LAGUERRE MULTIPLIER SEQUENCES AND SECTOR PROPERTIES OF ENTIRE FUNCTIONS

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Abstract. We show that the class of multiplier sequences which can be interpolated by rational functions is the same as the class of multiplier sequences which can be interpolated by polynomials. In addition, a result of Obreschkoff is used to show that Jensen polynomials related to the Riemann ξ-function have only real zeros up to degree 10^{17}. Partial results are stated concerning the problem of characterizing linear transformations which take polynomials whose zeros are constrained to lie in a sector to polynomials of the same type.

1. Introduction

Two distinct problems are investigated in this paper, yielding new results about multiplier sequences and n-sequences (see Definition 1.1). The first problem is to characterize all rationally interpolated multiplier sequences. The second problem is to characterize all linear transformations on polynomials which preserve the property that all of the zeros of a polynomial lie in a given sector with vertex at the origin. We begin with the definitions required for studying these problems.

Definition 1.1. ([2]) Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers which define a linear operator T, with the action $T[x^k] = \gamma_k x^k$. If for any polynomial p, of degree n or less, having only real zeros, the polynomial $T[p]$ also has only real zeros or is identically zero, then $\{\gamma_k\}_{k=0}^{\infty}$ is called an *n-sequence*. If $\{\gamma_k\}_{k=0}^{\infty}$ is an n-sequence for all $n \in \mathbb{N}$, then it is called a *multiplier sequence*.

Definition 1.2. ([4]) If $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ is an arbitrary entire function, we call

$$g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k$$

the n^{th} *Jensen polynomial associated* with φ.

A sequence $\{\gamma_k\}_{k=0}^{\infty}$ is an n-sequence if and only if the n^{th} associated Jensen polynomial, $\sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k$, has only real zeros (T. Craven, G. Csordas [2]).

Definition 1.3. A real entire function $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ is in the *Laguerre-Pólya class*, written $\varphi \in \mathscr{L}\text{-}\mathscr{P}$, if it can be expressed in the form

$$\varphi(x) = c x^m e^{-ax^2+bx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right) e^{\frac{-x}{x_k}} \qquad (0 \le \omega \le \infty),$$

where $b, c, x_k \in \mathbb{R}$, m is a non-negative integer, $a \ge 0$, $x_k \neq 0$, and $\sum_{k=1}^{\omega} \frac{1}{x_k^2} < \infty$.

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An entire function belongs to the Laguerre-Pólya class *if and only if* it is a locally uniform limit of real polynomials having only real zeros [10, Chapter VIII]. Theorem 1.4 is a classical result of Pólya and Schur.

Theorem 1.4. (Transcendental Characterization of Multiplier Sequences [16]) *A sequence of non-negative real numbers* $\{\gamma_k\}_{k=0}^{\infty}$ *is a multiplier sequence if and only if* $\varphi(x) = T[e^x] = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathscr{L}\text{-}\mathscr{P}$.

One possible motivation for studying multiplier sequences is the Riemann hypothesis, which is equivalent to the statement that the ξ-function [15],

$$\xi(iz) := \frac{1}{2}\left(z^2 - \frac{1}{4}\right)\pi^{-z/2-1/4}\Gamma\left(\frac{z}{2}+\frac{1}{4}\right)\zeta\left(z+\frac{1}{2}\right), \tag{1}$$

has only real simple zeros. The ξ-function satisfies $\frac{1}{8}\xi(\frac{x}{2}) = \sum_{m=0}^{\infty} \frac{(-1)^m \hat{b}_m x^{2m}}{(2m)!}$, and therefore the zeros of

$$F(x) := \sum_{m=0}^{\infty} \frac{\hat{b}_m}{(2m)!} x^m \tag{2}$$

are real and negative if and only if ξ has only real zeros [7]. The coefficients $\hat{b}_m \geq 0$ in (2) are moments associated with the kernel whose (Fourier) cosine transform is ξ (see [7]).

