DISCRETE ANALOGUES OF THE LAGUERRE INEQUALITIES AND A CONJECTURE OF I. KRASIKOV

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Abstract. A conjecture of I. Krasikov is proved. Several discrete analogues of classical polynomial inequalities are derived, along with results which allow extensions to a class of transcendental entire functions in the Laguerre-Pólya class.

1. Introduction

The classical Laguerre inequality for polynomials states that a polynomial of degree n with only real zeros, $p(x) \in \mathbb{R}[x]$, satisfies $(n-1)p'(x)^2 - np''(x)p(x) \geq 0$ for all $x \in \mathbb{R}$ (see [3, 13]). Thus, the classical Laguerre inequality is a necessary condition for a polynomial to have only real zeros. Our investigation is inspired by an interesting paper of I. Krasikov [8]. He proves several discrete polynomial inequalities, including useful versions of generalized Laguerre inequalities [17], and shows how to apply them by obtaining bounds on the zeros of some Krawtchouk polynomials. In [8], I. Krasikov conjectures a new discrete Laguerre inequality for polynomials. After establishing this conjecture, we generalize the inequality to transcendental entire functions (of order $\rho < 2$, and minimal type of order $\rho = 2$) in the Laguerre-Pólya class (see Definition 1.1).

Definition 1.1. A real entire function $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ is said to belong to the *Laguerre-Pólya class*, written $\varphi \in \mathcal{L}\text{-}\mathcal{P}$, if it can be expressed in the form

$$\varphi(x) = cx^m e^{-ax^2+bx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right) e^{\frac{-x}{x_k}} \qquad (0 \leq \omega \leq \infty),$$

where $b, c, x_k \in \mathbb{R}$, m is a non-negative integer, $a \geq 0$, $x_k \neq 0$, and $\sum_{k=1}^{\omega} \frac{1}{x_k^2} < \infty$.

The significance of the Laguerre-Pólya class stems from the fact that functions in this class, *and only these*, are uniform limits, on compact subsets of $\mathbb{C}$, of polynomials with only real zeros [12, Chapter VIII].

Definition 1.2. We denote by $\mathcal{L}\text{-}\mathcal{P}_n$ the set of polynomials of degree n in the Laguerre-Pólya class; that is, $\mathcal{L}\text{-}\mathcal{P}_n$ is the set of polynomials of degree n having only real zeros.

The minimal spacing between neighboring zeros of a polynomial in $\mathcal{L}\text{-}\mathcal{P}_n$ is a scale that provides a natural criterion for the validity of discrete polynomial inequalities.

Definition 1.3. Suppose $p(x) \in \mathcal{L}\text{-}\mathcal{P}_n$ has zeros $\{\alpha_k\}_{k=1}^n$, repeated according to their multiplicities, and ordered such that $\alpha_k \leq \alpha_{k+1}$, $1 \leq k \leq n-1$. We define the *mesh size*, associated with the zeros of p, by

$$\mu(p) := \min_{1 \leq k \leq n-1} |\alpha_{k+1} - \alpha_k|.$$

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With the above definition of mesh size, we can now state a conjecture of I. Krasikov, which is proved in Section 2.

Conjecture 1.4. (I. Krasikov [8]) If $p(x) \in \mathcal{L}\text{-}\mathcal{P}_n$ and $\mu(p) \geq 1$, then

$$(n-1)[p(x+1)-p(x-1)]^2 - 4np(x)[p(x+1)-2p(x)+p(x-1)] \geq 0 \tag{1}$$

holds for all $x \in \mathbb{R}$.

The classical Laguerre inequality is found readily by differentiating the logarithmic derivative of a polynomial $p(x)$ with only real zeros $\{\alpha_i\}_{i=1}^n$, to give

$$\frac{p''(x)p(x)-(p'(x))^2}{(p(x))^2} = \left(\frac{p'(x)}{p(x)}\right)' = \left(\sum_{k=1}^{n} \frac{1}{(x-\alpha_k)}\right)' = -\sum_{k=1}^{n} \frac{1}{(x-\alpha_k)^2}. \tag{2}$$

Since the right-hand side is non-positive,

$$(p'(x))^2 - p''(x)p(x) \geq 0.$$

This inequality is also valid for an arbitrary function in $\mathcal{L}\text{-}\mathcal{P}$ [3]. A sharpened form of the Laguerre inequality for polynomials can be obtained with the Cauchy-Schwarz inequality,

$$\left(\sum_{k=1}^{n} \frac{1}{(x-\alpha_k)}\right)^2 \leq n \sum_{k=1}^{n} \frac{1}{(x-\alpha_k)^2}. \tag{3}$$

In terms of p, (3) becomes $\left(\frac{p'(x)}{p(x)}\right)^2 \leq n\sum_{k=1}^{n} \frac{1}{(x-\alpha_k)^2}$, and with (2) yields the sharpened version of the Laguerre inequality for polynomials on which Conjecture 1.4 is based,

$$(n-1)(p'(x))^2 - np''(x)p(x) \geq 0. \tag{4}$$

The inequality (1) is a finite difference version of the classical Laguerre inequality for polynomials. Indeed, let us define

$$f_n(x,h,p) := (n-1)[p(x+h)-p(x-h)]^2 - 4np(x)[p(x+h)-2p(x)+p(x-h)]. \tag{5}$$

Then (1) can be written as $f_n(x,1,p) \geq 0$ $(x \in \mathbb{R})$, and we recover the classical Laguerre inequality for polynomials by taking the following limit:

$$\begin{aligned} \lim_{h\to 0} \frac{f_n(x,h,p)}{4h^2} &= (n-1)\left(\lim_{h\to 0} \frac{p(x+h)-p(x-h)}{2h}\right)^2 \\ &\quad - np(x)\left(\lim_{h\to 0} \frac{p(x+h)-2p(x)+p(x-h)}{h^2}\right) \\ &= (n-1)p'(x)^2 - np''(x)p(x). \end{aligned}$$

As I. Krasikov points out, the motivation for inequalities of type (1) is that classical discrete orthogonal polynomials $p_k(x)$ satisfy a three-term difference equation (see [15, p. 27], [8])

$$p_k(x+1) = b_k(x)p_k(x) - c_k(x)p_k(x-1),$$

where $b_k(x)$ and $c_k(x)$ are continuous over the interval of orthogonality. Many of the classical discrete orthogonal polynomials satisfy the condition that $c_k(x) > 0$ on the interval of orthogonality, and this implies that $\mu(p) \geq 1$ (see [11]). Therefore, inequalities when $\mu(p) \geq 1$ are of interest and may help provide sharp bounds on the loci of zeros of discrete orthogonal polynomials [8, 5, 6]. Indeed, W. H. Foster, I. Krasikov, and A. Zarkh have found bounds on the extreme zeros of many orthogonal polynomials using discrete and continuous Laguerre and new Laguerre type inequalities which they discovered [5, 6, 7, 8, 9, 10, 11].

