AN INTRINSIC CONSTRUCTION OF THE PRINCIPAL COMPONENT OF THE HILBERT SCHEME

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Abstract. The principal component of the Hilbert scheme of n points on a scheme X is the closure of the open subset parameterizing n distinct points. In this article we construct the principal component as a certain blow-up of the symmetric product of X. Our construction is based on a local explicit analysis of étale families from where the appropriate universal property, needed to identify the principal component with the blow-up, is derived.

Introduction

The Hilbert scheme of n points in a variety X parameterizes finite subschemes of length n. When the ambient variety is a smooth surface Fogarty showed that the Hilbert scheme of n points is again smooth [Fog73]. However, in higher dimensions the situation is different. Already for the affine three-space Iarrobino showed that the Hilbert scheme is not even irreducible for high enough n [Iar72]. It is therefore natural to consider the irreducible component that contains the open subscheme $\mathscr{U}_X^n$ parameterizing n distinct points (see e.g. [ES04, EV10, CEVV08, Lee08]).

From now on, let X be a general scheme. We let $\mathscr{U}_X^n$ be the scheme parameterizing n distinct points in X, i.e., families of closed subschemes of X that are finite and étale of rank n. The principal component $\overline{\mathscr{U}_X^n}$ refers to the schematic closure of $\mathscr{U}_X^n$ inside the Hilbert scheme. Ekedahl and Skjelnes gave a blow-up construction of $\overline{\mathscr{U}_X^n}$ [ES04]. That work is related to the one presented here, but note that their construction uses in an essential way the existence of the Hilbert scheme. In contrast, our approach is explicit and elementary, and produces the principal component *without* using the Hilbert scheme.

By conducting a careful local analysis of étale families, we get explicit affine schemes and formulas for the product structure of the universal family. These affine schemes with their universal properties naturally

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patch together to form the space $\mathscr{U}_X^n$. Moreover, this analysis allows us to construct a larger space $\mathscr{G}_X^n$ containing $\mathscr{U}_X^n$ as an open subspace.

The space $\mathscr{U}_X^n$ sits naturally as an open subscheme of the n-fold symmetric product of X, and we construct $\mathscr{G}_X^n$ as a certain blow-up of the symmetric product. There is moreover a canonical closed subspace $\mathscr{Z}$ in $\mathscr{G}_X^n \times X$ that is flat and finite of rank n over $\mathscr{G}_X^n$. The discriminant of $\mathscr{Z} \longrightarrow \mathscr{G}_X^n$ is an effective Cartier divisor, and we refer to such a family as *generically étale*. The constructed family is furthermore universal. Specifically that means that for any closed subspace Z in $X \times T$ which is generically étale of rank n over the arbitrary base space T, there exists a unique morphism $f\colon T \longrightarrow \mathscr{G}_X^n$ such that the pull-back $f^*\mathscr{Z} = Z$.

The universal property of $\mathscr{Z} \longrightarrow \mathscr{G}_X^n$ immediately gives that $\mathscr{G}_X^n$ equals the principal component $\overline{\mathscr{U}_X^n}$ of the Hilbert scheme.

In order to describe the contents of the present paper in more detail let us assume that the ambient scheme X is affine. For each sequence $x_1, \dots, x_n$ of global sections of $\mathscr{O}_X$, there is an open subscheme of $\mathscr{U}_X^n$ that parameterizes closed subschemes Z in X that are finite and étale over the base, and such that the pull-back of the sections $x_1, \dots, x_n$ form a module basis of $\mathscr{O}_Z$. By varying the sequences one obtains an open covering of $\mathscr{U}_X^n$.

There is a well-known method to parameterize all closed subschemes in an affine scheme X having a fixed sequence of sections $x = x_1, \dots, x_n$ as a module basis (see e.g. [GLS07] and [Poo08]). That construction, however, does not suffice for our purposes. Instead we provide an explicit construction of a pair of algebras $\mathscr{A}(x) \longrightarrow \mathscr{R}(x)$, and we show that this pair parameterizes closed subschemes Z in X that are étale over the base, and where the sequence of global sections form a basis for the module $\mathscr{O}_Z$. In particular, we give a basis for $\mathscr{R}(x)$ and a formula for the coefficients in this basis for any element of $\mathscr{R}(x)$. As our presentation is elementary, although technical, it is a bit striking that this description was not to be found in the literature.