The following theorem of Laguerre gives sufficient conditions for a function in $\mathscr{L}\text{-}\mathscr{P}$ to interpolate a multiplier sequence or an n-sequence.

Theorem 1.5. (Laguerre's Theorem [3]) *Given* $p(x) \in \mathbb{R}[x]$*, let* $Z_c(p(x))$ *denote the number of non-real zeros of* $p(x)$ *counting multiplicities. Let* $f(x) = \sum_{k=0}^{n} a_k x^k$ *be an arbitrary real polynomial of degree n, let* $\varphi(x) \in \mathscr{L}\text{-}\mathscr{P}$*, and suppose that none of the zeros of* φ *lie in the interval* $(0, n)$*. Then*

$$Z_c\left(\sum_{k=0}^{n} \varphi(k) a_k x^k\right) \leq Z_c(f(x)).$$

Thus the sequence $\{\varphi(k)\}_{k=0}^{\infty}$, as described in Theorem 1.5, will not increase the number of non-real zeros when applied to polynomials of degree n or less. A multiplier sequence which does not increase the number of non-real zeros when applied to any real polynomial is called a *complex zero decreasing sequence (CZDS)*. If φ has only non-positive zeros, then $\{\varphi(k)\}_{k=0}^{\infty}$ is a multiplier sequence, and in addition it is a CZDS. Characterizing all CZDS is an open problem [3], as is the special case of characterizing all meromorphically interpolated CZDS.

In the spirit of extending Laguerre's Theorem, T. Craven and G. Csordas posed Problems 1.5 and 1.5.

Problem 1.5. ([5]) Characterize the meromorphic functions $Y(x) = \varphi(x)/\psi(x)$, where φ and ψ are entire functions such that the polynomial $\sum_{k=0}^{n} Y(k) a_k x^k$ has only real zeros, whenever the polynomial $\sum_{k=0}^{n} a_k x^k$ has only real zeros.

Problem 1.5. ([5]) Characterize the meromorphic functions $Y(x)$ with the property that $\sum_{k=0}^{\infty} Y(k) a_k x^k/k!$ is a transcendental entire function with only real zeros (or the zeros all lie in the half-plane $\operatorname{Re} z < 0$), whenever the entire function $\sum_{k=0}^{\infty} a_k x^k/k!$ has only real zeros.

Meromorphically interpolated multiplier sequences, whose characterization is sought in Problem 1.5, have been named *meromorphic Laguerre multiplier sequences* [5]. Theorem 1.6 exhibits some non-trivial meromorphic Laguerre multiplier sequences.

Theorem 1.6. (T. Craven, G. Csordas [5]) *For each positive integer m, the function*

$$\varphi_m(x) = \sum_{k=0}^{\infty} \frac{(2mk)!}{(3mk)!} \frac{x^k}{k!}$$

has only real negative zeros, and whence the sequence $\{(2mk)!/(3mk)!\}_{k=0}^{\infty}$ *is a multiplier sequence.*

We consider the special case of Problem 1.5, where both φ and ψ are polynomials. The following theorem is proved in Section 2.

Theorem 1.7. *Let* $\{\gamma_k\}_{k=0}^{\infty} = \left\{\frac{p(k)}{q(k)}\right\}_{k=0}^{\infty}$*, with relatively prime* $p, q \in \mathbb{R}[x]$*. If* $\{\gamma_k\}_{k=0}^{\infty}$ *is a multiplier sequence, then* $q(k)$ *must be a constant. If* $\deg(q) \geq 1$*, then for all* $n \in \mathbb{N}$*,* $\left\{\frac{p(k+n)}{q(k+n)}\right\}_{k=0}^{\infty}$ *is not a multiplier sequence.*

In Section 3, an elegant theorem of Obreschkoff is extended to transcendental entire functions and we obtain the following.

Theorem 1.8. *The first* 2×10^{17} *Jensen polynomials associated with the function F, defined by* (2)*, have only real zeros.*

At the end of Section 3, we briefly consider the problem of characterizing linear operators which map polynomials whose zeros are constrained to lie in a sector to polynomials of the same type.