In this paper, we prove I. Krasikov's conjecture (see Theorem 2.17), extend it to a class of transcendental entire functions in the Laguerre-Pólya class, and formulate several conjectures (cf. Conjecture 2.19, Conjecture 2.21, Conjecture 2.22, and Conjecture 3.5). In Section 2, we establish several preliminary results about polynomials which satisfy a zero spacing requirement. In Section 3, we establish the existence of a polynomial sequence which satisfies a zero spacing requirement and converges uniformly on compact subsets of $\mathbb{C}$ to the exponential function. We use this result to extend a version of (1) to transcendental entire functions in the Laguerre-Pólya class up to order $\rho = 2$ and minimal type, and conjecture that it is true for all functions in $\mathcal{L}$-$\mathcal{P}$.

2. Proof of I. Krasikov's Conjecture

In this section we develop some discrete analogues of classical inequalities, form some intuition about the effect of imposing a minimal zero spacing requirement on a polynomial in $\mathcal{L}$-$\mathcal{P}$, and prove Conjecture 1.4. First, note that one can change the zero spacing requirement in Conjecture 1.4 by simply rescaling in x. For example, the following conjecture is equivalent to Conjecture 1.4 of Krasikov.

Conjecture 2.1. Let $p(x) \in \mathcal{L}$-$\mathcal{P}_n$. Suppose that $\mu(p) \geq h > 0$. Then for all $x \in \mathbb{R}$,

$$f_n(x, h, p) = (n-1)[p(x+h) - p(x-h)]^2 - 4np(x)[p(x+h) - 2p(x) + p(x-h)] \geq 0. \tag{6}$$

For the sake of clarity, we will work with (1) directly ($h = 1$), and keep in mind that we can always make statements about polynomials with an arbitrary positive minimal zero spacing by rescaling $p(x)$ (in other words "measuring x in units of h").

Lemma 2.2. *A local minimum of a polynomial, $p(x) \in \mathcal{L}$-$\mathcal{P}_n$, with only real simple zeros, is negative. Likewise, a local maximum of $p(x)$ is positive.*

Proof. Because $p(x)$ is a polynomial on $\mathbb{R}$ with simple zeros, at a local minimum (x_{min}, $p(x_{min})$), we have that $p'(x_{min}) = 0$ and $p''(x_{min}) > 0$ (because $p''(x_{min}) = 0$ would imply that p' has a multiple zero at x_{min} which is not possible). The classical Laguerre inequality asserts that if $p(x) \in \mathcal{L}$-$\mathcal{P}$, then for all $x \in \mathbb{R}$, $(p'(x))^2 - p''(x)p(x) \geq 0$. At a local minimum this expression becomes $-p''(x_{min})p(x_{min}) \geq 0$. Therefore, at a local minimum we have $p(x_{min}) \leq 0$. Since the zeros of p are simple, $p(x_{min}) \neq 0$. Thus $p(x_{min}) < 0$. The second statement of the lemma can be proved the same way, or by considering $-p$ and using the first statement. □

A statement similar to Lemma 2.2 is proved by G. Csordas and A. Escassut [4, Theorem 5.1] for a class of functions whose zeros lie in a horizontal strip about the real axis.

Lemma 2.3. *Let $p(x) \in \mathcal{L}$-$\mathcal{P}_n$, $n \geq 2$, $\mu(p) \geq 1$.*

(i) *If $p(x-1) > p(x)$ and $p(x+1) > p(x)$, then $p(x) < 0$.*
(ii) *If $p(x-1) < p(x)$ and $p(x+1) < p(x)$, then $p(x) > 0$.*

Proof. (i) Fix an $x_0 \in \mathbb{R}$. Let $p(x_0 - 1) > p(x_0)$, $p(x_0 + 1) > p(x_0)$, and assume for a contradiction that $p(x_0) \geq 0$. There cannot be any zeros of $p(x)$ in the interval $[x_0 - 1, x_0]$, for if there were, $p(x_0)p(x_0 - 1) > 0$ implies that the number of zeros in $(x_0 - 1, x_0)$ must be even, and this violates the zero spacing $\mu(p) \geq 1$. Similarly, there cannot be any zeros of $p(x)$ in $[x_0, x_0 + 1]$. If $p(x_0) < p(x_0 - 1)$ and $p(x_0) < p(x_0 + 1)$ then there is a point in $(x_0 - 1, x_0 + 1)$ where p' changes sign from negative to positive. This implies p achieves a non-negative local minimum on $[x_0 - 1, x_0 + 1]$ which contradicts Lemma 2.2.

(ii) The second statement follows by replacing p with $-p$ in (i). □

Using Lemma 2.3 we can verify that if $p(x) < \min\{p(x+1), p(x-1)\}$, then $p(x) < 0$ and thus the function

$$\begin{aligned} f_n(x,1,p) &= (n-1)[p(x+1)-p(x-1)]^2 - 4np(x)[p(x+1)-2p(x)+p(x-1)] \\ &= (n-1)[p(x+1)-p(x-1)]^2 \\ &\quad -4np(x)[(p(x+1)-p(x)) + (p(x-1)-p(x))] \end{aligned} \tag{7}$$

has a non-negative second term and (1) is satisfied. Similarly, (1) is valid when $p(x) > \max\{p(x-1), p(x+1)\}$. The proof of Conjecture 1.4 is now reduced to the case where $\min\{p(x+1), p(x-1)\} \le p(x) \le \max\{p(x+1), p(x-1)\}$. It is easy to show that if for some $p(x) \in \mathcal{L}\text{-}\mathcal{P}_n$, $f_n(x,1,p) \ge 0$ for all $x \in \mathbb{R}$, then for all $m \ge n$, $f_m(x,1,p) \ge 0$ for all $x \in \mathbb{R}$. If $\mu(p) \ge 1$, but $m < \deg(p)$, then for some $x_0 \in \mathbb{R}$, $f_m(x_0,1,p)$ may be negative. Indeed, let $p(x) = x(x-1)(x-2)$, then $f_3(x,1,p) = 72(x-1)^2$ and $f_2(x,1,p) = -12(x-3)(x-1)^2(x+1)$. In particular, $f_2(4,1,p) = -540$.

