers (of infinite posets) previously appeared in the work of Stanley, see [18, Example 1.2].

The number of  $(n-2)$ -spheres in the wedge giving the homotopy type of  $\Delta(M_n \setminus \{\hat{0}, \hat{1}\})$ , or equivalently  $(-1)^n \mu(\hat{0}, \hat{1})$  where  $\mu(\hat{0}, \hat{1})$  is the value of the Möbius function over  $M_n$ , is equal to the number of pairs of permutations of  $[n]$  having no common ascent. This set of permutation-pairs is well studied (see [8]). In [8] one can find a recurrence relation for these numbers which is exactly the defining relation for the Möbius number of the Segre square of  $B_n$ .

A second way to obtain this enumerative result is via the theory of lexicographic shellability [5]. A natural labeling rule for  $B_n \circ_{\text{rk}} B_n$  is to give a covering  $(A_1, B_1) \subset (A_2, B_2)$  the label  $(a, b)$ , where  $a$  and  $b$  are the unique elements of  $A_2 - A_1$  and  $B_2 - B_1$ , respectively. This is clearly an EL-labeling, and the falling chains are labeled by pairs of permutations with no common ascent.

A third approach is via the rank-selected  $\alpha$ - and  $\beta$ -invariants  $\alpha_J$  and  $\beta_J$  of  $B_n$ , as defined by Stanley [19, p.131]. One gets that

$$(-1)^n \mu(\hat{0}, \hat{1}) = \sum_{J \subseteq [n-1]} \alpha_J \beta_J.$$

This expression for  $\mu(\hat{0}, \hat{1})$  of  $B_n \circ_{\text{rk}} B_n$  follows from [6, Theorem 5.1 (iii)], and is more generally true for Segre squares of all Gorenstein\* (i.e., Cohen-Macaulay and Eulerian) posets. Since the Boolean lattice is lexicographically shellable there is a simple interpretation of  $\alpha_J$  and  $\beta_J$ . For lexicographically shellable posets  $\beta_J$  counts the number of maximal chains whose descent set is equal to  $J$  and  $\alpha_J$  counts the number of maximal chains whose descent set is contained in  $J$ . If one uses the

labeling  $\lambda$  of cover relations in  $B_n$  where  $\lambda(A \subset B)$  is the unique element of  $B - A$ , then maximal chains correspond to permutations in  $S_n$ . Thus  $\sum_{J \subseteq [n-1]} \alpha_J \beta_J$  counts pairs of permutations  $(\sigma, \tau)$  such that the descent set of the first is contained in the descent set of the second. Equivalently, it counts pairs  $(\sigma, \tau)$  of permutations such that at a place where  $\sigma$  has a descent the permutation  $\tau$  has an ascent. Now, if we reverse the permutations (when written as words) this set bijects to pairs of permutations with no common ascent.

We now turn to the Rees product  $P * Q$  of two pure posets  $P$  and  $Q$ , defined in Section 1. Note that  $P * Q$  is also pure, and that  $\text{rk}_{P*Q}(p, q) = \text{rk}_P(p)$ , for all  $(p, q) \in P * Q$ . In particular,  $\text{rk}(P * Q) = \text{rk}(P)$ .

**Lemma 11.** *Let  $P$  and  $Q$  be pure posets. Furthermore, let  $\tilde{Q} := (Q \times C_n)_{[0, n]}$ , where  $n = \text{rk}(P)$ ,  $C_n$  is a chain of  $n + 1$  elements, and the subscript denotes rank-selection to the elements of rank at most  $n$  in the direct product. Then the Rees product  $P * Q$  is isomorphic to the (unweighted) Segre product  $P \circ \tilde{Q}$ .*