2. Rationally interpolated sequences

In this section, Theorem 1.7 is proved as an immediate consequence of the following two fundamental results. Let $D := \frac{d}{dx}$ throughout the rest of the paper.

Theorem 2.1. (Hermite-Poulain Theorem [10, Chapter VIII]) *If* $q(x) := \sum\limits_{j=0}^{n} b_j x^j$ *has only real zeros and if* $f(x) \in \mathbb{R}[x]$*, then*

$$Z_c(q(D)[f(x)]) \leq Z_c(f(x))$$

where $Z_c(f(x))$ *denotes the number of non-real zeros of* $f(x)$*, counting multiplicities.*

Proposition 2.2. (Turán Inequalities [4]) *Let* $\varphi \in \mathscr{L}\text{-}\mathscr{P}$*,* $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$*. For each k the Turán Inequalities hold for the Taylor coefficients of* φ*; that is*

$$T_k(\varphi) := \gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0, \qquad k = 1, 2, 3, \ldots. \tag{3}$$

Lemma 2.3 is equivalent to the statement that if a sequence is interpolated by a rational function, where the denominator's degree is larger than that of the numerator, then that sequence can not be a multiplier sequence.

Lemma 2.3. *Let* $\{\eta_k\}_{k=0}^{\infty} = \left\{\frac{p(k)}{q(k)}\right\}_{k=0}^{\infty}$*, where* $p(x), q(x) \in \mathbb{R}[x]$*,* $\deg(p) < \deg(q)$*,* $p \not\equiv 0$*, and* $q(k) \neq 0$*, for* $k = 0, 1, 2, \ldots$*. Then every derivative of the entire function* $\varphi(x) = \sum_{k=0}^{\infty} \eta_k \frac{x^k}{k!}$ *has non-real zeros.*

Proof. Let $n = deg(p) < deg(q) = m$, then

$$\frac{1}{q(x)} = \frac{1}{a_m}\prod_{j=1}^{m}\frac{1}{x-\alpha_j} = \frac{1}{a_m}\frac{1}{x^m}\prod_{j=1}^{m}\left(\sum_{k=0}^{\infty}\left(\frac{\alpha_j}{x}\right)^k\right), \qquad |x| > \max_{1\le j\le m}|\alpha_j|,$$

where $\{\alpha_j\}_{j=1}^m$ are the zeros of $q(x)$ and a_m is the leading coefficient of $q(x)$. Then

$$Y(k) = \frac{p(k)}{q(k)} = \frac{A}{k^d} + \frac{B}{k^{d+1}} + \frac{C}{k^{d+2}} + O\left(\frac{1}{k^{d+3}}\right), \qquad (k\to\infty), \tag{4}$$

where $d = m - n > 0$, and $A, B, C \in \mathbb{R}$. Note that $A \neq 0$; if we multiply both sides of Equation (4) by k^d and let $k \to \infty$, we obtain that A is equal to the ratio of the leading coefficients of p and q. Using (4) to compute the Turán expression for large k, we obtain

$$Y(k)^2 - Y(k+1)Y(k-1) = \frac{-A^2 d}{k^{2d+2}} - \frac{2AB(d+1)}{k^{2d+3}} + O\left(\frac{1}{k^{2d+4}}\right), \qquad (k\to\infty). \tag{5}$$

The first term on the right-hand side of (5) is always negative and thus for sufficiently large k, $Y(k)$ fails to satisfy the Turán inequalities. Therefore, φ must have non-real zeros. Since the derivatives satisfy $\varphi^{(\ell)}(x) = \sum_{k=0}^{\infty}\eta_{k+\ell}\frac{x^k}{k!} = \sum_{k=0}^{\infty}Y(k+\ell)\frac{x^k}{k!}$, the asymptotic formula (5) still holds, and therefore every derivative of φ has non-real zeros. □

Definition 2.4. For any real sequence $\{\gamma_k\}_{k=0}^{\infty}$ we define $\Delta^n\gamma_p = \sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}\gamma_{p+j}$ for $n, p = 0, 1, 2, \ldots$, where $\Delta^0\gamma_p = \gamma_p$. For a real valued function $\gamma(k)$ we define $\Delta\gamma(k) := \gamma(k+1) - \gamma(k)$ and $\Delta^n\gamma(k)$ analogously, where the function argument replaces the index.