We next obtain inequalities and relations that are analogous to those used in deriving the continuous version of the classical Laguerre inequality for polynomials.

Definition 2.4. Let $p(x) \in \mathcal{L}\text{-}\mathcal{P}_n$ have only simple real zeros $\{\alpha_k\}_{k=1}^n$. Define forward and reverse "discrete logarithmic derivatives" associated with $p(x)$ by

$$F(x) := \frac{p(x+1)-p(x)}{p(x)} =: \sum_{k=1}^n \frac{A_k}{(x-\alpha_k)} \tag{8}$$

$$\text{and} \qquad R(x) := \frac{p(x)-p(x-1)}{p(x)} =: \sum_{k=1}^n \frac{B_k}{(x-\alpha_k)}. \tag{9}$$

Note that $\deg(p(x+1)-p(x)) < \deg(p(x))$ and $\deg(p(x)-p(x-1)) < \deg(p(x))$ permits unique partial fraction expansions of the rational functions F and R. Define the sequences $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ associated with $p(x)$ by requiring that they satisfy the equation above.

Remark 2.5. For an arbitrary finite difference, h, the scaled versions of the functions in Definition 2.4 are $F(x) := \frac{p(x+h)-p(x)}{hp(x)}$ and $R(x) := \frac{p(x)-p(x-h)}{hp(x)}$.

Lemma 2.6. *For $p(x) \in \mathcal{L}\text{-}\mathcal{P}_n$, $n \ge 2$, with $\mu(p) \ge 1$ and zeros $\{\alpha_k\}_{k=1}^n$, the associated sequences $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ satisfy $A_k \ge 0$ and $B_k \ge 0$, for all k, $1 \le k \le n$.*

Proof. From Definition 2.4 we have

$$p(x+1)-p(x) = \sum_{k=1}^n \frac{A_k}{(x-\alpha_k)} p(x) = \sum_{k=1}^n \left[A_k \prod_{j \ne k} (x-\alpha_j) \right].$$

Evaluating this at a zero of p yields $p(\alpha_k+1) = A_k \prod_{j\ne k}(\alpha_k - \alpha_j) = A_k p'(\alpha_k)$.

Thus,

$$A_k = \frac{p(\alpha_k+1)}{p'(\alpha_k)} \quad \text{and similarly} \quad B_k = \frac{-p(\alpha_k-1)}{p'(\alpha_k)}.$$

Since the zeros of p are simple, for some neighborhood of α_k, $U(\alpha_k)$,

$$\begin{aligned} & x \in U(\alpha_k),\ x < \alpha_k \quad \text{implies} \quad p(x)p'(x) < 0 \\ \text{and} \quad & x \in U(\alpha_k),\ x > \alpha_k \quad \text{implies} \quad p(x)p'(x) > 0. \end{aligned}$$

Since the zeros are spaced at least 1 unit apart, $p(\alpha_k+1)$ is either 0 or has the same sign as $p(x)$ for $x > \alpha_k$ on $U(\alpha_k)$. So for all $\varepsilon > 0$ sufficiently small, $p(\alpha_k+1)p'(\alpha_k+\varepsilon) \ge 0$, and by continuity $p(\alpha_k+1)p'(\alpha_k) \ge 0$. Thus $A_k = \frac{p(\alpha_k+1)}{p'(\alpha_k)} \ge 0$. Note $p'(\alpha_k) \ne 0$ since α_k is simple. Likewise, $p(\alpha_k-1)$ is either 0 or has the same sign as $p'(x)$ for $x < \alpha_k$ on

$U(\alpha_k)$. Hence for all $\varepsilon > 0$ sufficiently small, $p(\alpha_k - 1)p'(\alpha_k - \varepsilon) \le 0$. By continuity, $p(\alpha_k - 1)p'(\alpha_k) \le 0$, whence $B_k \ge 0$.

□

Example 2.7. If the zero spacing requirement in Lemma 2.6 is violated then some A_k or B_k may be negative. Indeed, consider $p(x) = x(x + 1 - \varepsilon)$. Then $\frac{p(x+1)-p(x)}{p(x)} = \frac{A_1}{x} + \frac{A_2}{x+1-\varepsilon}$, where

$$A_1 = \frac{2-\varepsilon}{1-\varepsilon} \qquad A_2 = \frac{-\varepsilon}{1-\varepsilon}.$$

For any positive $\varepsilon < 1$, $\mu(p) = 1 - \varepsilon$, and A_2 is negative.

Corollary 2.8. *For $p(x) \in \mathcal{L}$-$\mathcal{P}_n$, $n \ge 2$, with $\mu(p) \ge 1$, the associated functions $F(x)$ and $R(x)$ (see Definition 2.4) satisfy $F'(x) < 0$ and $R'(x) < 0$ on their respective domains.*

Proof. This corollary is a direct result of differentiating the partial fraction expressions for F and R and applying Lemma 2.6. □

Note that the degree of the numerator of $F(x)$ is $n - 1$. If $\mu(p) \ge 1$, then $F(x)$ has $n - 1$ real zeros, because $F(x)$ is strictly decreasing between any two consecutive poles of $F(x)$. This proves the following lemma.

Lemma 2.9. (Pólya and Szegö [18, vol. II, p. 39]) *For $p(x) \in \mathcal{L}$-$\mathcal{P}_n$, $n \ge 2$, with $\mu(p) \ge 1$, $F(x)$ and $R(x)$ have only real simple zeros.*

In the sequel (see Lemma 2.16), we show that if $\mu(p(x)) \ge 1$, then $\mu(p(x+1) - p(x)) \ge 1$, and the zeros of $F(x)$ and $R(x)$ are spaced at least one unit apart.