*Proof.* The elements of  $P \circ \tilde{Q}$  are of the form  $(p, q, i)$ , where  $\text{rk}(p) = \text{rk}(q) + i$ ,  $0 \leq i \leq n$ . In particular,  $\text{rk}(q) \leq \text{rk}(p)$ . Now we have  $(p, q, i) \leq (p', q', i')$  if and only if  $p \leq p'$ ,  $q \leq q'$  and  $i \leq i'$ . Thus by  $\text{rk}(p) = \text{rk}(q) + i$  and  $\text{rk}(p') = \text{rk}(q') + i'$  we infer that  $\text{rk}(p) - \text{rk}(q) \leq \text{rk}(p') - \text{rk}(q')$ . Thus the projection map onto the first two coordinates is an isomorphism from  $P \circ \tilde{Q}$  to the Rees product  $P * Q$ .  $\square$

*Proof of Corollary 2.* Let  $C_n$  denote a chain of  $n + 1$  elements, where  $n = \text{rk}(P)$ . By results of Baclawski [3] and Walker [21], a direct product of two posets which are Cohen-Macaulay over  $k$  (resp. homotopically Cohen-Macaulay) is again Cohen-Macaulay over  $k$  (resp. homotopically Cohen-Macaulay) if both posets are acyclic over  $k$  (resp. contractible). Thus  $Q \times C_n$  is Cohen-Macaulay if  $Q$  is Cohen-Macaulay and acyclic over  $k$  (resp. contractible). By Proposition 5 then also  $\tilde{Q}$  is Cohen-Macaulay, and finally it follows from Theorem 1 that  $P \circ \tilde{Q} \cong P * Q$  is Cohen-Macaulay.  $\square$

**Example 12.** Let  $B_n \setminus \{\emptyset\}$  be the Boolean lattice of all subsets of  $[n]$  with the empty set removed. Let  $C_n$  be a chain of  $n$  elements. Both  $B_n \setminus \{\emptyset\}$  and  $C_n$  are homotopically Cohen-Macaulay and contractible. By Corollary 2 we therefore know that  $R_n = (B_n \setminus \{\emptyset\}) * C_n$  is homotopically Cohen-Macaulay.

Attempts to compute the exact homotopy type of the poset  $R_n$  have led to a problem that we state at the end of Section 5.

#### 4. AFFINE SEMIGROUP RINGS

Our initial motivation for this work comes from the study of the Koszul property for affine semigroup rings in commutative algebra. In this section we explain this motivation and the ring-theoretic consequences of our main results.

Let  $\Lambda \subseteq \mathbb{N}^d$  be an affine semigroup (i.e., a finitely generated additive sub-semigroup containing  $\mathbf{0}$ ). For  $\lambda = (\lambda_1, \dots, \lambda_d) \in \Lambda$  we set  $\mathbf{x}^\lambda = x_1^{\lambda_1} \cdots x_d^{\lambda_d}$ . For a field  $k$  the semigroup-ring  $k[\Lambda] \subseteq k[x_1, \dots, x_d]$  is the subalgebra of the polynomial ring  $k[\mathbb{N}^d] = k[x_1, \dots, x_d]$  generated by all monomials  $\mathbf{x}^\lambda$  for  $\lambda \in \Lambda$ .

The semigroup  $\Lambda$  is equipped with the structure of a partially ordered set by setting  $\lambda \leq \gamma$  if there is a  $\rho \in \Lambda$  such that  $\lambda + \rho = \gamma$ . Clearly,  $\mathbf{0}$  is the unique minimal element of the semigroup  $\Lambda$  regarded as a poset. In the sequel we will always assume that the elements of  $\Lambda$  span  $\mathbb{R}^d$  as a vector space. Then as posets all intervals in  $\Lambda$  are pure if all elements of a minimal generating set lie on an affine hyperplane; in this situation we also say  $\Lambda$  is *homogeneous*. Note, that by the commutativity of  $\Lambda$  it follows that every lower interval  $[\mathbf{0}, \lambda]$  is self-dual as a poset. Also, in the poset  $\Lambda$  every interval is isomorphic to a lower interval:  $[\mu, \lambda] \cong [\mathbf{0}, \lambda - \mu]$ .

We call a  $k$ -algebra  $A$  *standard graded* if as a  $k$ -vector space  $A \cong \bigoplus_{i \in \mathbb{N}} A_i$ ,  $A_0 = k$ ,  $A_i A_j \subseteq A_{i+j}$  and  $A$  is as an algebra generated by  $A_1$ . If  $\Lambda$  is homogeneous then  $k[\Lambda]$  is a standard graded algebra.