We now use Lemma 2.3 and the Hermite-Poulain Theorem to prove Theorem 1.7.

Proof. (Proof of Theorem 1.7) For the first assertion assume that $\deg(q) \geq 1$ and that $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence. We arrive at a contradiction as follows. By the division algorithm, there exist unique $g, r \in \mathbb{R}[x]$ such that

$$\frac{p(k)}{q(k)} = g(k) + \frac{r(k)}{q(k)}, \tag{6}$$

where $\deg(r) < \deg(q)$. Let T be the linear operator defined on monomials by

$$T[x^k] = \gamma_k x^k = \frac{p(k)}{q(k)}x^k = \left(g(k) + \frac{r(k)}{q(k)}\right)x^k.$$

Assuming that $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence, $(D-1)^m T[e^x]$ will have only real zeros for all m by the Hermite-Poulain Theorem (where $D := \frac{d}{dx}$). But an application of $(D-1)$ to $T[e^x]$ yields,

$$(D-1)T[e^x] = (D-1)\sum_{k=0}^{\infty}\gamma_k\frac{x^k}{k!} = \sum_{k=0}^{\infty}\Delta\gamma_k\frac{x^k}{k!} \tag{7}$$

where,

$$\Delta\gamma_k = \left(g(k+1) + \frac{r(k+1)}{q(k+1)}\right) - \left(g(k) + \frac{r(k)}{q(k)}\right) = \Delta g(k) + \Delta\left(\frac{r(k)}{q(k)}\right), \tag{8}$$

and $\deg(\Delta g) < \deg(g)$. Continuing in the same way, $(D-1)^m$ replaces γ_k with $\Delta^m\gamma_k$ when it operates on $T[e^x]$, and

$$\Delta^m\gamma_k = \Delta^m g(k) + \Delta^m\left(\frac{r(k)}{q(k)}\right).$$

Choose the positive integer $m > \deg(g) = \deg(p) - \deg(q)$. Then an induction argument easily shows that $\Delta^m g(k) = 0$ and $\Delta^m\left(\frac{r(k)}{q(k)}\right) = \frac{n(k)}{d(k)}$, where $\deg(n) < \deg(d)$. Therefore,

$$(D-1)^m T[e^x] = \sum_{k=0}^{\infty} \frac{n(k)}{d(k)} \frac{x^k}{k!}.$$

Thus, by Lemma 2.3, $(D-1)^m T[e^x]$ has non-real zeros. Hence, $\{\gamma_k\}_{k=0}^{\infty}$ is not a multiplier sequence and thus we have obtained the desired contradiction. The last assertion follows immediately, since $p(k+n)/q(k+n)$ is still a rational function satisfying the hypotheses in the above argument. □

Corollary 2.5. *For any $h(x) \in \mathbb{R}[x]$, $\varphi(x) = \sum_{k=0}^{\infty} \frac{p(k)}{q(k)} \frac{x^k}{k!}$ with $p, q \in \mathbb{R}[x]$ relatively prime, $\deg(q) \geq 1$, the sum $\psi(x) = \varphi(x) + h(x)$ has infinitely many non-real zeros.*

Proof. Let $m = \deg(h) + 1$, then $D^m\psi(x) = D^m\varphi(x)$. By Theorem 1.7, every derivative of φ has non-real zeros, thus the same holds for ψ. To show that ψ has an infinite number of non-real zeros, assume instead that ψ has a finite number of non-real zeros. Then, by the the Pólya-Wiman Theorem [6], there is an $n \in \mathbb{N}$ such that $D^n\psi$ has only real zeros. This is a contradiction. □

It is worth noting that the shift n in Theorem 1.7 can be any real number. The last assertion of Theorem 1.7 is made in sharp contrast to the following result.

