LINEAR PRESERVERS AND ENTIRE FUNCTIONS WITH RESTRICTED ZERO LOCI

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By
Matthew Negus Chasse

Dissertation Committee:

George Csordas, Chairperson
Thomas Craven
Michelle Manes
Wayne Smith
Istvan Szapudi

We certify that we have read this dissertation and that, in our opinion, it is satisfactory in scope and quality as a dissertation for the degree of Doctor of Philosophy in Mathematics.

DISSERTATION COMMITTEE

Chairperson

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ABSTRACT

Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator such that $T[x^k] = \gamma_k x^k$ for all $k = 0, 1, 2, \ldots$, where $\gamma_k \in \mathbb{R}$. The real sequence $\{\gamma_k\}_{k=0}^{\infty}$ is called a *multiplier sequence* if for any $p \in \mathbb{R}[x]$, having only real zeros, $T[p]$ also has only real zeros. A characterization of all multiplier sequences that can be interpolated by rational functions is given. This partially solves a problem of G. Csordas and T. Craven, who asked for a characterization of all the meromorphic functions, $Y(k)$, such that $\{Y(k)\}_{k=0}^{\infty}$ is a multiplier sequence.

An eight-year-old conjecture of I. Krasikov is proved. Several discrete analogues of classical inequalities for polynomials with only real zeros are obtained, along with results which allow extensions to a class of transcendental entire functions in the Laguerre-Pólya class. A study of finite difference operators which preserve reality of zeros is initiated, and new results are proved.

Composition theorems and inequalities for polynomials having their zeros in a sector are obtained. These are analogs of classical results by Pólya, Schur, and Turán. In addition, a result of Obreschkoff is used to show that the Jensen polynomials related to the Riemann ξ-function have only real zeros up to degree 10^{17}. Sufficient conditions are established for a linear transformation to map polynomials having zeros only in a sector to polynomials of the same type, and some multivariate extensions of these results are presented. A complete characterization is given for linear operators which preserve closed ("strict") half-plane stability in the univariate Weyl algebra. These results provide new information about a general stability problem posed formally by G. Csordas and T. Craven. In his 2011 AMS Bulletin article, D. G. Wagner describes recent activity in multivariate stable polynomial theory as "exciting work—elementary but subtle, and with spectacular consequences." Wagner points out that many of the recent advancements in the theory of multivariate stable polynomials are due to the pioneering work of J. Borcea and P. Brändén. These results play an important role in the investigation of linear stability preservers in this dissertation.

Several different approaches to characterizing linear transformations which map polynomials having zeros only in one region of the complex plane to polynomials of the same type are explored. In addition, an open problem of S. Fisk is solved, and several partial results pertaining to open problems from the 2007 AIM workshop "Pólya-Schur-Lax problems: hyperbolicity and stability preservers" are obtained.

Contents

Chapter 1

Preamble

Let $\mathcal{P}$ denote the set of all hyperbolic polynomials; that is, all polynomials that have only real zeros. A transformation, $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is said to be a *real zero preserver* or a *hyperbolicity preserver* if $T(\mathcal{P}) \subset T(\mathcal{P}) \cup \{0\}$. In their seminal 1914 paper, G. Pólya and J. Schur [86] classified all diagonal linear hyperbolicity preservers T such that $T[x^k] = \gamma_k x^k$, where $\gamma_k \in \mathbb{R}$, for all $k \in \mathbb{N} \cup \{0\}$. This result of Pólya and Schur followed important contributions of Gauss, Lucas, Hermite, Poulain, Turán, and Laguerre. A multivariate polynomial, p, is called *stable* with respect to a given region $\Omega \subset \mathbb{C}^n$, if p is non-zero whenever all of its variables are in Ω. A resurgence in the study of hyperbolicity and stability preserving transformations has begun a renaissance in the theory [11, 12, 29, 30]. In particular, J. Borcea and P. Brändén [9] recently characterized all linear hyperbolicity and open half-plane stability preservers on the spaces of real and complex multivariate polynomials (for the univariate case see Chapter 4 or [11]). An excellent 2011 *Bulletin* article by D. G. Wagner [97] details the emerging theory of stability, which has already found numerous applications [7,15,16]. Zero localization problems which are ultimately related to questions of stability and hyperbolicity arise in many different areas; we list a few of these below.

i. **Control of Linear Systems** ([2, p. 465], [69]). The transfer function of a linear system is a rational function, where the locations of the zeros and poles describe the behavior of the system. In particular, the transfer function for a stable linear system possesses poles only in the left half-plane.

ii. **Matrix Theory** ([83, p. 136], [57, p. 345, 406]). If A is an $n \times n$ Hermitian matrix or a strictly totally positive matrix, then $p(\lambda) = \det(A - \lambda I)$ has only real zeros. If $K(x)$ is a density function, then necessary and sufficient conditions for $K(x-y)$ to be a totally positive kernel can be stated by requiring the reciprocal of the bilateral Laplace transform of K belong to an appropriate class of entire functions having only real zeros. The polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ has only real negative zeros if and only if the Toeplitz matrix $\{a_{i-j}\}_{i,j}$, where $a_k = 0$ for $k < 0$, is totally positive.

iii. **The Lee-Yang Theory of Phase Transitions** ([8, 67, 72, 100]). In statistical mechanics, the Lee-Yang program for analyzing phase transitions relies directly on the zero loci of the partition functions associated with the thermodynamic systems under consideration.

iv. **Combinatorics and Graph Theory** ([17–20, 44, 96]). The spanning tree polynomial of a finite connected graph has only real zeros. Sequences that are unimodal or log-concave can be generated by entire functions having only real zeros. Weighted planar networks share an intimate connection with totally positive matrices, which in turn are related to entire functions having only real zeros.

The theory of stable and hyperbolic polynomials is especially well-suited for addressing current problems in matrix theory and combinatorics. For example, several long-standing conjectures—Lieb's "permanent-on-top" (POT) conjecture [13, 56, 70] in matrix theory, the Bessis-Moussa-Villani (BMV) conjecture [71] in quantum statistical mechanics, and the Van der Waerden conjecture (which is proved, but has led to new conjectures) [52, 53]—can be reformulated by means of stable polynomials.

Many recent results concerning stable and hyperbolic polynomials are proved using the Grace-Walsh-Szegő Theorem on symmetric linear forms as the main building block.

1 Grace's theorem and other classical theorems on polynomials

In this section we briefly cite some basic results that will be needed in the investigation of stability preserving operators. To begin with, we state Grace's Theorem, which applies to circular regions. A *circular region* is either an open or closed disk, the complement of an open or closed disk, or a half-plane in $\mathbb{C}$. Thus, the boundary of a circular region is either a circle or a line (the possible images of a circle under a Möbius transformation). Well-known proofs of Grace's Theorem are in [92, Chapter 5] and [88, p. 60, Problem 145], and there have been several recent proofs [16, 97], including a multivariate generalization [10]. The book of Rahman and Schmeisser [89, p. 107] contains a number of statements equivalent to Grace's Theorem.

Definition 1. Two polynomials $f(x) := \sum_{k=0}^{n} a_k \binom{n}{k} x^k$ and $g(x) := \sum_{k=0}^{n} b_k \binom{n}{k} x^k$ are said to be *apolar* if $a_n \neq 0$, $b_n \neq 0$, and $\sum_{k=0}^{n} (-1)^k a_k b_{n-k} \binom{n}{k} = 0$.

Theorem 2 (Grace's Apolarity Theorem [92, p. 181])**.** *Let $A(z)$ and $B(z)$ be apolar polynomials. If $A(z)$ has all its zeros in a circular region D, then $B(z)$ has at least one zero in D.*

Grace's Theorem can be used to prove the following classical composition theorem for polynomials.

Theorem 3 (Malo-Schur-Szegő Theorem [29])**.** *Let*

$$A(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k \text{ and } B(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k,$$

and set

$$C(z) = \sum_{k=0}^{n} \binom{n}{k} a_k b_k z^k.$$

i. (Szegő) *If the zeros of the polynomial $A(z)$ lie in a circular region K, and if $\beta_1, \beta_2, \ldots, \beta_n$ are the zeros of $B(z)$, then every zero of $C(z)$ is of the form $\zeta = -w\beta_j$, for some i, $1 \leq j \leq n$, and some $w \in K$.*

ii. (Schur) *If all the zeros of $A(z)$ lie in a convex region K containing the origin and if the zeros of $B(z)$ lie in the interval $(-1,0)$, then the zeros of $C(z)$ also lie in K.*

iii. If the zeros of $A(z)$ lie in the interval $(-a,a)$ and if the zeros of $B(z)$ lie in the interval $(-b,0)$ (or in $(0,b)$), where $a,b>0$, then the zeros of $C(z)$ lie in $(-ab,ab)$.

iv. (Malo, Schur) *If all the zeros of the polynomial $f(z) = \sum_{k=0}^{\mu} a_k z^k$ are real and all the zeros of the polynomial $g(z) = \sum_{k=0}^{\nu} b_k z^k$ are real and of the same sign, then all the zeros of the polynomials $h(z) = \sum_{k=0}^{m} k! a_k b_k z^k$ and $p(z) = \sum_{k=0}^{m} a_k b_k z^k = \frac{1}{2\pi i}\int_{|w|=r} f\left(\frac{z}{w}\right) g(w) \frac{dw}{w}$ are all real, where ($m = \min\{\mu,\nu\}$).*

We refer to polynomials whose zeros interlace in Chapters 3 and 4.

Definition 4. Let f and g be polynomials with only real simple zeros $(a_1, a_2, \dots, a_m)$ and $(b_1, b_2, \dots, b_n)$ respectively, where $a_i < a_{i+1}$ and $b_i < b_{i+1}$. Without loss of generality, suppose that $a_m \geq b_n$. Then zeros of f and g are said to *interlace* if $|\deg(f) - \deg(g)| = |m-n| \leq 1$ and either:

i. $b_1 \leq a_1 \leq b_2 \leq a_2 \leq \cdots \leq b_n \leq a_m \qquad (m = n)$, or

ii. $a_1 \leq b_1 \leq a_2 \leq b_2 \leq \cdots \leq b_n \leq a_m \qquad (m-1 = n)$.

If the inequalities are strict, then the zeros of f and g are said to *strictly interlace.*

The following theorem gives conditions for the sum of two hyperbolic polynomials to be hyperbolic.

Theorem 5 (Hermite-Kakeya-Obreschkoff [89, p. 198])**.** *Let p and q be non-constant polynomials with real coefficients. Then p and q have strictly interlacing zeros if and only if, for all $\lambda, \mu \in \mathbb{R}$ such that $\lambda^2 + \mu^2 \neq 0$, the polynomial $g(z) := \lambda p(z) + \mu q(z)$ has simple real zeros.*

A perturbation argument provides an analogous theorem when the interlacing of the zeros is not strict.

Remark 6. If $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is a linear hyperbolicity preserver, and the zeros of p and q interlace, then

$T[\lambda p(z)+\mu q(z)] = \lambda T[p(z)]+\mu T[q(z)]$ has only real zeros for all $\lambda, \mu \in \mathbb{R}, \lambda^2+\mu^2 \neq 0$.

This implies that T preserves (non-strict) interlacing.

Polynomials whose zeros all lie in the upper or lower half-plane can be characterized by interlacing polynomials.

Theorem 7 (Hermite-Biehler [89, p. 197], [68, Chapter VII])**.** *Let*

$$f(z) = p(z) + iq(z) = c\prod_{k=1}^{n}(z - \alpha_k) \qquad (0 \neq c \in \mathbb{C}),$$

where $p(z)$, $q(z)$ are real polynomials. Then $p(z)$, $q(z)$ have strictly interlacing zeros if and only if the zeros of $f(z)$ are located in either the open upper half-plane or the open lower half-plane. If at some point x_0 of the real axis $q'(x_0)p(x_0) - q(x_0)p'(x_0) > 0$, then all the zeros of f lie in the open upper half-plane.

A polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ is said to be *affine* in a parameter z_j, if when all the other variables z_k $(k \neq j)$ are fixed, f is polynomial in z_j of the first degree. If f is affine in all variables, it is called *multi-affine*. A polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ is *symmetric* if it is independent of any permutation of the n variables. We define the k^{th} *elementary symmetric polynomial*, denoted $e_k(z_1, \ldots, z_n)$, to be the $(n-k)^{th}$ coefficient of the polynomial $\prod_{j=1}^{n}(x + z_j) =: \sum_{k=0}^{n} e_k(z_1, \ldots, z_n)x^{n-k}$. Each e_k is then multi-affine in $z_1, \ldots, z_n$. A polynomial that is symmetric and multi-affine in $z_1, \ldots, z_n$ has been termed a *symmetric linear form*, and can always be written as a linear combination of elementary symmetric polynomials. The Hermite-Biehler Theorem and a variant of the following theorem (see Chapter 4) were the main results used to characterize linear stability preservers in [11].

Theorem 8 (Grace-Walsh-Szegő Theorem on Symmetric Linear Forms [92, p. 182])**.** *Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be symmetric and multi-affine with exact degree n and let D be a circular region containing the points $\zeta_1, \ldots, \zeta_n$. Then there exists at least one point $\zeta \in D$ such that*

$$f(\zeta_1, \ldots, \zeta_n) = f(\zeta, \ldots, \zeta).$$

Proof. Since for all $w \in \mathbb{C}$, $f - w$ is a symmetric linear form, it is sufficient to show that if for $\zeta_1, \ldots, \zeta_n \in D$, $f(\zeta_1, \ldots, \zeta_n) = 0$ then there exists a $\zeta \in D$ such that $f(\zeta, \zeta, \ldots, \zeta) = 0$. Let $Q(z) = \prod_{k=1}^{n}(z - \zeta_k) = \sum_{k=0}^{n} e_{n-k}(\zeta_1, \ldots, \zeta_n)(-1)^k z^k$. If the expansion of f in elementary symmetric polynomials is

$$f(z_1, \ldots, z_n) = \sum_{k=0}^{n} A_k e_k(z_1, \ldots, z_n), \qquad A_k \in \mathbb{C}$$

then $f(z, z, \ldots, z) := \sum_{k=0}^{n} \binom{n}{k} A_k z^k$. We can now observe that $Q(z)$ and $f(z, z, \ldots, z)$ are apolar:

$$\sum_{k=0}^{n} (-1)^k \frac{(-1)^k e_k(\zeta_1, \ldots, \zeta_n) A_k \binom{n}{k}}{\binom{n}{k}} = f(\zeta_1, \ldots, \zeta_n) = 0.$$

Thus by Grace's Apolarity Theorem (Theorem 2), the polynomial $f(z, z, \ldots, z)$ has a zero ζ in D, since all the zeros of Q lie in D. □

We will frequently use the following fundamental theorem.

Theorem 9 (Hurwitz [22, p. 152])**.** *Let G be an open connected subset of $\mathbb{C}$ and suppose the sequence $\{f_n\}$ of analytic functions in G converges locally uniformly to f. If $f \not\equiv 0$, $\overline{B}(a, R) \subset G$ is a closed ball with center a and radius R in G, and $f(z) \neq 0$ for $|z - a| = R$, then there is an integer N such that for $n \geq N$, f and f_n have the same number of zeros in the open ball $B(a, R) \subset G$.*

Remark 10. This statment of Hurwitz's theorem immediatly implies that if f is non-zero on a compact set K, then there is an integer N such that for all $n \geq N$, $f_n \neq 0$ on K as well.

2 Multiplier sequences and the Laguerre-Pólya class

The characterization of multiplier sequences requires the following definitions.

Definition 11. ([25]) Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers and $T : \mathbb{R}[z] \to \mathbb{R}[z]$ be the corresponding (diagonal) linear operator given by $T[x^k] = \gamma_k x^k$, $k \in \mathbb{N} \cup \{0\}$. If for any hyperbolic polynomial p, of degree n or less, the polynomial $T[p]$ either hyperbolic or identically zero, $\{\gamma_k\}_{k=0}^{\infty}$ is called an *n-sequence.* If $\{\gamma_k\}_{k=0}^{\infty}$ is an n-sequence for all $n \in \mathbb{N}$, it is called a *multiplier sequence.*

Definition 12 ([26])**.** If $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ is an arbitrary entire function, we call

$$g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k$$

the n^{th} *Jensen polynomial associated* with φ. The n^{th} Jensen polynomial associated with the p^{th} derivative of φ, $\varphi^{(p)}(x)$, is denoted by

$$g_{n,p}(x) := \sum_{k=0}^{n} \binom{n}{k} \gamma_{k+p} x^k \qquad (n, p = 0, 1, 2, \ldots).$$

The n^{th} Jensen polynomial associated with a sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ is the result of applying the sequence to the binomial $(x+1)^n$; that is, $T[(x+1)^n] = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k$.

Definition 13. A real entire function $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ is in the *Laguerre-Pólya class*, written $\varphi \in \mathscr{L}\text{-}\mathscr{P}$, if it can be expressed in the form

$$\varphi(x) = c x^m e^{-ax^2+bx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right) e^{\frac{-x}{x_k}} \qquad (0 \le \omega \le \infty),$$

where $b, c, x_k \in \mathbb{R}$, m is a non-negative integer, $a \ge 0$, $x_k \ne 0$, and $\sum_{k=1}^{\omega} \frac{1}{x_k^2} < \infty$. Let $\mathscr{L}\text{-}\mathscr{P}^+$ denote the class of functions in the Laguerre-Pólya class that have non-negative Taylor coefficients, and $\mathscr{L}\text{-}\mathscr{P}(-\infty, 0]$ denote the set of functions in the Laguerre-Pólya class that have only non-positive zeros.

An entire function is in the Laguerre-Pólya class *if and only if* it is a locally uniform limit of real polynomials having only real zeros [68, Chapter VIII]. In particular, the set $\mathscr{L}\text{-}\mathscr{P}$ includes the identically zero function. Note that $\mathscr{L}\text{-}\mathscr{P}(-\infty, 0] \ne \mathscr{L}\text{-}\mathscr{P}^+$; for example, the function $1/\Gamma(x)$, where Γ is the gamma function, has only non-positive zeros, but has some negative Taylor coefficients. Theorem 14 is a classical result of Pólya and Schur, which was proved using Theorem 3 and an approximating property of the Jensen polynomials (see [26, Lemma 2.2]).

Theorem 14. (Characterization of Multiplier Sequences [86]) *Let* $\{\gamma_k\}_{k=0}^{\infty}$ *be a sequence of real numbers and* $T : \mathbb{R}[x] \to \mathbb{R}[x]$ *be the corresponding (diagonal) linear operator given by* $T[x^k] = \gamma_k x^k$, $k \in \mathbb{N} \cup \{0\}$. *The following are equivalent:*

i. $\{\gamma_k\}_{k=0}^{\infty}$ *is a multiplier sequence.*

ii. Either $T[e^x] \in \mathscr{L}\text{-}\mathscr{P}^+$, $T[e^{-x}] \in \mathscr{L}\text{-}\mathscr{P}^+$, $-T[e^x] \in \mathscr{L}\text{-}\mathscr{P}^+$,
or $-T[e^{-x}] \in \mathscr{L}\text{-}\mathscr{P}^+$.

iii. For all $n \in \mathbb{N}$, *the Jensen polynomials* $g_n(x) = T[(1+x)^n]$ *have only real zeros of the same sign or are identically zero.*

The classical Turán inequalities (see Proposition 26) imply that the terms of a multiplier sequence or n-sequence must either have the same sign, or alternate in sign. Theorem 3 immediately yields a characterization of n-sequences, which is due to T. Craven and G. Csordas [24], and is stated below for a single sign configuaration.

Theorem 15. (Algebraic Characterization of n-sequences [24]) *A non-negative sequence* $\{\gamma_k\}_{k=0}^{\infty}$ *is an n-sequence if and only if the* n^{th} *Jensen polynomial,* $g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k$, *has only real non-positive zeros.*

Proof. Let $t(x) = \sum_{k=0}^{n} b_k x^k$ be an arbitrary polynomial with only real zeros of degree $m \leq n$. If $g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k$ has only real non-positive zeros, then by Theorem 3, $\sum_{k=0}^{n} \gamma_k b_k x^k = \sum_{k=0}^{m} \gamma_k b_k x^k$ has only real zeros. Conversely, if $\{\gamma_k\}_{k=0}^{\infty}$ is an n-sequence, its associated Jensen polynomial has only real non-positive zeros, since it is the result of applying the sequence to $(1+x)^n$. □

One possible motivation for studying multiplier sequences is the Riemann hypothesis, which is equivalent to the statement that the ξ-function [86], as defined by Riemann,

$$\xi(iz) := \frac{1}{2}\left(z^2 - \frac{1}{4}\right)\pi^{-z/2-1/4}\,\Gamma\left(\frac{z}{2} + \frac{1}{4}\right)\zeta\left(z + \frac{1}{2}\right), \tag{1.1}$$

has only real simple zeros. The ξ-function satisfies $\frac{1}{8}\xi(\frac{x}{2}) = \sum_{m=0}^{\infty} \frac{(-1)^m \hat{b}_m x^{2m}}{(2m)!}$, and therefore the zeros of

$$F(x) := \sum_{m=0}^{\infty} \frac{\hat{b}_m}{(2m)!} x^m \tag{1.2}$$

are real and negative if and only if ξ has only real zeros. The coefficients $\hat{b}_m \geq 0$ in (1.2) are moments associated with the kernel whose (Fourier) cosine transform is ξ (see [35]).

If f is an entire function with a finite number of non-real zeros, we will denote the number of non-real zeros, counting multiplicities, by $Z_c(f)$. A operator T such that $Z_c(T[f]) \leq Z_c(f)$ for all $f \in \mathbb{R}[x]$ is called a *complex zero decreasing operator.* A multiplier sequence which acts as a complex zero decreasing operator when applied to polynomials in $\mathbb{R}[x]$ is called a *complex zero decreasing sequence (CZDS).* An alternate formulation of the Riemann Hypothesis, due to G. Csordas [32], is that the sequence interpolated by F in (1.2), $\{F(k)\}_{k=0}^{\infty}$, is a CZDS [13, Problem 21]. Rewriting F in terms of the gamma and zeta functions (see [13, 32]),

$$F(k) = \frac{\pi^{-1/4}}{64}(k-1)\pi^{-\sqrt{k}/4}\,\Gamma\left(\frac{1}{4}+\frac{\sqrt{k}}{4}\right)\zeta\left(\frac{1}{2}+\frac{\sqrt{k}}{2}\right). \tag{1.3}$$

3 Outline and summary of results

The results of this dissertation pertain to three areas of investigation: interpolation of multiplier sequences, discrete inequalities for polynomials having only real zeros, and the characterization of stability preserving operators. Material from Chapter 3 has been accepted for publication [33], and a manuscript containing results from Chapter 2 has been submitted, and is being considered for publication, pending revisions. Chapter 2 on meromophically interpolated multiplier sequences and Chapter 3 on discrete inequalities reference only material in Chapter 1. Chapters 4 and 5 both focus on the problem of characterizing linear preservers of polynomials whose zeros are confined to lie in a sector or some region of the complex plane. The main results are summarized below.

Theorem 23

(Characterization of rationally interpolated multiplier sequences)

The class of the multiplier sequences that can be interpolated by rational functions is characterized, and it is shown to coincide with the class of the multiplier sequences which can be interpolated by polynomials.

Theorem 84 (Proof of Krasikov's conjecture) and **Theorem 93**
(Extension of Krasikov's conjecture to transcendental entire functions)
A discrete version of a the classical Laguerre inequality is proved, which settles a conjecture of I. Krasikov. The result is subsequently extended to a class transcendental entire functions. In the process of proving the main theorem, we obtain other discrete analogs of classical polynomial inequalities.

Theorems 116 and **143**
(Sufficient conditions for linear sector preservers)
Sufficient conditions are given for a linear operator to preserve the set of polynomials whose zeros all lie in a closed sector. A different set of sufficient conditions are given for a linear operator T to preserve the set of polynomials whose zeros all lie in an open or closed sector, whenever T can be represented as differential operator of finite order.

Theorem 152
(Characterization of strict stability preservers in the univariate Weyl algebra)
A univariate polynomial is said to be *strictly* stable, if all of its zeros lie in the open lower half-plane. Characterizing linear operators that preserve strict stability is an open problem that has been discussed in recent literature [12, 13]. Theorem 116 provides a sufficient condition for a differential operator of finite order to preserve strict stability. It is shown that this condition is also necessary, thus providing a foothold for the classification of all strict half-plane stability preservers.

Theorem 177
This theorem states that the Jensen polynomials associated with the function F in (1.2), related to the Riemann ξ-function, have only real zeros up to degree 10^{17}. Theorem 177 is based on the observation that the zeros of the function F are constrained to lie in a sector symmetric about the negative real axis.

Theorem 204

(Proof of Fisk's conjecture)

A conjecture of S. Fisk is proved, yielding a transformation which preserves polynomials whose zeros lie in the interval $[-1, 1]$. This provides a new example of a linear preserver on a bounded region which possesses a representation as a differential operator of infinite order. The proof is based on a method of A. Iserles, E. B. Saff, and S. P. Nørsett, who have obtained similar results.

In addition to the theorems listed above, Chapter 5 contains preliminary results that have relevance to recently posed open problems [6,13]. The results in this dissertation that are either new, or appear to be new, are highlighted below.

All results

Chapter 2: Theorem 23, Lemma 27, Proposition 35, Proposition 40, Proposition 44, Corollary 45, Lemma 48, and Lemma 51.

Chapter 3: Lemma 70, Lemma 73, Corollary 75, Lemma 77, Lemma 78, Lemma 80, Lemma 81, Lemma 82, Lemma 83, Theorem 84, Theorem 88, Lemma 91, Lemma 92, Theorem 93, Theorem 95, Theorem 102, and Theorem 103.

Chapter 4: Corollary 115, Theorem 116, Lemma 127, Theorem 130, Theorem 131, Theorem 133, Theorem 134, Theorem 135, Proposition 138, Theorem 141, Theorem 143, Theorem 146, Theorem 148, Theorem 152, Theorem 154, Lemma 155, Proposition 156, Corollary 157, Theorem 162, Theorem 163, Theorem 170, Theorem 172, Theorem 173, and Theorem 177.

Chapter 5: Theorem 182, Proposition 185, Proposition 186, Lemma 190, Theorem 191, Theorem 193, Proposition 194, Theorem 204, Proposition 214, Proposition 216, and Proposition 217.

The numbers of new conjectures, problems, and questions are: 56, 57, 62, 86, 88, 89, 221, 222, 223, 224, 225, 226, and 227.

Chapter 2

Meromorphically interpolated sequences

1 Introduction

In this chapter we prove that the class of multiplier sequences which can be interpolated by rational functions is the same as the class of multiplier sequences which can be interpolated by polynomials. This immediately yields information about the location of zeros for special cases of the Fox-Wright functions. We note that rationally interpolated sequences may be useful in identifying when an entire function has non-real zeros. This idea garners some significance when considered in conjunction with impressive results of W. Bergweiler, A. Eremenko, and J. K. Langley, who have proved conjectures of Wiman and Pólya which give conditions for the derivatives of entire functions to have non-real zeros [4, 5]. A closely related result, proved by T. Craven, G. Csordas, and W. Smith, resolved the longstanding Pólya-Wiman conjecture (see Theorem 31) [31, 60]. This result is used in Section 2 to observe that any real polynomial interpolates a multiplier sequence for a sufficiently large shift of index. Section 4 contains suggestions on how to continue the investigation and lists several new problems concerning interpolating sequences in addition to others from the literature.

We now turn our attention to the following theorem of Laguerre, which gives sufficient conditions for a multiplier sequence (see Chapter 1) to be interpolated by a function in $\mathscr{L}$-$\mathscr{P}$. For $p(x) \in \mathbb{R}[x]$, let $Z_c(p(x))$ denote the number of non-real zeros of $p(x)$, counting multiplicities. A multiplier sequence which does not increase the number of non-real zeros when applied to any real polynomial is called a *complex zero decreasing sequence (CZDS).*

Theorem 16 (Laguerre's Theorem [27])**.**

i. Let $f(x) = \sum_{k=0}^{n} a_k x^k$ *be an arbitrary real polynomial of degree* n *and let* $h(x)$ *be a polynomial with only real zeros, none of which lie in the interval* $(0, n)$*. Then*

$$Z_c\left(\sum_{k=0}^{n} h(k) a_k x^k\right) \leq Z_c\left(f(x)\right).$$

ii. Let $f(x) = \sum_{k=0}^{n} a_k x^k$ *be an arbitrary real polynomial of degree* n*, let* $\varphi(x) \in \mathscr{L}$-$\mathscr{P}$*, and suppose that none of the zeros of* φ *lie in the interval* $(0, n)$*. Then*

$$Z_c\left(\sum_{k=0}^{n} \varphi(k) a_k x^k\right) \leq Z_c(f(x)).$$

iii. If $\varphi \in \mathscr{L}$-$\mathscr{P}(-\infty, 0]$*, then the sequence* $\{\varphi(k)\}_{k=0}^{\infty}$ *is a CZDS.*

Thus, the sequence $\{\varphi(k)\}_{k=0}^{\infty}$, as described in the first part of Theorem 16, does not increase the number of non-real zeros when applied to polynomials of degree n or less. If φ has only non-positive zeros, then $\{\varphi(k)\}_{k=0}^{\infty}$ is a multiplier sequence, and in addition it is a CZDS. Characterizing all CZDS is an open problem [29], as is the special case of characterizing all meromorphically interpolated CZDS.

T. Craven and G. Csordas initiated an extension of Laguerre's theorem when they posed the following problems, which still remain open.

Problem 17 ([30])**.** Characterize the meromorphic functions $Y(x) = \varphi(x)/\psi(x)$, where φ and ψ are entire functions such that the polynomial $\sum_{k=0}^{n} Y(k) a_k x^k$ has only real zeros whenever the polynomial $\sum_{k=0}^{n} a_k x^k$ has only real zeros.

Problem 18 ([30])**.** Characterize the meromorphic functions $Y(x)$ with the property that $\sum_{k=0}^{\infty} Y(k) a_k x^k / k!$ is a transcendental entire function with only real zeros (or the zeros all lie in the half-plane Re $z < 0$) whenever the entire function $\sum_{k=0}^{\infty} a_k x^k/k!$ has only real zeros.

Let $\mathcal{P}_\Omega$ be the set of polynomials, all of whose zeros lie in $\Omega \subset \mathbb{C}$. In connection with recent investigations of linear operators which map $\mathcal{P}_\Omega$ to $\mathcal{P}_\Omega$ for a given region $\Omega \subset \mathbb{C}$, we add the following two questions.

Problem 19. Given $\Omega \subset \mathbb{C}$, characterize the meromorphic functions $Y(x)$ such that the polynomial $\sum_{k=0}^{n} Y(k) a_k x^k$ has zeros only in Ω whenever the polynomial $\sum_{k=0}^{n} a_k x^k$ has zeros only in Ω.

Problem 20. Given $\Omega \subset \mathbb{C}$, characterize the meromorphic functions $Y(x)$ with the property that $\sum_{k=0}^{\infty} Y(k) a_k x^k/k!$ is a transcendental entire function with zeros only in Ω whenever the entire function $\sum_{k=0}^{\infty} a_k x^k/k!$ has zeros only in Ω.

Meromorphically interpolated multiplier sequences, whose characterization is sought in Problem 17, have been termed *meromorphic Laguerre multiplier sequences* [30]. Theorem 21 and Example 22 exhibit some non-trivial meromorphic Laguerre multiplier sequences.

Theorem 21 (T. Craven, G. Csordas [30])**.** *For each positive integer m, the function*

$$\varphi_m(x) = \sum_{k=0}^{\infty} \frac{(2mk)!}{(3mk)!} \frac{x^k}{k!}$$

has only real negative zeros, and whence the sequence $\{(2mk)!/(3mk)!\}_{k=0}^{\infty} = \{\Gamma(2mk+1)/\Gamma(3mk+1)\}_{k=0}^{\infty}$ is a multiplier sequence, where $\Gamma(x)$ denotes the gamma function [90, p. 8].

Example 22 ([30])**.** Let

$$L_{\alpha,\beta} := \left\{ \frac{\Gamma(k+1)}{\Gamma(\alpha k + \beta)} \right\}_{k=0}^{\infty}.$$

Then for $1 \le \beta < 3$, $L_{2,\beta}$ is a Laguerre multiplier sequence [41], [40, Theorem 1.1]. For all $\alpha \ge 2$, $L_{\alpha,1}$ and $L_{\alpha,2}$ are meromorphic Laguerre multiplier sequences [80, Corollary 3]. (See [30] for more explanation).

We consider a special case of Problem 17, where both φ and ψ are polynomials. The following theorem is proved in Section 2.

Theorem 23. *Let* $\{\gamma_k\}_{k=0}^{\infty} = \left\{\frac{p(k)}{q(k)}\right\}_{k=0}^{\infty}$, *with relatively prime* $p, q \in \mathbb{R}[x]$, *and* $q(x) \neq 0$ *for* $x \in \mathbb{N} \cup \{0\}$. *If* $\{\gamma_k\}_{k=0}^{\infty}$ *is a multiplier sequence, then* $q(k)$ *must be a constant. If* $\deg(q) \geq 1$, *then for all* $n \in \mathbb{N}$, $\left\{\frac{p(k+n)}{q(k+n)}\right\}_{k=0}^{\infty}$ *is not a multiplier sequence.*

Theorem 23 immediately implies the following result about Fox-Wright functions (see [50, 61]).

Corollary 24. *Given the Fox-Wright function,*

$$ {}_p\Psi_q(x) := \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j k + b_j)}{\prod_{j=1}^{q} \Gamma(c_j k + d_j)} \frac{x^k}{k!}, \tag{2.1}$$

if $p = q$, $a_j = c_j$, $(b_j - d_j) \in \mathbb{Z}$ *for all* $j = 1, \ldots, p$, *and* $(\prod_{j=1}^{p} \Gamma(a_j k + b_j))/(\prod_{j=1}^{q} \Gamma(c_j k + d_j))$ *is not equal to a polynomial, then every derivative of* ${}_p\Psi_q$ *has an infinite number of non-real zeros.*

If $b_j = 1$ $(j = 1, \ldots, p)$, and $d_j = 1$ $(j = 1, \ldots, q)$, in (2.1), then the Fox-Wright function reduces to the generalized hypergeometric function [90, p. 45].

In Section 3, we consider some additional consequences of Theorem 23. New problems concerning interpolated multiplier sequences are stated in Section 4, along with problems from recent publications.

2 Rationally interpolated sequences

In this section, Theorem 23 is proved as an immediate consequence of Theorem 25 and Proposition 26. Throughout the rest of the chapter, we let $D := \frac{d}{dx}$. If $g(x) = \sum_{k=0}^{\infty} a_k x^k$ and $f(x)$ are entire functions, we use the notation $g(D)f(x) := \sum_{k=0}^{\infty} a_k f^{(k)}(x)$, provided $g(D)f(x)$ converges to an entire function.

Theorem 25 (Hermite-Poulain-Pólya)**.**

i. ([68, Chapter VIII]) *If* $q(x) := \sum_{j=0}^{n} b_j x^j$ *has only real zeros and if* $f(x) \in \mathbb{R}[x]$, *then*

$$Z_c(q(D)f(x)) \leq Z_c(f(x)),$$

where $Z_c(f(x))$ denotes the number of non-real zeros of $f(x)$, counting multiplicities.

ii. ([85, p. 142]) *If $\varphi \in \mathscr{L}\text{-}\mathscr{P}$ and $\psi \in \mathscr{L}\text{-}\mathscr{P}^+$ (see Definition 13), then $\psi(D)\varphi(x) \in \mathscr{L}\text{-}\mathscr{P}$ and $\varphi(D)\psi(x) \in \mathscr{L}\text{-}\mathscr{P}$.*

Proposition 26 (Turán inequalities [26]). *Let $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathscr{L}\text{-}\mathscr{P}$. The Turán inequalities hold for the Taylor coefficients of φ; that is,*

$$T_k(\varphi) := \gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0, \qquad k = 1, 2, 3, \ldots. \tag{2.2}$$

The following lemma is equivalent to the statement that if a sequence is interpolated by a rational function, where the denominator's degree is larger than that of the numerator, then every shift of that sequence is not a multiplier sequence.