A standard graded  $k$ -algebra  $A = \bigoplus_{i \in \mathbb{N}} A_i$  is called *Koszul* if  $k$  has a linear resolution over  $A$ ; or equivalently, if  $\text{Tor}_i^A(k, k)_j = 0$  for  $i \neq j$  (see [13] for a comprehensive survey on Koszul rings). Via the bar resolution and work of Laudal and Sletsjøe [15], Peeva, Reiner and Sturmfels [16] observe the following relation between the Koszul property and Cohen-Macaulayness for affine semigroup rings:

**Proposition 13** ([16]). *For an affine semigroup  $\Lambda$  and a field  $k$  the following are equivalent:*

- (i) *the ring  $k[\Lambda]$  is Koszul;*
- (ii) *the interval  $(\mathbf{0}, \lambda)$  is a Cohen-Macaulay poset over  $k$ , for all  $\lambda \in \Lambda$ ;*
- (iii) *the interval  $(\mathbf{0}, \lambda)$  is pure and has homology concentrated in dimension  $\text{rk}(\lambda) - 2$ , for all  $\lambda \in \Lambda$ .*

Using this lemma, we now draw the ring-theoretic conclusions of our work in earlier section. For this we first review the required ring-theoretic concepts.

**Weighted Segre products:** Let  $A = \bigoplus_{i \geq 0} A_i$  and  $B = \bigoplus_{i \geq 0} B_i$  be two graded  $k$ -algebras. Also, let  $B = \bigoplus_{i \geq 0} B'_i$  be another grading of  $B$  as a  $k$ -algebra; i.e.,  $B'_i B'_j \subseteq B'_{i+j}$ . Assume that as  $k$ -vector spaces  $B_i = \bigoplus_{j \geq 0} B_i \cap B'_j$ . The *weighted Segre product*  $A \circ' B$  of  $A$  and  $B$  with respect to the grading  $B = \bigoplus_{i \geq 0} B'_i$  is the  $k$ -subalgebra of  $A \otimes B$  generated by the elements  $a \otimes b \in A \otimes B$  such that  $a \in A_i$  and  $b \in B'_i$  for  $i \in \mathbb{N}$ . If  $A$  and  $B$  are standard graded  $k$ -algebras then  $A \circ' B$  is generated as a  $k$ -algebra by  $\bigoplus_{i \geq 0} A_i \otimes (B'_i \cap B_1)$ . The concept of a weighted Segre product first appeared in work of Crona [11], where weighted Segre products of polynomial rings are considered. Since our results on weighted Segre products apply to affine semigroup rings only, we from now on we confine ourselves to this setting.

Let  $k[\Lambda]$  and  $k[\Gamma]$  be two affine semigroup rings for the homogeneous affine semigroups  $\Lambda \subseteq \mathbb{N}^d$  and  $\Gamma \subseteq \mathbb{N}^e$ . Let  $f$  be the standard grading for  $\Lambda$ . Let  $g : \Gamma \rightarrow \mathbb{N}$  be some grading (i.e., semigroup map with  $g(\gamma) > 0$  for all  $\gamma \neq 0$ ).

The *weighted Segre product* of the affine semigroups  $\Lambda, \Gamma$ , with respect to the grading  $g$ , is the affine semigroup  $\Lambda \circ_g \Gamma \subseteq \mathbb{N}^{d+e}$  of all pairs  $(\lambda, \gamma)$  with  $f(\lambda) = g(\gamma)$ . One easily sees that the semigroup-ring  $k[\Lambda \circ_g \Gamma] \subseteq k[\mathbb{N}^{d+e}]$  is (isomorphic to) the weighted Segre product  $k[\Lambda] \circ_g k[\Gamma]$  (in the sense of the previous paragraph) of the affine semigroup rings  $k[\Lambda]$  and  $k[\Gamma]$ . The semigroup ring  $k[\Lambda] \circ_g k[\Gamma]$  is again

homogeneous with grading induced by  $(\lambda, \gamma) \mapsto h(\gamma)$ , where  $h$  is the standard grading for  $\Gamma$ .