Proposition 2.6. *For any $p(x) \in \mathbb{R}[x]$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\{p(k + n)\}_{k=0}^{\infty}$ is a multiplier sequence.*

Proof. Without loss of generality assume that the leading coefficient of $p(x)$ is positive. Then, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, $p(n)$ is positive. Let T be the linear operator with the action $T[x^k] = p(k)x^k$. Then, it can be shown that

$$T[e^x] = g(x)e^x$$

for some $g \in \mathbb{R}[x]$. By the Pólya-Wiman Theorem [6], there exists $N \in \mathbb{N} \cup \{0\}$, $N \geq M$, such that

$$\sum_{k=0}^{\infty} p(k+N)\frac{x^k}{k!} = D^N T[p(k)] = D^N g(x)e^x$$

has only real zeros. Therefore, if $n \geq N$, then $\sum_{k=0}^{\infty} p(k+n)\frac{x^k}{k!}$ has only real zeros and positive coefficients, since $N \geq M$. Hence, $p(k + n)$ is a multiplier sequence by Theorem 1.4. □

Remark 2.6. Let $\mathscr{L}\text{-}\mathscr{P}^*$ denote the class of real entire functions that can be expressed in the form $\psi(x) = p(x)\varphi(x)$, where $p(x) \in \mathbb{R}[x]$ and $\varphi \in \mathscr{L}\text{-}\mathscr{P}$. If $\psi \in \mathscr{L}\text{-}\mathscr{P}^*$, and ψ has a finite number of positive zeros, then Lemma 2.6 and Theorem 1.5 imply that $\{\psi(k + n)\}_{k=0}^{\infty}$ is a multiplier sequence for sufficiently large n.

Let T be the linear operator with the action $T[x^k] = \frac{1}{k!}x^k$, and S be the linear operator with the action $S[x^k] = \frac{1}{k+1}x^{k+1} = \int_0^x [y^k]\, dy$. We have seen that rationally interpolated sequences may not be multiplier sequences, as is the case for the sequence $\{1/(k + 1)\}_{k=0}^{\infty}$. Similarly, the operator S does not preserve reality of zeros. On the other hand, S^nT preserves the reality of zeros for every non-negative integer n (S^nT is equivalent to multiplication by x^n followed by operating with T). The operator $x^k \mapsto \frac{x^k}{k+\beta} = \frac{\Gamma(k+\beta)}{\Gamma(k+\beta+1)}x^k$, $\beta > 0$, has a nice integral operator representation [14],

$$\frac{x^k}{k+\beta} = \frac{1}{x^\beta}\int_0^x y^{\beta-1}[y^k]\,dy \qquad (\beta > 0). \tag{9}$$

In light of representation (9), it is natural to raise the following question.

Problem 2.6. If R is a rational function, when is

$$\left\{\frac{1}{k!}R(k)\right\}_{k=0}^{\infty} \tag{10}$$

a multiplier sequence?

Note that if $R(k) \geq 0$ for $k \geq 0$ in Problem 2.6, then (10) is a multiplier sequence if and only if $\sum_{k=0}^{\infty} \frac{1}{k!k!}R(k)x^k \in \mathscr{L}\text{-}\mathscr{P}$.

Proposition 2.7. *If β is a positive integer, then the sequence*

$$\left\{\frac{1}{k!(k+\beta)}\right\}_{k=0}^{\infty} \tag{11}$$

is a multiplier sequence.

