A QUESTION ON THE CRITICAL POINTS OF A REAL POLYNOMIAL

MATTHEW CHASSE

ABSTRACT. A question is posed for the locations of critical points of a real polynomial. It has been answered in the negative by two Swedish high-school students: Cesar Höjeberg (Stockholm) and Lisa Lokteva (Borås).

1. STATEMENT OF QUESTION

For a polynomial $f \in \mathbb{R}[z]$, let $Z_\lambda(p)$ be the number of zeros of p which lie in the half-plane $\{z \in \mathbb{C} \,:\, \text{Im } z > \lambda\}$

Question 1. *For any $f \in \mathbb{R}[z]$, is it true that $Z_\lambda(p') \leq Z_\lambda(p)$ for all $\lambda > 0$?*

By Rolle's Theorem, the answer to Question 1 holds for the case $\lambda = 0$, so it has not been included. If the answer to Question 1 is true, it is clear that the symmetry of the zeros of f must play a role in its proof.

Example 1. The polynomial $p(z) = z(z^2 - 1)(z - 4i)$ has only one zero in the half-plane Im $z > 1/2$, while its derivative $p'(z)$ has three. Thus, the statement in Question 1 does not hold for an arbitrary polynomial in $\mathbb{C}[z]$.

The answer to Question 1 is yes when p has at most one pair of complex conjugate zeros by the Gauss-Lucas Theorem and Rolle's Theorem.

Answer: Cesar Höjeberg and Lisa Lokteva have found that the answer to Question 1 is no, with the following (counter) example

$$p(x) = -78 - 6x - 98x^2 + 21x^3 - 82x^4 - 67x^5 + 15x^6 - 64x^7.$$

DEPARTMENT OF MATHEMATICS, ROYAL INSTITUTE OF TECHNOLOGY, SE-100 44 STOCKHOLM, SWEDEN

E-mail address: chasse@kth.se