We can now derive Corollary 3 from Theorem 1.

**Proof of Corollary 3.** It is easily seen that if  $(\lambda, \gamma) \in \Lambda \circ_g \Gamma$  then the lower interval  $[\underline{\mathbf{0}}, (\lambda, \gamma)]$  in  $\Lambda \circ_g \Gamma$  is isomorphic to the  $g$ -weighted Segre product of posets  $[\underline{\mathbf{0}}, \lambda] \circ_g [\underline{\mathbf{0}}, \gamma]$ . Hence, Theorem 1 implies Corollary 3 via Proposition 13.  $\square$

We describe some special cases.

- **Segre product:** If  $g$  is the standard grading of  $k[\Gamma]$  then  $k[\Lambda] \circ_g k[\Gamma]$  is the usual Segre product  $k[\Lambda] \circ k[\Gamma]$  of rings. It is known that in general the Segre product of two Koszul rings is again Koszul (Backelin & Fröberg [2]).
- **Veronese-ring:** If  $\Gamma = \mathbb{N}$  and  $g(1) = s$  then  $k[\Lambda] \circ_g k[\Gamma]$  is the  $s$ -th Veronese ring of  $k[\Lambda]$ . Again it is known that in general a Veronese ring of a Koszul ring is Koszul (Backelin & Fröberg [2]).
- **Polynomial rings:** If  $\Lambda = \mathbb{N}^d$  and  $\Gamma = \mathbb{N}^e$  then for a grading  $g : \Gamma \rightarrow \mathbb{N}$  such that  $\Gamma$  is generated in a fixed  $g$ -degree the ring  $k[\Lambda] \circ_g k[\Gamma]$  is Koszul (Crona [11]).

**Rees products:** Let  $A$  be a ring and  $I$  an ideal in  $A$ . Then the *Rees ring*  $\mathcal{R}[A, I]$  is the direct sum  $\bigoplus_{i \geq 0} t^i I^i$ , where  $t$  is an additional indeterminate and  $I^0 = A$ . Here we consider the case when  $A = \bigoplus_{i \geq 0} A_i$  is a standard graded  $k$ -algebra and  $I = \mathfrak{m}_A = \bigoplus_{i \geq 1} A_i$ . We also generalize the construction in the following way. Let  $B = \bigoplus_{i \geq 0} B_i$  be another standard graded  $k$ -algebra. Then we define the *Rees product*  $A * B$  as the  $k$ -algebra  $\bigoplus_{i \geq 0} \mathfrak{m}_A^i \otimes_k B_i$ . If  $B = k[t]$  is the polynomial ring in a single variable the Rees product  $A * B$  is the Rees ring  $\mathcal{R}[A, \mathfrak{m}_A]$ .

Essentially the same arguments that show that Rees rings of a Koszul algebra with respect to the maximal ideal are Koszul also show that the Rees product  $A * B$  preserves Koszulness.

**Proposition 14.** *Let  $A$  and  $B$  be Koszul standard graded  $k$ -algebras. Then  $A * B$  is Koszul.*

*Proof.* Consider the Segre product  $R = A \circ (B \otimes k[t])$ . It is easily seen that the projection on  $A * B$  is a  $k$ -algebra isomorphism. Moreover, by [2] we know that Segre products preserve Koszulness, as do tensor products. Thus  $R$  is a Koszul  $k$ -algebra.  $\square$

We consider the case when  $A = k[\Lambda]$  and  $B = k[\Gamma]$  are standard graded affine semigroup rings for semigroups  $\Lambda \subseteq \mathbb{N}^d$  and  $\Gamma \subseteq \mathbb{N}^e$ . One checks that  $k[\Lambda] * k[\Gamma]$  is the affine semigroup ring  $k[\Lambda * \Gamma]$ , where  $\Lambda * \Gamma \subseteq \mathbb{N}^{d+e}$  is the affine semigroup generated by  $(\lambda, \underline{\mathbf{0}})$  and  $(\lambda, \gamma)$  for elements  $\lambda \in \Lambda$  and  $\gamma \in \Gamma$  of degree 1. Clearly, Proposition 14 implies that for Koszul  $k[\Lambda]$  and  $k[\Gamma]$  the Rees product  $k[\Lambda] * k[\Gamma]$  is Koszul as well. But we want to present an alternative derivation of this fact by using the poset Rees product in order to give the motivation for our poset theoretic construction.