Lemma 27. *Let $\{\eta_k\}_{k=0}^{\infty} = \left\{\frac{p(k)}{q(k)}\right\}_{k=0}^{\infty}$, where $p(x), q(x) \in \mathbb{R}[x]$, $\deg(p) < \deg(q)$, and $q(x) \neq 0$ for $x \in \mathbb{N} \cup \{0\}$. Then every derivative of the entire function $\varphi(x) = \sum_{k=0}^{\infty} \eta_k \frac{x^k}{k!}$ has non-real zeros.*

Proof. If $p \equiv 0$ the statement of theorem is trivial, so we assume otherwise. Let $n = \deg(p) < \deg(q) = m$, then

$$\frac{1}{q(x)} = \frac{1}{a_m} \prod_{j=1}^{m} \frac{1}{x - \alpha_j} = \frac{1}{a_m} \frac{1}{x^m} \prod_{j=1}^{m} \left(\sum_{k=0}^{\infty} \left(\frac{\alpha_j}{x} \right)^k \right), \qquad |x| > \max_{1 \leq j \leq m} |\alpha_j|,$$

where $\{\alpha_j\}_{j=1}^{m}$ are the zeros of $q(x)$ and a_m is the leading coefficient of $q(x)$. Then

$$Y(k) := \frac{p(k)}{q(k)} = \frac{A}{k^d} + \frac{B}{k^{d+1}} + \frac{C}{k^{d+2}} + \frac{F}{k^{d+3}} + O\left(\frac{1}{k^{d+4}}\right), \qquad (k \to \infty), \tag{2.3}$$

where $d = m - n > 0$, $A, B, C, F \in \mathbb{R}$, and

$$A = \lim_{k \to \infty} k^d \frac{p(k)}{q(k)}, \qquad B = \lim_{k \to \infty} \left[k^{d+1} \frac{p(k)}{q(k)} - Ak \right], \ \ldots.$$

Note that $A \neq 0$; if we multiply both sides of Equation (2.3) by k^d and let $k \to \infty$, we obtain that A is equal to the ratio of the leading coefficients of p and q. Using (2.3) to compute the Turán expression (cf. Proposition 26) for large k, we obtain

$$Y(k)^2 - Y(k+1)Y(k-1) = \frac{-A^2 d}{k^{2d+2}} - \frac{2AB(d+1)}{k^{2d+3}} + O\left(\frac{1}{k^{2d+4}}\right), \qquad (k \to \infty). \tag{2.4}$$

The first term on the right-hand side of (2.4) is always negative and thus for sufficiently large k, $Y(k)$ fails to satisfy the Turán inequalities. Suppose that φ has only real zeros. Since φ has order ≤ 1 (see (2.25)), $\varphi \in \mathscr{L}\text{-}\mathscr{P}$ and therefore the Turán inequalities hold; but this contradicts (2.4). Therefore, φ must have non-real zeros. Since the ℓ^{th} derivative of φ is $\varphi^{(\ell)}(x) = \sum_{k=0}^{\infty} \eta_{k+\ell} \frac{x^k}{k!} = \sum_{k=0}^{\infty} Y(k+\ell) \frac{x^k}{k!}$, the asymptotic formula (2.4) still holds for $\varphi^{(\ell)}$, and therefore every derivative of φ has non-real zeros. $\square$

Definition 28. For any real sequence $\{\gamma_k\}_{k=0}^{\infty}$ we define $\Delta^0 \gamma_p = \gamma_p$,

$$\Delta^n \gamma_p = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \gamma_{p+j} = \Delta(\Delta^{n-1}\gamma_p)$$

for $n, p = 0, 1, 2, \ldots$. For a real valued function $f(k)$, we define $\Delta f(k) := f(k+1) - f(k)$ and $\Delta^n f(k)$ analogously.

We establish the following preparatory results concerning the Δ-operator for use in the proof of Theorem 23.

Lemma 29. *If $g(x) \in \mathbb{C}[x]$ and $\deg(g) = \ell$, then $\Delta^j g(x) = 0$ for all $j > \ell$.*

Proof. Since $\Delta 0 = 0$, we only need to show that $\Delta^{m+1} g(x) = 0$ when $\deg(g) = m$, for all $m \in \mathbb{N} \cup \{0\}$. We prove the lemma by induction on m. Assume the lemma is true when the degree of g is $m-1$. If the degree of g is m, $g(x) = \sum_{j=0}^{m} a_j x^j$, and $b_j = \sum_{k=0}^{m} \binom{k}{j} a_k$, then

$$\begin{aligned}
\Delta^{m+1} g(x) &= \Delta^m \left[\sum_{j=0}^{m} a_j (x+1)^j - \sum_{j=0}^{m} a_j x^j \right] \\
&= \Delta^m \left[\left(a_m x^m + \sum_{j=0}^{m-1} b_j x^j \right) - \left(a_m x^m + \sum_{j=0}^{m-1} a_j x^j \right) \right] \\
&= \Delta^m \left[\sum_{j=0}^{m-1} (b_j - a_j) x^j \right] = 0,
\end{aligned}$$

where the last step follows from the induction hypothesis. $\square$

Lemma 30. *If $n_0(x), d_0(x) \in \mathbb{C}[x]$, with $\deg(n_0) < \deg(d_0)$, and for $j \in \mathbb{N}$, $\frac{n_j(x)}{d_j(x)} := \Delta^j \left(\frac{n_0(x)}{d_0(x)} \right)$, then $\deg(n_j) < \deg(d_j)$ for all $j \in \mathbb{N} \cup \{0\}$.*

Proof. The claim holds trivially for $j = 0$. Assume Lemma 30 holds for $j = m$. We proceed by induction on m.

$$\begin{aligned}\Delta^{m+1}\frac{n_0(x)}{d_0(x)} &= \Delta^m\left[\frac{n_0(x+1)}{d_0(x+1)} - \frac{n_0(x)}{d_0(x)}\right] \\ &= \Delta^m\frac{n_0(x+1)d_0(x) - n_0(x)d_0(x+1)}{d_0(x+1)d_0(x)} \\ &= \Delta^m\frac{n_1(x)}{d_1(x)}, \qquad \text{where } \deg(n_1) < \deg(d_1).\end{aligned}$$

Then, by induction hypothesis

$$\Delta^m\frac{n_1(x)}{d_1(x)} = \frac{n_{m+1}(x)}{d_{m+1}(x)}, \qquad \text{where } \deg(n_{m+1}) < \deg(d_{m+1}).$$

This proves the lemma. □

We next use Lemmas 29, 30, 27 and the Hermite-Poulain Theorem to prove Theorem 23.

Proof of Theorem 23. Let $\{\gamma_k\}_{k=0}^\infty = \left\{\frac{p(k)}{q(k)}\right\}_{k=0}^\infty$, with relatively prime $p, q \in \mathbb{R}[x]$, and $q(x) \neq 0$ for $x \in \mathbb{N} \cup \{0\}$. For the first assertion assume that $\deg(q) \geq 1$ and that $\{\gamma_k\}_{k=0}^\infty$ is a multiplier sequence. We arrive at a contradiction as follows. By the division algorithm, there exist unique $g, r \in \mathbb{R}[x]$ such that

$$\frac{p(k)}{q(k)} = g(k) + \frac{r(k)}{q(k)}, \tag{2.5}$$

where $\deg(r) < \deg(q)$. Let T be the linear operator defined on monomials by

$$T[x^k] = \gamma_k x^k = \frac{p(k)}{q(k)}x^k = \left(g(k) + \frac{r(k)}{q(k)}\right)x^k \text{ for all } k \in \mathbb{N} \cup \{0\}.$$

Since by assumption, $\{\gamma_k\}_{k=0}^\infty$ is a multiplier sequence, $(D-1)^m T[e^x]$ has only real zeros for all m by Theorem 25 (where $D := \frac{d}{dx}$). But an application of $(D-1)$ to $T[e^x]$ yields

$$(D-1)T[e^x] = (D-1)\sum_{k=0}^\infty \gamma_k\frac{x^k}{k!} = \sum_{k=0}^\infty \Delta\gamma_k\frac{x^k}{k!}, \tag{2.6}$$

where

$$\Delta\gamma_k = \left(g(k+1) + \frac{r(k+1)}{q(k+1)}\right) - \left(g(k) + \frac{r(k)}{q(k)}\right) = \Delta g(k) + \Delta\left(\frac{r(k)}{q(k)}\right), \tag{2.7}$$

and $\deg(\Delta g) < \deg(g)$. Continuing in the same way, γ_k is replaced by $\Delta^m \gamma_k$ when $(D-1)^m$ operates on $T[e^x]$, and

$$\Delta^m \gamma_k = \Delta^m g(k) + \Delta^m \left(\frac{r(k)}{q(k)} \right). \tag{2.8}$$

Choose the positive integer $m > \deg(g) = \deg(p) - \deg(q)$. Then by Lemmas 29 and 30, $\Delta^m g(k) = 0$ and $\Delta^m \left(\frac{r(k)}{q(k)}\right) = \frac{n(k)}{d(k)}$, where $\deg(n(x)) < \deg(d(x))$. Therefore, by (2.6) and (2.8),

$$(D-1)^m T[e^x] = \sum_{k=0}^{\infty} \frac{n(k)}{d(k)} \frac{x^k}{k!}. \tag{2.9}$$

Thus, by (2.9) and Lemma 27, the transcendental entire function $(D-1)^m T[e^x]$ has non-real zeros; this contradicts the assumption that $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence. The last assertion of Theorem 23 follows from the above argument, since $p(k+n)/q(k+n)$ is still a rational function which satisfies the hypotheses in Theorem 23. $\square$

Note that for $\{\gamma_k\}_{k=0}^{\infty} = \left\{\frac{p(k+\alpha)}{q(k+\alpha)}\right\}_{k=0}^{\infty}$, the conclusions of Theorem 23 still hold for any real number α, provided it does not introduce a pole at a non-negative integer value of k. With the aid of the following theorem, we next prove a result (Proposition 35) which is in sharp contrast to Theorem 23.

Theorem 31 (Craven-Csordas-Pólya-Smith-Wiman [31]). *If a real entire function of order less than two has a finite number of non-real zeros, then there is a positive integer m_0, such that if $m \geq m_0$, then $f^{(m)}$ has only real zeros.*

Definition 32. Denote the *falling factorial* [17] by

$$(x)_j := x(x-1)\cdots(x-j+1). \tag{2.10}$$

Lemma 33. *If $p(x) \in \mathbb{R}[x]$ and T is the linear operator such that $T[x^k] = p(k)x^k$ for all $k \in \mathbb{N} \cup \{0\}$, then $T[e^x] = g(x)e^x$, where $g(x) \in \mathbb{R}[x]$.*

Proof. Note that,

$$x^j e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^{k+j} = \sum_{k=j}^{\infty} \frac{1}{(k-j)!} x^k = \sum_{k=0}^{\infty} \frac{k(k-1)\cdots(k-j+1)}{k!} x^k. \tag{2.11}$$

Then, using Definition 32, (2.11) can be written as

$$x^j e^x = \sum_{k=0}^{\infty} (k)_j \frac{x^k}{k!}. \tag{2.12}$$

If n is the degree of p, then p has an expansion $p(x) = \sum_{j=0}^{n} a_j (x)_j$, $a_j \in \mathbb{R}$, and

$$\begin{aligned} T[e^x] &= \sum_{k=0}^{\infty} \sum_{j=0}^{n} a_j (k)_j \frac{x^k}{k!} \\ &= \sum_{j=0}^{n} a_j \sum_{k=0}^{\infty} (k)_j \frac{x^k}{k!} \\ &= \sum_{j=0}^{n} a_j x^j e^x \qquad \text{by (2.12)} \\ &= g(x) e^x. \end{aligned}$$

This proves the lemma. □

Remark 34. Moreover, the proof of Lemma 33 shows that $g(x)$ is precisely the polynomial given by performing the basis change $(x)_j \to x^j$ on $p(x)$. Note also that in Lemma 33,

$$T = p(xD). \tag{2.13}$$

Let the zeros of p be $\{\alpha_j\}_{j=1}^{n}$, $\alpha_j \in \mathbb{C}$. Then, (2.13) is verified by checking that the operators $(xD + \alpha_j)$, commute, and that

$$(xD - \alpha)x^k = (k - \alpha)x^k \qquad \text{for all } \alpha \in \mathbb{C}.$$

Hence, for each $k = 0, 1, 2, \ldots$,

$$p(xD)x^k = \prod_{j=1}^{n} (xD - \alpha_j)x^k = \prod_{j=1}^{n} (k - \alpha_j)x^k = p(k)x^k = T[x^k],$$

and $T = p(xD)$ by linearity. Another way to prove Lemma 33 is to use $T = p(xD)$ and an induction argument.

Proposition 35. *For any $p(x) \in \mathbb{R}[x]$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\{p(k + n)\}_{k=0}^{\infty}$ is a multiplier sequence.*

Proof. Without loss of generality assume that the leading coefficient of $p(x)$ is positive. Then, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, $p(n)$ is positive. Let T be the linear operator with the action $T[x^k] = p(k)x^k$ for all $k \in \mathbb{N} \cup \{0\}$. Then by Lemma 33,

$$T[e^x] = g(x)e^x$$

for some $g \in \mathbb{R}[x]$. By Theorem 31, there exists $N \in \mathbb{N}$, $N \geq M$, such that

$$\sum_{k=0}^{\infty} p(k+N)\frac{x^k}{k!} = D^N T[p(k)] = D^N g(x)e^x \tag{2.14}$$

has only real zeros. Therefore, if $n \geq N$, then $\sum_{k=0}^{\infty} p(k+n)\frac{x^k}{k!}$ has only real zeros and positive coefficients, since $N \geq M$. Hence, $p(k+n)$ is a multiplier sequence by Theorem 14. $\square$

Remark 36. Let $\psi = q(x)f(x)$, where $f \in \mathscr{L}\text{-}\mathscr{P}$ has a finite number of real positive zeros, and $q(x) \in \mathbb{R}[x]$. Then $\{\psi(k+n)\}_{k=0}^{\infty}$ is a multiplier sequence for sufficiently large n, since it is a composition of multiplier sequences $\{q(k+n)\}_{k=0}^{\infty}$ (by Proposition 35) and $\{f(k+n)\}_{k=0}^{\infty}$ (by Theorem 16) for n sufficiently large.

As another example of a sequence which becomes a multiplier sequence for a sufficiently large shift of index, consider a special case of the Fox-Wright function: the *generalized Mittag-Leffler* function, defined by

$$E_{\alpha,\beta}(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha, \beta > 0. \tag{2.15}$$

When $\beta = 1$ in (2.15) we obtain the classical *Mittag-Leffler* function, $E_\alpha(x) := E_{\alpha,1}(x)$. T. Craven and G. Csordas used detailed information about the zero locus of $E_{\alpha,\beta}(x)$ and Theorem 31 to prove the following.

Theorem 37 ([30]). *Let $\alpha > 2$ and $\beta > 0$. Then there exists a positive integer $m_0 := m_0(\alpha, \beta)$ such that*

$$E_{\alpha,\beta}^{(m)}(x) = \sum_{k=0}^{\infty} \frac{\Gamma(m+k+1)}{\Gamma(\alpha(k+m)+\beta)} \frac{x^k}{k!} \in \mathscr{L}\text{-}\mathscr{P}^+, \qquad m \geq m_0. \tag{2.16}$$

Thus the sequence $\{\Gamma(m+k+1)/\Gamma(\alpha(k+m)+\beta)\}_{k=0}^{\infty}$ is a meromorphic Laguerre multiplier sequence, for all nonnegative integers m sufficiently large.

We now consider the introduction of a factorial, and more generally a gamma function, to turn a sequence interpolated by a simple pole into a multiplier sequence. Let T be the linear operator with the action $T[x^k] = \frac{1}{k!}x^k$, and S be the linear operator with the action $S[x^k] = \frac{1}{k+1}x^{k+1} = \int_0^x y^k\, dy$. By Theorem 23, the sequence $\{1/(k+1)\}_{k=0}^{\infty}$ is not a multiplier sequence. Similarly, the operator S does not preserve reality of zeros. On the other hand, S^nT preserves the reality of zeros for every non-negative integer n (S^nT is equivalent to multiplication by x^n followed by operating with T). In fact, even more is true.

Theorem 38 ([49, Theorem 11 (b)])**.** *Let $S, T : \mathbb{R}[x] \to \mathbb{R}[x]$ be linear operators such that $T[x^k] = \frac{1}{k!}x^k$ and $S[x^k] = \frac{1}{k+1}x^{k+1}$ for all $k \in \mathbb{N} \cup \{0\}$. If $f(x) \in \mathbb{R}[x]$ has only non-positive zeros, then for all $n \in \mathbb{N} \cup \{0\}$, $S^nT[f(x)]$ has only non-positive zeros, which are all simple except possibly for the origin.*

The operator $x^k \mapsto \frac{x^k}{k+\beta} = \frac{\Gamma(k+\beta)}{\Gamma(k+\beta+1)}x^k$, $\beta > 0$, has a nice integral operator representation [82],

$$\frac{x^k}{k+\beta} = \frac{1}{x^\beta}\int_0^x y^{\beta-1}y^k\, dy \qquad (\beta > 0). \tag{2.17}$$

In light of representation (2.17), it is natural to raise the following question.

Question 39. If R is a rational function, then what are the conditions under which

$$\left\{\frac{1}{k!}R(k)\right\}_{k=0}^{\infty} \tag{2.18}$$

is a multiplier sequence?

Note that (2.18) is a multiplier sequence if and only if $\sum_{k=0}^{\infty} \frac{1}{k!k!}R(k)x^k \in \mathscr{L}\text{-}\mathscr{P}^+$.

Proposition 40. *If β is a positive integer, then the sequence*

$$\left\{\frac{1}{k!(k+\beta)}\right\}_{k=0}^{\infty} \tag{2.19}$$

is a multiplier sequence.

Proof. If $\beta = 1$ the sequence $\{1/(k+1)!\}_{k=0}^{\infty}$ is a multiplier sequence by Theorem 16. If $\beta > 1$, the sequences $\{1/(k+\beta)!\}_{k=0}^{\infty}$ and $\{(k+1)(k+2)\cdots(k+\beta-1)\}_{k=0}^{\infty}$ are known multiplier sequences by Theorem 16. Hence their composition, (2.19), must also be a multiplier sequence. □

Remark 41. The argument used to prove Proposition 40 also shows that for any real $\beta > 0$, $n \in \mathbb{N}$, the sequence

$$\left\{ \frac{1}{\Gamma(k+\beta)(k+\beta+n)} \right\}_{k=0}^{\infty}$$

is a multiplier sequence.

Computer experiments suggest that when β is not an integer, the sequence (2.19) in Proposition 40, is not a multiplier sequence.

Example 42. The fourth degree Jensen polynomial

$$g_4(x) = \sum_{k=0}^{4} \binom{4}{k} \frac{1}{k!(k+1/2)} x^k,$$

associated with the sequence

$$\left\{ \frac{1}{k!(k+1/2)} \right\}_{k=0}^{\infty}, \tag{2.20}$$

has two non-real zeros. Thus, the sequence (2.20) fails to satisfy the conditions in Theorem 14, and therefore is not a multiplier sequence.

In the proof of Proposition 2.22 below, we will refer to the following elementary property of Laguerre operators (see Remark 34).

Proposition 43. *If $h \in \mathbb{C}[x]$ has a zero at $z_0 \in \mathbb{C} \setminus \{0\}$ of multiplicity $n \in \mathbb{N}$, and $\alpha \in \mathbb{C}$, then $(xD+\alpha)h(x)$ has a zero of multiplicity $n-1$ at z_0; that is, the operator $(xD+\alpha)$ strictly decreases the multiplicity of any zero which does not occur at the origin. Therefore, if p(x) is a real polynomial having only real negative zeros, then the multiplier sequence $\{p(k)\}_{k=0}^{\infty}$ strictly decreases the multiplicity of any zero which does not occur at the origin when it is applied to a real polynomial.*

Proof. We first show the operator $(xD+\alpha)$ strictly decreases the multiplicity of any zero which does not occur at the origin. Let $\alpha \in \mathbb{C}$, $(x-z_0) \nmid q(x)$, and consider

$$\begin{aligned}(xD+\alpha)(x-z_0)^n q(x) &= nx(x-z_0)^{n-1}q(x) + (x-z_0)^n xq'(x) + \alpha(x-z_0)^n q(x) \\ &= (x-z_0)^{n-1}[(x-z_0)(xq'(x)+\alpha q(x)) + nxq(x)]. \qquad (2.21)\end{aligned}$$

The factor on the right in square brackets is not divisible by $(x - z_0)$ for $z_0 \neq 0$. Therefore, the multiplicity of the zero z_0 in (2.21) is $n-1$. This shows that $(xD+\alpha)$ strictly decreases the multiplicity of any zero which does not occur at the origin. The statement about $\{p(k)\}_{k=0}^{\infty}$ follows immediately from Remark 34 and Theorem 16. $\square$

Proposition 44. *Fix $n, \beta \in \mathbb{N}$. Then, there is a real $\varepsilon = \varepsilon(n,\beta)$, $|\varepsilon| > 0$, such that $\left\{\frac{1}{k!(k+\beta+\varepsilon)}\right\}_{k=0}^{\infty}$ is an n-sequence.*

Proof. Fix $n, \beta \in \mathbb{N}$, and let $f_k(\varepsilon) := 1/[k!(k+\beta+\varepsilon)]$. The Taylor series of $f_k(\varepsilon)$ about $\varepsilon = 0$ is

$$\begin{aligned} f_k(\varepsilon) &= \frac{1}{k!(k+\beta)} - \frac{1}{k!(k+\beta)^2}\varepsilon + O(\varepsilon^2) \\ &= \frac{1}{k!(k+\beta)}\left[1 - \frac{1}{(k+\beta)}\varepsilon + O(\varepsilon^2)\right], \quad (\varepsilon \to 0). \end{aligned} \tag{2.22}$$

When β is a positive integer, the sequence $\left\{\frac{1}{k!(k+\beta)}\right\}_{k=0}^{\infty}$ corresponding to the leading factor of (2.22) is a multiplier sequence by Proposition 40, while the sequence $\left\{-\frac{1}{(k+\beta)}\varepsilon\right\}_{k=0}^{\infty}$ corresponding to the correction term in the brackets is not a multiplier sequence by Theorem 23. For $\beta = 1$, the sequence $\{1/(k+1)!\}_{k=0}^{\infty}$ takes the polynomials $(1+x)^n$ to polynomials having *only simple negative zeros* by Theorem 38. In addition, for $\beta \geq 2$, the multiplier sequence

$$\left\{\frac{1}{k!(k+\beta)}\right\}_{k=0}^{\infty} = \left\{(k+1)\cdots(k+\beta-1)\frac{1}{(k+\beta)!}\right\}_{k=0}^{\infty} \tag{2.23}$$

takes the polynomials $(1+x)^n$ to polynomials having *only simple negative zeros* (using Theorem 38 for the special case where the domain polynomial has only negative zeros and Lageurre's Theorem—see Proposition 43, Theorem 16). Hence, for the Jensen polynomial

$$\begin{aligned} g_n(x) &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k!(k+\beta+\varepsilon)} x^k \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k!(k+\beta)} x^k + O(\varepsilon), \qquad (\varepsilon \to 0), \end{aligned}$$

there is a real $\varepsilon = \varepsilon_0$ sufficiently small, such that $g_n(x)$ has only negative (simple) zeros. Therefore, the sequence $\left\{\frac{1}{k!(k+\beta+\varepsilon_0)}\right\}_{k=0}^{\infty}$ is an n-sequence by Theorem 15. $\square$

3 Meromorphically interpolated functions having non-real zeros

In this section we investigate properties of the transcendental entire functions $\varphi = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!}$, where the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is interpolated by a non-polynomial rational function. Let us begin by noting that an entire function

$$\varphi(x) = \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} \frac{p(k)}{q(k)} \frac{x^k}{k!}, \tag{2.24}$$

where $p(x), q(x) \in \mathbb{R}[x]$, $p(x) \neq 0$, has order ([68, Chapter I])

$$\begin{aligned} \overline{\lim_{k\to\infty}} \frac{k \ln k}{\ln \frac{1}{|c_k|}} &= \overline{\lim_{k\to\infty}} \frac{k \ln k}{\ln |k!| - \ln |p(k)/q(k)|} \\ &= \overline{\lim_{k\to\infty}} \frac{k \ln k}{k \ln(k) + O(k) - \ln |p(k)/q(k)|} = 1. \end{aligned} \tag{2.25}$$

We have seen that φ has non-real zeros (cf. Theorem 23). The next corollary states that φ, defined by 2.24, has non-real zeros if it is perturbed by a polynomial.

Corollary 45. *For any $h(x) \in \mathbb{R}[x]$, and entire $\varphi(x) = \sum_{k=0}^{\infty} \frac{p(k)}{q(k)} \frac{x^k}{k!}$, with $p, q \in \mathbb{R}[x]$ relatively prime,* $\deg(q) \geq 1$, *the sum $\psi(x) = \varphi(x) + h(x)$ has infinitely many non-real zeros.*

Proof. Let $m = \deg(h)+1$, then $D^m\psi(x) = D^m\varphi(x)$. By Theorem 23, every derivative of φ has non-real zeros (see Lemma 47), thus the same holds for ψ. To show that ψ has an infinite number of non-real zeros, assume instead that ψ has a finite number of non-real zeros. Then, by Theorem 31, there is an $n \in \mathbb{N}$ such that $D^n\psi$ has only real zeros. This is a contradiction. $\square$

R. Duffin and A. C. Schaeffer have shown that if an entire function of order 1 and mean type is bounded on the real axis, then it has only real zeros when perturbed by a cosine function of appropriate frequency and amplitude.

Theorem 46 ([39])**.** *Let $f(z)$ be an entire function, satisfying the conditions*

i. $|f(z)| \leq 1$ *on the real axis ($z \in \mathbb{R}$), and* *ii.* $|f(z)| = O(e^{\lambda|z|}), \lambda > 0$, *uniformly over the entire plane.*

Then for every real α *the function*

$$\cos(\lambda z + \alpha) - f(z)$$

has only real zeros, or vanishes identically. Moreover, all the zeros are simple, except perhaps at points on the real axis where $f(x) = \pm 1$.

Note that S. Adams and D. Cardon have obtained results on sums of entire functions which preserve reality of zeros [1]. The following lemma is the observation that every derivative of φ in (2.24) has non-real zeros.

Lemma 47. *Let* T *be the linear operator such that for all* $k \in \mathbb{N} \cup \{0\}$, $T[x^k] = \frac{p(k)}{q(k)} x^k$, *where* $p(x), q(x) \in \mathbb{R}[x]$, *relatively prime, and* $\deg(q(x)) \geq 1$. *Then* $D^j T[e^x]$ *has infinitely many non-real zeros for every* $j \in \mathbb{N} \cup \{0\}$.

Proof. $D^j T[e^x] = \sum_{k=0}^{\infty} \frac{p(k+n)}{q(k+n)} \frac{x^k}{k!}$. By Theorem 23, any shift of T is not a multiplier sequence, and by Corollary 45 $D^j T[e^x]$ has infinitely many non-real zeros. □

The next proposition shows that replacing differentiation with an arbitrary Hermite-Poulain operator, $(D + \alpha)$, $\alpha \in \mathbb{R}$, (see Theorem 25) in Lemma 47 is not enough to produce an entire function with real zeros.

Proposition 48. *Let* T *be a linear operator defined on monomials by* $T[x^k] = \frac{p(k)}{q(k)} x^k$, *where* $p(x), q(x) \in \mathbb{R}[x]$, *relatively prime, and* $\deg(q(x)) \geq 1$. *Then* $g(D)T[e^x]$ *has an infinite number of non-real zeros for any polynomial* $g \in \mathbb{R}[x] \cap \mathscr{L}\text{-}\mathscr{P}$.

Proof. It is sufficient to consider the action of the operator $(D + \alpha)$, $\alpha \in \mathbb{R}$, acting on $T[e^x]$. We can then sequentially apply these operators, since they commute, to produce $g(D)$ (in the same manner as $p(xD)$ is applied in Remark 34). Note that $T[e^x] \neq e^{-\alpha x}$, for any $\alpha \in \mathbb{R}$, and therefore $(D + \alpha)T[e^x] \neq 0$.

$$\begin{aligned} (D+\alpha)T[e^x] &= (D+\alpha)\sum_{k=0}^{\infty} \frac{p(k)}{q(k)} \frac{x^k}{k!} \\ &= \sum_{k=0}^{\infty} \left(\frac{p(k+1)}{q(k+1)} + \alpha \frac{p(k)}{q(k)} \right) \frac{x^k}{k!} \\ &= \sum_{k=0}^{\infty} \left(\frac{p(k+1)q(k) + \alpha p(k)q(k+1)}{q(k+1)q(k)} \right) \frac{x^k}{k!}. \end{aligned}$$

If $\alpha = 0$, then $(D + \alpha)T[e^x]$ has an infinite number of non-real zeros by Lemma 47. If $\alpha \neq 0$, then because $q(k) \nmid q(k+1)$, and p and q are relatively prime, $q(k) \nmid [p(k+1)q(k)+\alpha p(k)q(k+1)]$. Similarly $q(k+1) \nmid q(k)$, and p and q are relatively prime, implies $q(k+1) \nmid [p(k+1)q(k)+\alpha p(k)q(k+1)]$. Then, $p(k+1)q(k)+\alpha p(k)q(k+1)$ and $q(k+1)q(k)$ are relatively prime, hence by Corollary 45, $(D + \alpha)T[e^x]$ has infinitely many non-real zeros. □

The asymptotic behavior of an interpolating function seems important in determining whether or not it interpolates a multiplier sequence. Proposition 51 addresses the asymptotic behavior of sequences.

Proposition 49 ([23])**.** *Let* $\Phi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!}$, $0 \leq \gamma_0 \leq \gamma_1 \leq \cdots$, *be a transcendental entire function in* $\mathscr{L}\text{-}\mathscr{P}^+$. *Then*

$$\Delta^n \gamma_p \geq 0 \qquad n, p = 0, 1, 2, \ldots.$$

Lemma 50 ([23])**.** *Let* $\Phi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!}$, $0 \leq \gamma_0 \leq \gamma_1 \leq \cdots$, *be a transcendental entire function in* $\mathscr{L}\text{-}\mathscr{P}^+$.

i. If, for some nonnegative integer p, $\gamma_p = \gamma_{p+1} \neq 0$, *then* $\gamma_0 = \gamma_1 = \cdots$ *and* $\Phi = \gamma_0 e^x$.

ii. If for $p > 0$, $\gamma_0 = \gamma_1 = \cdots = \gamma_{p-1} = 0$, *but* $\gamma_p \neq 0$, *then* $0 < \gamma_p < \gamma_{p+1} < \gamma_{p+2} < \cdots$.

Proposition 51. *Let* $\varphi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathscr{L}\text{-}\mathscr{P}^+$, *where* $0 < \gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_n \leq \cdots$. *If* φ *is not of the form* Ce^x, *then* $\lim_{k\to\infty} \gamma_k = \infty$.

Proof. We first note that if for some non-negative integer p, $\gamma_p = \gamma_{p+1}$, then it follows from Lemma 50 that the transcendental entire function φ is equal to $\gamma_0 e^x$. Since by assumption φ is not of the form Ce^x, the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is strictly increasing. Moreover, by Proposition 49, the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is convex; that is, $\gamma_{p+2} - \gamma_{p+1} \geq \gamma_{p+1} - \gamma_p$ for $p = 0, 1, 2, \ldots$. We claim that $\gamma_k \to \infty$ as $k \to \infty$. Suppose that $\{\gamma_k\}_{k=0}^{\infty}$

does not tend to infinity. Then, since the sequence is increasing and bounded, it must converge. Let $\varepsilon = \gamma_{m+1} - \gamma_m$. Since $\{\gamma_k\}_{k=0}^{\infty}$ is convergent, there is an $n > m$ such that

$$\gamma_{n+1} - \gamma_n < \varepsilon = \gamma_{m+1} - \gamma_m \leq \cdots \leq \gamma_{n+1} - \gamma_n.$$

this is a contradiction, and therefore $\gamma_k \to \infty$ as $k \to \infty$. □

4 Conjectures and Open Problems

In this section we present several open problems related to results established in this chapter. We conclude with examples of sequences of rational functions which converge locally uniformly (in the right half-plane) to functions which interpolate multiplier sequences, but where the approximating rational functions do not interpolate sequentially higher degree n-sequences. The first group of conjectures and problems is from a paper of T. Craven and G. Csordas.

We introduced the generalized Mittag-Leffler function, $E_{\alpha,\beta}$, in Section 2 (see (2.15)). The following conjecture naturally arises.

Conjecture 52 ([30])**.** For each positive integer m, $E_{2m,1/2}(x) \in \mathscr{L}\text{-}\mathscr{P}^{+}$.

Conjecture 53 (Charalambides [30])**.** If m and n are positive integers, then

$$L_{m,n} = \left\{ \frac{(mk)!(nk)!}{((m+n+1)k)!} \right\}_{k=0}^{\infty}$$

is a multiplier sequence and whence

$$\varphi_{m,n} := \sum_{k=0}^{\infty} \frac{\Gamma(mk+1)\Gamma(nk+1)}{\Gamma((m+k+1)k+1)} \frac{x^k}{k!} \in \mathscr{L}\text{-}\mathscr{P}^{+}.$$

In relation to Corollary 24, we cite the following open problem on Fox-Wright functions.

Problem 54. ([30]) Consider

$${}_2\Psi_1(x) = \sum_{k=0}^{\infty} \frac{\Gamma(ak+1)\Gamma(bk+1)}{\Gamma(ck+d+1)} \frac{x^k}{k!},$$

where $a, b, c, d \geq 0$ and $c \geq a+b$. Under what additional restrictions on the parameters a, b, c, d is it true that ${}_2\Psi_1(x) \in \mathscr{L}\text{-}\mathscr{P}^{+}$?

Even the special case of Problem 54, with $b = d = 0$, $a = m$, $c = m + 1$, leads to the following conjecture.

Conjecture 55. ([30]) For each positive integer m the sequence $\{\Gamma(mk+1)/\Gamma((m+1)k+1)\}_{k=0}^{\infty}$ is a multiplier sequence.

The asymptotic behavior of an interpolating function played a pivotal role in the proof of Theorem 23. Below are some related questions that may be significant.

In reference to Proposition 35, we pose the following question.

Question 56. Given a polynomial $p(x) \in \mathbb{R}[x]$, how can one determine n such that $\{p(k+n))\}_{k=0}^{\infty}$ is a multiplier sequence?

Note that Question 56 is equivalent to determining a sufficient number of differentiations in (2.14) to produce a function with only real zeros. The relevance of the next question stems from the recently proved Fisk-McNamara-Sagan-Stanley conjecture [15].

Question 57. If $p(x) \in \mathbb{R}[x]$, for which $n \in \mathbb{N}$ is $\{(p(k+n))^2 - p(k+n+1)p(k+n-1)\}_{k=0}^{\infty}$ a multiplier sequence?

Problem 58 ([13, Problem 34])**.** Is

$$\left\{(k+m)^{\sqrt{k+m}}\right\}_{k=0}^{\infty}$$

a multiplier sequence for all sufficiently large m?

Problem 58 was recently stated in the literature. Its solution would shed some light on the distribution of zeros of entire functions of order 1 and maximal type. Of special interest is the Riemann ξ-function (1.1), which has order 1 and maximal type. We state below some questions related to Problem 58.

Question 59. For what values of $\alpha, c \in \mathbb{R}$ is $\{(k+\alpha)^c\}_{k=0}^{\infty}$ a multiplier sequence?

We can give a partial answer to Question 59.

Proposition 60. *If $c \in \mathbb{Q}$, $c < 0$, then $\{(k+\alpha)^c\}_{k=0}^{\infty}$ is not a multiplier sequence for $\alpha \in \mathbb{R}$, where $-\alpha \notin \mathbb{N} \cup \{0\}$.*

Proof. First consider the case where $c = -\frac{n}{d} \in \mathbb{Q}$, $n, d \in \mathbb{N}$. Assume that $\{\gamma_k\}_{k=0}^{\infty} = \{(k+\alpha)^c\}_{k=0}^{\infty}$ is a multiplier sequence. Then we may compose $\{\gamma_k\}_{k=0}^{\infty}$ with itself d times, and conclude that $\{(k+\alpha)^{-n}\}_{k=0}^{\infty}$ is a multiplier sequence. This contradicts Theorem 23, and thus $\{(k+\alpha)^c\}_{k=0}^{\infty}$ can not be a multiplier sequence when c is a negative rational number. $\square$

Question 61. For what values of $\alpha, c \in \mathbb{R}$, $\alpha > 0$, is $\{\alpha^{k^c}\}_{k=0}^{\infty}$ a multiplier sequence?