From the local analysis of $\mathscr{U}_X^n$ we then construct a pair of algebras $\mathscr{A}_+(x) \longrightarrow \mathscr{R}_+(x)$ that parameterizes closed subschemes Z in X that are generically étale and where the sections $x_1, \dots, x_n$ form a basis for the module $\mathscr{O}_Z$. The algebra $\mathscr{A}_+(x)$ will give an open affine chart of the space $\mathscr{G}_X^n$. That the new pair $\mathscr{A}_+(x) \longrightarrow \mathscr{R}_+(x)$ has the desired universal properties follows almost immediately from the usual properties of the Rees algebra, the existence of the Grothendieck–Deligne norm map, and the explicit description of $\mathscr{A}(x) \longrightarrow \mathscr{R}(x)$ discussed above. In particular, we construct the space $\mathscr{G}_X^n$ without reference to the Hilbert scheme.

Due to the naturality of our construction it is not surprising that it is possible to pass from the affine situation to the global situation with algebraic spaces. However, the gluing of the canonical ideals involved requires some work, which we do in detail at the end of the article.

Using the local analysis described above, combined with patching arguments, we obtain, for any separated algebraic space $X \longrightarrow S$, that the principal component $\mathscr{G}^n_X = \overline{\mathscr{U}^n_X}$ is the blow-up of the n-fold symmetric product in a canonical closed subspace.

1. The alternator and symmetric tensors

In this section we will set up some notation and establish the identity given in Proposition (1.7) that we will use in the next section.

1.1. **Notation.** Let R be an A-algebra. For each non-negative integer n we let $\mathrm{T}^n_A R = R \otimes_A \cdots \otimes_A R$ denote the n-fold tensor product of R over A. The A-algebra $\mathrm{T}^n_A R$ can be viewed as an R-algebra in several different ways. For each $p = 1, \dots, n$, we let $\varphi_p \colon R \longrightarrow \mathrm{T}^n_A R$ denote the co-projection on the p^{th} factor.

The symmetric group $\mathfrak{S}_n$ of n letters acts naturally on $\mathrm{T}^n_A R$ by permuting the factors. We let $\mathrm{TS}^n_A R := (\mathrm{T}^n_A R)^{\mathfrak{S}_n}$ denote the A-algebra of invariants.

1.2. **The alternator.** For any n-tuple of elements $x = x_1, \dots, x_n$ in R we have the alternating tensor (called norm vector in [ES04])

$$\alpha(x_1, \dots, x_n) = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{|\sigma|} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \quad \text{in} \quad \mathrm{T}^n_A R.$$

We will often write $\alpha(x)$ instead of $\alpha(x_1, \dots, x_n)$. Let X denote the $(n \times n)$ matrix with coefficients $X_{p,q} := \varphi_p(x_q)$, where $\varphi_p \colon R \longrightarrow \mathrm{T}^n_A R$ are the co-projections. Then we have the determinantal expression of the alternating tensor as

$$\alpha(x) = \det(X) = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{|\sigma|} \varphi_{\sigma(1)}(x_1) \cdots \varphi_{\sigma(n)}(x_n).$$

The *alternator* is the induced map of A-modules

$$\alpha \colon \mathrm{T}^n_A R \longrightarrow \mathrm{T}^n_A R. \tag{1.2.1}$$

Proposition 1.3. *The alternator* (1.2.1) *is* $\mathrm{TS}^n_A R$*-linear.*

Proof. Let x and y be elements of $\mathrm{T}^n_A R$, with y a symmetric tensor. We need to check that $\alpha(xy) = \alpha(x) \cdot y$. As the map α is A-linear and in particular respects sums, we may assume that $x = x_1 \otimes \cdots \otimes x_n$. Write $y = \sum_\gamma y_{\gamma_1} \otimes \cdots \otimes y_{\gamma_n}$. We have

$$\begin{aligned}
\alpha(xy) &= \sum_\gamma \alpha(x_1 y_{\gamma_1}, \dots, x_n y_{\gamma_n}) \\
&= \sum_\gamma \sum_{\sigma \in \mathfrak{S}_n} (-1)^{|\sigma|} x_{\sigma(1)} y_{\gamma_{\sigma(1)}} \otimes \cdots \otimes x_{\sigma(n)} y_{\gamma_{\sigma(n)}} \\
&= \sum_{\sigma \in \mathfrak{S}_n} (-1)^{|\sigma|} \big(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}\big) \cdot \big(\sum_\gamma y_{\gamma_{\sigma(1)}} \otimes \cdots \otimes y_{\gamma_{\sigma(n)}}\big).
\end{aligned}$$

As y is symmetric we have that $y = \sum_\gamma y_{\gamma_{\sigma(1)}} \otimes \cdots \otimes y_{\gamma_{\sigma(n)}}$ for any permutation $\sigma \in \mathfrak{S}_n$. And consequently that the latter expression obtained above, is $\alpha(x) \cdot y$. ☐