**Proof of Corollary 4.** Let  $\text{rk}_\Lambda$  and  $\text{rk}_\Gamma$  be the rank functions of  $\Lambda$  and  $\Gamma$ , and let  $(\lambda', \gamma') \in \Lambda * \Gamma$ . Since  $\Lambda * \Gamma$  is generated by elements  $(\lambda, \gamma)$  where  $\text{rk}_\Lambda \lambda \geq \text{rk}_\Gamma \gamma$ , it follows that  $\text{rk}_\Lambda \lambda'' \geq \text{rk}_\Gamma \gamma''$  for all  $(\lambda'', \gamma'') \leq (\lambda', \gamma')$ . Moreover, this also implies that  $(\lambda'', \gamma'') \leq (\lambda', \gamma')$  if and only if  $\text{rk}_\Lambda \lambda' - \text{rk}_\Lambda \lambda'' \geq \text{rk}_\Gamma \gamma' - \text{rk}_\Gamma \gamma''$ . Thus

$$[\underline{\mathbf{0}}, (\lambda', \gamma')] \cong [\underline{\mathbf{0}}, \lambda'] * [\underline{\mathbf{0}}, \gamma'].$$

Now, by Proposition 13 we get that  $[\underline{0}, \lambda]$  and  $[\underline{0}, \gamma]$  are Cohen-Macaulay over  $k$ . Having a least and a maximal elements implies that  $[\underline{0}, \gamma]$  is contractible. Thus by Corollary 2 it follows that  $[\underline{0}, \lambda] * [\underline{0}, \gamma]$  is Cohen-Macaulay over  $k$ . Another application of Proposition 13 then proves the assertion.  $\square$

### 5. SOME OPEN PROBLEMS

The connection between topological combinatorics and ring theory via semigroup posets offers several interesting open problems. In closing we list a few.

In the following  $\Lambda \subseteq \mathbb{N}^d$  is a homogeneous affine semigroup, ordered in the usual way. Let  $\text{rk}$  be the rank function of  $\Lambda$  as a poset. Then the dimension  $\dim([\underline{0}, \lambda])$  of the order complex of the open interval  $(\underline{0}, \lambda)$  is  $\text{rk}(\lambda) - 2$ .

Let us call  $\Lambda$  Cohen-Macaulay over a fixed field  $k$  (resp. homotopically Cohen-Macaulay), if all intervals  $(\underline{0}, \lambda)$  in  $\Lambda$  are Cohen-Macaulay over  $k$  (resp. homotopically Cohen-Macaulay). Clearly, Cohen-Macaulayness over  $k$  depends on the characteristic of the field  $k$  only, and this property is equivalent to Koszulness of the ring  $k[\Lambda]$  (by Proposition 13). (*Note:* To say that *the semigroup*  $\Lambda$  is Cohen-Macaulay over a field  $k$  is not equivalent to saying that *the ring*  $k[\Lambda]$  is Cohen-Macaulay.)

- (1) Ring-theoretic work of Avramov and Peeva [1] implies the following fact: Suppose that there exists a  $\lambda \in \Lambda$  and an  $i > 0$  such that

$$\tilde{H}_{\text{rk}(\lambda)-2-i}([\underline{0}, \lambda]; k) \neq 0.$$

Then for all  $j > 0$  there exists some  $\lambda' \in \Lambda$  and some  $j' \geq j$  such that

$$\tilde{H}_{\text{rk}(\lambda')-2-j'}([\underline{0}, \lambda']; k) \neq 0.$$

Question: Does this have a combinatorial explanation?

