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Metric Geometry and Random Discrete Morse Theory

Bruno Benedetti
(joint work with Karim Adiprasito, Frank Lutz)

The exposition is in two parts. In the first part, I sketch the main result of the recent preprint [1] on metric geometry and collapsibility (with Karim Adiprasito). In the second part, I present a possible computational approach (with Frank Lutz).

1. Metric geometry on simplicial complexes

How to put a metric on a given simplicial complex? One way is to declare all edges to have unit length, and to regard all triangles as equilateral triangles in the Euclidean plane. This yields the *equilateral flat* metric, also known as *regular* metric. Many other options are possible; for example, one can assign different lengths to the various edges. The metric is called *acute* (resp. *non-obtuse*) if all dihedral angles in each simplex are less than 90 degrees (resp. at most 90 degrees). Clearly, equilateral implies acute, which in turn implies non-obtuse.

CAT(0) spaces. Once endowed with such metric, a simplicial complex becomes a *geodesic space*, i.e. a locally compact metric space in which distances can be measured along shortest paths. A geodesic space is called CAT(0) if any triangle formed by three shortest paths looks not fatter than the "corresponding" triangle (=with same edge lengths) in $\mathbb{R}^2$. For example, the cylinder $S^1 \times [0,1]$ is *not* CAT(0): Any non-degenerate triangle formed by three points on $S^1 \times \{0\}$ and by the shortest paths connecting them, is fatter than the corresponding Euclidean triangle. (Spaces where every point has a CAT(0) neighborhood, like the cylinder, are called *non-positively curved.*) All CAT(0) spaces are contractible. Apart from dimension one, the converse is false, as shown by the Dunce Hat.

Collapsible complexes. A well known combinatorial strengthening of contractibility was introduced in 1939 by Whitehead. A *free face* in a complex is a face properly contained only in one other face. (Not all complexes have free faces.) A complex is called *collapsible* if it can be reduced to a point by recursively deleting a free face. The deletion of a free face (and of the other face containing it) is topologically a deformation retract. Hence, all collapsible complexes are contractible. Apart from dimension one, the converse is false, as shown by the Dunce Hat.

It is easy to see that the stellar subdivision of a triangle is collapsible. However, it is not CAT(0) with the equilateral flat metric, basically because of the central degree-3 vertex. In 2008, Crowley found a first non-trivial relation between the aforementioned properties:

Theorem 1 (Crowley)**.** *Every* 3*-dimensional simplicial pseudomanifold that is* CAT(0) *with the equilateral flat metric, is collapsible.*

It turns out that a more general fact is true:

Main Result 1 (Adiprasito–B. [1])**.** *Every d-dimensional polytopal complex that is* CAT(0) *with a metric for which all vertex stars are convex, is collapsible.*

Under the equilateral flat metric, it is easy to see that all vertex stars are convex. Actually, even in any *non-obtuse* flat metric, all vertex stars are convex. This simple observation has three main consequences. Recall that CAT(0) *cube complexes* are CAT(0) spaces obtained from cubical complexes by giving each cube the metric of a regular Euclidean cube.

Consequence 1. *All* CAT(0) *cube complexes are collapsible.*

Consequence 2. *There is a smooth* 5*-manifold (with boundary) M so that*
- (i) *M is not homeomorphic to the d-ball.*
  Hence, every PL triangulation of M is not collapsible.
- (ii) *M is homeomorphic to a (compact)* CAT(0) *cube complex.*
  Hence, some non-PL triangulation of M is collapsible.

Consequence 3. *Discrete Morse inequalities may be sharper than (smooth) Morse inequalities, in bounding the homology of a manifold. Also, non-PL structures may be more efficient than PL structures, from a computational point of view.*

2. A computational approach: Random discrete Morse theory.

After proving that not all collapsible manifolds are balls, a natural question is whether it is possible to construct one example explicitly. This raises complexity issues. In principle, collapsibility is algorithmically decidable; one could just try all possible sequences of free-face-deletions. However, already for 3-balls with, say, 20 facets, the number of all possible sequences is beyond the computational limit.

A related problem is how to tell whether a triangulation is 'nice', and how to quantify its nastyness. If we decided to regard collapsible (or shellable) balls as the nicest triangulations, what is 'far from being nice'?