Proof. If $\beta = 1$ the sequence $\{1/(k+1)!\}_{k=0}^{\infty}$ is a multiplier sequence by Laguerre's Theorem. If $\beta > 1$, the sequences $\{1/(k+\beta)!\}_{k=0}^{\infty}$ and $\{(k+1)(k+2)\cdots(k+\beta-1)\}_{k=0}^{\infty}$ are known multiplier sequences by Laguerre's Theorem. Hence their composition, (11), must also be a multiplier sequence. □

Computer experiments suggest that when β is not an integer, the sequence (11) in Proposition 2.7, is not a multiplier sequence.

Example 2.8. For the sequence

$$\left\{\frac{1}{k!(k+1/2)}\right\}_{k=0}^{\infty}, \tag{12}$$

the fourth degree Jensen polynomial

$$g_4(x) = \sum_{k=0}^{4}\binom{4}{k}\frac{1}{k!(k+1/2)}x^k,$$

has two non-real zeros. Hence, (12) is not a multiplier sequence.

3. Sector Properties

In this section we state some results related to n-sequences and entire functions whose zeros lie in a given sector. Theorem 1.8 is proved, and we consider linear transformations which map polynomials whose zeros lie only in a sector to polynomials of the same type. The proofs of Theorem 3.9, Proposition 3.10, and Theorem 3.11 will appear elsewhere.

We introduce a notation for sectors to simplify the discussion in this section.

Definition 3.1. For $0 < \delta \leq \frac{\pi}{2}$, let

$$S(\phi,\delta) = \{z \in \mathbb{C}\setminus\{0\} \;:\; |\arg(z) - \phi| < \delta\}.$$

Let $\overline{S(\phi,\delta)}$ denote the closure of $S(\phi,\delta)$, and

$$\overline{S(\phi,0)} = \{z \in \mathbb{C} \;:\; z = 0 \text{ or } \arg(z) = \phi\}.$$

In particular, $\overline{S(\pi,\delta)}$ denotes the closed sector symmetric about the negative real axis with half angle δ.

Definition 3.2. Given two entire functions, $\varphi = \sum_{k=0}^{\infty} a_k z^k$ and $\psi = \sum_{k=0}^{\infty} b_k z^k$, the *Schur composition of φ and ψ*, denoted by $\varphi \odot \psi$, is given by

$$(\varphi \odot \psi)(z) := \sum_{k=0}^{\infty} k! a_k b_k z^k,$$

provided the series converges for all $z \in \mathbb{C}$.

Theorem 3.3. (N. Obreschkoff [13]) *Let the real polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$ have only real zeros and let the zeros of $g(x) = \sum_{k=0}^{m} b_k x^k$ lie in $\overline{S(0,\delta) \cup S(\pi,\delta)}$, where* $\sin\delta = \frac{1}{\sqrt{n}}$. *Then the polynomials*

$$g(D)f(x) = b_m f^{(m)}(x) + b_{m-1} f^{(m-1)}(x) + \cdots + b_0 f^{(0)}(x), \qquad b_k \in \mathbb{R},$$

and

$$(g \odot f)(x) = a_0 b_0 + 1! a_1 b_1 x + 2! a_2 b_2 x^2 + \cdots + k! a_k b_k x^k, \qquad b_k \geq 0,$$

have only real zeros, where $k = \min\{m, n\}$.

The following is a well-known characterization of entire functions which are locally uniform limits of polynomials having all of their zeros in a sector.

Theorem 3.4. ([10, Chapter VIII]) *In order that the entire function $\psi(z)$ can be uniformly approximated in every bounded domain by polynomials, all of whose zeros lie in $\overline{S(\theta,\delta)}$ ($\delta < \frac{\pi}{2}$), it is necessary and sufficient that the function can be represented in the form*

$$\psi(z) = c z^q e^{-\sigma z} \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right), \tag{13}$$

where

$$\sum_{k=1}^{\infty} |a_k|^{-1} < \infty, \qquad a_k \in \overline{S(\theta,\delta)}, \qquad \sigma \in \overline{S(\theta,\delta)}. \tag{14}$$

Definition 3.5. Denote by $C(S(\theta,\delta))$ the class of real entire functions ψ that can be represented in the form (13).