The next problem asks if Corollary 45 holds for an arbitrary function in $\mathscr{L}$-$\mathscr{P}$.

Problem 62. Let T be a linear operator defined on monomials by $T[x^k] = \frac{p(k)}{q(k)}x^k$ where $p(x), q(x) \in \mathbb{R}[x]$, relatively prime, and $\deg(q(x)) \geq 1$. Does $\varphi(x) + T[e^x]$ have non-real zeros for every $\varphi \in \mathscr{L}$-$\mathscr{P}$?

Even though non-polynomial rational functions do not interpolate multiplier sequences, it seems reasonable to investigate rationally interpolated sequences as they approach a multiplier sequence. Both interpolating functions in Examples 63 and 64 approach interpolating functions for multiplier sequences locally uniformly in the right half-plane, but fail to interpolate 2-sequences as they approach the limit function. This is verified by showing the $k = 1$ Turán inequality, $\gamma_1^2 - \gamma_0\gamma_2 \geq 0$, is violated, which implies the sequence is not a 2-sequence (the 2^{nd} Jensen polynomial associated with a sequence $\{\gamma_k\}_{k=0}^{\infty}$ has only real zeros precisely when $\gamma_1^2 - \gamma_0\gamma_2$ is non-negative).

Example 63. Consider the sequence $\{\gamma_k\}_{k=0}^{\infty} = \left\{\frac{k+1+\frac{1}{n}}{k+1-\frac{1}{n}}\right\}_{k=1}^{\infty}$ for $n \geq 1$. The Turán expression is

$$\gamma_1^2 - \gamma_2\gamma_0 = -\frac{8n^3}{12n^4 - 28n^3 - 8n + 1},$$

and is negative for all $n \geq 2$, therefore $\{\gamma_k\}_{k=0}^{\infty}$ is not a 2-sequence for all $n \geq 2$.

Example 64. Consider the sequence $\{\gamma_k\}_{k=0}^{\infty} = \left\{\frac{1}{(1+\frac{k}{n})^n}\right\}_{k=0}^{\infty}$ for $n \geq 2$. This approaches the multiplier sequence $\{\gamma_k\}_{k=0}^{\infty} = \{e^{-k}\}_{k=0}^{\infty}$. The Turán expression,

$$\gamma_1^2 - \gamma_0\gamma_2 = -\frac{1}{e^2}\frac{1}{n} + \frac{1}{2e^2}\frac{1}{n^2} - \frac{11}{24e^4}\frac{1}{n^4} + O\left(\frac{1}{n^5}\right), \qquad (n \to \infty), \tag{2.26}$$

is negative for sufficiently large n, and therefore $\{\gamma_k\}_{k=0}^{\infty}$ is not a 2-sequence for sufficiently large n. The Taylor expansion of the first Turán expression looks similar for $\{\gamma_k\}_{k=0}^{\infty} = \left\{\frac{(1+\frac{k}{2n})^n}{(1+\frac{k}{n})^n}\right\}_{k=0}^{\infty}$.

Chapter 3

Discrete Laguerre inequalities and a conjecture of I. Krasikov

1 Introduction

The classical Laguerre inequality for polynomials states that a polynomial of degree n with only real zeros, $p(x) \in \mathbb{R}[x]$, satisfies $(n-1)p'(x)^2 - np''(x)p(x) \geq 0$ for all $x \in \mathbb{R}$ (see [26, 73]). Thus, the classical Laguerre inequality is a necessary condition for a polynomial to have only real zeros. Our investigation is inspired by an interesting paper of I. Krasikov [64]. He proves several discrete polynomial inequalities, including useful versions of generalized Laguerre inequalities [81], and shows how to apply them by obtaining bounds on the zeros of some Krawtchouk polynomials. In [64], I. Krasikov conjectures a new discrete Laguerre inequality for polynomials. After establishing this conjecture, we generalize the inequality to transcendental entire functions (of order $\rho < 2$, and minimal type of order $\rho = 2$) in the Laguerre-Pólya class (see Definition 13). At the end of this chapter, we apply the techniques we have developed to obtain a finite difference complex zero decreasing operator.

Definition 65. We denote by $\mathscr{L}\text{-}\mathscr{P}_n$ the set of polynomials of degree n in the Laguerre-Pólya class; that is, $\mathscr{L}\text{-}\mathscr{P}_n$ is the set of polynomials of degree n having only real zeros.

The minimal spacing between neighboring zeros of a polynomial in $\mathscr{L}\text{-}\mathscr{P}_n$ is a scale that provides a natural criterion for the validity of discrete polynomial inequalities.

Definition 66. Suppose $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$ has zeros $\{\alpha_k\}_{k=1}^n$, repeated according to their multiplicities, and ordered such that $\alpha_k \leq \alpha_{k+1}$, $1 \leq k \leq n-1$. We define the *mesh size*, associated with the zeros of p, by

$$\mu(p) := \min_{1 \leq k \leq n-1} |\alpha_{k+1} - \alpha_k|.$$

With the above definition of mesh size, we can now state a conjecture of I. Krasikov, which is proved in Section 2.

Conjecture 67 (I. Krasikov [64])**.** If $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$ and $\mu(p) \geq 1$, then

$$(n-1)[p(x+1) - p(x-1)]^2 - 4np(x)[p(x+1) - 2p(x) + p(x-1)] \geq 0 \qquad (3.1)$$

holds for all $x \in \mathbb{R}$.

The classical Laguerre inequality is found readily by differentiating the logarithmic derivative of a polynomial $p(x)$, with only real zeros $\{\alpha_i\}_{i=1}^n$, to give

$$\frac{p''(x)p(x) - (p'(x))^2}{(p(x))^2} = \left(\frac{p'(x)}{p(x)}\right)' = \left(\sum_{k=1}^{n} \frac{1}{(x-\alpha_k)}\right)' = -\sum_{k=1}^{n} \frac{1}{(x-\alpha_k)^2}. \qquad (3.2)$$

Since the right-hand side is non-positive,

$$(p'(x))^2 - p''(x)p(x) \geq 0.$$

This inequality is also valid for an arbitrary function in $\mathscr{L}\text{-}\mathscr{P}$ [26]. A sharpened form of the Laguerre inequality for polynomials can be obtained with the aid of the Cauchy-Schwarz inequality,

$$\left(\sum_{k=1}^{n} \frac{1}{(x-\alpha_k)}\right)^2 \leq n \sum_{k=1}^{n} \frac{1}{(x-\alpha_k)^2}. \qquad (3.3)$$

In terms of p, (3.3) becomes $\left(\frac{p'(x)}{p(x)}\right)^2 \leq n\sum_{k=1}^{n}\frac{1}{(x-\alpha_k)^2}$. Then, (3.3) and (3.2) yield the sharpened version of the Laguerre inequality for polynomials on which Conjecture 67 is based,

$$(n-1)(p'(x))^2 - np''(x)p(x) \geq 0. \tag{3.4}$$

The inequality (3.1) is a finite difference version of the classical Laguerre inequality for polynomials. Indeed, let us define

$$f_n(x,h,p) := (n-1)[p(x+h)-p(x-h)]^2 - 4np(x)[p(x+h)-2p(x)+p(x-h)]. \tag{3.5}$$

Then (3.1) can be written as $f_n(x,1,p) \geq 0$ $(x \in \mathbb{R})$, and we recover the classical Laguerre inequality for polynomials by taking the following limit:

$$\begin{aligned}\lim_{h\to 0}\frac{f_n(x,h,p)}{4h^2} &= (n-1)\left(\lim_{h\to 0}\frac{p(x+h)-p(x-h)}{2h}\right)^2 \\ &\quad - np(x)\left(\lim_{h\to 0}\frac{p(x+h)-2p(x)+p(x-h)}{h^2}\right) \\ &= (n-1)p'(x)^2 - np''(x)p(x).\end{aligned}$$

As I. Krasikov points out, the motivation for inequalities of type (3.1) is that classical discrete orthogonal polynomials $p_k(x)$ satisfy a three-term difference equation (see [76, p. 27], [64])

$$p_k(x+1) = b_k(x)p_k(x) - c_k(x)p_k(x-1),$$

where $b_k(x)$ and $c_k(x)$ are continuous over the interval of orthogonality. Many of the classical discrete orthogonal polynomials satisfy the condition that $c_k(x) > 0$ on the interval of orthogonality, and this implies that $\mu(p) \geq 1$ (see [65]). Therefore, inequalities when $\mu(p) \geq 1$ are of interest, and may help provide sharp bounds on the loci of zeros of discrete orthogonal polynomials [46,48,64]. Indeed, W. H. Foster, I. Krasikov, and A. Zarkh have found bounds on the extreme zeros of many orthogonal polynomials using discrete and continuous Laguerre and new Laguerre type inequalities which they discovered [46–48, 62–65].

In this chapter, we prove I. Krasikov's conjecture (see Theorem 84), extend it to a class of transcendental entire functions in the Laguerre-Pólya class, and formulate several conjectures (cf. Conjecture 86, Conjecture 89, Conjecture 94). In Section 2, we establish several preliminary results about polynomials which satisfy a zero spacing requirement. In Section 3, we establish the existence of a polynomial sequence which satisfies a very special zero spacing requirement and converges uniformly on compact subsets of $\mathbb{C}$ to the exponential function. We use this result to extend a version of (3.1) to transcendental entire functions in the Laguerre-Pólya class up to order $\rho = 2$ and minimal type, and conjecture that (3.1) holds for all functions in $\mathscr{L}$-$\mathscr{P}$. In section 4, we extend the definition of zero spacing to all of $\mathbb{R}[x]$, and establish the existence of a finite difference complex zero decreasing operator.

2 Proof of I. Krasikov's conjecture

In this section we develop some discrete analogues of classical inequalities, form some intuition about the effect of imposing a minimal zero spacing requirement on a polynomial in $\mathscr{L}$-$\mathscr{P}$, and prove Conjecture 67. First, note that one can change the zero spacing requirement in Conjecture 67 by simply rescaling in x. For example, the following conjecture is equivalent to Conjecture 67 of Krasikov.

Conjecture 68. Let $p(x) \in \mathscr{L}$-$\mathscr{P}_n$. Suppose that $\mu(p) \geq h > 0$. Then for all $x \in \mathbb{R}$,

$$f_n(x,h,p) = (n-1)[p(x+h)-p(x-h)]^2 - 4np(x)[p(x+h)-2p(x)+p(x-h)] \geq 0. \quad (3.6)$$

For the sake of clarity, we will work with (3.1) directly ($h = 1$), and keep in mind that we can always make statements about polynomials with an arbitrary positive minimal zero spacing by rescaling $p(x)$ (in other words "measuring x in units of h").

Lemma 69. *A local minimum of a polynomial, $p(x) \in \mathscr{L}$-$\mathscr{P}_n$, with only real simple zeros, is negative. Likewise, a local maximum of $p(x)$ is positive.*

Proof. Since $p(x)$ has simple zeros, at a local minimum $(x_{min}, p(x_{min}))$, it follows that $p'(x_{min}) = 0$ and $p''(x_{min}) > 0$. (Note that $p''(x_{min}) = 0$ implies that p' has a multiple zero at x_{min} which is not possible). The classical Laguerre inequality asserts that if $p(x) \in \mathscr{L}$-$\mathscr{P}$, then for all $x \in \mathbb{R}$, $(p'(x))^2 - p''(x)p(x) \geq 0$. At a local minimum this expression becomes $-p''(x_{min})p(x_{min}) \geq 0$. Therefore, at a local minimum we have $p(x_{min}) \leq 0$. Since the zeros of p are simple, $p(x_{min}) \neq 0$. Thus $p(x_{min}) < 0$. The second statement of the lemma can be proved the same way, or by considering $-p$ and using the first statement. $\square$

A statement similar to Lemma 69 is proved by G. Csordas and A. Escassut [34, Theorem 5.1] for a class of functions whose zeros lie in a horizontal strip about the real axis. With the aid of Lemma 69 we obtain the following result in a discrete setting.

Lemma 70. *Let* $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$*,* $n \geq 2$*,* $\mu(p) \geq 1$*.*

i. If $p(x-1) > p(x)$ *and* $p(x+1) > p(x)$*, then* $p(x) < 0$*.*

ii. If $p(x-1) < p(x)$ *and* $p(x+1) < p(x)$*, then* $p(x) > 0$*.*

Proof. (i) Fix an $x_0 \in \mathbb{R}$. Let $p(x_0 - 1) > p(x_0)$, $p(x_0 + 1) > p(x_0)$, and assume for a contradiction that $p(x_0) \geq 0$. There cannot be any zeros of $p(x)$ in the interval $[x_0 - 1, x_0]$, for if there were, $p(x_0)p(x_0 - 1) > 0$ implies that the number of zeros in $(x_0 - 1, x_0)$ must be even, and this violates the zero spacing $\mu(p) \geq 1$. Similarly, there cannot be any zeros of $p(x)$ in $[x_0, x_0 + 1]$. If $p(x_0) < p(x_0 - 1)$ and $p(x_0) < p(x_0 + 1)$ then there is a point in $(x_0 - 1, x_0 + 1)$ where p' changes sign from negative to positive. This implies p achieves a non-negative local minimum on $[x_0 - 1, x_0 + 1]$ which contradicts Lemma 69.

(ii) The second statement follows by replacing p with $-p$ in (i). $\square$

Using Lemma 70 we can verify that if $p(x) < \min\{p(x+1), p(x-1)\}$, then $p(x) < 0$ and thus the function

$$\begin{aligned} f_n(x,1,p) &= (n-1)[p(x+1) - p(x-1)]^2 - 4np(x)[p(x+1) - 2p(x) + p(x-1)] \\ &= (n-1)[p(x+1) - p(x-1)]^2 \\ &\qquad -4np(x)[(p(x+1) - p(x)) + (p(x-1) - p(x))] \qquad (3.7) \end{aligned}$$

has a non-negative second term and (3.1) is satisfied. Similarly, (3.1) is valid when $p(x) > \max\{p(x-1), p(x+1)\}$. The proof of Conjecture 67 is now reduced to the case where $\min\{p(x+1), p(x-1)\} \leq p(x) \leq \max\{p(x+1), p(x-1)\}$. It is easy to show that if for some $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$, $f_n(x, 1, p) \geq 0$ for all $x \in \mathbb{R}$, then for all $m \geq n$, $f_m(x, 1, p) \geq 0$ for all $x \in \mathbb{R}$. If $\mu(p) \geq 1$, but $m < \deg(p)$, then for some $x_0 \in \mathbb{R}$, $f_m(x_0, 1, p)$ may be negative. Indeed, let $p(x) = x(x-1)(x-2)$, then $f_3(x, 1, p) = 72(x-1)^2$ and $f_2(x, 1, p) = -12(x-3)(x-1)^2(x+1)$. In particular, $f_2(4, 1, p) = -540$.

We next obtain inequalities and relations that are analogous to those used in deriving the continuous version of the classical Laguerre inequality for polynomials.

Definition 71. Let $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$ have only simple real zeros $\{\alpha_k\}_{k=1}^n$. Define forward and backward "discrete logarithmic derivatives" associated with $p(x)$ by

$$F(x) := \frac{p(x+1)-p(x)}{p(x)} =: \sum_{k=1}^{n} \frac{A_k}{(x-\alpha_k)} \tag{3.8}$$

$$\text{and} \qquad R(x) := \frac{p(x)-p(x-1)}{p(x)} =: \sum_{k=1}^{n} \frac{B_k}{(x-\alpha_k)}. \tag{3.9}$$

Note that $\deg(p(x+1)-p(x)) < \deg(p(x))$ and $\deg(p(x)-p(x-1)) < \deg(p(x))$ permits unique partial fraction expansions of the rational functions F and R. Define the sequences $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ associated with $p(x)$ by requiring that they satisfy the equations above.

Remark 72. For an arbitrary finite difference, $h > 0$, the scaled versions of the functions in Definition 71 are $F(x) := \frac{p(x+h)-p(x)}{hp(x)}$ and $R(x) := \frac{p(x)-p(x-h)}{hp(x)}$.

Lemma 73. *For $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$, $n \geq 2$, with $\mu(p) \geq 1$ and zeros $\{\alpha_k\}_{k=1}^n$, the associated sequences $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ satisfy $A_k \geq 0$ and $B_k \geq 0$, for all k, $1 \leq k \leq n$.*

Proof. From Definition 71,

$$p(x+1)-p(x) = \sum_{k=1}^{n} \frac{A_k}{(x-\alpha_k)} p(x) = \sum_{k=1}^{n} \left[A_k \prod_{j\neq k} (x-\alpha_j) \right].$$

Evaluating this at the zero α_k of p, yields $p(\alpha_k+1) = A_k \prod_{j\neq k}(\alpha_k - \alpha_j) = A_k p'(\alpha_k)$. Thus,

$$A_k = \frac{p(\alpha_k+1)}{p'(\alpha_k)}, \quad \text{and similarly} \quad B_k = \frac{-p(\alpha_k-1)}{p'(\alpha_k)}.$$

For some real $\delta > 0$,

$$x \in (\alpha_k - \delta, \alpha_k) \quad \text{implies} \quad p(x)p'(x) < 0, \text{ and}$$
$$x \in (\alpha_k, \alpha_k + \delta) \quad \text{implies} \quad p(x)p'(x) > 0.$$

Since the real zeros of g are spaced at least 1 unit apart, $p(\alpha_k+1)$ is either 0 or has the same sign as $p(x)$ for all $x \in (\alpha_k, \alpha_k+\delta)$. So for all positive $\varepsilon < \delta$, $p(\alpha_k+1)p'(\alpha_k+\varepsilon) \geq 0$, and by continuity $p(\alpha_k + 1)p'(\alpha_k) \geq 0$. Thus $A_k = \frac{p(\alpha_k+1)}{p'(\alpha_k)} \geq 0$. Note $p'(\alpha_k) \neq 0$ since α_k is simple. Likewise, $p(\alpha_k - 1)$ is either 0 or has the same sign as $p(x)$ for $x \in (\alpha_k - \delta, \alpha_k)$. Hence for all positive $\varepsilon < \delta$, $p(\alpha_k - 1)p'(\alpha_k - \varepsilon) \leq 0$. By continuity, $p(\alpha_k - 1)p'(\alpha_k) \leq 0$, whence $B_k \geq 0$. □

Example 74. If the zero spacing requirement in Lemma 73 is violated, then some A_k or B_k may be negative. Indeed, consider $p(x) = x(x + 1 - \varepsilon)$. Then $\frac{p(x+1)-p(x)}{p(x)} = \frac{A_1}{x} + \frac{A_2}{x+1-\varepsilon}$, where

$$A_1 = \frac{2-\varepsilon}{1-\varepsilon} \qquad A_2 = \frac{-\varepsilon}{1-\varepsilon}.$$

For any positive $\varepsilon < 1$, $\mu(p) = 1 - \varepsilon$, and A_2 is negative.

Corollary 75. *For $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$, $n \geq 2$, with $\mu(p) \geq 1$, the associated functions $F(x)$ and $R(x)$ (see Definition 71) satisfy $F'(x) < 0$ and $R'(x) < 0$ on their respective domains.*

Proof. This corollary is a direct result of differentiating the partial fraction expressions for F and R and applying Lemma 73. □

Note that the degree of the numerator of $F(x)$ is $n-1$. If $\mu(p) \geq 1$, then $F(x)$ has $n-1$ real zeros, because $F(x)$ is strictly decreasing between any two consecutive poles of $F(x)$. This proves the following lemma.

Lemma 76. (Pólya and Szegö [88, vol. II, p. 39]) *For $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$, $n \geq 2$, with $\mu(p) \geq 1$, $F(x)$ and $R(x)$ have only real simple zeros.*

In the sequel (see Lemma 83), we show that if $\mu(p(x)) \geq 1$, then $\mu(p(x + 1) - p(x)) \geq 1$, and the zeros of $F(x)$ and $R(x)$ are spaced at least one unit apart.

Lemma 77. *If $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$, then the associated sequences $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ satisfy $\sum_{k=1}^n A_k = n$ and $\sum_{k=1}^n B_k = n$.*

Proof. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathscr{L}\text{-}\mathscr{P}_n$ and denote the zeros of $p(x)$ by $\{\alpha_k\}_{k=1}^n$. Observe that

$$\lim_{|z|\to\infty} zF(z) = \lim_{|z|\to\infty} z\left(\frac{p(z+1)-p(z)}{p(z)}\right) = \lim_{|z|\to\infty} z\sum_{k=1}^n \frac{A_k}{(z-\alpha_k)} = \sum_{k=1}^n A_k. \quad (3.10)$$

Then (3.10) and

$$\begin{aligned} p(z+1)-p(z) &= a_n(z+1)^n + a_{n-1}(z+1)^{n-1} + \ldots + a_0 - [a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0] \\ &= na_n z^{n-1} + O(z^{n-2}),\ |z|\to\infty, \end{aligned}$$

imply that

$$\begin{aligned} \sum_{k=1}^n A_k = \lim_{|z|\to\infty} zF(z) &= \lim_{|z|\to\infty} z\left(\frac{p(z+1)-p(z)}{p(z)}\right) \\ &= \lim_{|z|\to\infty} z\left(\frac{na_n z^{n-1} + O(z^{n-2})}{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0)}\right) = n. \end{aligned}$$

A similar argument shows that $\sum_{k=1}^n B_k = n$. □

Lemma 78. *Given $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$, $n \geq 2$, with $\mu(p) \geq 1$, the associated functions $F(x)$ and $R(x)$ satisfy $(F(x))^2 \leq -nF'(x)$ and $(R(x))^2 \leq -nR'(x)$, for all $x \in \mathbb{R}$, where $p(x) \neq 0$.*

Proof. From Definition 71, $F(x) = \sum_{k=1}^n \frac{A_k}{x-\alpha_k}$ and therefore $F'(x) = \sum_{k=1}^n \frac{-A_k}{(x-\alpha_k)^2}$. By Lemma 73, $\mu(p) \geq 1$ implies the constants $A_k \geq 0$. Using the the Cauchy-Schwarz inequality,

$$\begin{aligned} (F(x))^2 = \left(\sum_{k=1}^n \frac{A_k}{x-\alpha_k}\right)^2 &= \left(\sum_{k=1}^n \sqrt{A_k}\left(\frac{\sqrt{A_k}}{x-\alpha_k}\right)\right)^2 \\ &\leq \left(\sum_{k=1}^n A_k\right)\sum_{k=1}^n \frac{A_k}{(x-\alpha_k)^2} = -nF'(x), \end{aligned}$$

where Lemma 77 has been used in the last equality. An identical argument shows $(R(x))^2 \leq -nR'(x)$ for all $x \in \mathbb{R}$. □

Remark 79. Simple examples show that the inequalities in Lemma 78 are sharp (consider $p(x) = x(x+1-\varepsilon)$).

Lemma 80. *Let $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$, $n \geq 2$, with $\mu(p) \geq 1$, and let $\{\beta_k\}_{k=1}^{n-1}$ be the zeros of $p(x+1) - p(x)$. Let $y \in \mathbb{R}$ be such that $\min\{p(y+1), p(y-1)\} < p(y) < \max\{p(y+1), p(y-1)\}$. Then if the interval $[y-1, y]$ does not contain any β_k,*

$$\frac{1}{n} F(y)R(y) \leq \frac{(p(y))^2 - p(y+1)p(y-1)}{(p(y))^2}.$$

Proof. If no β_k is in $[y-1, y]$, then

$$\frac{F'(x)}{(F(x))^2} = \frac{(p'(x+1)p(x) - p(x+1)p'(x))(p(x))^2}{(p(x+1) - p(x))^2 (p(x))^2}$$

can be extended to be continuous and bounded on $[y-1, y]$. By Lemma 78, $(F(x))^2 \leq -nF'(x)$. Dividing both sides of this inequality by $n(F(x))^2$ and integrating from $y-1$ to y, we obtain

$$\frac{1}{n} \leq \frac{1}{F(y)} - \frac{1}{F(y-1)} = \frac{p(y)}{p(y+1) - p(y)} - \frac{p(y-1)}{p(y) - p(y-1)}.$$

Using $\min\{p(y+1), p(y)\} < p(y) < \max\{p(y+1), p(y-1)\}$, it follows that either $p(y-1) < p(y) < p(y+1)$ or $p(y+1) < p(y) < p(y-1)$. In both cases, $(p(y+1) - p(y))(p(y) - p(y-1)) > 0$, and therefore

$$\begin{aligned} \frac{1}{n}(p(y+1) - p(y))(p(y) - & p(y-1)) \\ &\leq p(y)(p(y) - p(y-1)) - p(y-1)(p(y+1) - p(y)) \\ &\leq (p(y))^2 - p(y+1)p(y-1). \end{aligned}$$

Dividing by $(p(y))^2$ gives the result. □

Lemma 81. *For $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$, the associated functions $F(x)$ and $R(x)$ from Definition 71 satisfy*

$$F(x)R(x) = (F(x) - R(x)) + \frac{(p(x))^2 - p(x+1)p(x-1)}{(p(x))^2}$$

for all $x \in \mathbb{R}$, where $p(x) \neq 0$.

Proof. This lemma is verified by direct calculation using the definitions of $F(x)$ and $R(x)$ in terms of $p(x)$. $\square$

Lemma 82. *Let $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$, $n \geq 2$, with $\mu(p) \geq 1$.*

i. If $p(\beta) = p(\beta+1) > 0$, then for all $x \in (\beta, \beta+1)$, $p(x) > p(\beta)$ and $p(x) > \max\{p(x+1), p(x-1)\}$.

ii. If $p(\beta) = p(\beta+1) < 0$, then for all $x \in (\beta, \beta+1)$, $p(x) < p(\beta)$ and $p(x) < \min\{p(x+1), p(x-1)\}$.

iii. If $p(\beta) = p(\beta+1) = 0$, then for all $x \in (\beta, \beta+1)$, either $p(x) > \max\{p(x+1), p(x-1)\}$ or $p(x) < \min\{p(x+1), p(x-1)\}$.

Proof. Note that by Lemma 76, any β which satisfies $p(\beta) = p(\beta+1)$ under the hypotheses stated in Lemma 82 must be real and simple since β is a zero of $F(x)$.

For case (i), assume for a contradiction that there exists $x_0 \in (\beta, \beta+1)$ such that $p(x_0) \leq p(\beta)$. There can not be any zeros of p on $(\beta, \beta+1)$. For otherwise, $p(\beta)p(\beta+1) > 0$ implies that $p(x)$ must have at least two zeros on $(\beta, \beta+1)$, which contradicts $\mu(p) \geq 1$. Thus, for all $x \in (\beta, \beta+1)$, $p(x) > 0$. Specifically $p(x_0) > 0$.

By the mean value theorem there exists $a \in (\beta, \beta+1)$ with $p'(a) = 0$. Since $p(x)$ does not change sign on $(\beta, \beta+1)$, the interval $(\beta, \beta+1)$ must lie between two neighboring zeros of $p(x)$, call them α_1 and α_2, such that $(\beta, \beta+1) \subset (\alpha_1, \alpha_2)$. The zeros of $p(x)$ and $p'(x)$ interlace, and in order to preserve the interlacing a must be the only zero of $p'(x)$ in (α_1, α_2), hence $p'(\beta), p'(\beta+1) \neq 0$. Because the zeros are simple, for some $\varepsilon > 0$, for all $x \in (\alpha_1, \alpha_1 + \varepsilon)$, $p'(x)p(x) > 0$, and for all $x \in (\alpha_2 - \varepsilon, \alpha_2)$, $p'(x)p(x) < 0$. Since p' and p do not change sign on (α_1, β) or $(\beta+1, \alpha_2)$, this gives us that $p'(\beta) > 0$ and $p'(\beta+1) < 0$. Then if $p(x_0) \leq p(\beta)$, p' must change signs at least twice on (α_1, α_2) (actually three times), at least once on (β, x_0) and at least once on $(x_0, \beta+1)$, and this contradicts the uniqueness of a. Thus for all $x \in (\beta, \beta+1)$ we have $p(x) > p(\beta)$.

To show $p(x) > p(\beta)$ implies $p(x) > \max\{p(x+1), p(x-1)\}$ for all $x \in (\beta, \beta+1)$, notice that since $p'(y) < 0$ for all $y \in (\beta+1, \alpha_2)$, $p(\beta+1) > p(y)$

for all $y \in (\beta+1, \alpha_2)$, and due to the zero spacing $p \leq 0$ on (α_2, α_2+1), hence $p(\beta+1) > p(x+1)$ for all $x \in (\beta, \alpha_2)$. Thus, for all $x \in (\beta, \beta+1)$, $p(x) > p(\beta+1) > p(x+1)$. In the same way, $p'(y) > 0$ for $y \in (\alpha_1, \beta)$ and $p \leq 0$ on $(\alpha_1 - 1, \beta)$ imply that $p(\beta) > p(x)$ for all $x \in (\alpha_1 - 1, \beta)$ and therefore $p(x) > p(x-1)$ for all $x \in (\beta, \beta+1)$. Hence, for all $x \in (\beta, \beta+1)$, $p(x) > p(x-1)$ and $p(x) > p(x+1)$, therefore $p(x) > \max\{p(x+1), p(x-1)\}$.

Consider case (iii). If $p(\beta) = p(\beta+1) = 0$, then p does not change sign on $(\beta, \beta+1)$ since $\mu(p) \geq 1$. It suffices to consider the case when p is positive on $(\beta, \beta+1)$. Then for all $x \in (\beta, \beta+1)$, $p(x) > 0 = p(\beta)$. The conclusion $p(x) > \max\{p(x+1), p(x-1)\}$ ($p(x) < \min\{p(x+1), p(x-1)\}$) is a consequence of $p(x) > p(\beta)$ ($p(x) < p(\beta)$) by the same argument given in the proof of case (i).

To prove (ii), let $g(x) = -p(x)$ and apply (i).

□

Lemma 83. *If $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$, $n \geq 2$, $\mu(p) \geq 1$, and $g(x) = p(x+1) - p(x)$, then $\mu(g) \geq 1$.*

Proof. (*Reductio ad Absurdum*) If $\mu(g) < 1$, then there exist $\beta_1, \beta_2 \in \mathbb{R}$ such that $0 < \beta_2 - \beta_1 < 1$ and $g(\beta_1) = g(\beta_2) = 0$. In the proof of Lemma 82 we have shown that $p(x)$ does not change sign on (β_1, β_1+1). Without loss of generality assume that p is positive on (β_1, β_1+1). Observe that $\beta_2 \in (\beta_1, \beta_1+1)$, and thus by Lemma 82, $p(\beta_2) > \max\{p(\beta_2+1), p(\beta_2-1)\} \geq p(\beta_2+1)$. But this yields $p(\beta_2+1) - p(\beta_2) < 0$, and therefore $g(\beta_2) < 0$ contradicting $g(\beta_2) = 0$. □

Note that Lemma 83 is equivalent to the statement that if $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$ with $\mu(p) \geq 1$, then the associated functions $F(x)$ and $R(x)$ also have zeros spaced at least 1 unit apart. Preliminaries aside, we prove Conjecture 67 of I. Krasikov.

Theorem 84. *If $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$ and $\mu(p) \geq 1$, then*

$$f_n(x,1,p) = (n-1)[p(x+1)-p(x-1)]^2 - 4np(x)[p(x+1)-2p(x)+p(x-1)] \geq 0 \quad (3.11)$$

holds for all $x \in \mathbb{R}$.

Proof. Since (3.11) is true when $\deg(p(x))$ is 1 or 2, we assume $n \geq 2$. Fix $x = x_0 \in \mathbb{R}$. If $p(x_0 - 1) = p(x_0) = p(x_0 + 1)$, or if $p(x_0) = 0$, then $f_n(x, 1, p) \geq 0$. Thus, we may assume $p(x_0) \neq 0$. If $p(x_0) < \min\{p(x_0 + 1), p(x_0 - 1)\}$, or if $p(x_0) > \max\{p(x_0 + 1), p(x_0 - 1)\}$, then $f_n(x_0, 1, p) \geq 0$ (use (3.7) and Lemma 70).

We next consider the case when

$$\min\{p(x_0 - 1), p(x_0 + 1)\} < p(x_0) < \max\{p(x_0 - 1), p(x_0 + 1)\} \tag{3.12}$$

(thus $x_0 \neq \beta$ or $\beta + 1$, where $p(\beta + 1) = p(\beta)$), and show

$$\frac{f_n(x_0, 1, p)}{(p(x_0))^2} = (n-1)(F(x_0) + R(x_0))^2 - 4n(F(x_0) - R(x_0)) \geq 0,$$

where $F(x)$ and $R(x)$ are defined by (3.8) and (3.9) respectively. By Lemma 81,

$$\begin{aligned} \frac{f_n(x_0, 1, p)}{(p(x_0))^2} =& (n-1)(F(x_0) - R(x_0))^2 \\ & - 4n\left(\frac{1}{n}F(x_0)R(x_0) - \frac{(p(x_0))^2 - p(x_0+1)p(x_0-1)}{(p(x_0))^2}\right). \end{aligned} \tag{3.13}$$

By Lemma 83, $\mu(p(x+1) - p(x)) \geq 1$, and thus the zeros $\{\beta_k\}_{k=1}^{n-1}$ of $F(x)$ ($p(\beta_k + 1) = p(\beta_k)$) are spaced at least one unit apart. If $[x_0 - 1, x_0]$ does not contain any β_k, $\frac{f_n(x_0,1,p)}{(p(x_0))^2} \geq 0$ holds by Lemma 80 (see (3.13)) . If, on the other hand, $\beta_j \in (x_0 - 1, x_0)$ (recall $\beta_j \neq x_0, x_0 - 1$), then $x_0 \in (\beta_j, \beta_j + 1)$ and by Lemma 82 either $p(x_0) > \max\{p(x_0 - 1), p(x_0 + 1)\}$ or $p(x_0) < \min\{p(x_0 - 1), p(x_0 + 1)\}$. Both of these cases contradict our assumption (see (3.12)). We have now shown $f_n(x_0, 1, p)) \geq 0$ for all $x_0 \in \mathbb{R}$, except for the isolated points where $x_0 = \beta_j$ or $x_0 = \beta_j + 1$ for some j, but by continuity of $f_n(x, 1, p)$, (3.11) will hold.

□

The converse of Theorem 84 is false in general. Indeed, the following example shows that there are polynomials with arbitrary minimal zero spacing that still satisfy $f_n(x, 1, p) \geq 0$ for all $x \in \mathbb{R}$.

Example 85. Let $p(x) = (x + 3 + a)(x + 2)(x + 1)$, where $a \in \mathbb{R}$. Then

$$f_3(x, 1, p) = 288 + 144a + 24a^2 + (288 + 120a + 24a^2)x + (72 + 24a + 8a^2)x^2$$

The discriminant of $f_3(x,1,p)$ is $D = -192a^2(3+a)^2 \leq 0$. Thus $f_3(x,1,p)$ does not change sign and is always positive (this is verified by showing that the coefficient of x^2 is positive when considered as a quadratic in a). Therefore, $f_3(x,1,p) \geq 0$ for all $x \in \mathbb{R}$.

In general, a polynomial p may satisfy $f_n(p,1,x) \geq 0$ for all $x \in \mathbb{R}$, even if p has multiple zeros. If $p(x) = x^2(x+1)$, which has $\mu(p) = 0$, then $f_3(x,1,p) = 56x^2 + 32x + 8$ is non-negative for all $x \in \mathbb{R}$. A polynomial p with non-real zeros may also satisfy $f_n(p,1,x) \geq 0$ for all $x \in \mathbb{R}$. For example, let $p(x) = (x^2+1)(x+1)$, then $f_3(x,1,p) = 32x^2 - 32x + 8 \geq 0$ for all $x \in \mathbb{R}$.

It is known that a polynomial $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$ with only real zeros satisfies $\mu(p) \leq \mu(p')$; that is, $p'(x)$ will have a minimal zero spacing which is larger than that of $p(x)$ (N. Obreschkoff [78, p. 13, Satz 5.3], P. Walker [98]). In light of Lemma 83 and the aforementioned result, we suggest the following conjecture.

Conjecture 86. If $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$, $n \geq 2$, $\mu(p) \geq d \geq 1$, and $g(x) = p(x+1) - p(x)$, then $\mu(g) \geq d$.