Lemma 2.10. *If $p(x) \in \mathcal{L}$-$\mathcal{P}_n$, then the associated sequences $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ satisfy $\sum_{k=1}^n A_k = n$ and $\sum_{k=1}^n B_k = n$.*

Proof. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathcal{L}$-$\mathcal{P}_n$ and denote the zeros of $p(x)$ by $\{\alpha_k\}_{k=1}^n$. Observe that

$$\lim_{|z|\to\infty} zF(z) = \lim_{|z|\to\infty} z\left(\frac{p(z+1) - p(z)}{p(z)}\right) = \lim_{|z|\to\infty} z \sum_{k=1}^n \frac{A_k}{(z - \alpha_k)} = \sum_{k=1}^n A_k. \tag{10}$$

Then (10) and

$$\begin{aligned} p(z+1) - p(z) &= a_n(z+1)^n + a_{n-1}(z+1)^{n-1} + \ldots + a_0 - [a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0] \\ &= n a_n z^{n-1} + O(z^{n-2}),\ |z| \to \infty, \end{aligned}$$

imply that

$$\sum_{k=1}^n A_k = \lim_{|z|\to\infty} zF(z) = \lim_{|z|\to\infty} z\left(\frac{p(z+1) - p(z)}{p(z)}\right) = \lim_{|z|\to\infty} z\left(\frac{n a_n z^{n-1} + O(z^{n-2})}{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0)}\right) = n.$$

A similar argument shows that $\sum_{k=1}^n B_k = n$. □

Lemma 2.11. *Given $p(x) \in \mathcal{L}$-$\mathcal{P}_n$, $n \ge 2$, with $\mu(p) \ge 1$, the associated functions $F(x)$ and $R(x)$ satisfy $(F(x))^2 \le -nF'(x)$ and $(R(x))^2 \le -nR'(x)$, for all $x \in \mathbb{R}$, where $p(x) \ne 0$.*

Proof. From Definition 2.4, $F(x) = \sum_{k=1}^n \frac{A_k}{x-\alpha_k}$ and therefore $F'(x) = \sum_{k=1}^n \frac{-A_k}{(x-\alpha_k)^2}$. By Lemma 2.6, $\mu(p) \ge 1$ implies the constants $A_k \ge 0$. Using the the Cauchy-Schwarz inequality,

$$(F(x))^2 = \left(\sum_{k=1}^n \frac{A_k}{x - \alpha_k}\right)^2 \le \left(\sum_{k=1}^n A_k\right) \sum_{k=1}^n \frac{A_k}{(x - \alpha_k)^2} = -nF'(x),$$

where Lemma 2.10 has been used in the last equality. An identical argument shows $(R(x))^2 \leq -nR'(x)$ for all $x \in \mathbb{R}$. □

Remark 2.12. Simple examples show that the inequalities in Lemma 2.11 are sharp (consider $p(x) = x(x + 1 - \varepsilon)$).

Lemma 2.13. *Let $p(x) \in \mathcal{L}\text{-}\mathcal{P}_n$, $n \geq 2$, with $\mu(p) \geq 1$, and let $\{\beta_k\}_{k=1}^{n-1}$ be the zeros of $p(x+1)-p(x)$. Let $y \in \mathbb{R}$ be such that $\min\{p(y+1), p(y-1)\} < p(y) < \max\{p(y+1), p(y-1)\}$. Then if the interval $[y-1, y]$ does not contain any β_k,*

$$\frac{1}{n}F(y)R(y) \leq \frac{(p(y))^2 - p(y+1)p(y-1)}{(p(y))^2}.$$

Proof. If no β_k is in $[y-1, y]$, then $\frac{F'(x)}{(F(x))^2} = \frac{(p'(x+1)p(x)-p(x+1)p'(x))(p(x))^2}{(p(x+1)-p(x))^2(p(x))^2}$ can be extended to be continuous and bounded on $[y-1, y]$. By Lemma 2.11 $(F(x))^2 \leq -nF'(x)$. Dividing both sides of this inequality by $n(F(x))^2$ and integrating from $y-1$ to y we have

$$\frac{1}{n} \leq \frac{1}{F(y)} - \frac{1}{F(y-1)} = \frac{p(y)}{p(y+1)-p(y)} - \frac{p(y-1)}{p(y)-p(y-1)}.$$

Using $\min\{p(y+1), p(y)\} < p(y) < \max\{p(y+1), p(y-1)\}$, we have that either $p(y-1) < p(y) < p(y+1)$ or $p(y+1) < p(y) < p(y-1)$. In both cases, $(p(y+1)-p(y))(p(y)-p(y-1)) > 0$ and therefore

$$\begin{aligned} \frac{1}{n}(p(y+1)-p(y))(p(y)-p(y-1)) &\leq p(y)(p(y)-p(y-1)) - p(y-1)(p(y+1)-p(y)) \\ &\leq (p(y))^2 - p(y+1)p(y-1). \end{aligned}$$

Dividing both sides by $(p(y))^2$ gives the result. □

Lemma 2.14. *For $p(x) \in \mathcal{L}\text{-}\mathcal{P}_n$, the associated functions $F(x)$ and $R(x)$ from Definition 2.4 satisfy*

$$F(x)R(x) = (F(x) - R(x)) + \frac{(p(x))^2 - p(x+1)p(x-1)}{(p(x))^2}$$

for all $x \in \mathbb{R}$, where $p(x) \neq 0$.

Proof. This lemma is verified by direct calculation using the definitions of $F(x)$ and $R(x)$ in terms of $p(x)$. □

Lemma 2.15. *Let $p(x) \in \mathcal{L}\text{-}\mathcal{P}_n$, $n \geq 2$, with $\mu(p) \geq 1$.*

(i) *If $p(\beta) = p(\beta+1) > 0$, then for all $x \in (\beta, \beta+1)$, $p(x) > p(\beta)$ and $p(x) > \max\{p(x+1), p(x-1)\}$.*
(ii) *If $p(\beta) = p(\beta+1) < 0$, then for all $x \in (\beta, \beta+1)$, $p(x) < p(\beta)$ and $p(x) < \min\{p(x+1), p(x-1)\}$.*
(iii) *If $p(\beta) = p(\beta+1) = 0$, then for all $x \in (\beta, \beta+1)$, either $p(x) > \max\{p(x+1), p(x-1)\}$ or $p(x) < \min\{p(x+1), p(x-1)\}$.*

Proof. Note that by Lemma 2.9, any β which satisfies $p(\beta) = p(\beta+1)$ under the hypotheses stated in Lemma 2.15 must be real and simple since β is a zero of $F(x)$.

For case (i), assume for a contradiction that there exists $x_0 \in (\beta, \beta+1)$ such that $p(x_0) \leq p(\beta)$. There can not be any zeros of p on $(\beta, \beta+1)$, if there were, $p(\beta)p(\beta+1) > 0$ implies that $p(x)$ must have at least two zeros on $(\beta, \beta+1)$, which contradicts $\mu(p) \geq 1$. Thus, for all $x \in (\beta, \beta+1)$, $p(x) > 0$. Specifically $p(x_0) > 0$.