Remark 1.4. By the usual properties of the determinant we have that the alternator (1.2.1) factors as a map of A-modules

$$\wedge^n_A R \longrightarrow \mathrm{T}^n_A R.$$

When two is not a zero divisor in R then $\wedge^n_A R$ is in the natural way a $\mathrm{TS}^n_A R$-module, hence we get an induced $\mathrm{TS}^n_A R$-linear map from the exterior product (see also [LT07]). In general, however, the exterior product $\wedge^n_A R$ is not a $\mathrm{TS}^n_A R$-module (see [Lun08]). We thank D. Laksov for drawing our attention towards these differences.

1.5. **Alternators of different degrees.** We will compare the alternator maps of different degrees, and in the sequel we will let α_n denote the alternator map whose source is $\mathrm{T}^n_A R$. We let the group $\mathfrak{S}_{n-1}$ act on $\mathrm{T}^n_A R$ by permuting the first $n-1$ factors, and let

$$\mathrm{TS}^{n-1,1}_A R = (\mathrm{T}^n_A R)^{\mathfrak{S}_{n-1}}$$

denote the invariant ring. It is furthermore convenient to introduce the following notation $\alpha_{n-1,1} = \alpha_{n-1} \otimes_A \mathrm{id}_R$. An immediate relation between the alternators of different degrees is

$$\alpha(x) = \sum_{j=1}^{n} (-1)^{|\tau_{j,n}|} \alpha_{n-1,1}(\tau_{j,n}(x)), \tag{1.5.1}$$

where $\tau_{j,n} \in \mathfrak{S}_n$ is the transposition of the factors j and n.

Lemma 1.6. *The map* $\alpha_{n-1,1} \colon \mathrm{T}^n_A R \longrightarrow \mathrm{T}^n_A R$ *is* $\mathrm{TS}^{n-1,1}_A R$*-linear.*

Proof. Reasoning as in the proof of Proposition (1.3) the result follows. ☐

Proposition 1.7. *Let* $x = x_1, \dots, x_n$ *be an* n*-tuple of elements in an* A*-algebra* R*. For each integer* $i = 1, \dots, n$ *we let*

$$x_{[i]} = x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes \cdots \otimes x_n \otimes 1 \quad \text{in} \quad \mathrm{T}^n_A R$$

denote the n*-tensor we get by removing the* i^{th} *factor and inserting* 1 *on the last factor. We have, for any* $y \in \mathrm{TS}^{n-1,1}_A R$*, the identity*

$$\alpha(x)y = \sum_{i=1}^{n} (-1)^{n-i} \alpha\big(x_{[i]} y\big) \varphi_n(x_i) \quad \text{in} \quad \mathrm{T}^n_A R.$$

Proof. Using the identity (1.5.1) we get that

$$\alpha(x_{[i]} y) = \sum_{j=1}^{n} (-1)^{|\tau_{j,n}|} \alpha_{n-1,1}(\tau_{j,n}(x_{[i]} y)).$$

The element $\varphi_n(x_i) = 1 \otimes \cdots \otimes 1 \otimes x_i$ is $\mathfrak{S}_{n-1}$-invariant. And as $\alpha_{n-1,1}$ is $\mathrm{TS}_A^{n-1,1}$ R-linear we have that the sum appearing in the proposition is the sum of

$$\sum_{i=1}^{n}(-1)^{n-i}\alpha_{n-1,1}\big(x_{[i]}\varphi_n(x_i)y\big) \tag{1.7.1}$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n-1}(-1)^{n-i+1}\alpha_{n-1,1}\big(\tau_{j,n}(x_{[i]}y)\varphi_n(x_i)\big). \tag{1.7.2}$$

In the first sum (1.7.1) we can take out the $\mathfrak{S}_{n-1}$-invariant element y by the $\mathrm{TS}_A^{n-1,1}$ R-linearity of $\alpha_{n-1,1}$. It is readily checked that

$$\sum_{i=1}^{n}(-1)^{n-i}\alpha_{n-1,1}(x_{[i]}\varphi_n(x_i)) = \alpha(x).$$

Thus (1.7.1) equals $\alpha(x)y$ and we need to check that the remaining expression (1.7.2) is zero. The summation in (1.7.2) runs over the set of pairs (i,j) with $i = 1,\dots,n$ and $j = 1,\dots,n-1$, which is the disjoint sum of $\mathscr{C}_> = \{(i,j) \mid i > j\}$ and $\mathscr{C}_\leq = \{(i,j) \mid i \leq j\}$. For fixed $(i,j) \in \mathscr{C}_>$ we have the corresponding summand