The derivation of the classical Laguerre inequality relies on properties of the logarithmic derivative of a polynomial. In the same way, Conjecture 67 was proved using a discrete version of the logarithmic derivative. The analogy between the discrete and continuous logarithmic derivatives motivates Theorem 88 and Conjecture 89, based on Theorem 87 and its converse (B. Muranaka [75]).

Theorem 87 (P. B. Borwein and T. Erdélyi [14, p. 345])**.** *If $p \in \mathscr{L}\text{-}\mathscr{P}_n$, then*

$$m\left(\left\{x \in \mathbb{R} : \frac{p'(x)}{p(x)} \geq \lambda\right\}\right) = \frac{n}{\lambda} \qquad \textit{for all } \lambda > 0,$$

where m denotes Lebesgue measure.

Theorem 88. *If $p \in \mathscr{L}\text{-}\mathscr{P}_n$, $n \geq 2$, $\mu(p) \geq 1$, then*

$$m\left(\left\{x \in \mathbb{R} : \frac{p(x+1) - p(x)}{p(x)} \geq \lambda\right\}\right) = \frac{n}{\lambda} \qquad \textit{for all } \lambda > 0,$$

where m denotes Lebesgue measure.

Proof. Without loss of generality let $p(x) = \sum_{k=1}^{n} a_k x^k$ be monic. By Corollary 75,

$$\frac{p(x+1)-p(x)}{p(x)} = F(x) \tag{3.14}$$

is decreasing on its domain. For fixed $\lambda > 0$, let $\beta_1, \ldots, \beta_n$ denote the zeros of

$$p(x) - \frac{1}{\lambda}\left(p(x+1)-p(x)\right) = x^n + \left(a_{n-1} - \frac{n}{\lambda}\right) x^{n-1} + \cdots + a_0.$$

Note that $\beta_1, \ldots, \beta_k$ are also the zeros of $F(x) - \lambda$. Let α_k be zeros of p, and $\ell[a,b]$ denote the length of the interval from a to b. Then,

$$\begin{aligned}
m\left(\left\{x \in \mathbb{R} : \frac{p(x+1)-p(x)}{p(x)} \geq \lambda\right\}\right) &= \ell[\alpha_1,\beta_1] + \ell[\alpha_2,\beta_2] + \cdots + \ell[\alpha_n,\beta_n] \\
&= \sum_{k=1}^{n} (\beta_k - \alpha_k) \\
&= \sum_{k=1}^{n} \beta_k - \sum_{k=1}^{n} \alpha_k \\
&= -\left(a_{n-1} - \frac{n}{\lambda}\right) - (-a_{n-1}) \\
&= \frac{n}{\lambda}.
\end{aligned}$$

□

Conjecture 89. If $p(x)$ is a real polynomial of degree $n \geq 2$, and if

$$m\left(\left\{x \in \mathbb{R} : \frac{p(x+1)-p(x)}{p(x)} \geq \lambda\right\}\right) = \frac{n}{\lambda} \qquad \text{for all } \lambda > 0,$$

where m denotes Lebesgue measure, then $p \in \mathscr{L}\text{-}\mathscr{P}_n$ with $\mu(p) \geq 1$.

3 Extension to a class of transcendental entire functions

In analogy with (3.5) we define, for a real entire function φ,

$$f_\infty(x,h,\varphi) := [\varphi(x+h) - \varphi(x-h)]^2 - 4\varphi(x)[\varphi(x+h) - 2\varphi(x) + \varphi(x-h)]. \tag{3.15}$$

For $\varphi \in \mathscr{L}\text{-}\mathscr{P}$, with zeros $\{\alpha_i\}_{i=1}^{\omega}$, $\omega \le \infty$, we introduce the mesh size

$$\mu_\infty(\varphi) := \inf_{i \ne j} |\alpha_i - \alpha_j|. \tag{3.16}$$

We remark that if $\psi \notin \mathscr{L}\text{-}\mathscr{P}$, then ψ need not satisfy $f_\infty(x,h,\psi) \ge 0$ for all $x \in \mathbb{R}$. A calculation shows that if $\psi(x) = e^{x^2}$, then $f_\infty(0,1,\psi) = -8(e-1) < 0$. When $\varphi \in \mathscr{L}\text{-}\mathscr{P}_n$, $f_\infty(x,h,\varphi) \ge 0$ for all $x \in \mathbb{R}$ by Theorem 84. In order to extend Theorem 84 to transcendental entire functions, we require the following preparatory result to ensure that the approximating polynomials we use will satisfy a zero spacing condition.

Lemma 90. *For any $a \in \mathbb{R}$, $n \in \mathbb{N}$, $n \ge 2$,*

$$\lim_{n\to\infty} \sum_{k=1}^{n^n} \frac{1}{n\ln(n)(k+n)+a} = 1.$$

Proof. Fix $a \in \mathbb{R}$. Since the terms $\frac{1}{n\ln(n)(k+n)+a}$ are decreasing with k for n sufficiently large, we obtain

$$\int_1^{n^n+1} \frac{1}{n\ln(n)(k+n)+a}dk \le \sum_{k=1}^{n^n} \frac{1}{n\ln(n)(k+n)+a} \le \int_0^{n^n} \frac{1}{n\ln(n)(k+n)+a}dk,$$

for n sufficiently large, by considering the approximating Riemann sums for the integrals. Thus,

$$\begin{aligned} \frac{1}{n\ln(n)} \ln\left(\frac{n^n+1+\frac{a}{n\ln(n)}}{n+1+\frac{a}{n\ln(n)}}\right) &\le \sum_{k=1}^{n^n} \frac{1}{n\ln(n)(k+n)+a} \\ &\le \frac{1}{n\ln(n)} \ln\left(\frac{n^n+\frac{a}{n\ln(n)}}{n+\frac{a}{n\ln(n)}}\right). \end{aligned} \tag{3.17}$$

As $n \to \infty$, both the left and right sides of (3.17) approach 1, and whence the sum in the middle approaches 1. □

Lemma 91. *The set of polynomials $\left\{q_n(z) = \prod_{k=1}^{n^n}\left(1+\frac{z}{n\ln(n)(k+n)}\right) : n \in \mathbb{N},\ n \ge 2\right\}$ has a subsequence which converges uniformly on compact subsets of $\mathbb{C}$ to e^z.*

Proof. Let $K \subset \mathbb{C}$ be any compact set and let $R = \sup_{z \in K} |z|$. Recall the inequality

$$\frac{1}{2}|z| \leq |\ln(1+z)| \leq \frac{3}{2}|z| \qquad \text{for } |z| < \frac{1}{2}$$

[22, p. 165]. Then for $n > \max\{2R, 3\}$, $\left|\frac{z}{n\ln(n)(k+n)}\right| < \frac{1}{2}$, hence, for $k \geq 1$ and $z \in K$,

$$\frac{1}{2}\frac{|z|}{n\ln(n)(k+n)} \leq \left|\ln\left(1 + \frac{z}{n\ln(n)(k+n)}\right)\right| \leq \frac{3}{2}\frac{|z|}{n\ln(n)(k+n)},$$

and therefore

$$\frac{1}{2}\sum_{k=1}^{n^n}\frac{|z|}{n\ln(n)(k+n)} \leq \sum_{k=1}^{n^n}\left|\ln\left(1 + \frac{z}{n\ln(n)(k+n)}\right)\right| \leq \frac{3}{2}\sum_{k=1}^{n^n}\frac{|z|}{n\ln(n)(k+n)}.$$

As $n \to \infty$ the sums on the left and right sides of the inequality converge by Lemma 90 to $\frac{1}{2}|z|$ and $\frac{3}{2}|z|$ respectively. In particular, for some $\varepsilon > 0$ and $N > 2R$ sufficiently large, for all $n \geq N$ and for all $z \in K$,

$$\sum_{k=1}^{n^n}\left|\ln\left(1 + \frac{z}{n\ln(n)(k+n)}\right)\right| \leq \frac{3}{2}R + \varepsilon.$$

Then for all $n \geq N$, for all $z \in K$,

$$|q_n(z)| \leq \exp\left(\sum_{k=1}^{n^n}\left|\ln\left(1 + \frac{z}{n\ln(n)(k+n)}\right)\right|\right) \leq e^{\frac{3}{2}R+\varepsilon}.$$

So for $n > N$ sufficiently large, the sequence $\{q_n(z)\}_{n=2}^{\infty}$ is uniformly bounded on compact subsets $K \subset \mathbb{C}$ and thus form a normal family by Montel's theorem [22, p. 153]. Thus, there is a subsequence of $\{q_n(z)\}_{n=2}^{\infty}$ which converges uniformly on compact subsets of $\mathbb{C}$ to a function f, and therefore satisfies

$$\frac{f'(x)}{f(x)} = \lim_{n\to\infty}\frac{q_n'(x)}{q_n(x)} = \lim_{n\to\infty}\sum_{k=1}^{n^n}\frac{1}{n\ln(n)(k+n)+x} = 1, \tag{3.18}$$

for a fixed $x \in \mathbb{R}$, where the last equality is by Lemma 90. Equation (3.18) and $f(0) = 1$, imply $f(x) = e^x$ on $\mathbb{R}$, and thus f is the exponential function. □

Lemma 92. *If $\varphi(x) = p(x)e^{bx}$, $b \in \mathbb{R}$, $p \in \mathscr{L}\text{-}\mathscr{P}_n$, $n \geq 2$, and $\mu(p) \geq 1$, then $f_\infty(x, 1, \varphi) \geq 0$ for all $x \in \mathbb{R}$.*

Proof. By Lemma 91, there is a subsequence of

$$\left\{ q_j(x) = \prod_{k=1}^{j^j} \left(1 + \frac{x}{j\ln(j)(k+j)} \right) \right\}_{j=2}^{\infty},$$

call it $\{q_{j_m}(x)\}_{m=1}^{\infty}$, such that $q_{j_m}(x) \to e^x$ uniformly on compact subsets of $\mathbb{C}$, as $m \to \infty$. Let $\{\alpha_k\}_{k=1}^{n}$ be the zeros of $p(x)$, and $R = \max_{1\leq k\leq n} |\alpha_k|$. The zero of least magnitude of $q_{j_m}(bx)$, z_{j_m}, satisfies $|z_{j_m}| = \frac{j_m \ln(j_m)(1+j_m)}{b}$, $b \neq 0$. Both $\mu(q_{j_m}(bx)) \to \infty$ as $m \to \infty$ and $|z_{j_m}| \to \infty$ as $m \to \infty$. Thus, there is an M such that for all $m > M$, $|z_{j_m}| > R + 1$, and the sequence of polynomials $h_m(x) = p(x)q_{j_{M+m}}(bx)$, $m \geq 1$, is in $\mathscr{L}\text{-}\mathscr{P}_\ell$ for some ℓ, and satisfies $\mu(h_m) \geq 1$. By Theorem 84, $f_\infty(x, 1, h_m) \geq 0$ for all $x \in \mathbb{R}$, for all m. Since $h_m \to p(x)e^{bx}$ by construction,

$$\lim_{m\to\infty} f_\infty(x, 1, h_m) = f_\infty(x, 1, p(x)e^{bx}) \geq 0.$$

□

Theorem 93. *If $\varphi \in \mathscr{L}\text{-}\mathscr{P}$ has order $\rho < 2$, or if φ is of minimal type of order $\rho = 2$, and $\mu_\infty(\varphi) \geq 1$, then $f_\infty(x, 1, \varphi) \geq 0$ for all $x \in \mathbb{R}$.*

Proof. By the Hadamard factorization theorem, φ has the representation

$$\varphi(x) = cx^m e^{bx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{a_k} \right) e^{-\frac{x}{a_k}} \qquad (\omega \leq \infty),$$

where $a_k, b, c \in \mathbb{R}$, m is a non-negative integer, $a_k \neq 0$, and $\sum_{k=1}^{\omega} \frac{1}{a_k^2} < \infty$. Let

$$g_n(x) = cx^m e^{bx} \prod_{k=1}^{n} \left(1 + \frac{x}{a_k} \right) e^{-\frac{x}{a_k}}.$$

Then, $g_n(x) = c\exp\left(bx - \sum_{k=1}^{n} \frac{x}{a_k}\right) x^m \prod_{k=1}^{n} \left(1 + \frac{x}{a_k}\right)$ has the form $p(x)e^{\gamma x}$, $\gamma \in \mathbb{R}$, $p \in \mathscr{L}\text{-}\mathscr{P}_n$, and thus by Lemma 92, $f_\infty(x, 1, g_n) \geq 0$ for all $x \in \mathbb{R}$, and for all n. Since we also have $g_n \to \varphi$ by construction, $\lim_{n\to\infty} f_\infty(x, 1, g_n) = f_\infty(x, 1, \varphi) \geq 0$ for all $x \in \mathbb{R}$. □

In light of Theorem 93, we make the following conjecture.

Conjecture 94. If $\varphi \in \mathscr{L}\text{-}\mathscr{P}$ and $\mu_\infty(\varphi) \geq 1$ then $f_\infty(x, 1, \varphi) \geq 0$ for all $x \in \mathbb{R}$.

If Conjecture 94 is true, it can not be proved using a sequence of polynomials which satisfy a zero spacing requirement, as the following theorem shows.

Theorem 95. *Let $h : \mathbb{N} \to \mathbb{N}$, where $h(n) \to \infty$ as $n \to \infty$. There does not exist a sequence of polynomials of the form*

$$p_n(x) = \prod_{k=1}^{h(n)} \left(1 - \frac{x^2}{(f(k,n))^2}\right) \qquad k, n \in \mathbb{N},$$

where $\mu(p_n(x)) \geq 1$ for all $n \in \mathbb{N}$, and $f(k,n) : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is strictly positive and monotonically increasing in k for fixed n, such that $p_n(x) \to e^{-ax^2}$ locally uniformly, where $a \in \mathbb{R} \setminus \{0\}$.

Proof. Suppose there exists a sequence $\{p_n\}_{n=1}^{\infty}$ satisfying the above hypotheses such that $p_n \to e^{-ax^2}$ locally uniformly. For a fixed n, the hypotheses on p_n imply $f(1,n) > 0$, $f(k+1,n) - f(k,n) \geq 1$, and therefore

$$f(k,n) \geq f(1,n) + k - 1. \tag{3.19}$$

A computation yields

$$\frac{p_n'(x)}{p_n(x)} = \sum_{k=1}^{h(n)} \left(\frac{1}{f(k,n)+x} - \frac{1}{f(k,n)-x}\right). \tag{3.20}$$

Since by assumption $p_n \to e^{-ax^2}$ locally uniformly, by Hurwitz's Theorem (see Theorem 9) for an arbitrary $R \in \mathbb{R}$ and sufficiently large n, there are no zeros of p_n on the closed ball $\overline{B}(0,R)$, thus $f(1,n) \to \infty$ as $n \to \infty$. This implies that for sufficiently large n and for a fixed real x, both fractions in the sum (3.20) are monotonically decreasing with k, and each term in the sum has the same sign. In order that $p_n \to e^{-ax^2}$, it is necessary that $\frac{p_n'(x)}{p_n(x)} \to -2ax$. However, for n sufficiently large, and for a fixed positive x,

$$\begin{aligned} \left|\frac{p_n'(x)}{p_n(x)}\right| &\leq \left|\int_1^{h(n)} \left(\frac{1}{k+f(1,n)-1+x} - \frac{1}{k+f(1,n)-1-x}\right) dk\right| \\ &= \left|\ln\left(\frac{(h(n)+x)(f(1,n)-x)}{(h(n)-x)(f(1,n)+x)}\right)\right|, \end{aligned} \tag{3.21}$$

where the inequality follows from (3.19). Since $f(1,n) \to \infty$ as $n \to \infty$, the right-hand side of (3.21) converges to 0. This is a contradiction.

$\square$

In [42], it has been assumed that zero spacing does not make sense for functions of order 2 with mean type or greater. Conjecture 94 suggests there may be exceptional functions in $\mathscr{L}$-$\mathscr{P}$ to which the concept of zero spacing applies.

4 Complex zero decreasing finite difference operators

By the Hermite-Poulain Theorem (see Theorem 25), the operator $(D - \alpha)$, where $\alpha \in \mathbb{R}$, and more generally $\varphi(D)$, where $\varphi \in \mathscr{L}$-$\mathscr{P}$, are complex zero decreasing operators. In this section we show the finite difference operator $(\Delta - \alpha)$, where $\Delta p(x) := p(x+1) - p(x)$, is a complex zero decreasing operator with an appropriate zero spacing hypothesis. We will see that $f(\Delta)$, where $f \in \mathscr{L}$-$\mathscr{P}_n$ with $n \geq 2$, is not a complex zero decreasing operator (see Examples 99 and 100). First, we extend the definition of zero spacing to all polynomials in $\mathbb{R}[x]$.

Definition 96. Suppose $g(x) = h(x)p(x) \in \mathbb{R}[x]$, where $h(x)$ has only non-real zeros and $p(x)$ has only real zeros. If $\deg(p) \geq 2$, then we define the *mesh size*, associated with the zeros of g by $\mu^*(g) := \mu(p)$ (see Definition 66), otherwise $\mu^*(g) = 0$.

We now extend Definition 71 to include the case when $p(x)$ may have non-real zeros.

Definition 97. Let $g(x) \in \mathbb{R}[x]$ have real zeros $\{\alpha_k\}_{k=1}^n$ and 2ℓ non-real zeros. We extend the definition of forward and backward "discrete logarithmic derivatives" associated with g by

$$\begin{aligned} F(x) &:= \frac{g(x+1)-g(x)}{g(x)} =: \sum_{k=1}^{n} \frac{A_k}{(x-\alpha_k)} + \sum_{j=1}^{\ell} \frac{E_j x + D_j}{(x^2+b_j x+c_j)} \\ \text{and} \qquad R(x) &:= \frac{g(x)-g(x-1)}{g(x)} =: \sum_{k=1}^{n} \frac{B_k}{(x-\alpha_k)} + \sum_{j=1}^{\ell} \frac{F_j x + G_j}{(x^2+b_j x+c_j)}, \end{aligned}$$

where $(x^2 + b_j x + c_j)$ is an irreducible factor of g.

Remark 98. See Definition 71 and Remark 72 for additional comments relevant to Definition 97.

The examples below show that the operator $(\Delta + \alpha)$, $\alpha \in \mathbb{R}$, can only be applied to a polynomial once with the guarantee that the non-real zeros of the resulting polynomial will not be increased.

Example 99. Let $p(x) = (x-1)(x-2)(x-3)$. A computation shows that

$$(\Delta + 3)p(x) = p(x+1) - p(x) + 3p(x) = 3(x-2)^2(x-1).$$

Thus, $(\Delta+3)p(x)$ has no more non-real zeros than $p(x)$, but the mesh size is $\mu^*((\Delta+3)p(x)) = 0$. Another computation shows that $(\Delta+3)^2 p(x) = 3(x-1)(3x^2-9x+8)$, and this has two non-real zeros.

Remark 100. When $p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$ and $\mu^*(p) = \mu(p) \geq 1$, the interlacing of the zeros of $p(x+1)$ and $p(x)$ implies $(\Delta+3)p(x) \in \mathscr{L}\text{-}\mathscr{P}_n$ by Theorem 5 and a perturbation argument. A stronger statement is proved here (Theorem 103) .

Example 101. Let $p(x) = (x^2+1)(x-2)(x-3)$. Then, $\mu^*(p) = 1$ and

$$(\Delta + 3)p(x) = p(x+1) - p(x) + 3p(x) = (x-2)^2(3x^2 + x + 4).$$

Thus, $(\Delta+3)p(x)$ has no more non-real zeros than $p(x)$, but the mesh size is $\mu^*((\Delta+3)p(x)) = 0$. Another computation shows that $(\Delta+3)^2 p(x) = 9x^4 - 21x^3 + 21x^2 - 33x + 40$, which has four non-real zeros.

The following proof of Theorem 102 is essentially the proof of Lemma 73 with some modifications.

Theorem 102. *For $g(x) \in \mathbb{R}[x]$, with $\mu^*(g) \geq 1$ and real zeros $\{\alpha_k\}_{k=1}^n$, $n \in \mathbb{N}$, the associated sequences $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ satisfy $A_k \geq 0$ and $B_k \geq 0$, for all k, $1 \leq k \leq n$.*

Proof. From Definition 97,

$$\begin{aligned} g(x+1)-g(x) &= \sum_{k=1}^{n} \frac{A_k}{(x-\alpha_k)} g(x) + \sum_{j=1}^{\ell} \frac{E_j x + D_j}{(x^2+b_j x + c_j)} g(x) \\ &= \sum_{k=1}^{n} \left[A_k \prod_{j\neq k} (x-\alpha_j) \prod_{j=1}^{\ell} (x^2 + b_j x + c_j) \right] + \sum_{j=1}^{\ell} \frac{E_j x + D_j}{(x^2+b_j x + c_j)} g(x). \end{aligned}$$

Evaluating this at a real zero α_k of g yields

$$g(\alpha_k+1) = A_k \prod_{j\neq k} (\alpha_k - \alpha_j) \prod_{j=1}^{\ell} (\alpha_k^2 + b_j \alpha_k + c_j) = A_k g'(\alpha_k).$$

Thus,

$$A_k = \frac{g(\alpha_k+1)}{g'(\alpha_k)}, \quad \text{and similarly} \quad B_k = \frac{-g(\alpha_k-1)}{g'(\alpha_k)}.$$

For some real $\delta > 0$,

$$\begin{aligned} x \in (\alpha_k - \delta, \alpha_k) &\quad \text{implies} \quad g(x)g'(x) < 0, \text{ and} \\ x \in (\alpha_k, \alpha_k + \delta) &\quad \text{implies} \quad g(x)g'(x) > 0. \end{aligned}$$

Since the real zeros of g are spaced at least 1 unit apart, $g(\alpha_k+1)$ is either 0 or has the same sign as $g(x)$ for all $x \in (\alpha_k, \alpha_k+\delta)$. So for all positive $\varepsilon < \delta$, $g(\alpha_k+1)g'(\alpha_k+\varepsilon) \geq 0$ and by continuity $g(\alpha_k+1)g'(\alpha_k) \geq 0$. Thus $A_k = \frac{g(\alpha_k+1)}{g'(\alpha_k)} \geq 0$. Note $g'(\alpha_k) \neq 0$ since α_k is simple. Likewise, $g(\alpha_k - 1)$ is either 0 or has the same sign as $g(x)$ for $x \in (\alpha_k + \delta, \alpha_k)$. Hence for all positive $\varepsilon < \delta$, $g(\alpha_k - 1)g'(\alpha_k - \varepsilon) \leq 0$. By continuity, $g(\alpha_k-1)g'(\alpha_k) \leq 0$, whence $B_k \geq 0$. □

Theorem 103. *If $g(x) \in \mathbb{R}[x]$ with $\mu^*(g) \geq 1$, $\alpha \in \mathbb{R}$, then*

$$Z_c\left((\Delta - \alpha)g(x)\right) \leq Z_c(g(x))$$

where $Z_c(g(x))$ denotes the number of non-real zeros of g, counting multiplicities.

Proof. Let $\{x_k\}_{k=1}^n$ be the real zeros of $g(x)$, and fix $\alpha \in \mathbb{R}$. By Theorem 102, the associated sequence $\{A_k\}_1^n$ satisfies $A_k \geq 0$ for $k = 1, \ldots, n$, and this implies that for $F(x) = \frac{\Delta g}{g}$, and $k = 1, \ldots, n$,

$$\lim_{x\to x_k^-} F(x) - \alpha = -\infty \qquad \text{and} \qquad \lim_{x\to x_k^+} F(x) - \alpha = \infty.$$

Therefore, in between any two consecutive zeros of $g(x)$ there is at least one real zero of

$$F(x) - \alpha = \frac{1}{g(x)}([g(x+1) - g(x)] - \alpha g(x)).$$

Hence, $([g(x+1) - g(x)] - \alpha g(x))$ has at least $n-1$ real zeros. If $\alpha \neq 0$, by degree considerations, $([g(x+1) - g(x)] - \alpha g(x))$ must have at least n real zeros. Thus, $Z_c\left([g(x+1) - g(x)] - \alpha g(x)\right) \leq Z_c(g(x))$. $\square$

Remark 104. With g as in the hypotheses of Theorem 103, a similar argument shows $Z_c([g(x) - g(x-1)] - \alpha g(x)) \leq Z_c(g(x))$.

Chapter 4

Linear sector preservers and sector properties of entire functions

1 Introduction

In their groundbreaking 2009 paper, J. Borcea and P. Brändén characterized all linear transformations $T : \mathbb{R}[x] \to \mathbb{R}[x]$ which preserve reality of zeros, and in addition provided characterizations of linear preservers of polynomials whose zeros lie in closed circular domains [11]. This solved a problem that was first formally stated by T. Craven and G. Csordas [29], and has been a theme in the work of many mathematicians, including Hermite, Laguerre, and Pólya (see Theorem 14). Craven and Csordas also asked for a characterization of linear operators T, such that whenever the zeros of the polynomial p lie in a given sector, $T[p]$ has all of its zeros in the same sector. We state a refined version of this problem below, and obtain sufficient conditions for a large class of linear preservers of polynomials whose zeros lie in a sector. In the course of our investigation, we obtain several composition theorems and results pertaining to transcendental entire functions whose zeros all lie in a sector. We begin by introducing a notation for sectors to simplify the discussion in this chapter.

Definition 105. For $\delta > 0$, let

$$\mathbb{S}(\theta, \delta) = \{z \in \mathbb{C} \setminus \{0\} \;:\; |\arg(z) - \theta| < \delta\}.$$

Let $\overline{\mathbb{S}}(\theta,\delta)$ denote the closure of $\mathbb{S}(\theta,\delta)$, and $\overline{\mathbb{S}}(\theta,0)=\{z\in\mathbb{C}\ :\ z=0 \text{ or } \arg(z)=\theta\}$. In particular, $\overline{\mathbb{S}}(\pi,\delta)$ denotes the closed sector, symmetric about the negative real axis, with half angle δ.

Our main goal is to investigate the following problems stated in recent papers [9, 11, 26] (for more detail see Chapter 5).

Problem 106. Let $\mathcal{P}\subset\mathbb{C}[z]$ (or $\mathbb{R}[z]$) be the set of univariate polynomials having zeros only in $\overline{\mathbb{S}}(\theta,\delta)$, $(0\le\delta\le\frac{\pi}{2})$. Classify all linear operators $T:\mathbb{C}[z]\to\mathbb{C}[z]$ (or $\mathbb{R}[z]\to\mathbb{R}[z]$) such that $T(\mathcal{P})\subset\mathcal{P}$.

Problem 107. Let $\mathcal{P}\subset\mathbb{C}[z_1,\dots,z_n]$ (or $\mathbb{R}[z_1,\dots,z_n]$) be the set of multivariate polynomials p, such that $p(z_1,\dots,z_n)\neq 0$ whenever $z_1,\dots,z_n\notin\overline{\mathbb{S}}(\theta,\delta)$, $(0\le\delta\le\frac{\pi}{2})$. Classify all linear operators $T:\mathbb{C}[z_1,\dots,z_n]\to\mathbb{C}[z_1,\dots,z_n]$ (or $\mathbb{R}[z_1,\dots,z_n]\to\mathbb{R}[z_1,\dots,z_n]$) such that $T(\mathcal{P})\subset\mathcal{P}$.

While we do not solve these problems completely, we obtain sufficient conditions on an operator T in Problem 106 (and Problem 107) such that $T(\mathcal{P})\subset\mathcal{P}\cup\{0\}$. In addition, we gain insight about stability problems on non-circular regions. The condition $\delta\le\frac{\pi}{2}$ has been imposed in Problems 106 and 107 to make them more manageable. For a domain G which is not convex, even differentiation may produce new zeros outside G.

Definition 108. Let $\Omega\subset\mathbb{C}$. If $f\in\mathbb{C}[z_1,\dots,z_n]$ and $f(z_1,\dots,z_n)\neq 0$ for all $z_1,\dots,z_2\in\Omega$, then f is said to be Ω-*stable*. Note that a univariate polynomial, $p(z)$, which is Ω-stable, has all of its zeros in Ω^c. We denote by $\mathcal{S}_n(\theta,\delta)$ the set of all $f\in\mathbb{C}[z_1,\dots,z_n]$ which are $\overline{\mathbb{S}}(\theta,\delta)^c$-stable; that is, $\mathcal{S}_n(\theta,\delta)$ contains all the polynomials $f\in\mathbb{C}[z_1,\dots,z_n]$ such that $f(z_1,\dots,z_n)\neq 0$, for all $z_1,\dots,z_n\notin\overline{\mathbb{S}}(\theta,\delta)$. A polynomial is *stable* if it is non-zero in the open upper half-plane. Denote by $\mathcal{H}_n$ the set of stable polynomials in $\mathbb{C}[z_1,\dots,z_n]$. Thus, a polynomial $p(z_1,\dots,z_n)$ in $\mathcal{H}_n$ is non-zero whenever $\operatorname{Im} z_k>0$ for all $k=1,\dots,n$. A real polynomial is called *hyperbolic* if it has only real zeros. We denote by $\mathcal{H}_n(\mathbb{R})$ the set of real stable polynomials in n variables (*i.e.* $\mathcal{H}_n(\mathbb{R})=\mathcal{H}_n\cap\mathbb{R}[x]$). When the dimension $n=1$, we omit the subscript and write $\mathcal{H}$, $\mathcal{H}(\mathbb{R})$, and $\mathcal{S}(\theta,\delta)$.

Definition 109. An operator T is said to be a *stability preserver* (in n dimensions), if $T(\mathcal{H}_n) \subset \mathcal{H}_n \cup \{0\}$. T is a *hyperbolicity preserver* (in n dimensions) if $T(\mathcal{H}_n(\mathbb{R})) \subset \mathcal{H}_n(\mathbb{R}) \cup \{0\}$. If $\mathcal{P}$ is a set of polynomials, an operator T is a *$\mathcal{P}$-preserver* if $T(\mathcal{P}) \subset \mathcal{P} \cup \{0\}$.

Definition 110. Denote by $\overline{\mathcal{P}}$ the set of functions which are locally uniform limits in $\mathcal{P}$.

Definition 111. For convenience, we define $\mathbb{N}_{\text{o}} = \mathbb{N} \cup \{0\}$, and $D = \frac{d}{dz}$, for the rest of the chapter.

For an operator on a univariate polynomial space, $T : \mathbb{C}[z] \to \mathbb{C}[z]$, we extend its action to multivariate spaces by treating the other variables as constants. For example,

$$T[e^{zw}] = \sum_{k=0}^{\infty} \frac{w^k}{k!} T[z^k] \in \mathbb{C}[z][[w]]$$

is a formal power series in w. The following theorems of Borcea and Brändén [11] provide algebraic and transcendental characterizations of all linear hyperbolicity preservers and stability preservers.

Theorem 112 ([11] Characterization of stability preservers)**.** *A linear operator $T : \mathbb{C}[z] \to \mathbb{C}[z]$ preserves stability if and only if either*

i. T has range of dimension at most one and is of the form

$$T(f) = \alpha(f)P, \qquad f \in \mathbb{C}[z]$$

where $\alpha : \mathbb{C}[z] \to \mathbb{C}$ is a linear functional and P is a stable polynomial, or

ii. $T[e^{-zw}] \in \overline{\mathcal{H}}_2$, or equivalently

iii. $T[(z+w)^n] \in \mathcal{H}_2 \cup \{0\}$ for all $n \in \mathbb{N}_{\text{o}}$.

Theorem 113 ([11] Characterization of hyperbolicity preservers)**.** *A linear operator $T : \mathbb{R}[z] \to \mathbb{R}[z]$ preserves hyperbolicity if and only if either* *i. T has range of dimension at most two and is of the form*

$$T(f) = \alpha(f)P + \beta(f)Q, \qquad f \in \mathbb{R}[z],$$

where $\alpha, \beta : \mathbb{R}[z] \to \mathbb{R}$ are linear functionals and P, Q are hyperbolic polynomials with interlacing zeros, or

ii. either $T[e^{-zw}] \in \overline{\mathcal{H}}_2(\mathbb{R})$, or $T[e^{zw}] \in \overline{\mathcal{H}}_2(\mathbb{R})$,or equivalently

iii. either $T[(z+w)^n] \in \mathcal{H}_2(\mathbb{R}) \cup \{0\}$ for all $n \in \mathbb{N}_o$, or $T[(z-w)^n] \in \mathcal{H}_2(\mathbb{R}) \cup \{0\}$ for all $n \in \mathbb{N}_o$.

Theorems 112 and 113 are beautiful generalizations of the characterization of diagonal hyperbolicity preservers (multiplier sequences) using Jensen polynomials and a transcendental entire function (see Theorem 14). Ideally, one would like to obtain similar characterizations of $\mathcal{S}(\theta, \delta)$-preservers. We begin our investigation of Problems 106 and 107 by finding sector versions of classical composition theorems that can be generalized. First, we look at the following types of standard compositions involving two entire functions.

Definition 114. Given two entire functions, $\varphi = \sum_{k=0}^{\infty} a_k x^k$ and $\psi = \sum_{k=0}^{\infty} b_k x^k$, the *Hadamard composition of φ and ψ*, denoted by $\varphi * \psi$, is given by

$$(\varphi * \psi)(x) := \sum_{k=0}^{\infty} a_k b_k x^k.$$

The *Schur composition of φ and ψ*, denoted by $\varphi \odot \psi$, is given by

$$(\varphi \odot \psi)(x) := \sum_{k=0}^{\infty} k! a_k b_k x^k,$$

provided the series converges for all $x \in \mathbb{C}$.

Note that if φ and ψ are entire functions, then $\varphi * \psi$ is automatically entire. The following result is proved in Section 3.

Corollary 115. *If the zeros of $f, g \in \mathbb{R}[z]$ lie in the sector $\mathbb{S}(\pi, \delta)$, and*

$$\min\{\deg(f), \deg(g)\} \leq \frac{1}{|\sin\delta|^2},$$

then all the zeros of the Schur composition $f \odot g$ lie in $\mathbb{S}(\pi, \delta)$.

It appears difficult to avoid a restriction on the degrees of the polynomials in Corollary 115. Indeed, we find a simple example of two real polynomials with zeros in $\mathbb{S}(\pi, 2\pi/5)$ whose Hadamard (and Schur) composition has zeros outside $\mathbb{S}(\pi, 2\pi/5)$ (Example 126). This is a bit of a surprise, since it has been shown that the Hadamard composition of polynomials whose zeros lie in the left half-plane, $\mathbb{S}(\pi, \frac{\pi}{2})$, again lies in the left half-plane [49].

We prove a sector analog of the Hermite-Poulain Theorem and extend it to transcendental entire functions (Theorems 131, 135). Hermite-Poulain type differential operators appear to have a closer analogy to half-plane stability preservers than the diagonal sector preservers given by the Schur composition. Any linear operator T acting on $\mathbb{C}[z]$ (or $\mathbb{C}[z_1, \ldots, z_n]$) can be represented uniquely as a differential operator of possibly infinite order (a proof of this is supplied in Chapter 5). In the univariate case this representation has the form $T = \sum_{k=0}^{\infty} Q_k(z)D^k$. Given an operator $T : \mathbb{C}[z] \to \mathbb{C}[z]$ with the representation $T = \sum_{k=0}^{\infty} Q_k(z)D^k$, we define its *symbol* to be $F_T(z, w) = \sum_{k=0}^{\infty} Q_k(z)w^k \in \mathbb{C}[z, w]$. The ring of finite order differential operators of the form $T = \sum_{k=0}^{N} Q_k(z)D^k$ is known as the (univariate) *Weyl algebra*, and is denoted $\mathcal{A}_1[\mathbb{C}]$ [66, p. 7].

In Section 4, we prove some sufficient conditions for an operator T to preserve sector stability, including the following (see Definition 139 for $\mathcal{H}^{\delta}$).