Moreover, given  $\lambda$ ,  $i$  and  $j$  it would be interesting to know lower and upper bounds on  $\text{rk}(\lambda')$ , and on  $j'$ .

- (2) Question: Is there an affine semigroup  $\Lambda$  which is Cohen-Macaulay over some field  $k$  but not Cohen-Macaulay over some other field  $k'$ ? Is there an affine semigroup  $\Lambda$  which is Cohen-Macaulay over some field  $k$  but not homotopically Cohen-Macaulay?

Moreover, in case the answer to the first question is yes, it is interesting to know whether either or both of the sets of characteristics for which  $\Lambda$  is or is not Cohen-Macaulay can be infinite.

- (3) Work of Conca, Herzog, Trung and Valla [10] shows that for every  $\Lambda$  and field  $k$  there exists  $r > 0$  such that the rank-selected subposet  $\Lambda_r = \{\lambda \in \Lambda \mid r \text{ divides } \text{rk}(\lambda)\}$  has the property that all lower intervals  $(\underline{0}, \lambda)$  in  $\Lambda_r$  are Cohen-Macaulay over  $k$ .

Question: Does this have a combinatorial explanation?

There is an analogous result for bigraded affine semigroups. Let  $f : \Lambda \rightarrow \mathbb{N}^2$  be a map of semigroups such that  $f^{-1}((0, 1)) \cup f^{-1}((1, 0))$  is a generating set of  $\Lambda$  consisting only of elements of rank 1. For  $\gamma' \in \mathbb{N}^2$  denote by  $\Lambda_{\gamma'}$  the affine semigroup of all  $\lambda \in \Lambda$  such that  $f(\lambda)$  is a multiple of  $\gamma'$ . Then (by [10]) there is a  $\gamma \in \mathbb{N}^2$  such that for all  $\gamma' \geq \gamma$  – this order relation is taken in  $\mathbb{N}^2$  –  $\Lambda_{\gamma'}$  is Cohen-Macaulay over  $k$ . Of course, in general  $k[\Lambda_{\gamma'}]$  does not even have to be homogeneous.

Question: Does this have a combinatorial explanation? Is there a version of this result for the property “homotopically Cohen-Macaulay”?

- (4) Let again  $f : \Lambda \rightarrow \mathbb{N}^2$  be a map of semigroups such that  $f^{-1}((0, 1)) \cup f^{-1}((1, 0))$  is a generating set of  $\Lambda$  consisting only of elements of rank 1. A result by Blum [7] says that if  $k[\Lambda]$  is Koszul then for  $\gamma \in \mathbb{N}^2$  the affine semigroup ring  $k[\Lambda_\gamma]$  is Koszul as well. (Here we use the notation from Problem (3).)

Question: Does this have a combinatorial explanation?

Let us see how this result relates to weighted Segre products. For weighted Segre products we are given maps  $\text{rk} : \Lambda_1 \rightarrow \mathbb{N}$  and  $g : \Lambda_2 \rightarrow \mathbb{N}$  such that  $\text{rk}$  is the rank function of  $\Lambda_1$  and  $g$  is a strictly monotone map of affine semigroups. These two together give a map  $f := (\text{rk}, g) : \Lambda_1 \times \Lambda_2 \rightarrow \mathbb{N}^2$ . Let  $\Gamma = f(\Lambda_1 \times \Lambda_2) \cap \{(a, a) | a \in \mathbb{N}\}$ . Then  $\Lambda_1 \circ_g \Lambda_2$  is the affine semigroup  $(\Lambda_1 \times \Lambda_2)_\Gamma := f^{-1}(\Gamma)$ . If  $\Gamma$  is generated by a single element  $\gamma$  then  $(\Lambda_1 \times \Lambda_2)_\Gamma$  is a diagonal in the sense of [10] and [7] — except that  $f$  usually does not fulfill that  $f^{-1}((0, 1)) \cup f^{-1}((1, 0))$  is a generating set of  $\Lambda_1 \times \Lambda_2$ .