Since $p(x)$ does not change sign on $(\beta, \beta+1)$, the interval $(\beta, \beta+1)$ must lie between two neighboring zeros of $p(x)$, call them α_1 and α_2, such that $(\beta, \beta+1) \subset (\alpha_1, \alpha_2)$. By the mean value theorem there exists $a \in (\beta, \beta+1)$ with $p'(a) = 0$. The zeros of $p(x)$ and $p'(x)$ interlace, and in order to preserve the interlacing a must be the only zero of $p'(x)$ in (α_1, α_2), hence $p'(\beta), p'(\beta+1) \neq 0$. Because the zeros are simple, for some $\varepsilon > 0$, for all $x \in (\alpha_1, \alpha_1 + \varepsilon)$, $p'(x)p(x) > 0$, and for all $x \in (\alpha_2 - \varepsilon, \alpha_2)$, $p'(x)p(x) < 0$. Since p' and p do not change sign on (α_1, β) or $(\beta+1, \alpha_2)$, this gives us that $p'(\beta) > 0$ and $p'(\beta+1) < 0$. Then if $p(x_0) \leq p(\beta)$, p' must change signs at least twice on (α_1, α_2) (actually three times), at least once on (β, x_0) and at least once on $(x_0, \beta+1)$, and this contradicts the uniqueness of a. Thus for all $x \in (\beta, \beta+1)$ we have $p(x) > p(\beta)$.

To show $p(x) > p(\beta)$ implies $p(x) > \max\{p(x+1), p(x-1)\}$ for all $x \in (\beta, \beta+1)$, notice that since $p'(y) < 0$ for all $y \in (\beta+1, \alpha_2)$, $p(\beta+1) > p(y)$ for all $y \in (\beta+1, \alpha_2)$, and due to the zero spacing $p \leq 0$ on (α_2, α_2+1), hence $p(\beta+1) > p(x+1)$ for all $x \in (\beta, \alpha_2)$. Thus, for all $x \in (\beta, \beta+1)$, $p(x) > p(\beta+1) > p(x+1)$. In the same way, $p'(y) > 0$ for $y \in (\alpha_1, \beta)$ and $p \leq 0$ on $(\alpha_1 - 1, \beta)$ imply that $p(\beta) > p(x)$ for all $x \in (\alpha_1 - 1, \beta)$ and therefore $p(x) > p(x-1)$ for all $x \in (\beta, \beta+1)$. Hence, for all $x \in (\beta, \beta+1)$, $p(x) > p(x-1)$ and $p(x) > p(x+1)$, therefore $p(x) > \max\{p(x+1), p(x-1)\}$.

Consider case (iii). If $p(\beta) = p(\beta+1) = 0$, then p does not change sign on $(\beta, \beta+1)$ since $\mu(p) \geq 1$. It suffices to consider the case when p is positive on $(\beta, \beta+1)$. Then for all $x \in (\beta, \beta+1)$, $p(x) > 0 = p(\beta)$. The conclusion $p(x) > \max\{p(x+1), p(x-1)\}$ ($p(x) < \min\{p(x+1), p(x-1)\}$) is a consequence of $p(x) > p(\beta)$ ($p(x) < p(\beta)$) by the same argument given in the proof of case (i).

To prove (ii), let $g(x) = -p(x)$ and apply (i).

□

Lemma 2.16. *If $p(x) \in \mathcal{L}\text{-}\mathcal{P}_n$, $n \geq 2$, $\mu(p) \geq 1$, and $g(x) = p(x+1) - p(x)$, then $\mu(g) \geq 1$.*

Proof. (*Reductio ad Absurdum*) If $\mu(g) < 1$, then there exist $\beta_1, \beta_2 \in \mathbb{R}$ such that $0 < \beta_2 - \beta_1 < 1$ and $g(\beta_1) = g(\beta_2) = 0$. In the proof of Lemma 2.15 we have shown that $p(x)$ does not change sign on (β_1, β_1+1). Without loss of generality assume that p is positive on (β_1, β_1+1). Observe that $\beta_2 \in (\beta_1, \beta_1+1)$, and thus by Lemma 2.15, $p(\beta_2) > \max\{p(\beta_2+1), p(\beta_2-1)\} \geq p(\beta_2+1)$. But this yields $p(\beta_2+1) - p(\beta_2) < 0$, and therefore $g(\beta_2) < 0$ contradicting $g(\beta_2) = 0$. □

Note that Lemma 2.16 is equivalent to the statement that if $p(x) \in \mathcal{L}\text{-}\mathcal{P}_n$ with $\mu(p) \geq 1$, then the associated functions $F(x)$ and $R(x)$ also have zeros spaced at least 1 unit apart. Preliminaries aside, we prove Conjecture 1.4 of I. Krasikov.

Theorem 2.17. *If $p(x) \in \mathcal{L}\text{-}\mathcal{P}_n$ and $\mu(p) \geq 1$, then*

$$f_n(x,1,p) = (n-1)[p(x+1) - p(x-1)]^2 - 4np(x)[p(x+1) - 2p(x) + p(x-1)] \geq 0 \tag{11}$$

holds for all $x \in \mathbb{R}$.

Proof. Since (11) is true when $\deg(p(x))$ is 1 or 2, we assume $n \geq 2$. Fix $x = x_0 \in \mathbb{R}$. If $p(x_0 - 1) = p(x_0) = p(x_0+1)$, or if $p(x_0) = 0$, then $f_n(x,1,p) \geq 0$. Thus, we may assume $p(x_0) \neq 0$. If $p(x_0) < \min\{p(x_0+1), p(x_0-1)\}$, or if $p(x_0) > \max\{p(x_0+1), p(x_0-1)\}$, then $f_n(x_0, 1, p) \geq 0$ (use (7) and Lemma 2.3).