$$(-1)^{n-i+1}\alpha_{n-1,1}(\tau_{j,n}(x_{[i]}y)\varphi_n(x_i)) \tag{1.7.3}$$

in (1.7.2). We claim that this summand is canceled by the corresponding summand over $(j,i-1) \in \mathscr{C}_\leq$. Let $\sigma_p^q \in \mathfrak{S}_n$, for any integers $p \leq q \leq n$, denote the cyclic, increasing permutation of the factors $p,\dots,q$. Since $j < i$ we have

$$\tau_{j,n}(x_{[i]}y)\varphi_n(x_i) = \sigma_j^{i-1}\big(\tau_{i-1,n}(x_{[j]})\varphi_n(x_j)\big)\tau_{j,n}(y). \tag{1.7.4}$$

Moreover, $\sigma_j^{i-1} \circ \tau_{i-1,n} = \tau_{j,n} \circ \sigma_j^{i-1}$, and as $\sigma_j^{i-1}(y) = y$ we can write (1.7.4) as $\sigma_j^{i-1}\big(\tau_{i-1,n}(x_{[j]}y)\varphi_n(x_j)\big)$. It then follows that (1.7.3) equals

$$(-1)^{n-i+1+|\sigma_j^{i-1}|}\alpha_{n-1,1}(\tau_{i-1,n}(x_{[j]}y)\varphi_n(x_j)).$$

As $\mid \sigma_j^{i-1} \mid = i-1-j$ we have that the summand (1.7.3) is canceled by the corresponding summand over $(j,i-1) \in \mathscr{C}_\leq$. Consequently (1.7.2) is zero and we have proven the proposition. □

1.8. **Linear span.** Let $z = \sum_{i=1}^n a_i x_i$ in R, with scalars $a_1,\dots,a_n$ in A. Then, as the alternator $\alpha\colon \mathrm{T}_A^n R \longrightarrow \mathrm{T}_A^n R$ is multi-linear and alternating, we have

$$\alpha(x_1,\dots,x_{i-1},z,x_{i+1},\dots,x_n) = a_i \cdot \alpha(x) \quad \text{in} \quad \mathrm{T}_A^n R. \tag{1.8.1}$$

In particular these identities hold for any $z \in R$ if the elements $x_1,\dots,x_n$ form an A-module basis of R. However these identities always hold formally in $\mathrm{T}_A^n R$. Indeed, as a special case of our previous result we have

Corollary 1.9. *Let $x = x_1, \dots, x_n$ be an n-tuple of elements in R. For any $z \in R$ we have the identity*

$$\alpha(x)\varphi_n(z) = \sum_{i=1}^{n} \alpha(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_n)\varphi_n(x_i)$$

in the tensor algebra $\mathrm{T}^n_A R$.

Proof. Specializing the proposition with $y = \varphi_n(z) \in \mathrm{TS}^{n-1,1}_A R$ gives

$$\alpha(x)\varphi_n(z) = \sum_{i=1}^{n} (-1)^{n-i}\alpha(x_{[i]}\varphi_n(z)) \cdot \varphi_n(x_i).$$

The result then follows since

$$(-1)^{n-i}\alpha(x_{[i]}\varphi_n(z)) = \alpha(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_n). \quad \square$$

Remark 1.10. There is a canonical map $\mathrm{TS}^{n-1}_A R \otimes_A R \longrightarrow \mathrm{TS}^{n-1,1}_A R$ but this is not always an isomorphism in positive characteristic. In particular the inclusion $\mathrm{TS}^n_A R \longrightarrow \mathrm{T}^n_A R$ that factors through the inclusion $\mathrm{TS}^{n-1,1}_A R \longrightarrow \mathrm{T}^n_A R$, does not always factorize through $\mathrm{TS}^{n-1}_A R \otimes_A R$. We thank T. Ekedahl for pointing this out and thereby correcting a mistake in an earlier version of this article.

2. Linear solution spaces

Let $x_1, \dots, x_n$ be a fixed n-tuple of elements in R. We will show how to obtain an R-algebra $\mathscr{R}$ and an A-subalgebra $\mathscr{A} \subseteq \mathscr{R}$ such that the elements $x_1, \dots, x_n$ in the R-algebra $\mathscr{R}$ form an $\mathscr{A}$-module basis. The universality of the constructed pair will be established in the section following this one.