We can also interpret this result as a result about the weighted Segre product of two affine semigroup rings  $k[\Lambda]$  and  $k[\Gamma]$ . Suppose that  $k[\Lambda]$  is standard bigraded (i.e. graded with grading in  $\mathbb{N}^2$  and generated by elements of degree  $(1, 0)$ , and  $(0, 1)$ ) and  $k[\Gamma]$  is bigraded in  $\mathbb{N}^2$  by some grading  $g$ . Then the obvious extension of the symbol  $k[\Lambda] \circ_g k[\Gamma]$  to this situation gives  $k[\Lambda_\gamma]$  if we take  $k[\Gamma] = k[t]$  and grade  $t$  by  $\Gamma$ .

Question: Is there a result about the preservation of Koszulness for weighted Segre products of bigraded Koszul algebras?

Question: Is there a result about the preservation of Cohen-Macaulayness for weighted Segre products of bigraded posets?

- (5) Let  $\Lambda_d$  denote the affine semigroup generated by all vectors  $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{N}^d$  such that  $\sum \lambda_i = d$ , except  $(1, \dots, 1)$ . It has been shown for  $d = 3$  that  $\Lambda_d$  is Cohen-Macaulay, or equivalently that  $k[\Lambda_d]$  is Koszul [9]. For  $d \geq 4$  the question is still open.

Question: Is  $\Lambda_d$  Cohen-Macaulay for all  $d$ ?

- (6) Define the Rees product  $R_n$  as in Example 12. Being homotopically Cohen-Macaulay we know that  $R_n$  is homotopy equivalent to a wedge of spheres of dimension  $n - 1$ . We conjecture that the number of spheres in this wedge is the derangement number  $D_n$ , i.e., the number of permutations in the symmetric group  $S_n$  without fixed points. For  $n \leq 7$  we have verified by computer that the homology of  $R_n$  is concentrated in top dimension and is free of rank  $D_n$ . Since the poset is homotopically Cohen-Macaulay this implies the conjecture for  $n \leq 7$ .

The evidence for this conjecture, other than computation for small cases, is a natural relationship with another poset, which has already been seen to have that homotopy type. Namely, let  $K_n$  be the set of words of pairwise distinct letters over  $[n]$  (i.e. an element of  $K_n$  is a sequence  $a_1 \cdots a_k$  where  $a_i \in [n]$  and  $a_i \neq a_j$  for  $1 \leq i < j \leq k$ ). We order  $K_n$  by subword order:  $a_1 \cdots a_k \leq b_1 \cdots b_l$  if and only if there are indices  $1 \leq i_1 < \cdots < i_k \leq l$  such that  $a_1 \cdots a_k = b_{i_1} \cdots b_{i_k}$ . By results of [12] and [5] it follows that  $K_n$  is homotopy equivalent to a wedge of  $D_n$  spheres of dimension  $n - 1$ . The two posets are related by the poset map  $\phi : K_n \rightarrow R_n$  which sends  $a_1 \cdots a_k$

to  $(\{a_1, \dots, a_k\}, j)$ , where  $j - 1$  is the number of descents in  $a_1 \cdots a_k$ . *Does this map relate the two posets homotopically?*

For  $(A, i) \in R_n$  the lower fiber  $\phi^{-1}((R_n)_{\leq(A,i)})$  is the the order ideal  $I_{A,i}$  generated by all words which use all letters in  $A$  and have  $i - 1$  descents. This ideal is, as examples show (see Table 1), in general *not* contractible. However, it seems to have reduced Euler-characteristic 0 – which would suffice since both our posets are Cohen-Macaulay. Clearly,  $I_{A,i}$  as a poset only depends on  $i$  and the cardinality of  $A$ . Thus it suffices to consider the case  $A = [n]$ . The following table lists the homology groups  $\tilde{H}_*(I_{[n],i}, \mathbf{Z})$ . Note that we only list explicitly those homology groups that are non-zero.