Theorem 116 (Sufficient conditions for closed and open sector preservers in the Weyl algebra)**.** *Let $T \in \mathcal{A}_1[\mathbb{C}]$. If either*

i. $F_T(z, e^{-i2(\theta+\delta+\pi/2)}w) \in \mathcal{H}^{\theta+\delta}$ and $F_T(z, e^{-i2(\theta-\delta-\pi/2)}w) \in \mathcal{H}^{\pi+\theta-\delta}$, $0 < \delta \leq \frac{\pi}{2}$, or

ii. $F_T(z, e^{-i2\theta}w) \in \mathcal{S}_2(\theta, \delta)$, $0 \leq \delta \leq \frac{\pi}{2}$,

then T is an $\mathcal{S}(\theta, \delta)$-preserver. Furthermore, if $0 < \delta \leq \frac{\pi}{2}$ and

$$F_T(z, e^{-i2\theta}w) = w^k H(z, e^{-i2\theta}w), \tag{4.1}$$

where $k \in \mathbb{N}_o$, $H \in \mathbb{C}[z, w]$, and $H(z, e^{-i2\theta}w) \neq 0$ whenever $z \notin \mathbb{S}(\theta, \delta)$ and $w \in \overline{\mathbb{S}}(\theta, \delta)^c \cup \{0\}$, then T preserves the set of polynomials whose zeros all lie in the open sector $\mathbb{S}(\theta, \delta)$.

The sufficient condition (ii) in Theorem 116 is not necessary when $\delta < \pi/2$, but it is analogous to the condition that characterizes both univariate and multivariate stability preservers which can be expressed as differential operators of finite order (see Theorem 150). When $\delta \neq 0$, (ii) implies (i); we have stated them separately not only due to the $\delta = 0$ case, but because they have been obtained by different methods of proof. The proof of (ii) uses only an argument involving differential operators, while the most direct proof of (i) relies implicitly on the Grace-Walsh-Szegő Theorem (Theorem 8). It seems desirable to obtain results for sectors that do not directly depend on the Grace-Walsh-Szegő Theorem, which appears unavoidably tied to circular regions. For this purpose, we point out in Section 4 that (ii) can be used to prove (i). Indeed, both of the conditions (ii) and (i) reduce to the conditions which characterize Weyl algebra stability preservers (Theorem 150) when $\theta = -\pi/2$ and $\delta = \pi/2$. The form for open sectors, (4.1), enables us to give a characterization of closed (also called *strict*) upper half-plane stability preservers in the univariate Weyl algebra (Theorem 152), which completes the univariate case of a characterization begun by J. Borcea and P. Brändén in [12]. We also obtain multivariate extensions of some of the univariate theorems in Section 4 and a sector analog of a result of E. Lieb and A. Sokal (Proposition 156), which applies to multivariate affine polynomials.

The motivation for solving Problems 106 and 107 is their potential for application in combinatorics, matrix theory and other areas. A satisfactory solution of Problem 106 would reduce to Theorem 112 in the case that the sector is a half-plane. Theorem 112, along with its multivariate versions, has already found a variety of significant applications [10, 12, 16].

We begin by reviewing some known composition theorems and properties of entire functions related to sectors in Section 2. Subsequently, some of these results are extended (Proposition 173, Theorems 171, 162), and applied to the Riemann ξ-function (Theorem 177). In Section 3, we find several composition theorems which appear to be new (see 135, 115). We address Problems 106 and 107 in Section 4, prove sufficient conditions for an operator T to be an $\mathcal{S}(\theta, \delta)$-preserver, and provide a characterization of strict stability preservers in the univariate Weyl algebra. In Section 5, we prove Turán-type inequalities for polynomials having their zeros in a sector. A connection is established between the locus of zeros of an entire function having order ≤ 1, and the zeros of its associated Jensen polynomials.

2 Classical results

As stated in Definition 110, $\overline{\mathcal{S}}(\theta, \delta)$ is closure of $\mathcal{S}(\theta, \delta)$ (Definition 108) under locally uniform limits. The following theorem gives a standard characterization of $\overline{\mathcal{S}}(\theta, \delta)$, when $0 \leq \delta < \pi/2$.

Theorem 117 ([68, Chapter VIII]). *An entire function $\psi(z) \in \overline{\mathcal{S}}(\theta, \delta)$ $(\delta < \frac{\pi}{2})$, if and only if $\psi(z)$ can be represented in the form*

$$\psi(z) = cz^q e^{-\sigma z} \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right), \tag{4.2}$$

where

$$\sum_{k=1}^{\infty} |a_k|^{-1} < \infty, \qquad a_k \in \overline{\mathbb{S}}(\theta, \delta), \qquad \sigma \in \overline{\mathbb{S}}(\theta, \delta). \tag{4.3}$$

The following theorem was first proved by N. G. DeBruijn [37], and an alternate proof is given by T. Craven and G. Csordas [29].

Theorem 118 (Generalized Malo-Schur-Szegő Composition Theorem). *If $A(z) \in \mathcal{S}(\theta_1, \delta_1)$ $(\delta_1 \leq \pi)$ and if $B(z) \in \mathcal{S}(\theta_2, \delta_2)$ $(\delta_2 \leq \pi)$, then $C(z) = (A \odot B)(z) \in \mathcal{S}(\theta_1 + \theta_2 + \pi, \delta_1 + \delta_2)$.*

Remark 119. Theorem 118 also holds when all of the zeros of the polynomials involved are required to lie in open sectors. Observe that Theorem 118 is the best result possible in the following sense. If $A(z) = z + e^{i\theta} \in \mathcal{S}(\pi, \theta)$, $B(z) = z + e^{i\phi+\varepsilon} \in \mathcal{S}(\pi, \phi + \varepsilon)$, where $\theta, \phi > 0$, then, $(A \odot B)(z) = z + e^{i\phi+\theta+\varepsilon} \in \mathcal{S}(\pi, \phi + \theta + \varepsilon)$ but $(A \odot B)(z) \notin \mathcal{S}(\pi, \phi + \theta)$ for $\varepsilon > 0$).

The following theorem of N. Obreschkoff [77] yields a criterion for the coefficients of a polynomial to form an n-sequence (see Definition 11). In addition, it is an extension of Schur's composition theorem (Theorem 3), as the special case $\delta = 0$ allows for f to have arbitrary degree. Its relevance to n-sequences was pointed out by T. Craven and G. Csordas [29].

Theorem 120 (N. Obreschkoff [77])**.** *Let the real polynomial* $f(x) = \sum_{k=0}^{n} a_k x^k$ *have only real zeros and let the zeros of* $g(z) = \sum_{k=0}^{m} b_k x^k$ *lie in* $\overline{\mathbb{S}}(0,\delta) \cup \overline{\mathbb{S}}(\pi,\delta)$*, where* $\sin\delta \leq \frac{1}{\sqrt{n}}$*. Then the polynomials*

$$g(D)f(z) = b_m f^{(m)}(z) + b_{m-1} f^{(m-1)}(z) + \cdots + b_0 f^{(0)}(z), \qquad b_k \in \mathbb{R}, \tag{4.4}$$

and

$$(g \odot f)(x) = a_0 b_0 + 1! a_1 b_1 z + 2! a_2 b_2 z^2 + \cdots + r! a_r b_r z^r, \qquad b_k \geq 0, \tag{4.5}$$

have only real zeros, where $r = \min\{m, n\}$*.*

In Obreschkoff's original proof of Theorem 120, (4.4) is proved in a clever, but cumbersome, three page argument. Theorem 120 can now be proved in a few lines using the characterization of linear hyperbolicity preservers on polynomials of fixed degree [11]. An argument of Pólya involving Sturm's Theorem (see [68, Chapter VIII]), shows (4.5) follows from (4.4) in Theorem 120. In Section 3, we will use Theorem 120 to give a new composition theorem for polynomials having their zeros in a sector. We extend Theorem 120 in Section 5 to transcendental entire functions, and use it to relate the sector locus of zeros of an entire function φ and the zeros of the Jensen polynomials associated with φ .

There is a close relationship between the zero locus of a polynomial p with respect to a sector, and the total positivity of the coefficients of p (Definition 121).

Definition 121. A sequence $\{a_k\}_{k=0}^{\infty}$ is $\mathbf{TP}_m$, if every $k \times k$ minor, $k \leq m$, of the infinite matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \ldots \\ 0 & a_0 & a_1 & \ldots \\ 0 & 0 & a_0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is non-negative. A sequence that is $\mathbf{TP}_m$ is called *m-times positive*, and if a sequence is m-times positive for every $m \in \mathbb{N}$, it is said to be *totally positive* ($\mathbf{TP}$).

Using the following theorem of I. J. Schoenberg on total positivity and a more recent result of O. Katkova [59], we establish a Turán-type inequality necessary for the zeros of a polynomial to lie in a given sector. Schoenberg's results are realized using the variation diminishing properties of totally positive matrices acting on sequences of real numbers.

Theorem 122 (I. J. Schoenberg [91])**.** *Let $f(z) = \sum_{k=0}^{n} a_k z^k$, $a_k \in \mathbb{R}$.*

i. If $\{a_k\}$ is $\mathbf{TP}_m$, $m < \infty$, then $f(z) \neq 0$ for $z \in \left\{ z : |\arg z| < \dfrac{\pi m}{n+m-1} \right\}$.

ii. If $f(0) > 0$ and $f(z) \neq 0$ for $z \in \left\{ z : |\arg z| < \dfrac{\pi m}{m+1} \right\}$, then $\{a_k\}$ is $\mathbf{TP}_m$.

In Chapter 5, we discuss totally positive matrices in connection with generating functions for orthogonal polynomial sequences.

The following known theorem is stated by D. Cardon and A. Rich in [21], who give a new proof, and a similar result is given by G. Csordas, A. Ruttan and R. S. Varga [36].

Theorem 123. *Let $\varphi(z) = \prod_n (1 + \rho_n z) = \sum_{k=0}^{\infty} \gamma_k \frac{z^k}{k!}$ be a real entire function of genus 0 and suppose that all the zeros of φ lie in $\mathbb{S}(0, \pi/4)$. Then, the strict Turán inequalities, $\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} > 0$, hold for all positive integers k.*

In Section 5, we prove a closed sector version of Theorem 123 for order 1 entire functions.

3 Composition theorems for sectors

In this section, we prove some new composition theorems, but with the exception of the known case of multiplier sequences, they will be dependent on the degree of the polynomials involved. A theorem involving differential operators (Theorem 131), appears to have a closer analogy to similar theorems for stability and hyperbolicity preservers. We use this observation to guide our investigation of linear sector preservers in Section 4.

We first make elementary observations regarding quadratic polynomials, before considering Schur and Hadamard compositions of polynomials in $\mathcal{S}(\theta, \delta)$.

Lemma 124. *The real polynomial* $p(z) = a_2z^2 + a_1z + a_0$ *has zeros in* $\overline{\mathbb{S}}(\pi, \delta)$ $(0 \leq \delta \leq \frac{\pi}{2})$ *if and only if* $a_1^2 \geq 4a_0a_2 \cos^2 \delta$ *and* $a_2a_1, a_1a_0, a_2a_0 \geq 0$.

Proof. If $a_2 = 0$, then $a_1a_0 \geq 0$ implies that p has a non-positive real zero. Assume that $a_2 > 0$. A polynomial with complex conjugate zeros $\alpha, \bar{\alpha}$ of argument θ has the form

$$a_2z^2 - a_2 2 \cos\theta|\alpha|z + a_2|\alpha|^2. \tag{4.6}$$

The zeros lie within $\overline{\mathbb{S}}(\pi, \delta)$ if and only if $\cos^2\theta \geq \cos^2\delta$ and $\cos\theta \leq 0$. Equating coefficients in (4.6), $a_1 = -2a_2\cos\theta|\alpha|$, and $a_0 = a_2|\alpha|^2$. Thus, $a_1^2 = 4a_0a_2\cos^2\theta \geq 4a_0a_2\cos^2\delta$. The same reasoning applies to the $a_2 < 0$ case. □

Proposition 125. *If* $p, q \in \mathbb{R}[z]$ *have degree* ≤ 2 *and all of their zeros lie in* $\overline{\mathbb{S}}(\pi, \delta)$ $(0 \leq \delta \leq \frac{\pi}{2})$, *then*

i. $p * q$ *has zeros in* $\overline{\mathbb{S}}(\pi, \delta)$ *if* $\delta \leq \frac{\pi}{3}$, *and*

ii. $p * q$ *has real zeros if* $\delta \leq \frac{\pi}{4}$.

Proof. Let $p(z) = a_2z^2 + a_1z + a_0$, $q(z) = b_2z^2 + b_1z + b_0$. By Lemma 124, $a_1^2 \geq 4a_0a_2\cos^2\delta$ and $b_1^2 \geq 4b_0b_2\cos^2\delta$. Therefore,

$$(a_1b_1)^2 \geq 16(a_0b_0)(a_2b_2)\cos^4\delta. \tag{4.7}$$

With (4.7), Lemma 124 asserts that $p * q$ has all its zeros in $\overline{\mathbb{S}}(\pi, \delta)$ when

$$4(a_0b_0)(a_2b_2)\cos^4\delta \geq (a_0b_0)(a_2b_2)\cos^2\delta,$$

which occurs when $|\cos\delta| \geq \frac{1}{2}$, and $\delta \leq \frac{\pi}{3}$. If the right hand side of (4.7) is greater than or equal to $4(a_0b_0)(a_2b_2)$, the discriminant of $p * q$ will be nonnegative, implying $p * q$ has real zeros. The case $\delta \leq \frac{\pi}{4}$ is therefore sufficient for $p * q$ to have real zeros. □

Two real polynomials with zeros in the sector $\overline{\mathbb{S}}(\pi, \delta)$ need not have a Hadamard product with zeros in the same sector, as the following example shows.

Example 126. Let
$$p(z) = z^2 + 2\cos\left(\frac{2\pi}{5}\right) z + 1.$$
Then p has its zeros in the sector $\overline{\mathbb{S}}(0, 2\pi/5)$, but $p * p$ has the zeros $z \approx -\frac{1}{5} \pm i$, which lie outside $\overline{\mathbb{S}}(0, 2\pi/5)$. The zeros of $p \odot p$, $z \approx -\frac{1}{10} \pm i\frac{7}{10}$, are also outside $\overline{\mathbb{S}}(0, 2\pi/5)$.

In light of Example 126, we look for additional conditions under which the Hadamard composition of two polynomials with zeros in $\overline{\mathbb{S}}(\pi, \delta)$ will have zeros in the same sector. We proceed to restrict the degrees of the polynomials. It is possible certain angular restrictions will work as well, as in the case where the sector is the left half-plane (see [49]).

Lemma 127. *Let the linear transformation $T : \mathbb{C}[z] \to \mathbb{C}[z]$ be given by $T[z^k] = \gamma_k z^k$, where $\{\gamma_k\}_{k=0}^{\infty}$ is an n-sequence and all γ_k are positive. If the zeros of $f(z)$ all lie in $\mathbb{S}(\theta, \delta)$, and deg$(f) \leq n$, then all the zeros of $T[f]$ lie in $\mathbb{S}(\theta, \delta)$. If $f(z) \in \mathcal{S}(\theta, \delta)$,* deg$(f) \leq n$*, then $T[f] \in \mathcal{S}(\theta, \delta)$ $(0 \leq \delta \leq \frac{\pi}{2})$.*

Proof. When $\deg(f) \leq 1$, the statement of the lemma is true trivially, so we assume that $\deg(f) \geq 2$. Suppose the zeros of $f(z)$ lie in $\mathbb{S}(\theta, \delta)$, then they lie between the angles $\alpha = \theta - \delta$ and $\beta = \theta + \delta$. Let $f(z) = \sum_{k=0}^{n} a_k z^k$, and let $R_\phi[g] := g(e^{-\phi} z)$ be the linear operator which rotates the zeros of a polynomial counterclockwise by an angle ϕ. Then $R_{-\alpha}[f](z) = f(e^{i\alpha} z) = \sum_{k=0}^{n} a_k e^{ik\alpha} z^k$ has all its zeros in the open upper half-plane. By the Hermite-Biehler theorem (Theorem 7), there exist $p, q \in \mathbb{R}[x]$, $p(z) = \sum_{k=0}^{n} c_k z^k$, $q(z) = \sum_{k=0}^{n} b_k z^k$, such that $R_{-\alpha}[f](z) = p(z) + iq(z)$, the zeros of p and q interlace, and $p'(0)q(0) - p(0)q'(0) = c_1 b_0 - c_0 b_1 > 0$. By the Hermite-Kakeya-Obreschkoff theorem (Theorem 5), the zeros of $p_2 = T[p]$ and $q_2 = T[q]$ interlace (see Remark 6). Since $p_2'(0)q_2(0) - p_2(0)q_2'(0) = \gamma_1\gamma_0(c_1 b_0 - c_0 b_1) > 0$, all the zeros of
$$T[R_{-\alpha}[f]](z)] = \sum_{k=0}^{n} a_k \gamma_k e^{ik\alpha} z^k = R_{-\alpha}[T[f]](z),$$
lie in the upper half-plane by the Hermite-Biehler theorem. This imples that $T[f](z)$ has no zeros in the half-plane equal to $\overline{\mathbb{S}}(\alpha - \pi/2, \pi/2)$. A similar argument shows that the zeros of $T[f](z)$ cannot lie in the half-plane equal to $\overline{\mathbb{S}}(\beta + \pi/2, \pi/2)$. This implies the zeros of $T[f](z)$ lie between the angles α and β, and thus in $\mathbb{S}(\theta, \delta)$. A

limiting argument proves the lemma for the case when some of the zeros of f lie on the boundary of $\mathbb{S}(\theta,\delta)$. If f has zeros on the boundary $\overline{\mathbb{S}}(\theta,\delta)\setminus\mathbb{S}(\theta,\delta)$, then define a new polynomial f_ε which has the same zeros as f, but with those on the boundary moved into the interior, $\mathbb{S}(\theta,\delta)$, by a distance ε, such that $f_\varepsilon \to f$ locally uniformly. By what we have already shown, $T[f_\varepsilon]$ has all of its zeros in $\mathbb{S}(\theta,\delta)$, and thus by Hurwitz's Theorem (Theorem 9) $T[f]\in\mathcal{S}(\theta,\delta)$. Note that the possibility $T[f]\equiv 0$ is excluded because all $\gamma_k>0$, hence the kernel of T is trivial. □

When $\{\gamma_n\}_{n=0}^{\infty}$ is a multiplier sequence, Lemma 127 implies the following theorem in [68, Chapter VIII], which is proved with essentially the same argument.

Theorem 128 ([68, Chapter VIII])**.** *If the zeros of $f(x)\in\mathbb{C}[x]$, lie in $\mathbb{S}(\theta,\delta)$ and the linear transformation $T:\mathbb{C}[z]\to\mathbb{C}[z]$, $T[z^n]=\gamma_n z^n$ is a positive multiplier sequence, then the zeros of $T[f]$ lie in $\mathbb{S}(\theta,\delta)$ $(0\le\delta\le\frac{\pi}{2})$.*

A similar result, where the polynomial has zeros in a double sector, was proved by L. Weisner [99].

Theorem 129 ([99])**.** *Let $T:\mathbb{C}[z]\to\mathbb{C}[z]$ be a linear transformation, such that $T[z^k]=\gamma_k z^k$, $\gamma_k\in\mathbb{R}$, for all $k=0,1,2,\ldots$. If T satisfies $T[e^z]\in\mathscr{L}\text{-}\mathscr{P}$, and the zeros of $f(x)\in\mathbb{C}[x]$, lie in $\mathbb{S}(\theta,\delta)$, then the zeros of $T[f]$ lie in the double sector $\mathbb{S}(\theta,\delta)\cup\mathbb{S}(\theta+\pi,\delta)$, provided $T[f]\not\equiv 0$.*

Theorem 130. *Let $g\in\mathbb{R}[z]$, $f\in\mathbb{C}[z]$, the zeros of g lie in the sector $\mathbb{S}(\pi,\phi)$, and the zeros of f lie in the sector $\mathbb{S}(\theta,\delta)$. If $\deg(f)\le\frac{1}{|\sin\phi|^2}$, then all the zeros of the Schur composition, $f\odot g$, lie in $\mathbb{S}(\theta,\delta)$ $(0\le\delta\le\frac{\pi}{2})$.*

Proof. Let $g(z)=\sum_{k=0}^{\nu} b_k z^k$ and $f(z)=\sum_{k=0}^{\mu} a_k z^k$. We know that $\{k!b_k\}_{k=0}^{m}$ is an n-sequence for $n\le\frac{1}{|\sin\delta|^2}$ by Theorem 120. Then by Lemma 127, all the zeros of the Schur composition $\displaystyle\sum_{k=0}^{\min\{\nu,\mu\}} k!a_kb_kz^k$ lie in $\mathbb{S}(\theta,\delta)$. □

Corollary 115 now follows immediately from Theorem 130.

We now prove a sector version of the Hermite-Poulain Theorem (Theorem 25). The method of proof employed is a modification of that used to prove the Gauss-Lucas Theorem (see [74, p. 22]).

Theorem 131. *Let $f, g \in \mathbb{C}[z]$, $g(z) = \sum_{k=0}^{n} a_k z^k$, and let the zeros of g lie in $\overline{\mathbb{S}}(\pi, \delta)$. If all the zeros of f lie in $\mathbb{S}(\pi, \delta)$, where $0 < \delta \leq \frac{\pi}{2}$, then all the zeros of*

$$g(D)f(z) = a_0 f(z) + a_1 f'(z) + a_2 f''(z) + \cdots + a_n f^{(n)}(z) \tag{4.8}$$

lie in $\mathbb{S}(\pi, \delta)$, provided $g(D)f(z) \not\equiv 0$. If $g(D)f(z) \equiv 0$, then $g(z) = z^j h(z)$ with $j > \deg(f)$.

Proof. First consider the case where $g(D)f(z) \equiv 0$. The only way this can occur is if $g(D) = D^j h(D)$ with $j > \deg(f)$; otherwise, the right-hand side of (4.8) has a term of highest degree which is not identically zero. We continue assuming $g(D)f(z) \not\equiv 0$. Since $g(D) = \prod_{\ell=1}^{n}(D + \alpha_\ell)$ where $\alpha_\ell \in \overline{\mathbb{S}}(0, \delta)$, and the operators $D + \alpha_\ell$ commute, it suffices to show that the zeros of $(D + \alpha)f(z) \not\equiv 0$, $\alpha \in \overline{\mathbb{S}}(0, \delta)$, all lie in $\mathbb{S}(\pi, \delta)$.

Let $\{\zeta_j\}_{j=1}^{m}$ be the the zeros of $f(z)$. If $(D + \alpha)g(z)$ has a zero which is not a zero of $f(z)$, then it will also be a zero of

$$\frac{(D + \alpha)f(z)}{f(z)} = \sum_{j=0}^{m} \frac{1}{z - \zeta_j} + \alpha = \sum_{j=0}^{m} \frac{\bar{z} - \bar{\zeta}_j}{|z - \zeta_j|^2} + \alpha. \tag{4.9}$$

Suppose that z_0 is a zero of $(D + \alpha)f(z)$ which lies outside of $\mathbb{S}(\pi, \delta)$, then z_0 is also a zero of left-hand side of (4.9). First consider the case where $\mathrm{Re}[z_0] \geq 0$. Then

$$\mathrm{Re}\left[\sum_{j=0}^{m} \frac{\bar{z}_0 - \bar{\zeta}_j}{|z_0 - \zeta_j|^2}\right] > 0,$$

and since $\mathrm{Re}[\alpha] \geq 0$ we find that z_0 can not be a zero of $(D + \alpha)f(z)$ by (4.9).

We have established that z_0 must lie in the open left half-plane. If $\delta = \pi/2$, the theorem is proved. We proceed assuming $\delta < \pi/2$ and that z_0 lies outside $\mathbb{S}(\pi, \delta)$ in the upper-half plane with $\mathrm{Re}[z_0] < 0$. Multiplying (4.9) by $e^{-i\delta}$ and taking the imaginary part yields

$$\mathrm{Im}\left[e^{-i\delta} \left.\frac{(D + \alpha)f(z)}{f(z)}\right|_{z=z_0}\right] = \mathrm{Im}\left[e^{-i\delta} \frac{\bar{z}_0 - \bar{\zeta}_j}{|z_0 - \zeta_j|^2}\right] + \mathrm{Im}\left[e^{-i\delta}\alpha\right].$$

The rotation places both $e^{-i\delta}\zeta_j$ and $e^{-i\delta}\bar{\zeta}_j$ in the open upper half-plane, while $e^{-i\delta}\bar{z}_0$ lies in the closed lower half-plane, therefore

$$\mathrm{Im}\left[\sum_{j=0}^{m} \frac{e^{-i\delta}(\bar{z}_0 - \bar{\zeta}_j)}{|z_0 - \zeta_j|^2}\right] < 0.$$

Since $\mathrm{Im}[e^{-i\delta}\alpha] \le 0$, z_0 can not be a zero of the right-hand side of (4.9). A similar argument handles the case when z_0 is in lower half-plane, outside $\mathbb{S}(\pi, \delta)$, with $\mathrm{Re}[z_0] < 0$. Thus $z_0 \notin \mathbb{S}(\pi, \delta)$ cannot be a zero of $(D + \alpha)f(z)$, and this proves the theorem. □

Results similar to Theorem 131 have been obtained by T. Takagi (see [74, p. 84]), S. Takahashi [94], and E. Benz [3]. Theorem 131 is sharp with respect to the angular width of the sector (consider $g(D)f(z)$ where $g(z) = z + e^{i(\delta+\varepsilon)}$, $f(z) = z + r$, for r sufficiently small). Finally, there does not appear to be a nice double sector analog of Theorem 131, as evinced by the following example.

Example 132. Let $f(z) = z^2+2z+4$, which has its zeros in $\overline{\mathbb{S}}\left(\pi, \frac{\pi}{3}\right)$, and $g(z) = z-1$, then $g(D)f(z) = -(z^2 + 2)$ has purely imaginary zeros.

Theorem 133. *If $g(z) = \sum_{k=0}^{n} a_k z^k \in \mathcal{S}(\theta, \delta)$ $(0 < \delta \le \pi/2)$, and $f \in \mathbb{C}[z]$ has all of its zeros in $\mathbb{S}(\theta, \delta)$, then $g\left(e^{i2\theta}D\right) f(z)$ has all of its zeros in $\mathbb{S}(\theta, \delta)$, provided $g\left(e^{i2\theta}D\right) f(z) \not\equiv 0$. If $g\left(e^{i2\theta}D\right) f(z) \equiv 0$, then $g(z) = z^j h(z)$, where $j > \deg(f)$, and $h(z) \in \mathcal{S}(\theta, \delta)$.*

Proof. With the given hypotheses $g\left(e^{-i(\pi-\theta)}z\right)$ has all of its zeros in $\overline{\mathbb{S}}(\pi, \delta)$ and $f\left(e^{-i(\pi-\theta)}z\right)$ has all of its zeros in $\mathbb{S}(\pi, \delta)$. Thus by Theorem 131, all the zeros of

$$g(e^{-i(\pi-\theta)}D)f\left(e^{-i(\pi-\theta)}z\right) = \sum_{k=0}^{n} a_k e^{-ik(\pi-\theta)} D^k f\left(e^{-i(\pi-\theta)}z\right) \tag{4.10}$$

$$= \sum_{k=0}^{n} a_k e^{-i2k(\pi-\theta)} f^{(k)}\left(e^{-i(\pi-\theta)}z\right) \not\equiv 0, \tag{4.11}$$

lie in $\mathbb{S}(\pi, \delta)$, and therefore

$$\sum_{k=0}^{n} a_k e^{-i2k(\pi-\theta)} f^{(k)}(z) = g(e^{i2\theta}D)f(z)$$

has all of its zeros in $\mathbb{S}(\theta, \delta)$ or is identically 0. If $g\left(e^{i2\theta}D\right) f(z) \equiv 0$, Theorem 131 shows $g(z) = z^j h(z)$ where $j > \deg(f)$. □

Theorem 134. *If $g, f \in \mathcal{S}(\theta, \delta)$ $(0 \le \delta \le \pi/2)$, then $g\left(e^{i2\theta}D\right) f(z) \in \mathcal{S}(\theta, \delta) \cup \{0\}$.*

Proof. If $\delta \neq 0$, then we may perturb any zeros of $f(z)$ lying on the boundary of $\overline{\mathbb{S}}(\theta,\delta)$ into its interior by a distance ε. Denote this perturbed polynomial by f_ε. By Theorem 133, the zeros of $g\left(e^{i2\theta}D\right) f_\varepsilon(z)$ lie in $\mathbb{S}(\theta,\delta)$ or $g\left(e^{i2\theta}D\right) f_\varepsilon(z) \equiv 0$. Letting $\varepsilon \to 0$, it follows by Hurwitz's theorem that $g\left(e^{i2\theta}D\right) f(z) \in \mathcal{S}(\theta,\delta) \cup \{0\}$.

If $\delta = 0$, let $\{\zeta_j\}_{j=1}^m$ be the zeros of $f(z)$. We may assume that $g(z) = z - \alpha$, where $\alpha \in \overline{\mathbb{S}}(\theta, 0)$. Then

$$\frac{g\left(e^{i2\theta}D\right) f(z)}{f(z)} = \sum_{j=0}^{m} \frac{e^{i2\theta}}{z-\zeta_j} - \alpha. \tag{4.12}$$

Without loss of generality, fix a point $z_0 \in \{z : \operatorname{Im}[e^{-i\theta}z] \geq 0\} \setminus \overline{\mathbb{S}}(\theta, 0)$. Then $\theta < \arg(z_0 - \zeta_j) \leq \theta + \pi$ for all $j = 1, \dots, n$ and consequently the sum in (4.12) has argument in the open interval $[\theta - \pi, \theta)$. Since $\arg\alpha = \theta$, the right hand side of (4.12) can not be zero at z_0. But any zero that is not a zero of left hand side of (4.12) must be a zero of f, and thus the proof is complete. $\square$

Theorem 135. *Let $0 \leq \delta \leq \pi/2$. If $h \in \overline{\mathcal{S}}(\theta,\delta)$ and $p \in \mathcal{S}(\theta,\delta)$, then $h(e^{i2\theta}D)p(z) \in \overline{\mathcal{S}}(\theta,\delta)$, and $p(e^{i2\theta}D)h(z) \in \overline{\mathcal{S}}(\theta,\delta)$.*

Proof. Let h_n be a sequence of polynomials such that $h_n \to h$ locally uniformly and each h_n has zeros only in $\overline{\mathbb{S}}(\theta,\delta)$. Then by Theorem 134 all of the zeros of $h_n(e^{i2\theta}D)p(z) \not\equiv 0$ lie in $\overline{\mathbb{S}}(\theta,\delta)$. Letting $n \to \infty$ and applying Hurwitz's Theorem, either $h(e^{i2\theta}D)p(z)$ has all its zeros in $\overline{\mathbb{S}}(\theta,\delta)$ or it is identically zero. Likewise any $p(e^{i2\theta}D)h(z) \not\equiv 0$ has zeros only in $\overline{\mathbb{S}}(\theta,\delta)$. $\square$

Remark 136. Note that if $\delta = \frac{\pi}{2}$ in Theorem 135 the class $\overline{\mathcal{S}}(\theta,\delta)$ may contain functions of order 2 and is not restricted to the form given in Theorem 117.

There exist sector analogies of both Sturm's theorem and Decarte's Rule of signs (see Marden [74, pp.189-191]). The following results of R. O'Donnell and S. Takahashi constrain the sector containing the zeros of a polynomial given the coefficients.

Theorem 137 ([79,94])**.** *Let $p(z) = \sum_{j=0}^n a_j z^j$, and for all pairs a_j,a_{j+s} of succesive nonzero coefficients, form the ratios $r_{j,s} = (-1)^s \frac{a_j}{a_{j+s}}$.*

i. Let $h \in \mathbb{N}$, $h \geq n$. If all the ratios $r_{j,s}$ lie in the sector $\overline{\mathbb{S}}(\theta, \frac{\pi}{h})$, then all the zeros of $p(z)$ lie in the sector $\overline{\mathbb{S}}(\theta+\pi, \frac{\pi}{n}-\frac{\pi}{h})$ [79] .

ii. If all the zeros of $p(z)$ lie in $\mathbb{S}(\theta, \delta)$, then so do all the ratios $r_{j,1}$ [94].

The results just mentioned, like Theorem 130, have hypotheses on the degrees of the polynomials involved and are therefore difficult to use when trying to classify linear operators on a space of polynomials of arbitrary degree and also elude extension to transcendental entire functions. The best analogy to a hyperbolicity preserving theorem is given by Theorem 131. Guided by this result, we pay special attention to differential operator representations of sector preservers in the next section.

4 Linear sector preservers

In this section we prove sufficient conditions for a linear operator T to be an $\mathcal{S}(\theta, \delta)$-preserver and consider some special cases of multivariate sector preservers. The following proposition states the diagonal $\mathcal{S}(\theta, \delta)$-preservers with trivial kernel are precisely the linear operators corresponding to positive multiplier sequences.

Proposition 138. *Let $T : \mathbb{C}[z] \to \mathbb{C}[z]$ be a linear operator, with $T[z^k] = c\gamma_k z^k$ for all $k \in \mathbb{N}_{\text{o}}$, where $c \in \mathbb{C}$ and $\gamma_k > 0$. Then T is an $\mathcal{S}(\theta, \delta)$-preserver if and only if $\{\gamma_k\}_{k=0}^{\infty}$ is a positive multiplier sequence.*

Proof. If $\{\gamma_k\}_{k=0}^{\infty}$ is a positive multiplier sequence, then by Theorem 128, T is an $\mathcal{S}(\theta, \delta)$-preserver. To prove the converse we argue by contradiction. Assume that $\{\gamma_k\}_{k=0}^{\infty}$ is not a multiplier sequence. Then by Theorem 14, there exists an $n \in \mathbb{N}$ such that for the polynomial $p(z) = (z+1)^n$, $T[p]$ has a pair of non-real zeros. Thus, $T[p]$ has zeros outside $\overline{\mathbb{S}}(\pi, 0)$, so T is not an $\mathcal{S}(\pi, 0)$-preserver. Similarly, we may consider for $\overline{\mathbb{S}}(\theta, \delta)$ the polynomial $p(e^{i(\pi-(\theta+\delta))}z)$ which has a multiple zero on the edge of the sector $\overline{\mathbb{S}}(\theta, \delta)$. Since T is diagonal, $T[p(e^{i(\pi-(\theta+\delta))}z)] = T[p](e^{i(\pi-(\theta+\delta))}z)$. Thus, $T[p(e^{i(\pi-(\theta+\delta))}z)]$ has at least one zero outside of $\overline{\mathbb{S}}(\theta, \delta)$. $\square$

If in Proposition 138, T is restricted to $\mathbb{R}[z]$, T may correspond to a diagonal operator with action $T[z^k] = \gamma_k z^k$, where $\{\gamma_k\}_{k=0}^{\infty}$ is not a multiplier sequence. This occurs in the case where the sector is the left half-plane; the sequence of coefficients $\{\gamma_0, \ldots, \gamma_n, 0, 0, 0, \ldots\}$ of a polynomial $\sum_{k=0}^{n} \gamma_k z^k$, whose zeros all lie in the left half-plane, define a linear operator $T[z^k] = \gamma_k z^k$ which is an $\mathcal{S}(\pi, \pi/2) \cap \mathbb{R}[z]$-preserver [49].

Definition 139. Let $\mathbb{H}^+$ denote the open upper half-plane . For $\delta \in \mathbb{R}$, let

$$\mathbb{H}_\delta = \left\{ z : ze^{-i\delta} \in \mathbb{H}^+ \right\}.$$

Denote the set of polynomials that are $\mathbb{H}_\delta$-stable in n variables by $\mathcal{H}_n^\delta$, where the subscript n is omitted for univariate polynomials.

Remark 140. Note that $\mathbb{H}_\delta$ refers to the open upper half-plane rotated counterclockwise by an angle δ; this is the opposite of the convention in [12].

With with a rotation of variables in Theorem 112 of J. Borcea and P. Brändén [11], we can obtain a sufficient condition for a linear operator to be a sector preserver.

Theorem 141 (Sufficient condition I)**.** *If $T : \mathbb{C}[z] \to \mathbb{C}[z]$ is a linear transformation, $0 < \delta \leq \frac{\pi}{2}$,*

i. $T[(z+w)^n] \neq 0$ or $T[(z+w)^n] \equiv 0$ for all $n \in \mathbb{N}_o$, $z, w \in \mathbb{H}_{\theta+\delta}$, and

ii. $T[(z+w)^n] \neq 0$ or $T[(z+w)^n] \equiv 0$ for all $n \in \mathbb{N}_o$, $z, w \in \mathbb{H}_{\pi+\theta-\delta}$.