Theorem 3.6. *If $\varphi \in C(S(\pi, \delta))$, then for all $g \in \mathbb{R}[x]$ with* $\deg(g) \leq \frac{1}{|\sin\delta|^2}$, $\phi \odot g$ *has only real zeros. In particular, if φ has order $\rho \leq 1$, positive Taylor coefficients, and all of the zeros of φ lie in $S(\pi, \delta)$, then the Jensen polynomials associated with φ up to degree $n \leq \frac{1}{|\sin\delta|^2}$ have only real zeros.*

Proof. If $\varphi \in C(S(\pi, \delta))$, then there exists a sequence $\{p_n(x)\}_{n=0}^{\infty}$ of real polynomials with zeros only in $\overline{S(\pi, \delta)}$, such that $p_n \to \varphi$ locally uniformly. By Theorem 3.3, $p_n \odot g$ has only real zeros for $\deg(g) \leq \frac{1}{|\sin\delta|^2}$, and thus by Hurwitz's Theorem $\varphi \odot g$ has only real zeros, provided it is not identically 0. Now assume that φ has order $\rho \leq 1$, positive Taylor coefficients, and all of its zeros lie in $\overline{S(\pi, \delta)}$. Then writing φ in the form (13) without any assumed parameter restrictions, one may compute $-\sigma = \lim_{x\to\infty} \frac{\varphi'(x)}{\varphi(x)} \geq 0$, which follows from the positivity of the Taylor coefficients. Hence, φ satisfies conditions (14) and therefore $\varphi \in C(S(\pi, \delta))$. The Schur composition property follows from the first argument, and thus the Jensen polynomials $g_n = \varphi \odot b_n$, $b_n(x) = (1 + x)^n$, will have only real zeros up to the degree indicated. □

An Application to the ξ-function. The Riemann hypothesis is equivalent to the statement that the Riemann ξ-function (see (1)) has only real zeros. It is well-known that the zeros of the ξ-function lie in the strip $|\mathrm{Im}(z)| < \frac{1}{2}$ and are symmetric about the origin and the imaginary axis. By (2), the function F associated with ξ satisfies

$$F(-4z^2) = \frac{1}{8}\xi(z), \tag{15}$$

and has order $\rho = \frac{1}{2}$. G. Csordas, T. S. Norfolk, and R. S. Varga [7] proved that the Taylor coefficients of F satisfy the Turán inequalities (3). The validity of these inequalities is equivalent to the statement that all the second degree Jensen polynomials associated with the derivatives $F^{(k)}(x)$, $k = 0, 1, 2, \ldots$, have only real zeros. We we next prove Theorem 1.8. This result shows that a large number of the Jensen polynomials associated with F have only real zeros.

Proof. (Proof of Theorem 1.8) Let t_{max} represent a value such that ξ has only real zeros in the rectangle

$$\left\{z : |\mathrm{Re}(z)| < t_{max} \text{ and } |\mathrm{Im}(z)| < \frac{1}{2}\right\}. \tag{16}$$

Then the zeros of ξ are constrained to lie within the double sector $S(0, \delta) \cup S(\pi, \delta)$, with $\delta = \arctan\left(\frac{1}{2t_{max}}\right)$. Equation (15) implies that the zeros of $F(z)$ must then lie in $S(\pi, 2\delta)$. Moreover, if we set

$$N = \left\lfloor \frac{1}{|\sin(2\delta)|^2} \right\rfloor = \left\lfloor \frac{1}{4\sin^2\delta\cos^2\delta} \right\rfloor = \left\lfloor \frac{(4t_{max}^2 + 1)^2}{16t_{max}^2} \right\rfloor, \tag{17}$$

where $\lfloor\cdot\rfloor$ is the floor function, Theorem 3.6 implies the Jensen polynomials associated with $F(z)$ up to degree N have only real zeros. It is known that ξ has only real zeros in the rectangle (16) for $t_{max} = 545, 439, 823$ [11]. This implies that at least the first $N = 2 \times 10^{17}$ Jensen polynomials of $F(z)$ have only real negative zeros. □

Note that since $C(S(\pi, \delta))$ is closed under differentiation, the Jensen polynomials associated with the derivatives $F^{(k)}$, $k = 0, 1, 2, \ldots$, also have only real zeros for degrees up to the lower bound given by (17). O. Katkova [9] has given a proof that the ξ-function is k-times totally positive for large k using the same type of argument and a theorem of Schoenberg [9, p. 5].