$n \setminus i$	1	2	3	4	5	6
1	0					
2	0	0				
3	0	$\tilde{H}_1 = \tilde{H}_2 = \mathbf{Z}$	0			
4	0	0	0	0		
5	0	$\tilde{H}_3 = \tilde{H}_4 = \mathbf{Z}$	$\tilde{H}_3 = \tilde{H}_4 = \mathbf{Z}^6$	$\tilde{H}_3 = \tilde{H}_4 = \mathbf{Z}$	0	
6	0	0	$\tilde{H}_4 = \tilde{H}_5 = \mathbf{Z}^{13}$	$\tilde{H}_4 = \tilde{H}_5 = \mathbf{Z}^{13}$	0	0

**Table 1.** Homology groups of  $I_{[n],i}$ .

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REFERENCES

1. Avramov, L.; Peeva, I.: Finite regularity and Koszul algebras. *Amer. J. Math.* **123** (2002), 275–281.
2. Backelin, J.; Fröberg, R.: Koszul algebras, Veronese subrings and rings with linear resolutions. *Rev. Roumaine Math. Pures Appl.* **30** (1985), no. 2, 85–97.
3. Baclawski, K.: Cohen-Macaulay ordered sets. *J. Algebra* **63** (1980), 226–258.
4. Björner, A.: Topological Methods. In: *Handbook of Combinatorics*, eds. R. Graham, M. Grötschel, L. Lovász. North-Holland, Amsterdam, 1995, pp. 1819–1872. *Advances Math.* **43** (1982), 87–100.
5. Björner, A.; Wachs, M.: On lexicographically shellable posets. *Trans. Amer. Math. Soc.* **277** (1983), 323–341.
6. Björner, A.; Wachs, M.; Welker, V.: Poset fiber theorems. *Trans. Amer. Math. Soc.* **xx** (2004), -. (Preprint 2002)

7. Blum, S.: Subalgebras of bigraded Koszul Algebras. *J. Algebra* **242** (2001), 795–809.
8. Carlitz, L.; Scoville, R.; Vaughan, T.: Enumeration of pairs of permutations and sequences. *Bull. Amer. Math. Soc.* **80** (1974), 881–884.
9. Caviglia, G.: The pinched Veronese is Koszul, (Preprint 2003).
10. Conca, A.; Herzog, J.; Trung, N. V.; Valla, G.: Diagonal subalgebras of bigraded algebras and embeddings of blow-ups of projective spaces. *Amer. J. Math.* **119** (1997), no. 4, 859–901.
11. Crona, K.: A new class of Koszul algebras. *C. R. Acad. Sci. Paris, Ser. I* **323** (1996), 705–710.
12. Farmer, F.D.: Cellular homology of posets. *Math. Japon.* **23** (1978/79), 607–613.
13. Fröberg, R.: Koszul algebras. In “Advances in commutative ring theory” (Fez, 1997), 337–350, *Lecture Notes in Pure and Appl. Math.*, 205, Dekker, 1999.
14. Fröberg, R.; Hoa, L.T.: Segre products and Rees algebras of face rings. *Comm. Alg.* **20** (1992), 3369–3380.
15. Laudal, A. O.; Sletsjøe, A.: Betti numbers of monoid algebras. Applications to 2-dimensional torus embeddings. *Math. Scand.* **56** (1985), 145–162.
16. Peeva, I.; Reiner, V.; Sturmfels, B.: How to shell a monoid. *Math. Ann.* **310** (1998), no. 2, 379–393.
17. Quillen, D.: Homotopy properties of the poset of non-trivial  $p$ -subgroups of a finite group. *Advances Math.* **28** (1978), 101–128.
18. Stanley, R.: Binomial posets, Möbius inversion, and permutation enumeration. *J. Combin. Theory, Ser. A* **20** (1976), 336–356.
19. Stanley, R.: *Enumerative Combinatorics, Vol. 1.* Wadsworth, Monterey, 1986. Second printing, Cambridge Univ. Press, 1997.
20. Stanley, R.: *Combinatorics and Commutative Algebra.* Second Edition. Birkhäuser, Boston, 1995.
21. Walker, J.W.: Canonical homeomorphisms of posets. *Europ. J. Combinatorics* **9** (1988), 97–107.

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