Then T is an $\mathcal{S}(\theta, \delta)$-preserver.

Proof. Let $p \in \mathbb{C}[z]$, and let all the zeros of p lie in $\overline{\mathbb{S}}(\theta, \delta)$. By (i) and Theorem 112, T preserves $\mathbb{H}_{\theta+\delta}$-stability, thus $T[p]$ has no zeros in $\mathbb{H}_{\theta+\delta}$. By (ii) and Theorem 112 again, T preserves $\mathbb{H}_{\pi+\theta-\delta}$-stability, thus $T[p]$ has no zeros in $\mathbb{H}_{\pi+\theta-\delta}$. Therefore, all the zeros of $T[p]$ lie in $(\mathbb{H}_{\theta+\delta} \cup \mathbb{H}_{\pi+\theta-\delta})^c = \overline{\mathbb{S}}(\theta, \delta)$. □

Remark 142. Note that the set of operators which satisfy (i) and (ii) in Theorem 141 is not trivial, as it contains the diagonal operators given by multiplier sequences in Proposition 138.

We may write conditions (i) and (ii) in Theorem 141 in terms of the equivalent transcendental characterizations of half-plane stability preservers in Theorem 112, and obtain the following equivalent result.

Theorem 143 (Sufficient condition II)**.** *If* $T : \mathbb{C}[z] \to \mathbb{C}[z]$ *is a linear transformation,* $0 < \delta \leq \frac{\pi}{2}$, $\zeta_1 = e^{-i2(\theta+\delta+\pi/2)}$, $\zeta_2 = e^{-i2(\theta-\delta-\pi/2)}$,

i. $T[e^{\zeta_1 zw}] \in \overline{\mathcal{H}}^{\theta+\delta}$, *and*

ii. $T[e^{\zeta_2 zw}] \in \overline{\mathcal{H}}^{\pi+\theta-\delta}$,

then T *is an* $\mathcal{S}(\theta, \delta)$*-preserver.*

Proof. Let $\Phi_\alpha[f(z,w)] := f(e^{-i\alpha}z, e^{-i\alpha}w)$. Then, $\Phi_\alpha : \mathbb{C}[z,w] \to \mathbb{C}[z,w]$ is a linear operator and $\Phi_\alpha(\mathcal{H}_2) = \mathcal{H}_2^\alpha$. A linear operator T is then an $\mathcal{H}^\alpha$-preserver if and only if $T_\alpha = \Phi_\alpha^{-1} T \Phi_\alpha$ is an $\mathcal{H}$-preserver. By Theorem 112, T_α is an $\mathcal{H}$-preserver if $T_\alpha[e^{-zw}] \in \overline{\mathcal{H}}_2$, which is equivalent to $T[\Phi_\alpha[e^{-zw}]] \in \Phi_\alpha[\overline{\mathcal{H}}_2]$. Re-expressing this condition by evaluating Φ_α,

$$T[\exp(e^{-i2(\alpha+\pi/2)}zw)] \in \overline{\mathcal{H}}_2^\alpha$$

implies that T is an $\mathcal{H}^\alpha$-preserver. The two conditions in Theorem 143 then imply that T is both an $\mathcal{H}^{\theta+\delta}$-preserver and an $\mathcal{H}^{\theta-\delta+\pi}$-preserver, and therefore T is an $\mathcal{S}(\theta, \delta)$-preserver. $\square$

Example 144. An example of an $\mathcal{S}(\pi, \pi/4)$-preserver, other than a multiplier sequence, which satisfies the hypotheses in Theorems 141 and 143 is $g(D)$ where,

$$g(z) = (z + 1 + i)(z + 1 - i) = z^2 + 2z + 2.$$

In Theorem 143, $\zeta_1 = e^{-i\frac{3\pi}{2}} = i$, and

$$g(D)e^{\zeta_1 zw} = g(\zeta_1 w)e^{\zeta_1 zw}.$$

Since $g(\zeta_1 w)e^{\zeta_1 zw} = ((iw)^2 + 2(iw) + 2)e^{\zeta_1 zw} \in \overline{\mathcal{H}}^{5\pi/4}$, condition (i) of Theorem 143 is satisfied, and in a similar fashion (ii) of Theorem 143 is satisfied. ($g(D)$ is also an $\mathcal{S}(\pi, \pi/4)$-preserver by Theorem 134).

An $\mathcal{S}(\theta, \delta)$-preserver which does not satisfy the conditions in Theorem 143 remains to be found. In the previous section we showed that a differential operator theorem for sectors exists in more or less perfect analogy to the Hermite-Poulain Theorem (Theorem 131). Motivated by Theorem 131, we now introduce the ring of finite order differential operators with polynomial coefficients, known as the Weyl algebra. For convenience, we also define the Weyl algebra in the general multivariate case, to be used in the sequel.

Definition 145. We adopt the notation that if $\alpha \in \mathbb{N}_o^n$, then $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$; that is, z is interpreted in the multivariate sense for a vector exponent. Similarly, $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}$, where $\partial_k = \frac{\partial}{\partial z_k}$, and $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$. For $\alpha, \beta \in \mathbb{N}_o^n$, $\alpha \leq \beta$ if $\alpha_k \leq \beta_k$ for all $k = 1, \ldots, n$, and $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$. The n^{th} *Weyl algebra*, $\mathcal{A}_n[\mathbb{C}]$, is the ring of finite order differential operators with polynomial coefficients in n variables [66, p. 7, 8, 14],

$$\mathcal{A}_n[\mathbb{C}] = \left\{ \sum_{\beta \in \mathbb{N}_o^n}^{\beta \leq N} Q_\beta(z_1, \ldots, z_n) \partial^\beta \;\middle|\; Q_\beta \in \mathbb{C}[z] \text{ and } N \in \mathbb{N}_o^n \right\}.$$

For an operator $T \in \mathcal{A}_n[\mathbb{C}]$, we define its *symbol* $F_T(z, w) \in \mathbb{C}[z_1, \ldots, z_n, w_1, \ldots, w_n]$ to be the following polynomial

$$F_T(z, w) := \sum_{\beta \in \mathbb{N}_o^n} Q_\beta(z) w^\beta. \tag{4.13}$$

Using Theorem 134, we now prove a sufficient condition for a linear operator to be an $\mathcal{S}(\theta, \delta)$-preserver in the univariate Weyl algebra.

Theorem 146 (Sufficient condition III)**.** *Let $T \in \mathcal{A}_1[\mathbb{C}]$ and $0 \leq \delta \leq \frac{\pi}{2}$. If the symbol of T satisfies $F_T(z, e^{-i2\theta} w) \in \mathcal{S}_2(\theta, \delta)$, then T is an $\mathcal{S}(\theta, \delta)$-preserver.*

Proof. First we show that if $F_T(z, 0) \not\equiv 0$, then $F_T(z, 0) \neq 0$ for all $z \notin \overline{\mathbb{S}}(\theta, \delta)$. Suppose instead that there exists a $z_0 \notin \overline{\mathbb{S}}(\theta, \delta)$, such that $F_T(z_0, 0) = 0$. Since by assumption $F_T(z, 0) \not\equiv 0$,

$$F_T(z, w) = wP_1(z, w) + P_0(z),$$

for polynomials P_1 and P_0, where $P_0 \not\equiv 0$ and $P_0(z_0) = 0$. Because $\overline{\mathbb{S}}(\theta, \delta)^c$ is open, by the implicit function theorem there exists a complex number $\varepsilon \notin \overline{\mathbb{S}}(\theta, \delta)$, sufficiently small, and $z_1 \notin \overline{\mathbb{S}}(\theta, \delta)$ in a neighborhood of z_0, such that $F_T(z_1, \varepsilon e^{-i2\theta}) = 0$. This contradicts the hypothesis that $F_T(z, e^{-i2\theta}w) \in \mathcal{S}_2(\theta, \delta)$. Therefore, if $F_T(z, 0) \not\equiv 0$, then $F_T(z, 0) \neq 0$ for all $z \notin \overline{\mathbb{S}}(\theta, \delta)$. Equivalently, if $F_T(z, 0) = 0$ for any $z \notin \overline{\mathbb{S}}(\theta, \delta)$, then $F_T(z, 0) \equiv 0$. Therefore, $F_T(z, w)$ has the form,

$$F_T(z, w) = H(z, w)w^k, \tag{4.14}$$

where $H(z, 0) \neq 0$ for all $z \notin \overline{\mathbb{S}}(\theta, \delta)$ and $k \in \mathbb{N}_o$. The power w^k in (4.14) corresponds to differentiation for the operator T, which is an $\mathcal{S}(\theta, \delta)$-preserver by the Theorem 134 (or by the Gauss-Lucas Theorem [74, p. 22]). It is therefore sufficient to consider the case where $k = 0$ in (4.14). Thus, we continue assuming that $F_T(z, 0) \neq 0$ for all $z \notin \overline{\mathbb{S}}(\theta, \delta)$. Fix $\zeta_0 \notin \overline{\mathbb{S}}(\theta, \delta)$ and let $f \in \mathcal{S}(\theta, \delta)$. Then, $F_T(\zeta_0, e^{-i2\theta}w) \in \mathcal{S}(\theta, \delta)$, and by Theorem 134, $F_T(\zeta_0, D)f(z) \in \mathcal{S}(\theta, \delta) \cup \{0\}$. If $F_T(\zeta_0, D)f(z) \equiv 0$, then by Theorem 134 $F_T(\zeta_0, 0) = 0$, which contradicts out current assumption on F_T. Therefore, $F_T(\zeta_0, D)f(z) \in \mathcal{S}(\theta, \delta)$, and since ζ_0 was arbitrary $F_T(\zeta, D)f(z) \in \mathcal{S}_2(\theta, \delta)$. Specializing $\zeta = z$ yields $T[f(z)] = F_T(z, D)f(z) \in \mathcal{S}(\theta, \delta)$. $\square$

Example 147. Since $\{k + 1\}_{k=0}^{\infty}$ is a multiplier sequence by Laguerre's Theorem (Theorem 16), the associated diagonal operator $T = z\frac{d}{dz} + 1$ is an $\mathcal{S}(\pi, \pi/4)$-preserver by Theorem 138. However, $F_T(z, e^{-i2\theta}w) = ze^{-i2\theta}w + 1 \notin \mathcal{S}_2(\theta, \delta)$ for $\theta = \pi$, $\delta = \pi/4$. For example, $z = w = i$ produces a zero of $F_T(z, w) = zw + 1$. This shows there are $\mathcal{S}(\theta, \delta)$-preservers in the Weyl algebra which do not satisfy the hypotheses of Theorem 146.

Theorem 148. (Sufficient condition for an open sector preserver)
Let $T = \sum_{k=0}^{N} Q(k)D^k \in \mathcal{A}_1[\mathbb{C}]$, $N \in \mathbb{N}_o$, and $0 < \delta \leq \frac{\pi}{2}$. If

$$F_T(z, e^{-i2\theta}w) = w^k H(z, e^{-i2\theta}w), \tag{4.15}$$

where $H \in \mathbb{C}[z, w]$, $k \in \mathbb{N}_o$ is such that $Q_j \equiv 0$ for $k \leq j$, and $H(z, e^{-i2\theta}w) \neq 0$ whenever $z \notin \mathbb{S}(\theta, \delta)$ and $w \in \overline{\mathbb{S}}(\theta, \delta)^c \cup \{0\}$, then T preserves $(\mathbb{S}(\theta, \delta))^c$-stability.

Proof. By Theorem 133 (or the Gauss-Lucas Theorem [74, p. 22]) differentiation preserves $(\mathbb{S}(\theta,\delta))^c$-stability. Since the factor of w^k in (4.15) corresponds to k differentiations in the operator T, it is sufficient to proceed assuming $k = 0$, and $F_T(z, e^{-i2\theta}w) \neq 0$ for all $z \notin \mathbb{S}(\theta,\delta)$ and $w \in \overline{\mathbb{S}}(\theta,\delta)^c \cup \{0\}$. Fix $\zeta_0 \notin \mathbb{S}(\theta,\delta)$ and let $f(z)$ have its zeros only in $\mathbb{S}(\theta,\delta)$. Then $F_T(\zeta_0, e^{-i2\theta}w) \in \mathcal{S}(\theta,\delta)$ and by Theorem 133, $F_T(\zeta_0, D)f(z)$ has all of its zeros in $\mathbb{S}(\theta,\delta)$ or else is identically 0. If it is identically 0, then Theorem 133 implies that $F_T(\zeta_0, 0) = 0$ which contradicts our assumption on F_T. Therefore, $F_T(\zeta_0, D)f(z)$ has all of its zeros in $\mathbb{S}(\theta,\delta)$. Since ζ_0 was arbitrary $F_T(\zeta, D)f(z)$ is $(\mathbb{S}(\theta,\delta))^c$-stable in ζ and z. Specializing $\zeta = z$ yields that $T[f(z)] = F_T(z, D)f(z)$ has zeros only in $\mathbb{S}(\theta,\delta)$. $\square$

We can convert between the operator symbol F_T and transcendental symbol $T[e^{wz}]$ using the following observation.

Proposition 149. ([11]) *If* $T : \mathbb{C}[z] \to \mathbb{C}[z]$ *is a linear operator then*

$$T[e^{\zeta wz}]e^{-\zeta wz} = F_T(z, \zeta w). \tag{4.16}$$

Proof. This follows immediately from writing T as a differential operator and evaluating the left hand side of (4.16). $\square$

Equation (4.16) allows us to transfer results about general linear operators to operators in the Weyl algebra. One may also wish to use Weyl algebra results as a guide for obtaining full characterizations of linear preservers.

We have now compiled all the results needed to give the proof of Theorem 116.

Proof of Theorem 116. Using (4.16) we may convert Theorem 143 into condition (i) for the Weyl algebra. Condition (ii) is a restatement of Theorem 146. The form of the operator symbol for open sectors, (4.1), is a restatement of Theorem 148. $\square$

Although they are not equivalent, we can use condition (ii) to prove (i) in Theorem 116 when $\delta \neq 0$.

Proof of Theorem 116, (i) *using condition* (ii) *for the case* $0 < \delta \leq \pi/2$.
When $\delta = \pi/2$ condition (ii) of Theorem 116 immediately implies the following:
If $F_T(z, e^{-i2(\theta \pm \frac{\pi}{2})}w) \in \mathcal{H}^\theta$*, then* T *is an* $\mathcal{H}^\theta$*-preserver.*
Note that we have included the $\pm\pi/2$ to indicate a sign ambiguity. Then if the conditions of (i) are satisfied, T is both an $\mathcal{H}^{\theta+\delta}$-preserver and an $\mathcal{H}^{\theta-\delta+\pi}$-preserver, and therefore T is an $\mathcal{S}(\theta, \delta)$-preserver. □

Using Theorem 112 and (4.16) we immediately obtain the following characterization of univariate (open upper half-plane) stability preservers in the Weyl algebra.

Theorem 150. ([12]) *An operator* $T \in \mathcal{A}_1[\mathbb{C}]$ *preserves stability if and only if* $F_T(z, -w) \in \mathcal{H}_2(\mathbb{C})$.

Theorem 150 was actually proved for the more general multivariate case in [12]. A sufficient condition for an operator in the Weyl algebra to preserve stability in a closed half-plane is also proved in [12], and there it is demonstrated that this condition is not necessary. For $\theta = -\pi/2$ and $\delta = \pi/2$, Theorem 148 has weaker hypotheses, and thus is a refinement of this result in the univariate case. J. Borcea and P. Brändén also prove a necessary condition for an operator in the Weyl algebra to preserve closed upper half-plane stability, which in the univariate case can be stated as follows.

Theorem 151 ([12])**.** *Let* $T \in \mathcal{A}_1[\mathbb{C}]$*. If* T *is preserves* $\overline{\mathbb{H}}^+$ *stability, then* $F_T(z, w) \neq 0$ *for* $z \in \overline{\mathbb{H}}^+$ *and* $w \in \mathbb{H}^+$.

Borcea and Brändén show that this condition is not sufficient, and comment that a characterization requires "intermediate" conditions between the sufficient and necessary conditions they supply. For at least the univariate case, we have found these conditions.

Theorem 152 (Characterization of strict stability preservers in the univariate Weyl algebra)**.** *Let* $T \in \mathcal{A}_1[\mathbb{C}]$*.* T *preserves* $\overline{\mathbb{H}}^+$*-stability if and only if*

$$F_T(z, -w) = w^k H(z, -w), \tag{4.17}$$

for some $k \in \mathbb{N}_o$*,* $H(z, w) \in \mathbb{C}[z, w]$*, and* $H(z, -w) \neq 0$ *whenever* $z \in \overline{\mathbb{H}}^+$ *and* $w \in \mathbb{H}^+ \cup \{0\}$.

Proof. Sufficiency of the condition on the operator symbol follows from Theorem 148. Theorem 151 shows it is necessary to have $F_T(z, -w) \neq 0$ whenever $z \in \overline{\mathbb{H}}^+$ and $w \in \mathbb{H}^+$. Thus, we only need to show that if for some $z_0 \in \overline{\mathbb{H}}^+$, $F_T(z_0, 0) = 0$, then $F_T(z, 0) \equiv 0$, and F_T has the form given by (4.17). Suppose on the contrary that

$$H(z, w) = wP_1(z, w) + P_0(z),$$

where $P_0(z_0) = 0$ and $P_0(z) \not\equiv 0$. Then the associated operator is $T = P_1(z, D)D^{k+1} + P_0(z)D^k$, and $T[(z + i)^k] = k!P_0(z)$ is not $\overline{\mathbb{H}}^+$ stable since it has a zero at $z_0 \in \overline{\mathbb{H}}^+$. This contradicts that T preserves $\overline{\mathbb{H}}^+$ stability, and therefore F_T must have the form given in (4.17). □

We now establish some multivariate versions of our results. To accomplish this, we require the following multivariate version of Theorem 112.

Theorem 153 ([9])**.** *Let $T : \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ be a linear operator. Then T preserves stability if and only if either*

(a) T has range of dimension at most one and is of the form

$$T(f) = \alpha(f)P,$$

where α is a linear functional on $\mathbb{C}[z_1, \ldots, z_n]$ and P is a stable polynomial, or

(b) $T[e^{-zw}] \in \overline{\mathcal{H}}_{2n}$.

From Theorem 153 we have the following sufficient condition for an $\mathcal{S}_n(\theta, \delta)$-preserver. The exponent n below denotes the n-fold Cartesian product of a set; that is, $S^n = S \times S \times \cdots \times S$ (n times) $= \{(s_1, \ldots, s_n) : s_1, \ldots, s_n \in S\}$.

Theorem 154. *Let $T : \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ be a linear operator. If for each $B \in \{(\theta + \delta), (\theta - \delta)\}^n$, $T[e^{\zeta zw}] \in \overline{\mathcal{H}}_{2n}^B$, where $\zeta = e^{i(2B+\pi)}$, then T is an $\mathcal{S}_n(\theta, \delta)$-preserver.*

Proof. For $\alpha \in \mathbb{R}^n$ we define $\Phi_\alpha[f] := f(e^{-i\alpha_1}z_1, \ldots, e^{-i\alpha_n}z_n, e^{-i\alpha_1}w_1, \ldots, e^{-i\alpha_n}w_n)$. Then, as in the proof of Theorem 143, $T_\alpha = \Phi_\alpha^{-1}T\Phi_\alpha$ is an $\mathcal{H}_n$-preserver if and only if T is an $\mathcal{H}_n^\alpha$-preserver. Applying Theorem 153 to T_α, the requirement for T to be an $\mathcal{H}_n^\alpha$-preserver is

$$T[\exp(e^{i2(\alpha+\pi/2)}zw)] \in \overline{\mathcal{H}}_{2n}^\alpha.$$

The 2^n conditions $T[e^{\zeta zw}] \in \overline{\mathcal{H}}_{2n}^B$ (one for each possible choice of B), force T to be a stability preserver on every member of the set $\{\mathbb{H}_{\theta+\delta}, \mathbb{H}_{\theta-\delta+\pi}\}^n$, and therefore T must be an $\mathcal{S}_{2n}(\theta,\delta)$-preserver. □

Let us now consider the possibility of extending the univariate Weyl algebra condition given by Theorem 146. In order to extend Theorem 146 to the multivariate setting, without using Theorem 154, we require a stronger differential operator theorem. Using essentially the same idea as in the proof of Theorem 131, we can obtain a closed sector version of a proposition of E. Lieb and A. Sokal for the half-plane [72].

Lemma 155. *Let $Q_0, Q_1 \in \mathbb{C}[z]$, $0 \le \delta \le \pi/2$, and $R(v,w) = Q_0(w) + vQ_1(w) \neq 0$ for $v, w \notin \overline{\mathbb{S}}(\pi,\delta)$. Then either $Q_0(z)+Q_1'(z) \neq 0$ for $z \notin \overline{\mathbb{S}}(\pi,\delta)$ or $Q_0(z)+Q_1'(z) \equiv 0$.*

Proof. Our proof depends on the following facts.

i. Either $Q_0(w) \neq 0$ whenever $w \notin \overline{\mathbb{S}}(\pi,\delta)$, or $Q_0(w) \equiv 0$.

ii. Either $Q_1(w) \neq 0$ whenever $w \notin \overline{\mathbb{S}}(\pi,\delta)$, or $Q_1(w) \equiv 0$.

iii. If $Q_1(w) \neq 0$ for $w \notin \overline{\mathbb{S}}(\pi,\delta)$, then $\frac{Q_0(w)}{Q_1(w)} \in \overline{\mathbb{S}}(0,\delta)$ for $w \notin \overline{\mathbb{S}}(\pi,\delta)$.

iv. If $Q_1(w) \neq 0$ for $w \notin \overline{\mathbb{S}}(\pi,\delta)$, then either $\frac{Q_1'(w)}{Q_1(w)} \notin \overline{\mathbb{S}}(\pi,\delta)$ for $w \notin \overline{\mathbb{S}}(\pi,\delta)$, or $\frac{Q_1'(w)}{Q_1(w)} \equiv 0$.

Both (i) and (ii) follow immediately from Hurwitz's theorem by letting $v \to 0$ and $v \to \infty$ in $Q_0(w) + vQ_1(w)$ and $\frac{1}{v}Q_0(w) + Q_1(w)$ respectively. If $Q_1(w) \neq 0$ then

$$\frac{R(v,w)}{Q_1(w)} = \frac{Q_0(w)}{Q_1(w)} + v \neq 0$$

for $v, w \notin \overline{\mathbb{S}}(\pi,\delta)$, this implies (iii) by the implicit function theorem. If Q_1 is a constant, then $Q_1' \equiv 0$, otherwise let $\{\beta_j\}_{j=0}^n$ be the zeros of Q_1. Then,

$$\frac{Q_1'(w)}{Q_1(w)} = \sum_{j=1}^n \frac{1}{w-\beta_j} \notin \overline{\mathbb{S}}(\pi,\delta),$$

since $\beta_j \in \overline{\mathbb{S}}(\pi, \delta)$ (this is statement (iv)). We can now prove the lemma. If $Q_1 \equiv 0$, then $Q_0(w) \neq 0$ for $w \notin \overline{\mathbb{S}}(\pi, \delta)$, and hence $Q_0(z) + Q_1'(z) \neq 0$ for $w \notin \overline{\mathbb{S}}(\pi, \delta)$. Otherwise, if $Q_1 \neq 0$, we first assume that Q_1 is not a constant. Then by (ii), $Q_1(w) \neq 0$ for $w \notin \overline{\mathbb{S}}(\pi, \delta)$. By (iii) and (iv), $\frac{Q_0(w)}{Q_1(w)} \in \overline{\mathbb{S}}(0, \delta)$ and $\frac{Q_1'(w)}{Q_1(w)} \notin \overline{\mathbb{S}}(\pi, \delta)$ for $w \notin \overline{\mathbb{S}}(\pi, \delta)$. Therefore,

$$Q_0(z) + Q_1'(z) = Q_1(z) \left[\frac{Q_0(w)}{Q_1(w)} + \frac{Q_1'(w)}{Q_1(w)} \right] \neq 0$$

for $w \notin \overline{\mathbb{S}}(\pi, \delta)$. If Q_1 is a constant then by (i) we have either $Q_0(z) + Q_1'(z) \equiv 0$ or $Q_0(z) + Q_1'(z) \neq 0$ for $w \notin \overline{\mathbb{S}}(\pi, \delta)$. This completes the proof. □

Proposition 156. *Let $Q_0, Q_1 \in \mathbb{C}[z]$, and $R(v, w) = Q_0(w) + vQ_1(w) \neq 0$ for $v, w \notin \overline{\mathbb{S}}(\theta, \delta)$. Then either $Q_0(z) + e^{i2\theta} Q_1'(z) \neq 0$ for $z \notin \overline{\mathbb{S}}(\theta, \delta)$ or $Q_0(z) + e^{i2\theta} Q_1'(z) \equiv 0$.*

Proof. This lemma follows immediately from a change of variables in Lemma 155. Let $v = e^{-i(\pi-\theta)} v_2$, $w = e^{-i(\pi-\theta)} w_2$. Then $v, w \notin \overline{\mathbb{S}}(\theta, \delta)$ precisely when $v_2, w_2 \notin \overline{\mathbb{S}}(\pi, \delta)$. By hypothesis,

$$Q_0(e^{-i(\pi-\theta)} w_2) + e^{-i(\pi-\theta)} v_2 Q_1(e^{-i(\pi-\theta)} w_2) \neq 0 \text{ for all } v_2, w_2 \notin \overline{\mathbb{S}}(\pi, \delta), \tag{4.18}$$

which implies, by Lemma 155, that

$$Q_0(e^{-i(\pi-\theta)} w_2) + e^{-i2(\pi-\theta)} Q_1'(e^{-i(\pi-\theta)} w_2) \neq 0 \text{ for all } w_2 \notin \overline{\mathbb{S}}(\pi, \delta), \tag{4.19}$$

or $Q_0 + e^{-i2(\pi-\theta)} Q_1' \equiv 0$. Letting $z = e^{-i(\pi-\theta)} w_2$ in (4.19) produces the conclusion in the lemma. □

Using Proposition 156 we can prove the following analog of a stability preserving operation [12] for sector stable affine polynomials.

Corollary 157. *Suppose that $1 \le i < j \le n$ and $F(z_1, \dots, z_n) \in \mathcal{S}_n(\theta, \delta)$ is affine in the variable z_i. Then $F(z_1, \dots, z_{i-1}, e^{i2\theta} \partial_j, z_{i+1}, \dots, z_j, \dots, z_n) \in \mathcal{S}_{n-1}(\theta, \delta) \cup \{0\}$.*

Proof. Fix $z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{j-1}, z_{j+1}, \dots, z_n \notin \overline{\mathbb{S}}(\theta, \delta)$. Then $F(z_1, \dots, z_n) = Q_0(z_j) + z_i Q_1(z_j) \in \mathcal{S}_2(\theta, \delta)$. Then, by Proposition 156,

$$F(z_1, \dots, z_{i-1}, e^{i2\theta} \partial_j, z_{i+1}, \dots, z_j, \dots, z_n) = Q_0(z_j) + \frac{\partial}{\partial z_j} Q_1(z_j) \in \mathcal{S}_1(\theta, \delta).$$

Because $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n \notin \overline{\mathbb{S}}(\theta, \delta)$ were chosen arbitrarily, the corollary is proved. $\square$

A common approach to extend Corollary 157 to the multivariate Weyl algebra in the case of a half-plane is to use the Grace-Walsh-Szegő theorem (Theorem 8), or an equivalent result, to convert statements about multi-affine polynomials into statements about arbitrary multivariate polynomials. In the case of a non-circular domain, as when the sector is not a half-plane, these techniques are not available, and our investigation, for this method of proof (see Remark 158), ends here for now.

Remark 158. We conclude this section by making the following observations.

i. The multivariate extensions of Theorem 116 parts (ii) and (i), for $\delta \neq 0$, quickly follow from Theorem 154 and equation (4.16). Likewise, the univariate Weyl algebra conditions for closed sectors, for $\delta \neq 0$, also follow from Theorem 143.

ii. In the same way that Theorem 116 part (ii) can be used to prove Theorem 116 part (i), the open sector conditions in Theorem 148 can be used to prove a version of Theorem 116 part (i) for open sectors.

iii. Corollary 157 may be proved for the non-multi-affine case using the the non-multi-affine half-plane version of the theorem (see [12] for the non-multi-affine half-plane theorem).

Note the approach using differential operators yields precise results about multivariate stability in open sectors and other regions to which Theorems 112 and 154 can not be readily applied. In addition, these alternate methods of proof provide intuition about the necessity of sufficient conditions obtained from Theorems 112 and 154 for regions that are intersections of half-planes.

5 Turán inequalities and Jensen polynomials

In this section some theorems are proved about extended Turán inequalities and how they relate to the sector locus of zeros of an entire function. The following lemma of D. Dimitrov provides inequalities sufficient for the degree 3 Jensen polynomials $g_{3,p}$ (see Definition 12) associated with an entire function φ to have only real zeros.

Lemma 159 ([38]). *Let $\varphi(z) = \sum_{k=0}^{\infty} \gamma_k \frac{z^k}{k!}$ be a real entire function. Then with $k \in \mathbb{N}$, the real polynomial*

$$g_{3,k-1}(x) = \gamma_{k-1} + 3\gamma_k x + 3\gamma_{k+1} x^2 + \gamma_{k+2} x^3$$

with nonzero leading coefficent γ_{k+2} is hyperbolic if and only if the inequalities

$$T_k(\varphi) := \gamma_{k+1}^2 - \gamma_k \gamma_{k+2} \geq 0,$$

and

$$J_k(\varphi) := 4(\gamma_k^2 - \gamma_{k-1}\gamma_{k+1})(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2}) - (\gamma_k\gamma_{k+1} - \gamma_{k-1}\gamma_{k+2})^2 \geq 0$$

hold simultaneously.

We prove a theorem that relates the location of zeros of a real entire function with respect to a sector to the zeros of its associated Jensen polynomials (Theorem 171). This yields a new sufficient condition that the inequalities in Lemma 159 are satisfied. At the end of this section we show that the Jensen polynomials associated with the function F in equation (1.2), related to the Riemann ξ-function, have only real zeros up through degree 10^{17} (Theorem 177). We also establish some Turán type inequalities for entire functions having their zeros confined to a sector, which we prove below (Theorems 163 and 170).

Definition 160. The *star* or *reverse* of a polynomial $p(x)$ of degree n is $p^*(x) := x^n p\left(\frac{1}{x}\right)$.

Remark 161. Note that the reverse of a polynomial is an involution only if $p(0) \neq 0$. It is readily verified that if the zeros of a polynomial p lie in a sector which is symmetric about the real axis then those of p^* must lie in the same sector. In particular, if the zeros of a polynomial p lie in a sector $\mathbb{S}(\pi, \delta)$ (thus $p(0) \neq 0$), then the zeros of p^* are confined to lie in $\mathbb{S}(\pi, \delta)$ as well.

Theorem 162. *If $h \in \overline{\mathcal{S}}(\pi, \delta)$ ($\overline{\mathcal{S}}(0, \delta)$), then all the zeros of the Jensen polynomials associated with h lie in $\overline{\mathbb{S}}(\pi, \delta)$.*

Proof. Applying Theorem 135, all the zeros of $h(D)z^n = g_n^*(z)$ are in $\overline{\mathbb{S}}(\pi, \delta)$. The operation * : $g_n^*(z) \mapsto g_n(z)$ preserves the property that the zeros lie in $\overline{\mathbb{S}}(\pi, \delta)$. Therefore, the n^{th} Jensen polynomial $g_n(x) = z^n g_n^*(\frac{1}{z})$ has all of its zeros in $\overline{\mathbb{S}}(\pi, \theta)$ as well. □

Theorem 163. *Suppose a real entire function $f = \sum_{k=0}^{\infty} \gamma_k \frac{z^k}{k!}$, of order at most 1, satisfies*

$$\lim_{z \to \infty} \frac{f'(z)}{f(z)} = c \in \overline{\mathbb{S}}(0, \delta). \tag{4.20}$$

A necessary condition for the zeros of f to lie in $\overline{\mathbb{S}}(\pi, \delta)(\delta < \frac{\pi}{2})$ is that

$$\gamma_k^2 - \cos^2 \delta \gamma_{k+1} \gamma_{k-1} \geq 0 \textit{ for all } k \in \mathbb{N}.$$

Proof. If all the zeros of f lie in $\overline{\mathbb{S}}(\pi, \delta)$ ($\delta < \frac{\pi}{2}$), f has order at most 1, and satisfies (4.20), then by Theorem 117, $f(z) \in \overline{\mathcal{S}}(\pi, \delta)$. By Theorem 162 all the zeros of the Jensen polynomials of f also lie in $\overline{\mathbb{S}}(\pi, \delta)$. The second Jensen polynomial associated with f is $g_{2,0}(z) = \gamma_2 z^2 + 2\gamma_1 z + \gamma_0$. By Lemma 124, $\gamma_1^2 - \cos^2 \delta \gamma_2 \gamma_0 \geq 0$. Theorem 135 implies that every derivative of f has all of its zeros in $\overline{\mathbb{S}}(\pi, \delta)$. Applying Lemma 124 to the second Jensen polynomial for the $(k-1)^{th}$ derivative of f gives the inequality for arbitrary $k \geq 1$. □

Lemma 164 ([74, p. 28])**.** *If f is any polynomial whose zeros are symmetric about the origin with zeros inside of $\mathbb{S}(0, \gamma) \cup \mathbb{S}(\pi, \gamma)$, with $\gamma < \frac{\pi}{4}$, then all the zeros of f' lie in $\mathbb{S}(0, \gamma) \cup \mathbb{S}(\pi, \gamma)$.*

Proof. If f has zeros which are symmetric with respect to the origin then we can write

$$f(z) = z^k \prod_{j=1}^{n} (z^2 - (\alpha_j)^2), \qquad |\arg[\alpha_j]| < \frac{\pi}{2}, k \in \mathbb{N} \cup \{0\}.$$

Let $F(z) = [f(z^{\frac{1}{2}})]^2 = z^k \prod_{j=1}^{n} (z - (\alpha_j)^2)^2$. By hypothesis all the α_j can be chosen to satisfy $|\arg[\alpha_j]| < \gamma < \frac{\pi}{4}$, so that $|\arg[\alpha_j^2]| < 2\gamma < \frac{\pi}{2}$ and the zeros of $F(z)$ lie in a convex sector. By the Gauss-Lucas theorem [74, p. 22], the zeros of

$$F'(z) = 2f(z^{\frac{1}{2}})f'(z^{\frac{1}{2}})\frac{1}{2\sqrt{z}}$$

lie in $\mathbb{S}(0, 2\gamma)$. This implies the zeros of $f(z)$ lie in $\mathbb{S}(0, \gamma) \cup \mathbb{S}(\pi, \gamma)$. $\square$

The next proposition is a version of the "no consecutive zero terms property" of multiplier sequences generalized to polynomials with zeros in a sector.

Proposition 165. *If the zeros of* $f(x) = \sum_{k=0}^{n} a_k x^k$ *lie in the sector* $\overline{\mathbb{S}}\left(\pi, \frac{\pi}{4}\right)$ *and* $a_k a_{k+j} \neq 0$*,* $j \geq 2$*, then* $a_{k+1}a_{k+2}\cdots a_{k+j-1} \neq 0$*. If the zeros of* $f(x) = \sum_{k=0}^{n} a_k x^k$ *lie in the open double sector* $\mathbb{S}\left(\pi, \frac{\pi}{4}\right) \cup \mathbb{S}\left(0, \frac{\pi}{4}\right)$ *and are symmetric with respect to the origin with* $a_k a_{k+j} \neq 0$*,* $j \geq 3$*, then the sequence* $a_{k+1}, a_{k+2}, \cdots, a_{k+j-1}$ *has no neighboring zero terms.*

Proof. Say the first statement were not true. Then there is an f with all its zeros in $\overline{\mathbb{S}}\left(\pi, \frac{\pi}{4}\right)$ and $a_k a_{k+j} \neq 0$, for some $j \geq 2$, such that $a_{k+1}, a_{k+2}, \ldots a_{k+j-1} = 0$.