We next consider the case when

$$\min\{p(x_0-1), p(x_0+1)\} < p(x_0) < \max\{p(x_0-1), p(x_0+1)\} \tag{12}$$

(thus $x_0 \neq \beta$ or $\beta+1$, where $p(\beta+1) = p(\beta)$), and show

$$\frac{f_n(x_0,1,p)}{(p(x_0))^2} = (n-1)(F(x_0)+R(x_0))^2 - 4n(F(x_0)-R(x_0)) \geq 0,$$

where $F(x)$ and $R(x)$ are defined by (8) and (9) respectively. By Lemma 2.14,

$$\frac{f_n(x_0,1,p)}{(p(x_0))^2} = (n-1)(F(x_0)-R(x_0))^2 \\ -4n\left(\frac{1}{n}F(x_0)R(x_0) - \frac{(p(x_0))^2 - p(x_0+1)p(x_0-1)}{(p(x_0))^2}\right). \tag{13}$$

By Lemma 2.16, $\mu(p(x+1)-p(x)) \geq 1$, and thus the zeros $\{\beta_k\}_{k=1}^{n-1}$ of $F(x)$ ($p(\beta_k+1) = p(\beta_k)$) are spaced at least one unit apart. If $[x_0-1, x_0]$ does not contain any β_k, $\frac{f_n(x_0,1,p)}{(p(x_0))^2} \geq 0$ holds by Lemma 2.13 (see (13)) . If, on the other hand, $\beta_j \in (x_0-1, x_0)$ (recall $\beta_j \neq x_0, x_0-1$), then $x_0 \in (\beta_j, \beta_j+1)$ and by Lemma 2.15 either $p(x_0) > \max\{p(x_0-1), p(x_0+1)\}$ or $p(x_0) < \min\{p(x_0-1), p(x_0+1)\}$, and both of these cases contradict our assumption (see (12)). We have now shown $f_n(x_0,1,p)) \geq 0$ for all $x_0 \in \mathbb{R}$, except for the isolated points where $x_0 = \beta_j$ or $x_0 = \beta_j + 1$ for some j, but by continuity of $f_n(x,1,p)$, (11) will hold.

□

The converse of Theorem 2.17 is false in general. Indeed, the following example shows that there are polynomials with arbitrary minimal zero spacing that still satisfy $f_n(x,1,p) \geq 0$ for all $x \in \mathbb{R}$.

Example 2.18. Let $p(x) = (x+n+a)\prod_{k=1}^{n-1}(x+k)$ with $n \geq 2$, $a \in \mathbb{R}$. Using a symbolic manipulator (we used Maple)

$$f_n(x,1,p) = C(x,n,a)\prod_{k=2}^{n-2}(x+k)^2$$

where

$$\begin{aligned} C(x,n,a) :=\ & (n-1)(-2n^3 - 4na + 4a^2 + n^2 + n^4)x^2 \\ & + (n-1)(6n^2a + 4n^4 - 8n^3a + 8a^2 - 12na + 4na^2 - 8n^3 + 2n^4a + 4n^2)x \\ & + (n-1)(-8na - 4na^2 + 4a^2 + 4n^4a - 8n^3 + 4n^4 + 4n^2 + 12n^2a \\ & + n^4a^2 + 13n^2a^2 - 16n^3a - 6n^3a^2). \end{aligned} \tag{14}$$

$C(x,n,a)$ is quadratic in x and its discriminant is $D = -16na^2(n-1)^2(n-2)^3(a-n)^2 \leq 0$. Therefore $C(x,n,a)$ does not change sign and is always positive (this is verified by showing that the coefficient of x^2 is positive when considered as a quadratic in a), whence $f_n(x,1,p) \geq 0$ for all $x \in \mathbb{R}$.

In general, a polynomial p may satisfy $f_n(p,1,x) \geq 0$ for all $x \in \mathbb{R}$, even if p has multiple zeros. If $p(x) = x^2(x+1)$, which has $\mu(p) = 0$, then $f_3(x,1,p) = 56x^2 + 32x + 8$ is non-negative for all $x \in \mathbb{R}$. A polynomial p with non-real zeros may also satisfy $f_n(p,1,x) \geq 0$ for all $x \in \mathbb{R}$. For example, let $p(x) = (x^2+1)(x+1)$, then $f_3(x,1,p) = 32x^2 - 32x + 8 \geq 0$ for all $x \in \mathbb{R}$.

It is known that a polynomial $p(x) \in \mathcal{L}\text{-}\mathcal{P}_n$ with only real zeros satisfies $\mu(p) \leq \mu(p')$; that is, $p'(x)$ will have a minimal zero spacing which is larger than that of $p(x)$ (N. Obreschkoff [16, p. 13, Satz 5.3], P. Walker [19]). In light of Lemma 2.16, the aforementioned result suggests the following conjecture.

Conjecture 2.19. If $p(x) \in \mathcal{L}\text{-}\mathcal{P}_n$, $n \geq 2$, $\mu(p) \geq d \geq 1$, and $g(x) = p(x+1) - p(x)$, then $\mu(g) \geq d$.

The derivation of the classical Laguerre inequality relies on properties of the logarithmic derivative of a polynomial. In the same way, Conjecture 1.4 was proved using a discrete version of the logarithmic derivative. The analogy between the discrete and continuous logarithmic derivatives motivates the following conjectures, based on Theorem 2.20 and its converse (B. Muranaka [14]).

Theorem 2.20. (P. B. Borwein and T. Erdélyi [1, p. 345]) *If $p \in \mathcal{L}\text{-}\mathcal{P}_n$, then*

$$m\left(\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda\right\}\right) = \frac{n}{\lambda} \qquad \textit{for all } \lambda > 0,$$

where m denotes Lebesgue measure.

Conjecture 2.21. If $p \in \mathcal{L}\text{-}\mathcal{P}_n$, $n \geq 2$, $\mu(p) \geq 1$, then

$$m\left(\left\{x \in \mathbb{R} : \frac{p(x+1) - p(x)}{p(x)} \geq \lambda\right\}\right) = \frac{n}{\lambda} \qquad \text{for all } \lambda > 0,$$

where m denotes Lebesgue measure.

Conjecture 2.22. If $p(x)$ is a real polynomial of degree $n \geq 2$, and if

$$m\left(\left\{x \in \mathbb{R} : \frac{p(x+1) - p(x)}{p(x)} \geq \lambda\right\}\right) = \frac{n}{\lambda} \qquad \text{for all } \lambda > 0,$$

where m denotes Lebesgue measure, then $p \in \mathcal{L}\text{-}\mathcal{P}_n$ with $\mu(p) \geq 1$.