A possible statistic approach is what we sketch below, called *random discrete Morse theory* [3]. The idea is elementary, and consists of the following algorithm:

Input: An arbitrary simplicial complex C, given by it list of facets (or by its face poset). Initialize $c_0 := 0$, $c_1 := 0$, $\ldots$, $c_{\dim C} := 0$.
(1) Is the complex empty? If yes, stop the algorithm; otherwise, go to (2).
(2) Are there free codimension-one faces? If yes, go to (3); if no, go to (4).
(3) (Elementary Collapse): Pick one free codimension-one face *uniformly at random* and delete it. Then go back to (1).
(4) (Storage of Critical Face): Pick one of the top-dimensional faces *uniformly at random* and delete it from the complex. If i is the dimension of the deleted face, increment c_i by 1 unit; then go back to (1).
Output: the "discrete Morse vector" $(c_0, c_1, c_2, \ldots, c_{\dim C})$. By construction, c_i counts critical faces of dimension i.

This algorithm requires no backtracking, and 'digests' the complex very rapidly. The output $(1, 0, 0, \ldots, 0)$ is a certificate of collapsibility. If the output is different, the complex could still be collapsible with a different sequence of free-face deletions. Every sequence has some positive probability to be the one picked up by the algorithm; unfortunately, this probability can be arbitrarily small. Nevertheless, when the certificate of collapsibility is not reached, the algorithm outputs something meaningful, namely, the f-vector of a homotopy equivalent cell complex. Intuitively, if this vector is close to $(1, 0, 0, \ldots, 0)$, we could still say that the complex is 'close to be collapsible'.

Since the output arrives quickly, we can re-launch the program, say, 10000 times, possibly on separate computers (independently). The distribution of the obtained outcomes yields an approximation of the *discrete Morse spectrum*, which is the distribution of all possible outcomes. This allows an empirical analysis of how complicated the complex is. For example, here is the data collected by running the algorithm 10000 times on Hachimori's triangulation `nc-sphere` [6]:

$\mathbb{Z}$-homology $= (\mathbb{Z}, 0, 0, \mathbb{Z})$, $\pi_1 = (0)$
f-vector $= (381, 2309, 3856, 1928)$

Time employed:
3.228 seconds (Hasse diagram)
+0.470 seconds per run

$(1, 1, 1, 1)$:	7902
$(1, 2, 2, 1)$:	1809
$(1, 3, 3, 1)$:	234
$(1, 4, 4, 1)$:	25
$\mathbf{(1, 0, 0, 1)}$:	**12**
$(2, 3, 2, 1)$:	9
$(1, 6, 6, 1)$:	3
$(2, 4, 3, 1)$:	3
$(2, 5, 4, 1)$:	2
$(1, 5, 5, 1)$:	1

The optimal Morse vector appears in 0.12% of cases, so `nc-sphere` minus a facet is somewhat 'barely collapsible'. In fact, our non-deterministic algorithm is the first to find a collapsing sequence for it. In contrast, on many polytopal 3-spheres (and also for Barnette's non-polytopal, shellable 3-sphere) the Morse vector (1, 0, 0, 1) appears basically 100% of the times.

This way we obtained optimal discrete Morse functions for many triangulations of various topologies and dimensions, among which Kühnel et al.'s triangulations of the K_3 surface [4, 8] and of $\mathbb{CP}^2$, or the Csorba–Lutz triangulation of the Hom-complex $\mathrm{Hom}(C_5, K_5)$ [5]. The spectra which one can intuit experimentally seem interesting per se; we hope future theoretical work can justify them. Here is the largest example on which the algorithm was successful:

Main Result 2 (Adiprasito–B.–Lutz [2])**.** *There is a collapsible* 5*-manifold different from the* 5*-ball with* f*-vector* $(5013, 72300, 290944, 495912, 383136, 110880)$.

The construction is as follows. Start with the 16-vertex triangulation of the Poincaré sphere; remove the star of a vertex; take the product with an interval; cone over the boundary; form a one-point suspension to achieve a non-PL 5-sphere with 32 vertices; take the barycentric subdivision; take the collar of the PL singular set. The resulting manifold is homeomorphic to the one in [1, Thm. 4.12].

References

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