If a $g \in \mathbb{C}[z]$ has all n of its zeros in the sector symmetric about the real axis, so does $z^n g(\frac{1}{z})$. Thus, both the transformation $R : g(z) \to z^n g(\frac{1}{z})$ and differentiation (by the Guass-Lucas theorem or Lemma 164) preserve the sector $\overline{\mathbb{S}}\left(\pi, \pi/4\right)$ (or double sector $\mathbb{S}\left(\pi, \pi/4\right) \cup \mathbb{S}\left(0, \pi/4\right)$) containing the zeros. Consider

$$\begin{aligned} &D^{n-k-j}R\left[D^k f(z)\right] \\ &\quad= D^{n-k-j}R\left[a_k k! + a_{k+j}\frac{(k+j)!}{j!}z^j + \cdots + a_n\frac{n!}{(n-k)!}z^{n-k}\right] \\ &\quad= D^{n-k-j}\left[a_n\frac{n!}{(n-k)!} + \cdots + a_{k+j}\frac{(k+j)!}{j!}z^{n-k-j} + a_k k! z^{n-k}\right] \\ &\quad= a_{k+j}\frac{(k+j)!(n-k-j)!}{j!} + a_k k!\frac{(n-k-j)!}{j!}z^j \end{aligned} \tag{4.21}$$

The zeros of the polynomial on the right-hand side of (4.21) lie outside of the sector $\overline{\mathbb{S}}\left(\pi, \frac{\pi}{4}\right)$ for $j \geq 2$. This contradicts our assumption on f. If $j \geq 3$, the right-hand side of (4.21) has zeros outside the open double sector $\mathbb{S}(\pi, \pi/4) \cup \mathbb{S}(0, \pi/4)$. This proves the double sector case. $\square$

Remark 166. If $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence, then all the zeros of the Jensen polynomials are real and have the same sign (Theorem 14). Proposition 165 then implies that if $\gamma_j\gamma_{j+n} \neq 0$, then $\gamma_k \neq 0$ for all $j \leq k \leq j+n$. This is a well-known property of multiplier sequences.

O. Katkova and A.Vishnyakova [59] have already found a Turán type inequality sufficient for m-times positivity of a matrix, which, together with Schoenberg's Theorem 122 provides a condition for polynomials to have all of their zeros in certain sectors (Theorem 170).

Definition 167. For $c \geq 1$ let $\mathbf{TP}_2(c)$ denote the class of all matrices $M = (a_{i,j})$ with positive entries such that

$$a_{i,j}a_{i+1,j+1} \geq c a_{i,j+1}a_{i+1,j} \text{ for all } i, j.$$

Theorem 168 (O. Katkova, A.Vishnyakova [59])**.** *If $M \in \mathbf{TP}_2(c)$ for $c \geq c_m = 4\cos^2\left(\frac{\pi}{m+1}\right)$ then $M \in \mathbf{TP}_m$.*

Remark 169. Theorem 168 was proved by extending an argument due to T. Craven and G. Csordas [28].

Theorem 170. *If $f(z) = \sum_{k=0}^{n} a_k z^k$, $a_k \geq 0$, satisfies*

$$a_k^2 - 4\cos^2\left(\frac{\pi}{m+1}\right) a_{k+1}a_{k-1} \geq 0 \text{ for all } k = 1, 2, 3, \ldots,$$

then the zeros of f lie in the sector $\mathbb{S}\left(\pi, \frac{\pi(n-1)}{n-1+m}\right)$.

Proof. By hypothesis, the infinite matrix of coefficients

$$\begin{pmatrix} a_0 & a_1 & a_2 & \ldots \\ 0 & a_0 & a_1 & \ldots \\ 0 & 0 & a_0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is

TP_m by Theorem 168. Then Schoenberg's Theorem 122, implies that f is nonzero in the region $\{z : |\arg z| < \frac{\pi m}{n+m-1}\}$, so any zeros of f must lie in $\mathbb{S}\left(\pi, \frac{\pi(n-1)}{n-1+m}\right)$. $\square$

Theorem 171. *If $\varphi \in \overline{\mathcal{S}}(\pi, \delta)$ $(0 \leq \delta < \frac{\pi}{2})$ is a real entire function, then for all $g \in \mathbb{R}[x]$ with $\deg(g) \leq \frac{1}{|\sin\delta|^2}$, $\phi \odot g$ has only real zeros. In particular, if φ has order $\rho \leq 1$, positive Taylor coefficients, and all of the zeros of φ lie in $\mathbb{S}(\pi, \delta)$, then the Jensen polynomials associated with φ up to degree $n \leq \frac{1}{|\sin\delta|^2}$ have only real zeros.*

Proof. If the real entire function $\varphi \in \overline{\mathcal{S}}(\pi, \delta)$, then there exists a sequence $p_n(x) \in \mathbb{R}[x]$ having zeros only in $\mathbb{S}(\pi, \delta)$, such that $p_n \to \varphi$ locally uniformly. By Theorem 120, $p_n \odot g$ has only real zeros for $\deg(g) \leq \frac{1}{|\sin\delta|^2}$, and thus by Hurwitz's Theorem $\varphi \odot g$ has only real zeros, provided it is not identically 0. Now assume that φ has order $\rho \leq 1$, positive Taylor coefficients, and all of its zeros lie in $\mathbb{S}(\pi, \delta)$. Then writing φ in the form (4.2) without any *a priori* restrictions on the parameter σ, one may compute $-\sigma = \lim_{x\to\infty} \frac{\varphi'(x)}{\varphi(x)} \geq 0$, which follows from the positivity of the Taylor coefficients. Hence, φ satisfies conditions (4.3) and therefore $\varphi \in \overline{\mathcal{S}}(\pi, \delta)$. The Schur composition property follows from the first argument, and thus the Jensen polynomials $g_n = \varphi \odot (1+x)^n$ will have only real zeros up to the degree indicated. □

The following theorems are similar to Theorem 123.

Theorem 172. *Let $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$ be a real entire function. If*

i. $\varphi(z) \in \overline{\mathcal{S}}\left(0, \frac{\pi}{4}\right)$, *or*

ii. if $\varphi(z) \in \overline{\mathcal{S}}\left(\pi, \frac{\pi}{4}\right)$

then $T_k(\varphi) = \gamma_k^2 - \gamma_{k+1}\gamma_{k-1} \geq 0$ for all $k = 1, 2, \ldots$.

Proof. Suppose $\varphi(z) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \in \overline{\mathcal{S}}(\pi, \pi/4)$. Therefore $\{\gamma_k\}_{k=0}^{\infty}$ is a 2-sequence by Theorem 171. This is equivalent to the statement $\gamma_k^2 - \gamma_{k+1}\gamma_{k-1} \geq 0$. The same argument can be applied to the $\varphi(s) \in \overline{\mathcal{S}}(0, \pi/4)$ case.

□

Theorem 173. *Let $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$ be a real entire function. If $\varphi(x) \in \overline{\mathcal{S}}(0, \delta)$, or if $\varphi(x) \in \overline{\mathcal{S}}(\pi, \delta)$, where*

$$|\sin\delta| \leq \frac{1}{\sqrt{3}},$$

then $T_k(\varphi) = \gamma_k^2 - \gamma_{k+1}\gamma_{k-1} \geq 0$ and $J_k(\varphi) \geq 0$, $k = 1, 2, 3, \ldots$ (see Lemma 159).

Proof. Say $\varphi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \in \overline{\mathcal{S}}(\pi, \delta)$. Then by Theorem 135, derivative of $\varphi(x)$ which is not identically zero, has zeros only in $\overline{\mathbb{S}}(\pi, \delta)$. Since $|\sin \delta| \leq \frac{1}{\sqrt{3}}$, each sequence $\{\gamma_j, \gamma_{j+1}, \gamma_{j+2}, \ldots\}$ is a 3-sequence by Theorem 171, which implies that third Jensen polynomial of $\varphi^{(j)}$ has only real zeros. Therefore, φ satisfies $T_j(\varphi) \geq 0$ and $J_j(\varphi) \geq 0$ by Lemma 159. □

The three examples below pertain to the inequalities in Lemma 159, for which Theorem 173 provides a sufficient condition. A polynomial which satisfies $T_k \geq 0$ and $J_k \geq 0$ for all $k = 0, 1, 2, \ldots$, is given in Example 174, and Example 175 defines polynomial for which $T_k \geq 0$ and $J_k \ngeq 0$ for some values of k .

Example 174. For the polynomial $q(z) = (1 + z)^2$,
$\{T_k(q)\}_{k=0}^{\infty} = \{1, 2, 4, 0, 0, 0, \ldots\}$, and
$\{J_k(q)\}_{k=0}^{\infty} = \{4, 16, 0, 0, 0, \ldots\}$.

Example 175. The polynomial $q(z) = z^3 + 7z^2 + 12z + 10$, has zeros at $z = -5, -1 \pm i$,
$\{T_k(q)\}_{k=0}^{\infty} = \{100, 4, 124, 36, 0, 0, 0 \ldots\}$, and
$\{J_k(q)\}_{k=0}^{\infty} = \{-12800, -9680, 10800, 0, 0, 0, \ldots\}$.

Example 176. Let $q(z) = 8z^4 + 20z^3 + 26z^2 + 19z + 5$. The zeros of q are at $z = -1/2, -1, -\frac{1}{2} \pm i$,
$\{T_k(q)\}_{k=0}^{\infty} = \{25, 101, 424, 4416, 36864, 0, 0, 0 \ldots\}$, and
$\{J_k(q)\}_{k=0}^{\infty} = \{1075, 20752, 771072, 120324096, 0, 0, 0, \ldots\}$.

Although satisfying the condition $0 \leq |\mathrm{Im}(z_k)| < -\mathrm{Re}(z_k)$ for each zero z_k is sufficient for the Turán inequalities to hold for a polynomial, it is not necessary as Example 176 shows. Example 176 also shows that $T_k \geq 0$, $J_k \geq 0$, can be satisfied by a degree 4 polynomial with non-real zeros.

An Application to the ξ-function

The Riemann hypothesis is equivalent to the statement that the Riemann ξ-function (see (1.1)) has only real zeros. It is well-known that the zeros of the ξ-function lie in the strip $|\mathrm{Im}(z)| < \frac{1}{2}$ and are symmetric about the origin and the imaginary axis. By (3.8), the function F associated with ξ satisfies

$$F(-4z^2) = \frac{1}{8}\xi\left(z\right), \tag{4.22}$$

and has order $\rho = \frac{1}{2}$. G. Csordas, T. S. Norfolk, and R. S. Varga [35] proved that the Taylor coefficients of F satisfy the Turán inequalities (2.2). The validity of these inequalities is equivalent to the statement that all the second degree Jensen polynomials associated with the derivatives $F^{(k)}(x)$, $k = 0, 1, 2, \ldots$, have only real zeros. We now show that a large number of the Jensen polynomials associated with F have only real zeros.

Theorem 177. *The first* 2×10^{17} *Jensen polynomials associated with the function* $F(z)$*, defined by* (1.2)*, have only real zeros.*

Proof. Let t_{max} represent a value such that ξ has only real zeros in the rectangle

$$\left\{z : |\mathrm{Re}(z)| < t_{max} \text{ and } |\mathrm{Im}(z)| < \frac{1}{2}\right\}. \tag{4.23}$$

Then the zeros of ξ are constrained to lie within the double sector $S(0, \delta) \cup S(\pi, \delta)$, with $\delta = \arctan\left(\frac{1}{2t_{max}}\right)$. Equation (4.22) implies that the zeros of $F(z)$ must then lie in $S(\pi, 2\delta)$. Moreover, if we set

$$N = \left\lfloor \frac{1}{|\sin(2\delta)|^2} \right\rfloor = \left\lfloor \frac{1}{4\sin^2\delta\cos^2\delta} \right\rfloor = \left\lfloor \frac{(4t_{max}^2 + 1)^2}{16t_{max}^2} \right\rfloor, \tag{4.24}$$

where $\lfloor \cdot \rfloor$ is the floor function, Theorem 171 implies the Jensen polynomials associated with $F(z)$ up to degree N have only real zeros. It is known that ξ has only real zeros in the rectangle (4.23) for $t_{max} = 545, 439, 823$ [95]. This implies that at least the first $N = 2 \times 10^{17}$ Jensen polynomials of $F(z)$ have only real negative zeros. □

Note that since $C(S(\pi, \delta))$ is closed under differentiation, the Jensen polynomials associated with the derivatives $F^{(k)}$, $k = 0, 1, 2, \ldots$, also have only real zeros for degrees up to the lower bound given by (4.24). O. Katkova [58] has given a proof that the ξ-function is k-times totally positive for large k using the same type of argument and a Theorem 122 of Schoenberg [58, p. 5].

Chapter 5

Stability preservers on arbitrary regions

1 Introduction

Let $\mathbb{C}_{\leq n}[z]$ denote the vector space of complex polynomials which have degree less than or equal to n. We will continue to use terminology from Chapter 4, along with the following notation.

Definition 178. Denote by $\mathcal{P}_\Omega$ the set of all complex polynomials whose zeros lie only in $\Omega \subset \mathbb{C}$.

T. Craven and G. Csordas [29] formally identified the following generalization of Pólya's problem on classifying linear operators which preserve reality of zeros.

Problem 179. (T. Craven, G. Csordas 2004 [29]) Let $\Omega \subseteq \mathbb{C}$. Characterize all linear transformations

$$T : \mathbb{C}_{\leq n}[z] \to \mathbb{C}_{\leq n}[z]$$

such that $T[\mathcal{P}_\Omega \cap \mathbb{C}_{\leq n}] \subset \mathcal{P}_\Omega \cap \mathbb{C}_{\leq n}$.

Csordas and Craven state that Problem 179 is unsolved in all but the simplest cases, and point out the special cases when (i) $\Omega = \mathbb{R}$, (ii) Ω is a half plane, (iii) Ω is a sector centered at the origin, (iv) Ω is a horizontal strip, and (v) Ω is a double sector centered at the origin. J. Borcea and P. Brändén considered Problem 179 along with several modified versions, where the domain is chosen to be either $\mathbb{C}[z]$, $\mathbb{C}_n[z]$, or $\mathbb{C}_{\leq n}[z]$, and the range is $\mathbb{C}[z]$. They solved these completely in the case where $\Omega = \mathbb{R}$ and Ω is a closed circular domain [11]. In their proof, Borcea and Brändén use the Grace-Walsh-Szegő Theorem (Theorem 8) on linear symmetric forms, and arrive at a number of related results, including Theorem 112 in Chapter 4.

An operator T such that $T(\mathcal{P}_\Omega) \subset \mathcal{P}_\Omega \cup \{0\}$ is sometimes referred to as a *preserver* on the region Ω (see also Definition 109). A set of polynomials, $\{q_k\}_{k=0}^\infty$, is said to be a *simple set*, if $\deg(q_k) = k$ for all $k = 0, 1, 2, \ldots$ [90, p. 147]. Let $\Omega \subset \mathbb{C}$. We obtain results relevant to (i) and (iii) of the following approaches to Problem 179.

i. Characterize the linear $\mathcal{P}_\Omega$-preservers $T : \mathbb{C}[z] \to \mathbb{C}[z]$, which possess a simple set of polynomials $\{q_n\}_{n=0}^\infty$, such that $T[q_n] = \lambda_n q_n$, $\lambda_n \in \mathbb{C}$, by establishing conditions on the polynomials $\{q_n\}_{n=0}^\infty$ and numbers $\{\lambda_n\}_{n=0}^\infty$.

ii. Suppose $T : \mathbb{C}[z] \to \mathbb{C}[z]$ has the representation

$$T[p] = \int_E p(w)K(w,z)d\sigma(w),$$

where K is a two parameter weight function, E is a measurable subset of the complex plane, and σ is a complex measure. Characterize the kernel functions $K(w,z)$ which correspond to stability preserving transformations.

iii. ([13, Problem 10]) Every linear operator $T : \mathbb{C}[z] \to \mathbb{C}[z]$ can be represented in the form

$$T = \sum_{k=0}^{\infty} Q_k(z)D^k. \tag{5.1}$$

Characterize all linear $\mathcal{P}_\Omega$-preservers, T, in terms of the polynomial coefficients Q_k.

Approach (i) aims to generalize the notion of a multiplier sequence, and is also motivated by change of basis transformations (cf. [6,17,54,55,84]). Approach (ii) is motivated by the Riesz representation theorem and properties of totally positive kernels (see [57] and Theorem 201). Approach (iii) is given as an open problem in the literature [13]. Change of basis transformations are exhibited in Theorems 180 and 181, below, and Theorem 204, which is proved in Section 3.

Theorem 180. (A. Iserles, E. B. Saff [55]) *Suppose the polynomial* $\sum_{k=0}^{n} a_k z^k \in \mathbb{R}[z]$ *has all of its zeros in the open unit disc* $\{z : |z| < 1\}$. *Then all the zeros of the polynomial* $\sum_{k=0}^{n} a_k T_k(z)$, *where* $T_k(z) = \cos(k \arccos(z))$ *is the* k^{th} *Chebyshev polynomial, lie in the interval* $(-1, 1)$.

The following known theorem supplies an example of a basis transformation which preserves $\mathscr{L}\text{-}\mathscr{P}^+ \cap \mathbb{R}[z]$, by using the polynomials $(z)_k$ (see Definition 32).

Theorem 181. *If* $p(x) = \sum_{k=0}^{n} a_k (z)_k \in \mathscr{L}\text{-}\mathscr{P}^+$, *then* $\sum_{k=0}^{n} a_k z^k \in \mathscr{L}\text{-}\mathscr{P}^+$.

Proof. Let $p \in \mathscr{L}\text{-}\mathscr{P}^+ \cap \mathbb{R}[z]$. Then $\{p(k)\}_{k=0}^{\infty}$ is a multiplier sequence, by Theorem 16, and thus,

$$\sum_{k=0}^{\infty} p(k) \frac{z^k}{k!} \in \mathscr{L}\text{-}\mathscr{P}^+.$$

If p is expanded in the basis $\{(z)_k\}_{k=0}^{\infty}$, then

$$\begin{aligned} \sum_{k=0}^{\infty} p(k) \frac{z^k}{k!} &= \sum_{k=0}^{\infty} \sum_{j=0}^{n} a_j (k)_j \frac{z^k}{k!} \\ &= \sum_{j=0}^{n} a_j \sum_{k=0}^{\infty} (k)_j \frac{z^k}{k!} \\ &= \left(\sum_{j=0}^{n} a_j z^j \right) e^z \in \mathscr{L}\text{-}\mathscr{P}^+, \end{aligned} \tag{5.2}$$

where we have used (2.12) to obtain (5.2). This proves the theorem. □

Our investigation starts with diagonalizable stability preservers. We provide a sufficient condition for a degree preserving linear operator to be diagonalizable. A multiplier sequence with no more than two non-zero terms is called *trivial*. We prove in Section 2 that any non-trivial multiplier sequence on a monic simple basis, where the basis polynomials have only simple real zeros, must be non-decreasing (see Definition 183, Theorem 193). In Section 3, we prove a conjecture of S. Fisk while establishing a new transformation that preserves polynomials having only real zeros in the interval $[-1, 1]$ (Theorem 204). New properties and representations of differential operators in the form (5.1) are presented in Section 4. We conclude with a list of conjectures, questions, and open problems.

2 Diagonalizable transformations

Change of basis transformations, such as the transformation in Theorem 180, are degree preserving and have trivial kernel. This section focuses on degree preserving transformations which are diagonalizable. Degree preserving transformations may serve as a tool for finding more general results.

Theorem 182. *Let Ω be a region of $\mathbb{C}$ homeomorphic to the closed unit disk or the line segment $[0,1]$. If T is a degree preserving linear operator with trivial kernel, such that $T[P_\Omega] \subset P_\Omega$, then the following claims hold.*

i. For all $n = 0, 1, 2, \ldots$, there exist monic $q_n \in \mathbb{C}[x]$ such that $T[q_n] = \lambda_n q_n$, $\deg(q_n) = n$, $\lambda_n \in \mathbb{C}$, and all the zeros of a given q_n lie in Ω.

ii. Furthermore, if $\lambda_k \neq \lambda_j$ whenever $j \neq k$, then the polynomials $\{q_n\}_{n=0}^{\infty}$ are unique.

Proof. Endow the vector space of all complex polynomials of degree n with the usual Euclidean norm. Therefore, T defines a continuous map $\varphi : \Omega^n \to \Omega^n$, given by $(z_1, z_2, \ldots, z_n) \mapsto (z_1', z_2', \ldots, z_n')$, where $\{z_k\}_{k=1}^n$ are the zeros of a degree n polynomial $p(z) \in \mathcal{P}_\Omega$ and $\{z_k'\}_{k=1}^n$ are the zeros of $T[p(z)]$. Since Ω is homeomorphic to the closed unit disk (or $[0,1]$), by Brouwer's fixed point theorem [51, p. 85], φ has a fixed point $(w_1, w_2, \ldots, w_n) \in \Omega^n$. Let $q_n(z) = (z - w_1)(z - w_2)\cdots(z - w_n)$. Because $(w_1, w_2, \ldots, w_n)$ is a fixed-point of the map φ, the image $T[q_n]$ must be a scalar multiple of q_n. This proves (i).

Suppose there exist monic polynomials q_n and q_n', such that $q_n \neq q_n'$, $T[q_n] = \lambda_n q_n$ and $T[q_n'] = \lambda_n' q_n'$. Then $\deg(q_n - q_n') = m \leq n - 1$. Thus,

$$T[q_n(z) - q_n'(z)] = \lambda_n q_n(z) - \lambda_n' q_n'(z)$$

has degree m, because T is degree preserving. This implies that $\lambda_n = \lambda_n'$. Therefore, the polynomial $q_m = q_n(z) - q_n'(z)$, which has degree strictly less that n, satisfies $T[q_m] = \lambda_m q_m$, where $\lambda_m = \lambda_n$. This proves (ii). $\square$

Definition 183. Let $T : \mathbb{C}[x] \to \mathbb{C}[x]$ be a linear operator. If there exists a simple set of polynomials $\{q_k\}_{k=0}^{\infty}$ such that $T[q_k] = \lambda_k q_k$, $\lambda_k \in \mathbb{C}$, then T is said to be *diagonalizable* and T is said to be *diagonal with respect to* q_k. The set $\{q_k\}_{k=0}^{\infty}$ is said to be a *diagonal basis* for T. If T is hyperbolicity preserving, and $\lambda_k \in \mathbb{R}$ for all $k \in \mathbb{N} \cup \{0\}$, we call $\{\lambda_k\}_{k=0}^{\infty}$ a *q-multiplier sequence.*

Definition 184. Let T be diagonalizable with respect to $\{q_k\}_{k=0}^{\infty}$, with $T[q_k] = \lambda_k q_k$. If

$$|\lambda_k| \leq |\lambda_{k+1}| \qquad \text{for all } k = 0, 1, 2, \ldots, \tag{5.3}$$

then T is said to be *attractive.*

Remark 185. If T is attractive, then for any $p \in \mathbb{R}[x]$ with degree precisely k and $T[p] \not\equiv 0$,

$$\lim_{n\to\infty} \left(\frac{1}{\lambda_k} T\right)^n [p] = q_k,$$

where $T[q_k] = \lambda_k q_k$.

Proposition 186. *Let $\Omega \subset \mathbb{C}$ be bounded, non-empty, and $\Omega \neq \{0\}$. If T is a $\mathcal{P}_\Omega$-preserver with trivial kernel, and is diagonalizable on a simple set $\{q_k\}_{k=0}^{\infty}$, then T is attractive.*

Proof. Suppose that there is a T satisfying the hypotheses which is not attractive. Then if $T[q_k] = \lambda_k q_k$ for all $k \in \mathbb{N} \cup \{0\}$, there is an $n \in \mathbb{N}$ such that $0 < |\lambda_n| < |\lambda_{n-1}|$ (all $\lambda_k \neq 0$ because T has trivial kernel). Consider $g(z) = \sum_{k=0}^{n} c_k q_k(z) = (z + \omega)^n \in \mathcal{P}_\Omega$, where $\omega \in \Omega \setminus \{0\}$, and thus $c_n \neq 0$ and $c_{n-1} \neq 0$. Since the normalizations of the q_k are arbitrary, we may assume they are monic. Iterating T ($T^m = T \circ T \circ \cdots \circ T$, m times), yields

$$T^m[g(z)] = \lambda_{n+1}^m c_n \left(q_n + \left(\frac{\lambda_{n-1}}{\lambda_n}\right)^m \frac{c_{n-1}}{c_n} q_{n-1} \ + \cdots + \left(\frac{\lambda_0}{\lambda_n}\right)^m \frac{c_0}{c_n} q_0 \right) \tag{5.4}$$

$$= \lambda_{n+1}^m c_n \left(x^n + \left(\frac{\lambda_{n-1}}{\lambda_n}\right)^m A_{n-1} x^{n-1} + \cdots + \left(\frac{\lambda_0}{\lambda_n}\right)^m A_0 \right), \tag{5.5}$$

where $A_{n-1}, \ldots, A_0$ are constants depending on the coefficients of the polynomials $\{q_k\}_{k=0}^{n}$. As $m \to \infty$, the magnitude of the sum of the zeros of $T^m[g(z)]$, given by $\left|\frac{\lambda_{n-1}}{\lambda_n}\right|^m |A_{n-1}|$, approaches ∞. This implies that at least one of the zeros of $T^m[g(z)]$ tends to ∞ as $m \to \infty$, which contradicts that T is a $\mathcal{P}_\Omega$ preserver. $\square$

Definition 187. Let a sequence of real numbers $\{\gamma_k\}_{k=0}^{\infty}$ be associated with the operator T which is diagonal on monomials and satisfies $T[x^k] = \gamma_k x^k$, $k \in \mathbb{N} \cup \{0\}$. The sequence $\{\gamma_k\}_{k=0}^{\infty}$ is said to possess the *Gauss-Lucas property* if for any polynomial $f(z) \in \mathbb{C}[z]$, $T[f]$ has all of its zeros in the convex hull containing the zeros of f and the origin.

Compare Proposition 186 to the following theorem, which is proved using the Schur-Malo-Szegő composition theorem (Theorem 3).

Theorem 188 (T. Craven, G. Csordas [23]). *Let $\Gamma = \{\gamma_k\}$ be a nonzero sequence of real numbers. Then Γ possesses the Gauss-Lucas property if and only if Γ is a multiplier sequence and either $0 \leq \gamma_n \leq \gamma_{n+1}$ for $n = 0, 1, 2, \ldots$, or $0 \geq \gamma_n \geq \gamma_{n+1}$ for $n = 1, 2, \ldots$.*

The problem of characterizing multiplier sequences on other bases (see Definition 183) has been examined in detail by A. Piotrowski. The *Hermite multiplier sequences*, hyperbolicity preserving linear operators which are diagonal with respect to the Hermite basis, have been completely characterized by Piotrowski [84].

Theorem 189 (A. Piotrowksi [84, p. 140]). *A real sequence $\{\gamma_k\}$ is a Hermite multiplier sequence if and only if $\{\gamma_k\}_{k=0}^{\infty}$ can be expressed in one of the following forms:*

$$\{\lambda_k\}_{k=0}^{\infty}\ ,\ \{-\lambda_k\}_{k=0}^{\infty}\ ,\ \{(-1)^k\lambda_k\}_{k=0}^{\infty}\ ,\ \{(-1)^{k+1}\lambda_k\}_{k=0}^{\infty}\ ,$$

where $\{\lambda_k\}_{k=0}^{\infty}$ is either a non-decreasing, non-negative multiplier sequence, or has the trivial form

$$\{0, 0, 0, \ldots, 0, \lambda_n, \lambda_{n+1}, 0, 0, 0, \ldots\}.$$

Theorem 189 provides the solution to a problem of T. Craven and G. Csordas in [6]. We will show that for any simple set of polynomials $\{q_k\}_{k=0}^{\infty}$, having only simple real zeros and positive leading coefficients, any non-trivial non-negative q-multiplier sequence must be increasing (Theorem 193). This result is relevant to a general problem about multiplier sequences in [6] (cf. Problem 219). The proof is a generalization of a proof of A. Piotrowski.

Lemma 190. *Let $\{q_k\}_{k=0}^{\infty}$ be a simple set of real polynomials, where each polynomial q_k has only real zeros and positive leading coefficient. Then for any $n \in \mathbb{N}$, there exists a positive $C \in \mathbb{R}$ such that for all $c > C$, $q_n(x) + cq_{n-2}(x)$ has at least two non-real zeros.*

Proof. Let the zeros of q_n and q_{n-2} be contained in the interval $[a, b]$. Then for a positive c, all the real zeros of $g(x) = q_n(x) + cq_{n-2}(x)$ must lie in $[a, b]$, since g is always either positive or negative outside of $[a, b]$. By Hurwitz's theorem (Theorem 9), as $c \to \infty$ the number and location of the zeros of $q_n(x)/c + q_{n-2}(x)$ on $[a, b]$ must approach those of $q_{n-2}(x)$. By degree considerations, the only way this can happen is for $q_n(x)/c + q_{n-2}(x)$ to have at least two non-real zeros for all c sufficiently large. $\square$

A. Piotrowski has also shown that for a simple set of polynomials $\{q_k\}_{k=0}^{\infty}$, any q-multiplier sequence is also a multiplier sequence on the stadard basis $\{x^k\}_{k=0}^{\infty}$ [84]. We use Piotrowski's method of proof below to establish a more general statement.

Theorem 191. *Let $\Omega \subset \mathbb{C}$ be a union of closed sectors with vertex at the origin, $\Omega = \bigcup_{j=1}^{n} \overline{\mathbb{S}}(\theta_j, \delta_j)$ (see Definition 105). If T is a $\mathcal{P}_\Omega$-preserver which is diagonal on the simple basis of polynomials $\{q_k\}_{k=0}^{\infty}$ with $T[q_k] = \lambda_k q_k$, then the operator $M[z^k] = \lambda_k z^k$ is also a $\mathcal{P}_\Omega$-preserver.*

Proof. Without loss of generality we may assume the polynomials q_k are monic. Let T be any $\mathcal{P}_\Omega$-preserver and define the linear operator $T^{(r)}$ by $T^{(r)}[p_k^{(r)}] = \lambda_k p_k^{(r)}$ where $p_k^{(r)}(z) = \frac{1}{r^k} q_k(rz)$, for real $r > 0$. We first show that for each fixed positive $r \in \mathbb{R}$, $T^{(r)}$ is a $\mathcal{P}_\Omega$-preserver. Let $f(z) = \sum_{k=0}^{n} c_k p_k^{(r)}(z) \in \mathcal{P}_\Omega$, then

$$T^{(r)}[f(z)] = \sum_{k=0}^{n} \lambda_k c_k p_k^{(r)}(z) \tag{5.6}$$

$$= \sum_{k=0}^{n} \lambda_k c_k \frac{1}{r^k} q_k(rz) \tag{5.7}$$

$$= \sum_{k=0}^{n} \lambda_k b_k q_k(rz), \tag{5.8}$$

where $b_k = \frac{c_k}{r^k}$, and $\sum_{k=0}^{n} b_k q_k(rz) = f(z) \in \mathcal{P}_\Omega$. Since Ω is a union of closed sectors, $\sum_{k=0}^{n} b_k q_k(rz) \in \mathcal{P}_\Omega$ if and only if $\sum_{k=0}^{n} b_k q_k(z) \in \mathcal{P}_\Omega$. Therefore, $T[\sum_{k=0}^{n} b_k q_k(z)] = \sum_{k=0}^{n} \lambda b_k q_k(z) \in \mathcal{P}_\Omega$. Again, because Ω is a union of sectors, we may replace z with rz for $r > 0$, thus $\sum_{k=0}^{n} \lambda b_k q_k(rz) = T^{(r)}[f] \in \mathcal{P}_\Omega$. This shows that $T^{(r)}$ is a $\mathcal{P}_\Omega$-preserver.

Note that $T^{(r)} \to M$ as $r \to \infty$. For any $f \in \mathcal{P}_\Omega$, $r > 0$, $T^{(r)}[f] \in \mathcal{P}_\Omega$, and by Hurwitz's Theorem (Theorem 9),

$$\lim_{r\to\infty} T^{(r)}[f] = M[f] \in \mathcal{P}_\Omega \cup \{0\},$$

since Ω is closed. Thus, M is a $\mathcal{P}_\Omega$-preserver. $\square$

Note the following special cases of Theorem 191.

Corollary 192. *Let Ω be a line through the origin, a closed sector with vertex at the origin, or a closed double sector with vertex at the origin. If T is any $\mathcal{P}_\Omega$-preserver which is diagonal on the basis $\{p_k\}_{k=0}^{\infty}$, with $T[p_k] = \lambda_k p_k$, then the operator diagonal on monomials, defined by $M[z^k] = \lambda_k z^k$ for all $k \in \mathbb{N} \cup \{0\}$, is also a $\mathcal{P}_\Omega$-preserver.*

Theorem 193. *Let $\{q_k\}_{k=0}^{\infty}$ be a simple set of real polynomials, with only simple real zeros and positive leading coefficients. If $\{\lambda_k\}_{k=0}^{\infty}$ is a non-negative q-multiplier sequence with at least 3 non-zero terms, then $\lambda_k \geq \lambda_{k-1}$ for all $k \in \mathbb{N}$.*

Proof. Let $\{\lambda_k\}_{k=0}^{\infty}$ be a non-negative q-multiplier sequence with at least 3 non-zero terms. Because any q-multiplier sequence $\{\lambda_k\}_{k=0}^{\infty}$ is a multiplier sequence by Corollary 192, it can have no internal zeros (see Remark 166), and it must satisfy the Turán inequalities (Proposition 26), $\lambda_k^2 - \lambda_{k+1}\lambda_{k-1} \geq 0$ for all $k = 1, 2, 3, \ldots$. Assume there is a $d \in \mathbb{N}$ such that $\lambda_{d-1} > \lambda_d > 0$. We obtain a contradiction as follows.

First suppose that $\lambda_{d+1} = 0$. Then $\lambda_{d-2} \neq 0$, because $\{\lambda_k\}_{k=0}^{\infty}$ has no internal zeros and at least 3 non-zero terms. Let $f(x) = cq_{d-2}(x) + q_d(x) + dq_{d+1}(x) \in \mathbb{R}[x]$, where c is chosen sufficiently large that $c\frac{\lambda_{d-2}}{\lambda_d} q_{d-2}(x) + q_d(x)$ has non-real zeros by Lemma 190, and d is chosen such that $(1/d)(cq_{d-2}(x) + q_d(x)) + q_{d+1}(x)$, and therefore $f(x)$ has only real zeros (d exists because the zeros of q_{d+1} are simple). Applying the sequence $\{\lambda_k\}_{k=0}^{\infty}$ to $f(x)$ results in $c\lambda_{d-2}q_{d-2}(x) + \lambda_d q_d(x) = \lambda_d(c\frac{\lambda_{d-2}}{\lambda_d} q_{d-2}(x) + q_d(x))$, which has non-real zeros by the choice of c. This contradicts that $\{\lambda_k\}_{k=0}^{\infty}$ is a q-multiplier sequence.

Otherwise, if $\lambda_{d+1} > 0$, then the Turán inequalities imply

$$\frac{\lambda_d}{\lambda_{d-1}} \geq \frac{\lambda_{d+1}}{\lambda_d}. \tag{5.9}$$

Our assumption that $\lambda_{d-1} > \lambda_d > 0$, with the aid of (5.9), implies that $\lambda_{d-1} > \lambda_d > \lambda_{d+1}$. Let $f(x) = cq_{d-1}(x) + q_{d+1}(x) \in \mathbb{R}[x]$, where $c > 0$ is chosen sufficiently small that f has only real zeros (c exists because the zeros of q_{d+1} are simple). Since by assumption, $\{\lambda_k\}_{k=0}^{\infty}$ is a Q-multiplier sequence, we may repeatedly apply the sequence to f and the resulting polynomial will have only real zeros. Applying $\{\lambda_k\}_{k=0}^{\infty}$ to f, n times, produces

$$\lambda_{d+1}^n \left(c \left(\frac{\lambda_{d-1}}{\lambda_{d+1}} \right)^n q_{d-1}(x) + q_{d+1}(x) \right). \tag{5.10}$$

Because $\lambda_{d-1} > \lambda_{d+1}$, for n sufficiently large the polynomial in (5.10) has non-real zeros by Lemma 190. This contradicts that $\{\lambda_k\}_{k=0}^{\infty}$ is a multiplier sequence, and proves the theorem. $\square$

T. Forgács and A. Piotrowski have obtained detailed results about $L^{(\alpha)}$-multiplier sequences [45], where $L_n^{(\alpha)}$ is the n^{th} generalized Laguerre polynomial [90, p. 200]. In particular, Forgács and Piotrowski have shown that the necessary condition given in Theorem 193, $\lambda_k \geq \lambda_{k-1}$, is not sufficient for a real sequence $\{\lambda_k\}_{k=0}^{\infty}$ to be a nontrivial $L^{(\alpha)}$-multiplier sequence. Therefore, $L^{(\alpha)}$-multiplier sequences form a proper subset of the Hermite multiplier sequences. The following proposition provides an example of a multiplier sequence on the Legendre polynomial basis, denoted $\{P_k\}_{k=0}^{\infty}$ (see Section 3).