Sector Preservers. We begin by stating a sector version of the Hermite-Poulain Theorem. This theorem is a corollary to a theorem proved by Takagi [12, p. 84].

Theorem 3.7. ([12, p. 84]) *Let $f, g \in \mathbb{C}[z]$, $f(z) = \sum_{k=0}^{n} a_k z^k$, and let the zeros of f lie in $S(\pi, \delta)$, $(\delta \leq \frac{\pi}{2})$. If all the zeros of g lie in $S(\pi, \delta)$, then all the zeros of*

$$f(D)g(x) = a_0 g(x) + a_1 g'(x) + a_2 g''(x) + \cdots + a_n g^{(n)}(x)$$

lie in $S(\pi, \delta)$, provided $f(D)g(x) \not\equiv 0$.

Definition 3.8. Let R be a subset of the complex plane. A linear transformation on a set of polynomials is called an *R-preserver* if it maps polynomials with zeros only in R to polynomials of the same type.

The problem of classifying all linear $S(\theta, \delta)$-preservers for convex $S(\theta, \delta)$ was formally identified in [3] along with the general linear R-preserver problem. J. Borcea and P. Brändén completely solved the linear R-preserver problem in the case that R is a closed half-plane or a circular region [1]. The proofs below are omitted but will appear elsewhere. Using Theorem 3.3, one can prove the following.

Theorem 3.9. *Let $f, g \in \mathbb{R}[z]$, the zeros of g lie in the sector $S(\pi, \phi)$, and the zeros of f lie in the sector $S(\theta, \delta)$, $(\delta < \frac{\pi}{2})$. If $deg(f) \leq \frac{1}{|sin\phi|^2}$, then all the zeros of the Schur composition $g \odot f$ lie in $S(\theta, \delta)$.*

Unfortunately, Theorem 3.9 has a hypothesis on the degree of the polynomial f, and is therefore difficult to use when trying to classify linear operators on a space of polynomials of arbitrary degree. Below are several partial results pertaining to the problem of characterizing linear sector preservers.

Proposition 3.10. *A linear operator is diagonal on the monomial basis, with $T[x^k] = \gamma_k x^k$, and preserves the property that all the zeros of a polynomial lie in a given sector $\overline{S(\theta, \delta)}$, if and only if $\{\gamma_k\}_{k=0}^{\infty}$ is a non-negative multiplier sequence (or a complex constant multiple of a non-negative multiplier sequence).*

If in Proposition 3.10, T is restricted to $\mathbb{R}[x]$, T may correspond to a diagonal operator with action $T[x^k] = \gamma_k x^k$, where $\{\gamma_k\}_{k=0}^{\infty}$ is not a multiplier sequence. This occurs in the case where the sector is the left half-plane [8].

Theorem 3.11. (Corrolary to [1, Theorem 4]) *Suppose that $T : \mathbb{C}[z] \to \mathbb{C}[z]$ is a linear transformation, such that*

(1) $T[(z+w)^n] \neq 0$ *for all $n \in \mathbb{N}$, $z, w \in S(\theta - \delta - \frac{\pi}{2}, \frac{\pi}{2})$, and*
(2) $T[(z+w)^n] \neq 0$ *for all $n \in \mathbb{N}$, $z, w \in S(\theta + \delta + \frac{\pi}{2}, \frac{\pi}{2})$.*

Then T is an $\overline{S(\theta, \delta)}$-preserver.

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