3. Extension to a Class of Transcendental Entire Functions

In analogy with (5) we define, for a real entire function φ,

$$f_\infty(x,h,\varphi) := [\varphi(x+h) - \varphi(x-h)]^2 - 4\varphi(x)[\varphi(x+h) - 2\varphi(x) + \varphi(x-h)]. \tag{15}$$

For $\varphi \in \mathcal{L}\text{-}\mathcal{P}$, with zeros $\{\alpha_i\}_{i=1}^{\omega}$, $\omega \leq \infty$, we introduce the mesh size

$$\mu_\infty(\varphi) := \inf_{i \neq j} |\alpha_i - \alpha_j|. \tag{16}$$

We remark that if $\psi \notin \mathcal{L}\text{-}\mathcal{P}$, then ψ need not satisfy $f_\infty(x,h,\psi) \geq 0$ for all $x \in \mathbb{R}$. A calculation shows that if $\psi(x) = e^{x^2}$, then $f_\infty(0,1,\psi) = -8(e-1) < 0$. When $\varphi \in \mathcal{L}\text{-}\mathcal{P}_n$, $f_\infty(x,h,\varphi) \geq 0$ for all $x \in \mathbb{R}$ by Theorem 2.17. In order to extend Theorem 2.17 to transcendental entire functions, we require the following preparatory result to ensure that the approximating polynomials we use will satisfy a zero spacing condition.

Lemma 3.1. *For any $a \in \mathbb{R}$, $n \in \mathbb{N}$, $n \geq 2$,*

$$\lim_{n\to\infty} \sum_{k=1}^{n^n} \frac{1}{n\ln(n)(k+n)+a} = 1.$$

Proof. Fix $a \in \mathbb{R}$. Since the terms $\frac{1}{n\ln(n)(k+n)+a}$ are decreasing with k for n sufficiently large, we obtain

$$\int_1^{n^n+1} \frac{1}{n\ln(n)(k+n)+a}dk \leq \sum_{k=1}^{n^n} \frac{1}{n\ln(n)(k+n)+a} \leq \int_0^{n^n} \frac{1}{n\ln(n)(k+n)+a}dk,$$

for n sufficiently large, by considering the approximating Riemann sums for the integrals. Thus

$$\frac{1}{n\ln(n)}\ln\left(\frac{n^n+1+\frac{a}{n\ln(n)}}{n+1+\frac{a}{n\ln(n)}}\right)\leq\sum_{k=1}^{n^n}\frac{1}{n\ln(n)(k+n)+a}\leq\frac{1}{n\ln(n)}\ln\left(\frac{n^n+\frac{a}{n\ln(n)}}{n+\frac{a}{n\ln(n)}}\right).\tag{17}$$

As $n\to\infty$, both the left and right sides of (17) approach 1, and whence the sum in the middle approaches 1. □

Lemma 3.2. *The set of polynomials* $\left\{q_n(x)=\prod_{k=1}^{n^n}\left(1+\frac{x}{n\ln(n)(k+n)}\right): n\in\mathbb{N},\ n\geq 2\right\}$, *forms a normal family on* $\mathbb{C}$. *There is a subsequence of* $\{q_n(x)\}_{n=2}^{\infty}$ *which converges uniformly on compact subsets of* $\mathbb{C}$ *to* e^x.

Proof. Let $K\subset\mathbb{C}$ be any compact set and let $R=\sup_{z\in K}|z|$. Recall the inequality

$$\frac{1}{2}|z|\leq|\ln(1+z)|\leq\frac{3}{2}|z|\qquad\text{for } |z|<\frac{1}{2}$$

[2, p. 165]. Then for $n>2R$, $\left|\frac{z}{n\ln(n)(k+n)}\right|<\frac{1}{2}$, hence, for $k\geq 1$ and $z\in K$

$$\frac{1}{2}\frac{|z|}{n\ln(n)(k+n)}\leq\left|\ln\left(1+\frac{z}{n\ln(n)(k+n)}\right)\right|\leq\frac{3}{2}\frac{|z|}{n\ln(n)(k+n)},$$

and therefore

$$\frac{1}{2}\sum_{k=1}^{n^n}\frac{|z|}{n\ln(n)(k+n)}\leq\sum_{k=1}^{n^n}\left|\ln\left(1+\frac{z}{n\ln(n)(k+n)}\right)\right|\leq\frac{3}{2}\sum_{k=1}^{n^n}\frac{|z|}{n\ln(n)(k+n)}.$$

As $n\to\infty$ the sums on the left and right sides of the inequality converge by Lemma 3.1 to $\frac{1}{2}|z|$ and $\frac{3}{2}|z|$ respectively. In particular, for some $\varepsilon>0$ and $N>2R$ sufficiently large, for all $n\geq N$ and for all $z\in K$,

$$\sum_{k=1}^{n^n}\left|\ln\left(1+\frac{z}{n\ln(n)(k+n)}\right)\right|\leq\frac{3}{2}R+\varepsilon.$$

Then for all $n\geq N$, for all $z\in K$,

$$|q_n(z)|\leq e^{\sum_{k=1}^{n^n}\left|\ln\left(1+\frac{z}{n\ln(n)(k+n)}\right)\right|}\leq e^{\frac{3}{2}R+\varepsilon}.$$

So for $n>N$ sufficiently large, the sequence $\{q_n(z)\}_{n=2}^{\infty}$ is uniformly bounded on compact subsets $K\subset\mathbb{C}$ and thus form a normal family by Montel's theorem [2, p. 153]. Thus, there is a subsequence of $\{q_n(z)\}_{n=2}^{\infty}$ which converges uniformly on compact subsets of $\mathbb{C}$ to a function f, and therefore satisfies

$$\frac{f'(x)}{f(x)}=\lim_{n\to\infty}\frac{q_n'(x)}{q_n(x)}=\lim_{n\to\infty}\sum_{k=1}^{n^n}\frac{1}{n\ln(n)(k+n)+x}=1,\tag{18}$$

for a fixed $x\in\mathbb{R}$, where the last equality is by Lemma 3.1. Equation (18) and $f(0)=1$, imply $f(x)=e^x$ on $\mathbb{R}$, and thus f is the exponential function. □

Lemma 3.3. *If* $\varphi(x)=p(x)e^{bx}$, $b\in\mathbb{R}$, $p\in\mathcal{L}\text{-}\mathcal{P}_n$, $n\geq 2$, *and* $\mu(p)\geq 1$, *then* $f_\infty(x,1,\varphi)\geq 0$ *for all* $x\in\mathbb{R}$.