Proposition 194. *The sequence $\{k(k+1)\}_{k=0}^{\infty}$ is a Legendre polynomial basis multiplier sequence.*

Proof. The Legendre polynomials satisfy the differential equation [90]

$$(1 - z^2)P_k''(z) - 2zP_k'(z) + k(k+1)P_k(z) = 0.$$

Thus, the linear operator $T := (z^2 - 1)D^2 + 2zD$ acts as a P-multiplier sequence, $T[P_k(z)] = k(k+1)P_k(z)$. By Theorem 113, it is sufficient to show that $T[(x+w)^n] \in \mathcal{H}_2(\mathbb{R}) \cup \{0\}$ for all $n \in \mathbb{N}$. Direct computation yields

$$T[(z+w)^n] = n(z+w)^{n-2}((n+1)z^2 + 2zw + (1-n)). \tag{5.11}$$

It is then sufficient to show the quadratic factor in (5.11) is upper half-plane stable or identically zero. A computation shows that $T[(z+w)^0] \equiv 0$, and $T[(z+w)^1] = 2z$ is upper half-plane stable. For $n \geq 2$, let the zeros of the quadratic factor in (5.11) be $\alpha, \beta \in \mathbb{C}$. Then

$$\alpha + \beta = \frac{-2w}{n+1} \quad \text{and} \quad \alpha\beta = \frac{1-n}{n+1} < 0 \quad (n \geq 2) \tag{5.12}$$

Since $\alpha\beta < 0$, $\arg(\alpha) = \pi - \arg(\beta)$. This implies $\mathrm{Im}[\alpha]\mathrm{Im}[\beta] \geq 0$. If $\mathrm{Im}[w] > 0$, then (5.12) implies $\mathrm{Im}[\alpha] < 0$ and $\mathrm{Im}[\beta] < 0$. Therefore, $T[(x+w)^n] \neq 0$ when $\mathrm{Im}[w] > 0$ and $\mathrm{Im}[x] > 0$. Since n was arbitrary the proposition is proved. □

Remark 195. The sequence $\{(k+1)(k+2)\}_{k=0}^{\infty}$ is not a Legendre basis multiplier sequence; $p(x) = (x+1)^3$ serves as a counterexample. Since $\{(k+1)(k+2)\}_{k=0}^{\infty}$ is an increasing multiplier sequence, the Legendre multiplier sequences form a proper subset of the Hermite multiplier sequences.

3 The monomial to Legendre polynomial basis transform

Let $P_n(x)$ represent the n^{th} Legendre polynomial, defined by the generating function [90, p. 157],

$$G(x, \mu) := (1 - 2x\mu + \mu^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)\mu^n. \tag{5.13}$$

Recall that the Legendre polynomials satisfy the orthogonality relation [90, p. 174-175]

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases} \frac{2}{2n+1}, & \text{for } m = n \\ 0, & \text{for } m \neq n. \end{cases} \tag{5.14}$$

In this section, we prove the following conjecture from S. Fisk's book [43], which is left as an open problem in the book's empirical tables of basis transformations.

Conjecture 196 (S. Fisk [43, p. 711])**.** If $f(x) = \sum_{k=0}^{n} a_n x^n \in \mathbb{R}[x]$ has all of its zeros in the interval $(-1, 1)$, then $T[f](x) = \sum_{k=0}^{n} a_n P_n(x)$ has all of its zeros in $[-1, 1]$.

In the proof of Conjecture 196, we rely on the strict sign regularity (Definition 197) of a generating function related to the transformation.

Definition 197 (Strict Sign Regularity)**.** The kernel function $G : X \times Y \to \mathbb{R}$ is said to be *strictly sign regular* (SSR), if given any two increasing m-tuples $x_1 < x_2 < \ldots < x_m$, $y_1 < y_2 < \ldots < y_m$, in X and Y respectively,

$$\varepsilon(m) \begin{vmatrix} G(x_1, y_1) & G(x_1, y_2) & \ldots & G(x_1, y_m) \\ G(x_2, y_1) & G(x_2, y_2) & \ldots & G(x_2, y_m) \\ \vdots & \vdots & \ddots & \vdots \\ G(x_m, y_1) & G(x_m, y_2) & \ldots & G(x_m, y_m) \end{vmatrix} > 0. \tag{5.15}$$

for all $m \in \mathbb{N}$, where $\varepsilon(m)$ is a function of m only and has the range $\{-1, 1\}$. If $\varepsilon(m) = 1$, the kernel $G(x, \mu)$ is said to be *strictly totally positive* (STP) (see [57]).

The following theorem of A. Iserles and E. B. Saff is required for our proof of Conjecture 196.

Theorem 198 (A. Iserles and E. B. Saff [55])**.** *Let $\{Q_n(x)\}_{n=0}^{\infty}$ be an orthogonal sequence of polynomials satisfying*

$$\int_{a}^{b} Q_n(x)Q_m(x)dx = \begin{cases} h_m, & \textit{for } m = n \\ 0, & \textit{for } m \neq n. \end{cases}$$

Given that the generating function $G(x,\mu) = \sum_{\mu=0}^{\infty} \delta_k Q_k(x)\mu^k$ *is SSR for all* $x \in (a,b)$ *and* $\mu \in (c,d)$ *and that all the zeros of the polynomial* $\sum_{k=0}^{n} q_k x^k$ *are in the interval* (c,d)*, all of the zeros of the polynomial* $\sum_{k=0}^{n} \frac{q_k}{\delta_k h_k} Q_k(x)$ *reside in* $[a,b]$.

Theorem 198 is proved using properties of biorthogonal polynomials [55]. For reference, we list some fundamental results on totally positive kernels here.

Theorem 199 ([57, p. 15])**.**

$$K(x,y) = e^{xy}$$

is STP, and therefore SSR, for $x, y \in \mathbb{R}$.

Theorem 200 ([57, p. 18])**.** *Let* $K(x,y)$ *be SSR* ($x \in X$ *and* $y \in Y$)*, and let* $\varphi(x)$ *and* $\psi(x)$ *maintain the same constant sign on* X *and* Y*, respectively. Then*

i. $L(x,y) = \varphi(x)\psi(y)K(x,y)$ *is SSR.*

ii. Now let $u = \varphi^{-1}(x)$ *and* $v = \psi^{-1}(x)$ *be strictly increasing functions mapping* X *and* Y *onto* U *and* V*, respectively, where* φ^{-1} *and* ψ^{-1} *are the inverse functions of* φ *and* ψ*, respectively. Consider*

$$L(u,v) = K[\varphi(u), \psi(x)] \qquad u \in U, v \in V.$$

Then $L(u,v)$ *is SSR and with signs* $\varepsilon_m(K) = \varepsilon_m(L)$*. If* $\phi(u)$ *is strictly increasing while* $\psi(v)$ *is strictly decreasing, then* $L(u,v)$ *is SSR and* $\varepsilon_m(K) = (-1)^{\frac{m(m-1)}{2}} \varepsilon_m(L)$*.*

Theorem 201 ([57, p. 16](Composition Rule))**.** *Let* K*,* L *and* M *be Borel-measurable functions of two variables satisfying*

$$M(x,y) = \int_Z K(x,\eta)L(\eta,y)d\sigma(\eta),$$

where the integral is assumed to converge absolutely. The variables x*,* y*, and* η *are in subsets of the real line* X*,* Y*, and* Z *respectively, and* $d\sigma(\eta)$ *is a sigma finite measure on* Z*. If both* K *and* L *are strictly sign regular, then so is* M*.*

Theorem 201 is a direct result of generalizing the Cauchy-Binet theorem for matrix products (G. Pólya and G. Szegő [87, p. 61]) and in it "strictly sign regular" may be replaced by a requirement for strict or non-strict sign regularity or total positivity of arbitrary order.

We now prove two preliminary lemmas.

Lemma 202.

$$K(x,y) = e^{-xy}$$

is strictly sign regular (SSR) for all $x, y \in \mathbb{R}$.

Proof. Given two arbitrary increasing m-tuples $x_1 < x_2 < \ldots < x_m$, $y_1 < y_2 < \ldots < y_m$, in $\mathbb{R}$, then $-x_m < -x_{m-1} < \ldots < -x_1$ is also an increasing m-tuple in X. Since, by Theorem 199, e^{xy} is stricly totally positive,

$$\begin{vmatrix} e^{-x_m y_1} & e^{-x_m y_2} & \ldots & e^{-x_m y_m} \\ e^{-x_{m-1} y_1} & e^{-x_{m-1} y_2} & \ldots & e^{-x_{m-1} y_m} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-x_1 y_1} & e^{-x_1 y_2} & \ldots & e^{-x_1 y_m} \end{vmatrix} > 0. \tag{5.16}$$

Let $\lfloor x \rfloor$ denote "the greatest integer less than or equal to x". After $\lfloor \frac{m}{2} \rfloor$ row exchanges (first row with last row, second row with second to last, etc...) on the determinant in (5.16), we obtain

$$(-1)^{\lfloor \frac{m}{2} \rfloor} \begin{vmatrix} e^{-x_1 y_1} & e^{-x_1 y_2} & \ldots & e^{-x_1 y_m} \\ e^{-x_2 y_1} & e^{-x_2 y_2} & \ldots & e^{-x_2 y_m} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-x_m y_1} & e^{-x_m y_2} & \ldots & e^{-x_m y_m} \end{vmatrix} > 0.$$

Thus $K(x,y) = e^{-xy}$ is SSR as claimed. Note this also follows immediately by the second part of Theorem 200. □

A statement similar to Lemma 203 is made by A. Iserles and E. B. Saff, but here a different statement and proof are given due to an apparent mistake in [55].

Lemma 203. *The function*

$$K(x,y) = \frac{1}{(x+y)^{\beta+1}},$$

where $\beta > -1$, is SSR for $x, y \in (0, \infty)$

Proof. By Lemma 202 both $e^{-x\eta}$ and $e^{-y\eta}$ are SSR for all $x, y \in \mathbb{R}$. Theorem 201 then implies

$$\begin{aligned} \frac{1}{\Gamma(\beta+1)} \int_0^\infty e^{-x\eta} e^{-y\eta} \eta^\beta d\eta &= \frac{1}{\Gamma(\beta+1)} \int_0^\infty e^{-(x+y)\eta} \eta^\beta d\eta \\ &= \frac{1}{\Gamma(\beta+1)} \frac{1}{(x+y)^{\beta+1}} \int_0^\infty z^\beta e^{-z} dz \\ &= \frac{1}{(x+y)^{\beta+1}}, \end{aligned}$$

is SSR for all $x, y \in (0, \infty)$, and $\beta > -1$ (both conditions are requirements for convergence of the integral). $\square$

We now prove Fisk's conjecture (Conjecture 196), which results in the following Theorem.

Theorem 204. *If $f(x) = \sum_{k=0}^n a_n x^n \in \mathbb{R}[x]$ has all of its zeros in the interval $[-1, 1]$, then $T[f](x) = \sum_{k=0}^n a_n P_n(x)$ has all of its zeros in $[-1, 1]$.*

Proof of Conjecture 196 and Theorem 204. First assume that $f(x)$ has all of its zeros in the interval $(-1, 1)$ as in Conjecture 196. The series on the right-hand side of (5.13) is known to converge for all $x, \mu \in (-1, 1)$. Operating on (5.13) with $\left(2\mu \frac{\partial}{\partial \mu} + 1\right)$, we obtain a second generating function which converges for the same range of parameters:

$$\begin{aligned} G_2(x,\mu) &= \left(2\mu \frac{\partial}{\partial \mu} + 1\right) G(x,\mu) \\ &= \frac{1-\mu^2}{(1-2x\mu+\mu^2)^{\frac{3}{2}}} = \sum_{n=0}^\infty (2n+1) P_n(x) \mu^n. \end{aligned}$$

To prove the conjecture, by Theorem 198 (with $\delta_k = 2k+1$, $h_k = \frac{2}{2k+1}$), it is sufficient to show that $G_2(x, \mu)$ is SSR for $x, \mu \in (-1, 1)$.

$$G_2(x,\mu) = (1-\mu^2) \frac{1}{(1+\mu^2)^{\frac{3}{2}} \left(1 - \frac{2x\mu}{1+\mu^2}\right)^{\frac{3}{2}}},$$

and applying the first part of Theorem 200 twice, it suffices to show $F(x,\mu) = \frac{1}{\left(1-\frac{2x\mu}{1+\mu^2}\right)^{\frac{3}{2}}}$ is SSR. Setting $\eta = \frac{2\mu}{\mu^2+1}$, it is then sufficient to show that

$$\frac{1}{(1-x\eta)^{\frac{3}{2}}}$$

is SSR for $x, \eta \in (-1,1)$. Replacing x with $\frac{x-1}{x+1}$ and η with $\frac{y-1}{y+1}$ we can equivalently show that the function

$$\left(\frac{(x+1)(y+1)}{2(x+y)}\right)^{\frac{3}{2}}$$

is SSR for $x, y \in (0,\infty)$. Applying the first part of Theorem 200 again, we can drop the factors $(x+1)^{\frac{3}{2}}$, $(y+1)^{\frac{3}{2}}$, and $1/2^{\frac{3}{2}}$, leaving $\frac{1}{(x+y)^{\frac{3}{2}}}$. By Lemma 203, $\frac{1}{(x+y)^{\frac{3}{2}}}$ is indeed SSR for $x, y \in (0,\infty)$, and this completes the proof of the conjecture when f has all of its zeros in $(-1,1)$. By continuity of the transformation T, $f(x)$ may have its zeros in the closed interval $[-1,1]$ as well. □

4 Differential operator representations

The following theorem is stated using the multivariate notation of Definition 145.

Theorem 205. *If $T : \mathbb{C}[z_1,\ldots,z_n] \to \mathbb{C}[z_1,\ldots,z_n]$ is a linear operator, then it has a unique representation*

$$T = \sum_{\beta\in\mathbb{N}_o^n} Q_\beta(z)\partial^\beta, \tag{5.17}$$

where $Q_\beta \in \mathbb{C}[z_1,\ldots,z_n]$, and the sum may be infinite.

It is instructive to consider the case $T : \mathbb{C}[z_1,z_2] \to \mathbb{C}[z_1,z_2]$. Suppose that T can be expressed in the form

$$T = Q_{(0,0)}(z) + Q_{(1,0)}(z)\partial_1 + Q_{(0,1)}(z)\partial_2 + Q_{(1,1)}(z)\partial_1\partial_2 + Q_{(2,0)}(z)\partial_1^2 + \cdots. \tag{5.18}$$

When T operates on the first few monomial basis polynomials it produces,

$$\begin{aligned}
T[1] &= Q_{(0,0)}(z), \\
T[z_1] &= Q_{(1,0)}(z) + z_1 Q_{(0,0)}(z), \\
T[z_2] &= Q_{(0,1)}(z) + z_2 Q_{(0,0)}(z), \\
T[z_1 z_2] &= Q_{(1,1)}(z) + z_2 Q_{(1,0)}(z) + z_1 Q_{(0,1)}(z) + z_1 z_2 Q_{(0,0)}(z), \text{ and} \\
T[z_1^2] &= 2Q_{(2,0)}(z) + 2z_1 Q_{(0,1)}(z) + z_1^2 Q_{(0,0)}(z).
\end{aligned}$$

These equations can then be used to recursively solve for the first few coefficients $Q_\beta(z)$, using only the values of T evaluated on the basis polynomials. For example,

$$Q_{(2,0)}(z) = \frac{1}{2}\left(T[z_1^2] - (2z_1 Q_{(0,1)}(z) + z_1^2 Q_{(0,0)}(z))\right).$$

It is clear that we can continue to solve for any Q_β in the same way, and we formalize this procedure in the following proof.

Proof of Theorem 205. One finds $Q_{(0,\ldots,0)}$ by operating on 1,

$$T[1] = Q_{(0,\ldots,0)}(z),$$

where $z = (z_1, \ldots, z_n)$. We then recursively define

$$Q_\beta(z) = \frac{1}{\beta!} T[z^\beta] - \sum_{\alpha<\beta} \frac{z^{\beta-\alpha}}{(\beta-\alpha)!} Q_\beta(z), \tag{5.19}$$

where $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$. The right hand side of (5.17) agrees with T on the basis z^β, $\beta \in \mathbb{N}_o^n$, by construction, and therefore by the linearity of T, (5.17) holds on $\mathbb{C}[z_1, \ldots, z_n]$. The uniqueness of the Q_β follows from the uniqueness of the solution to the system of recursive linear equations (5.19). $\square$

Remark 206. In the univariate case we may write (5.19) as

$$Q_{n+1}(z) = \frac{1}{(n+1)!}\left[T[z^{n+1}] - \sum_{k=0}^{n} \binom{n+1}{k} k! Q_k(z) z^{n+1-k}\right]. \tag{5.20}$$

For the remainder of this section Q_k and Q_β will denote to complex polynomials in the representation (5.17).

The Weyl algebra, $\mathcal{A}_n[\mathbb{C}]$, was introduced in Chapter 4 (see Definition 145). For reference, we list recently proved characterizations of Weyl algebra preservers in n variables.

Theorem 207 ([12]). *An operator $T \in \mathcal{A}_n[\mathbb{C}]$ is a stability preserver if and only if $F_T(z, -w) \in \mathcal{H}_{2n}$.*

Theorem 208 ([12]). *An operator $T \in \mathcal{A}_n[\mathbb{R}]$ is a hyperbolicity preserver if and only if $F_T(z, -w) \in \mathcal{H}_{2n}(\mathbb{R})$.*

Note that Theorems 207 and 208 follow immediately from Theorem 153 (and its hyperbolicity analog) and equation 4.16. Theorems 207 and 208 yield the following information about linear stability and hyperbolicity preservers which can be written as *finite order* differential operators.

Proposition 209. *If the linear operator $T : \mathbb{R}[z] \to \mathbb{R}[z]$ is a hyperbolicity perserver, and if T can be represented as a differential operator of finite order,*

$$T = \sum_{k=0}^{N} Q_k(z) D^k, \qquad (Q_k \in \mathbb{R}[z]), \tag{5.21}$$

then the polynomials Q_k have only real zeros.

Proof. By Theorem 208, the symbol of T satisfies

$$F_T(z, -w) = \sum_{k=0}^{N} Q_k(z)(-w)^k \in \mathcal{H}_2(\mathbb{R}).$$

It is known that $\partial_w := \frac{\partial}{\partial w}$ preserves multivariate stability as a simple consequence of the Gauss-Lucas Theorem [74, p. 22]. Therefore, for any $j = 0, 1, 2, \ldots,$

$$\partial_w^j F_T(z, -w) = \sum_{k=j}^{N} Q_k(z)(-1)^k \frac{k! w^{k-j}}{(k-j)!} \in \mathcal{H}_2(\mathbb{R}) \cup 0.$$

By Theorem 208, there is a hyperbolicity preserving operator $T^{(j)} = F_{T^{(j)}}(z, \partial)$ associated with the symbol $F_{T^{(j)}}(z, -w) = \partial_w^j F_T(z, -w)$. Then $T^{(j)}[1] = (-1)^j j! Q_j(z)$ is hyperbolic, or equal to 0. Since j is arbitrary, this completes the proof. $\square$

Proposition 210. *If the linear operator $T : \mathbb{C}[z] \to \mathbb{C}[z]$ preserves upper half-plane stability, and if T can be represented as a differential operator of finite order,*

$$T = \sum_{k=0}^{N} Q_k(z) D^k, \qquad (Q_k \in \mathbb{C}[z]),$$

then the polynomials Q_k are upper half-plane stable.

Proof. Similar to the proof of Proposition 209. $\square$

In a similar fashion the multivariate versions of Propositions 209 and 210 hold—this has also been observed in the recent paper [16]. As one might guess, a linear operator T possessing coefficients $Q_k(z)$ with only real zeros in representation (5.21) is not necessarily a hyperbolicity preserver.

Example 211. *The converses of Propositions 209 and 210 are false.* If $T = (z^2 - 1)D + 1$, $p(x) = z + 5$, then all Q_k associated with T have only real zeros. The polynomial $T[p(z)] = z^2 + z + 4$ does not have real zeros, so T does not preserve hyperbolicity or stability. Moreover, violating the hypotheses of Laguerre's Theorem (Theorem 16) is enough to show the converses of Propositions 209 and 210 are false. The operator $T = zD - 1$, and the polynomial $p(z) = z^2 - 7$, also produce a counter example.

Example 212. *Propositions 209 and 210 can not be extended to infinite order differential operators.* Let T_1, T_2 be the linear transformations given by $T_1[z^k] = \frac{z^k}{k!}$ and $T_2[z^k] = H_k(x)$, for all $k \in \mathbb{N}_o$, where H_k is the k^{th} Hermite polynomial. Both T_1 and T_2 preserve hyperbolicity (see Remark 213). Hence, $T = T_2 T_1$ preserves hyperbolicity, and in the representation

$$T = \sum_{k=0}^{\infty} Q_k(z) D^k,$$

the coefficient

$$Q_2(z) = -\frac{1}{2}(z^2 + 1)$$

has non-real zeros.

Remark 213. Note that $\{1/k!\}$ is a multiplier sequence by Laguerre's Theorem (Theorem 16). When T has the action of a multiplier sequence, such as $T_1[z^k] = z^k/k!$ for all $k \in \mathbb{N}_0$, the coefficients $Q_k(z)$ in (5.20) have only real zeros (see Remark 215). The linear transformation defined by $T_2[z^k] = H_k(z)$, where H_k is the k^{th} Hermite polynomial, can be represented as $T_2 = e^{-D^2/2}$ [84], which is hyperbolicity preserving by the Hermite-Poulain Theorem (Theorem 25).

Note that if $T = \sum_{k=0}^{\infty} Q_k(z)D^k$ maps polynomials of degree n to polynomials of degree n or less, each Q_k has at most degree k. This is the case when T is a degree preserving basis transformation.

Proposition 214. *If $T = \sum_{k=0}^{\infty} Q_k(z)D^k$ is the monomial to Chebyshev polynomial basis transformation, defined by $T[z^k] = \cos(k \arccos(z))$, for all $k \in \mathbb{N}_o$, then the coefficients $Q_k(z)$ have only real zeros. Furthermore,*

$$Q_{2k}(z) = \frac{1}{(2k)!}(z^2-1)^k, \qquad Q_{2k+1} = 0, \qquad k = 0, 1, 2, \ldots.$$

Proof. The Chebyshev polynomials are given by the formula [90, p. 301]

$$T_n(z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (z^2-1)^k z^{n-2k}.$$

For $k = 0$, $T[1] = 1 = Q_0(x)$, and $k = 1$, $Q_1(z) = T[z] - zQ_0(z) = 0$. Assume that the formulas for Q_k hold up to some even n. Then $n+1$ is odd, and by (5.20),

$$\begin{aligned}
Q_{n+1}(z) &= \frac{1}{(n+1)!}\left[T[z^{n+1}] - \sum_{k=0}^{n} Q_k(z)k!\binom{n+1}{k} z^{n+1-k}\right] \\
&= \frac{1}{(n+1)!}\left[T_{n+1}(z) - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} Q_{2k}(z)(2k)!\binom{n+1}{2k} z^{n+1-2k}\right] \\
&= \frac{1}{(n+1)!}\left[T_{n+1}(z) - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (z^2-1)^k \binom{n+1}{2k} z^{n+1-2k}\right] \\
&= 0.
\end{aligned}$$

If we show that $Q_{n+2}(z) = \frac{1}{(n+2)!}(z^2-1)^{\frac{n+2}{2}}$, the proposition will hold by induction on n. Since $Q_{n+1} \equiv 0$,

$$\begin{aligned}
Q_{n+2}(z) &= \frac{1}{(n+2)!}\left[T[z^{n+2}] - \sum_{k=0}^{n} Q_k(z)k!\binom{n+2}{k}z^{n+2-k}\right] \\
&= \frac{1}{(n+2)!}\left[T_{n+2}(z) - \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} Q_{2k}(z)(2k)!\binom{n+2}{2k}z^{n+2-2k}\right] \\
&= \frac{1}{(n+2)!}\left[\sum_{k=0}^{\frac{n}{2}+1}\binom{n+2}{2k}(z^2-1)^k z^{n+2-2k} - \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(z^2-1)^k\binom{n+1}{2k}z^{n+2-2k}\right] \\
&= \frac{1}{(n+2)!}(z^2-1)^{\frac{n}{2}+1} \\
&= \frac{1}{(n+2)!}(z^2-1)^{\frac{n+2}{2}}.
\end{aligned}$$

Hence the claim holds. □

Remark 215. Let $\{q_k\}_{k=0}^{\infty}$ be a simple set of polynomials. Using $F_T(z,w)$ as a generating function and (4.16) (see Proposition 216), one may readily deduce that for a change of basis transformation T, which maps $z^k \to q_k(z)$, the associated coefficients in representation (5.20) are

$$Q_k(z) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}q_j(z)(-z)^{k-j}. \tag{5.22}$$

In the case where T has the action of a multiplier sequence $\{\gamma_k\}_{k=0}^{\infty}$, (5.22) reduces to

$$Q_k(z) = \frac{z^k}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\gamma_k = \frac{z^k}{k!}g_k^*(-1), \tag{5.23}$$

where $g_k^*(z) = \sum_{j=0}^{k}\binom{k}{j}\gamma_k z^{k-j}$ is the reverse of the k^{th} Jensen polynomial (equation (5.23) for multiplier sequences first appears in the dissertation of A. Piotrowski [84, p. 35], where a different proof is given).

Proposition 216. *If $T : \mathbb{C}[z] \to \mathbb{C}[z]$ is a linear operator, then in the representation $T = \sum_{k=0}^{\infty} Q_k(z)D^k$, the coefficients are given by*

$$Q_k(z) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}T[z^j](-z)^{k-j} \tag{5.24}$$

Proof. Using (4.16), the operator symbol is given by

$$\begin{aligned}
F_T(z,w) &= T[e^{zw}]e^{-zw} \\
&= \left(\sum_{j=0}^{\infty} \frac{T[z^j]w^j}{j!}\right)\left(\sum_{\ell=0}^{\infty} \frac{(-z)^\ell w^\ell}{\ell!}\right) \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{T[z^j](-z)^{k-j}}{j!(k-j)!} w^k \\
&= \sum_{k=0}^{\infty}\left(\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}T[z^j](-z)^{k-j}\right) w^k.
\end{aligned}$$

Formula (5.24) follows immediately. □

An operator is *stability reversing* if it maps polynomials which are non-zero in the open upper half-plane to polynomials which are non-zero in the open lower half-plane. In [11], it is shown that any hyperbolicity preserver is either stability preserving or stability reversing. From Theorem 113, one can identify the conditions that characterize a hyperbolicity preserver T, which is stability reversing, as $T[(z-w)^n] \in \mathcal{H}_2(\mathbb{R}) \cup \{0\}$ for all $n \in \mathbb{N}_o$, or equivalently $T[e^{zw}] \in \overline{\mathcal{H}}(\mathbb{R})$. Example 212 shows not all hyperbolicity preservers have hyperbolic coefficients in the differential operator representation (5.20). The following proposition gives a sufficient condition for hyperbolic polynomial coefficients in (5.20).

Proposition 217. *Let $T : \mathbb{C}[z] \to \mathbb{C}[z]$ be a linear hyperbolicity preserver. If T is stability reversing, then the coefficients $Q_k(z)$ in the representation $T = \sum_{k=0}^{\infty} Q_k(z)D^k$ are either hyperbolic or identically zero.*

Proof. If T is stability reversing, then by Theorem 113 it satisfies

$$T[(z-w)^k] = \sum_{j=0}^{k}\binom{k}{j}T[z^j](-w)^{k-j} \in \mathcal{H}_2(\mathbb{R}) \cup \{0\}$$

for all $k \in \mathbb{N}_o$. Specializing $w = z$, and using (5.24), yields

$$k!Q_k(z) = \sum_{j=0}^{k}\binom{k}{j}T[z^j](-z)^{k-j} \in \mathcal{H}_1(\mathbb{R}) \cup \{0\}.$$

□

Remark 218. Note that the multivariate versions of Propositions (216) and (217) hold also, with essentially identical proofs. Formula (5.24) for representation (5.17), with n complex variables, is

$$Q_\beta(z) = \frac{1}{\beta!} \sum_{\kappa \in \mathbb{N}_o^n}^{\kappa \leq \beta} \binom{\beta}{\kappa} T[z^\kappa](-z)^{\beta-\kappa}, \tag{5.25}$$

where $\binom{\beta}{\kappa} := \frac{\beta!}{\kappa!(\beta-\kappa)!}$.

5 Conjectures and Open Problems

In this section we present some questions related to the stability preserving transformations we have discussed in Chapters 4 and 5, along with open problems from the literature. It was shown in Theorem 193 that any non-trivial multiplier sequence on a monic simple basis (where each basis polynomial has only simple zeros), is necessarily non-decreasing. This includes the case of multiplier sequences with respect to the classical orthogonal polynomial sets, such as Hermite, Laguerre, and Legendre polynomials.

Problem 219 ([6])**.** Let $\{q_k\}_{k=0}^{\infty}$ be a simple set of real polynomials in $\mathscr{L}$-$\mathscr{P}$. Characterize all real sequences $\{\gamma_k\}_{k=0}^{\infty}$ such that

$$\text{if } \ f(x) := \sum_{k=0}^{n} a_k q_k(z) \in \mathscr{L}\text{-}\mathscr{P} \text{ , then } \ \sum_{k=0}^{n} a_k \gamma_k q_k(z) \in \mathscr{L}\text{-}\mathscr{P} \tag{5.26}$$

Problem 220 ([84])**.** Classify all q-multiplier sequences where $\{q_k\}_{k=0}^{\infty}$ is an arbitrary orthogonal set.

We ask the following question as a generalization of Problem 220.

Question 221. Let γ be a continuous curve in the complex plane. One can construct a simple set of orthogonal polynomials $\{q_n\}_{n=0}^{\infty}$ with zeros in the convex hull of the curve γ [93]. What are q_n-multipliers $\{\lambda_n\}_{n=0}^{\infty}$ which preserve the property that all the zeros of a polynomial lie in the convex hull of γ?

In a fashion similar to Proposition 214, the following conjecture appears to hold.

Conjecture 222. If $T = \sum_{k=0}^{\infty} Q_k(z)D^k$ is the monomial to Legendre polynomial basis transformation then the $Q_k(z)$ have only real zeros. Furthermore,

$$Q_{2k}(z) = C_{2k}(z^2-1)^k \qquad Q_{2k+1} = 0 \qquad k = 0, 1, 2, \ldots,$$

where $C_0 = 1$ and

$$C_{2k} = \frac{(4k+1)!!}{((2k+1)!)^2} - \sum_{j=0}^{k-1} \frac{C_{2j}}{(2k-2j+1)!}.$$

Based on the observations in this chapter, we propose the following bold, and perhaps overly speculative, conjecture.

Conjecture 223. If $T : \mathbb{R}[z] \to \mathbb{R}[z]$ or $T : \mathbb{C}[z] \to \mathbb{C}[z]$ is a linear P_Ω-preserver, for some convex region $\Omega \subset \mathbb{C}$, and can be represented in the form

$$T = \sum_{k=0}^{N} Q_k(z)D^k,$$

then each $Q_k(z)$ has zeros only in Ω.

More conservatively, we ask the following questions.

Question 224. Let $\Omega_1 \subset \mathbb{C}$ and $\Omega_2 \subset \mathbb{C}$. What are the properties of the Q_k coefficients in representation (5.20) of a linear operator $T : \mathbb{C}[z] \to \mathbb{C}[z]$ which maps polynomials having zeros only in Ω_1 to polynomials whose zeros lie only in Ω_2?

Question 225. For $T = \sum_{k=0}^{\infty} Q_k(z)D^k$, what conditions on Q_k will guarantee that every truncation of T is a hyperbolicity preserver? Given an $n \in \mathbb{N}$, what conditions will guarantee there exists $m > n$ such that $\hat{T} = \sum_{k=0}^{m} Q_k(z)D^k$ is a hyperbolicity preserver?

In reference to approach (ii) listed in the introduction of this chapter, we ask the following question.

Question 226. Let $\Omega \subset \mathbb{C}$. Suppose $T : \mathbb{C}[z] \to \mathbb{C}[z]$ has the representation

$$T[p] = \int_{C(0,1)} p(w)K(w,z)dw,$$

where $C(0,1)$ is the unit circle and

$$K(w,z) = \sum_{k=0}^{\infty} \frac{k!}{2\pi i} \frac{Q_k(z)}{(w-z)^{k+1}}.$$

What properties characterize $K(w,z)$ when T is a $\mathcal{P}_\Omega$-preserver?

Seeking an extension of Proposition 138, we pose the following question.

Question 227. Let $T : \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ be a linear operator. If T is diagonal with $T[z^\alpha] = \lambda_\alpha z^\alpha$, where $\{\lambda_\alpha\}_{\alpha \in \mathbb{N}_0}$ is a positive multivariate multiplier sequence (a multivariate multiplier sequence is an $\mathcal{H}_n(\mathbb{R})$-preserver, $T : \mathcal{H}_n(\mathbb{R}) \to \mathcal{H}_n(\mathbb{R})$), is T an $\mathcal{S}_n(\theta, \delta)$-preserver?

Index of Notation

The closure of a set B of real or complex numbers is denoted $\overline{B}$. Additional notation is listed in the table below.

Notation	Description	Page
$B(a,R)$	open ball centered at a with radius R	6
B^n	n-fold Cartesian product of the set B	77
Δ	foward difference operator	18
$(x)_j$	falling factorial	20
F_T	symbol associated with the operator T	73
$g_{n,p}$	Jensen polynomial	7
$J_k(\varphi)$	k^{th} Turán expression for 3^{rd} Jensen polynomial	81
$\mathbb{H}^+$	open upper half-plane	71
$\mathbb{H}_\delta$	$\mathbb{H}^+$ rotated counterclockwise by δ	71
$\mathcal{H}$	set of univariate polynomials that are $\mathbb{H}^+$-stable	56
$\mathcal{H}_n$	set of polynomials which are $\mathbb{H}^+$-stable in n variables	56
$\mathcal{H}^\alpha$	set of univariate polynomials that are $\mathbb{H}_\alpha$-stable	71
$\mathcal{H}_n^\alpha$	set of polynomials which are $\mathbb{H}_\alpha^+$-stable in n variables	71
$\mathcal{H}(\mathbb{R})$	set of (univariate) hyperbolic polynomials	56
$\mathcal{H}_n(\mathbb{R})$	set of real stable polynomials in n variables	56
$\overline{\mathcal{H}}$	closure of $\mathcal{H}$ under locally uniform limits	57
$\overline{\mathcal{H}}_n^\alpha$	closure of $\mathcal{H}_n^\alpha$ under locally uniform limits	57
$\mathscr{L}\text{-}\mathscr{P}$	the Laguerre-Pólya class	7
$\mathscr{L}\text{-}\mathscr{P}^+$	set of functions in $\mathscr{L}\text{-}\mathscr{P}$ with positive Taylor coefficients	7
$\mathscr{L}\text{-}\mathscr{P}(-\infty,0]$	set of functions in $\mathscr{L}\text{-}\mathscr{P}$ whose zeros all lie in $(-\infty,0]$	7
$\mathbb{N}_o$	$\mathbb{N}_o = \mathbb{N}\cup\{0\}$	57
$\mathcal{P}_\Omega$	the set of polynomials whose zeros all lie in Ω	88
$\mathbb{S}(\theta,\delta)$	open sector with vertex at the origin	55
$\mathcal{S}(\theta,\delta)$	set of polynomials with zeros only in $\overline{\mathbb{S}}(\theta,\delta)$	56
$\overline{\mathcal{S}}(\theta,\delta)$	closure of $\mathcal{S}(\theta,\delta)$ under locally uniform limits	57, 61
$T_k(\varphi)$	k^{th} Turán expression for the coefficients of φ	17, 81

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