Proof. By Lemma 3.2, there is a subsequence of $\left\{q_j(x) = \prod_{k=1}^{j^j}\left(1 + \frac{x}{j\ln(j)(k+j)}\right)\right\}_{j=2}^{\infty}$, call it $\{q_{j_m}(x)\}_{m=1}^{\infty}$, such that $q_{j_m}(x) \to e^x$ uniformly on compact subsets of $\mathbb{C}$, as $m \to \infty$. Let $\{\alpha_k\}_{k=1}^n$ be the zeros of $p(x)$, and $R = \max_{1\le k\le n} |\alpha_k|$. The zero of least magnitude of $q_{j_m}(bx)$, z_{j_m}, satisfies $|z_{j_m}| = \frac{j_m \ln(j_m)(1+j_m)}{b}$, $b \neq 0$. Both $\mu(q_{j_m}(bx)) \to \infty$ as $m \to \infty$ and $|z_{j_m}| \to \infty$ as $m \to \infty$. Thus, there is an M such that for all $m > M$, $|z_{j_m}| > R + 1$, and the sequence of polynomials $h_m(x) = p(x)q_{j_{M+m}}(bx)$, $m \ge 1$, is in $\mathcal{L}$-$\mathcal{P}_\ell$ for some ℓ, and satisfies $\mu(h_m) \ge 1$. By Theorem 2.17, $f_\infty(x, 1, h_m) \ge 0$ for all $x \in \mathbb{R}$, for all m. Since $h_m \to p(x)e^{bx}$ by construction, $\lim_{m\to\infty} f_\infty(x, 1, h_m) = f_\infty(x, 1, p(x)e^{bx}) \ge 0$. □

Theorem 3.4. *If $\varphi \in \mathcal{L}$-$\mathcal{P}$ has order $\rho < 2$, or if φ is of minimal type of order $\rho = 2$, and $\mu_\infty(\varphi) \ge 1$, then $f_\infty(x, 1, \varphi) \ge 0$ for all $x \in \mathbb{R}$.*

Proof. By the Hadamard factorization theorem, φ has the representation

$$\varphi(x) = cx^m e^{bx} \prod_{k=1}^{\omega}\left(1 + \frac{x}{a_k}\right)e^{-\frac{x}{a_k}} \qquad (\omega \le \infty),$$

where $a_k, b, c \in \mathbb{R}$, m is a non-negative integer, $a_k \neq 0$, and $\sum_{k=1}^{\omega} \frac{1}{a_k^2} < \infty$. Let

$$g_n(x) = cx^m e^{bx} \prod_{k=1}^{n}\left(1 + \frac{x}{a_k}\right)e^{-\frac{x}{a_k}}.$$

Then, $g_n(x) = ce^{bx - \sum_{k=1}^n \frac{x}{a_k}} x^m \prod_{k=1}^n \left(1 + \frac{x}{a_k}\right)$ has the form $p(x)e^{\gamma x}$, $\gamma \in \mathbb{R}$, $p \in \mathcal{L}$-$\mathcal{P}_n$, and thus by Lemma 3.3, $f_\infty(x, 1, g_n) \ge 0$ for all $x \in \mathbb{R}$, and for all n. Since we also have $g_n \to \varphi$ by construction, $\lim_{n\to\infty} f_\infty(x, 1, g_n) = f_\infty(x, 1, \varphi) \ge 0$ for all $x \in \mathbb{R}$. □

In light of Theorem 3.4, we make the following conjecture.

Conjecture 3.5. If $\varphi \in \mathcal{L}$-$\mathcal{P}$ and $\mu_\infty(\varphi) \ge 1$ then $f_\infty(x, 1, \varphi) \ge 0$ for all $x \in \mathbb{R}$.

References

[1] P. B. Borwein and T. Erdélyi, *Polynomials and polynomial inequalities*, Springer-Verlag, New York, 1995.

[2] J. B. Conway, *Functions of One Complex Variable I*, Springer, New York, 1978.

[3] T. Craven and G. Csordas, *Jensen polynomials and the Turán and Laguerre inequalities*, Pacific J. Math., **136** (1989), 241–260.

[4] G. Csordas and A. Escassut, *The Laguerre inequality and the distribution of zeros of entire functions*, Ann. Math. Blaise Pascal, **12** (2005), 331–345.

[5] W. H. Foster and I. Krasikov, *Inequalities for real-root polynomials and entire functions*, Adv. in Appl. Math., **29** (2002), 102–114.

[6] W. H. Foster and I. Krasikov, *Bounds for the extreme roots of orthogonal polynomials*, Int. J. of Math. Algorithms, **2** (2000), 121–132.

[7] W. H. Foster and I. Krasikov, *Explicit bounds for Hermite polynomials in the oscillatory region*, LMS J. Comput. Math., **3** (2000), 307–314.

[8] I. Krasikov, *Discrete analogues of the Laguerre inequality*, Anal. Appl. (Singap.), **1** (2003), 189–197.

[9] I. Krasikov, *Bounds for the Christoffel-Darboux kernel of the binary Krawtchouk polynomials*, in Codes and Association Schemes (Pistcataway, NJ, 1999), 193–198, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., **56**, Amer. Math. Soc., Providence, RI, 2001.

[10] I. Krasikov, *Nonnegative quadratic forms and bounds on orthogonal polynomials*, J. Approx. Theory, **111** (2001), 31–49.

[11] I. Krasikov and A. Zarkh, *On the zeros of discrete orthogonal polynomials*, J. Approx. Theory, **156** (2009), 121–141.

[12] B. Ja. Levin, *Distribution of Zeros of Entire Functions, Transl. Math. Mono. Vol. 5*, Amer. Math. Soc., Providence, RI (1964); revised ed. 1980.

[13] J. B. Love, Problem E 1532, Amer. Math. Monthly, **69** (1962), 668.

[14] B. Muranaka, *The Laguerre inequality and the distribution of zeros of entire functions*, Master's thesis, University of Hawaii, Honolulu, Hawaii, December 2003.

[15] A. F. Nikiforov, S. K. Suslov, and V. B. Urarov, *Classical orthogonal polynomials of a discrete variable*, Springer-Verlag, Berlin (1991).

[16] N. Obreschkoff, *Verteilung und Berechnung der Nullstellen reeller Polynome*, Veb Deutscher Verlag der Wissenschaften, Berlin, 1963.

[17] M. L. Patrick, *Extension of inequalities of the Laguerre and Turán type*, Pacific J. Math., **44** (1973), 675–682.

[18] G. Pólya and G. Szegő, *Problems and Theorems in Analysis, vol. II*, Springer-Verlag, New York (1976).

[19] P. Walker, *Bounds for the separation of real zeros of polynomials*, J. Austral. Math. Soc. Ser. A, **59** (1995), 330–342.

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