2.1. **Localization of the square of an alternating tensor.** Let $x = x_1, \dots, x_n$ be an n-tuple of elements in R, and let $\alpha(x)$ denote the tensor described in 1.2. It is clear that a permutation $\sigma \in \mathfrak{S}_n$ sends the tensor $\alpha(x) \in \mathrm{T}^n_A R$ to $(-1)^{|\sigma|}\alpha(x)$. Consequently, if $y = y_1, \dots, y_n$ is another n-tuple of elements in R then $\alpha(x)\alpha(y)$ is a symmetric tensor, that is $\alpha(x)\alpha(y) \in \mathrm{TS}^n_A R$. In particular $\alpha^2(x)$, the square of the alternating tensor, is symmetric.

From Corollary (1.9) we see that the element $\alpha(x)$ has to be inverted in order to express $\varphi_n(z)$ in the form $\varphi_n(z) = \sum_{i=1}^n a_i\varphi_n(x_i)$ in $\mathrm{T}^n_A R$. We therefore introduce the notation

$$\mathscr{A} = \mathscr{A}(\alpha^2(x)) = \mathrm{TS}^n_A R[\alpha^2(x)^{-1}]$$

for the localization of $\mathrm{TS}^n_A R$ in the symmetric tensor $\alpha^2(x)$.

Lemma 2.2. *The elements $\varphi_n(x_1), \dots, \varphi_n(x_n)$ in the localized ring $\mathrm{T}^n_A R[\alpha(x)^{-1}]$ are linearly independent over $\mathscr{A}$.*

Proof. An element $\sum_{i=1}^n a_i'\varphi_n(x_i)$ in $\mathrm{T}^n_A R[\alpha(x)^{-1}]$ with $a_i' \in \mathscr{A}$ can be written as $\alpha^{-2p}(x)\sum_{i=1}^n a_i\varphi_n(x_i)$, with $a_i \in \mathrm{TS}^n_A R$, for some integer p. Assume therefore that we have a relation

$$a_1\varphi_n(x_1) + \cdots + a_n\varphi_n(x_n) = 0 \tag{2.2.1}$$

in $\mathrm{T}^n_A R$, with symmetric tensors $a_1, \ldots, a_n$ in $\mathrm{TS}^n_A R$. A permutation $\sigma \in \mathfrak{S}_n$ will transform the equation (2.2.1), to the identity

$$a_1\varphi_{\sigma(n)}(x_1) + \cdots + a_n\varphi_{\sigma(n)}(x_n) = 0.$$

In particular we can replace φ_n in (2.2.1) with φ_j, for any $j = 1, \ldots, n$. We then obtain the matrix equation

$$X \cdot A = 0,$$

where X is the $(n \times n)$-matrix with coefficients $X_{p,q} = \varphi_p(x_q)$, and A is the $(n \times 1)$-matrix with coefficients $a_1, \ldots, a_n$. The coefficients of the matrices are in $\mathrm{T}^n_A R$. The determinant of the matrix X is $\alpha(x_1, \ldots, x_n)$, and it follows that the coefficients $a_1, \ldots, a_n$ are all zero in $\mathrm{T}^n_A R[\alpha(x)^{-1}]$. □

2.3. **The linear solution spaces.** We have the inclusion of rings

$$\mathrm{TS}^n_A R \xrightarrow{\ i\ } \mathrm{TS}^{n-1,1}_A R \xrightarrow{\ j\ } \mathrm{T}^n_A R. \tag{2.3.1}$$

We have the n-tuple $x = x_1, \ldots, x_n$ fixed, and we localize the above sequence of rings with respect to the symmetric tensor $\alpha^2(x) \in \mathrm{TS}^n_A R$. We have earlier (2.1) introduced the notation $\mathscr{A} := \mathrm{TS}^n_A R[\alpha^2(x)^{-1}]$, and now we introduce

$$\mathscr{R} = \mathscr{R}(\alpha^2(x)) = \mathrm{TS}^{n-1,1}_A R[\alpha^2(x)^{-1}] \tag{2.3.2}$$

for the localization of $\mathrm{TS}^{n-1,1}_A R$ with respect to the element $\alpha^2(x) \in \mathrm{TS}^n_A R$. The R-algebra structure on $\mathscr{R}$ is the one induced from the co-projection map $\varphi_n \colon R \longrightarrow \mathrm{T}^n_A R$ on the last factor. We will also denote the structure map with $\varphi_n \colon R \longrightarrow \mathscr{R}$. Moreover, as the first map i of (2.3.1) is injective, we have that $\mathscr{A} \subseteq \mathscr{R}$ is a ring extension.

Theorem 2.4. *Let $x = x_1, \ldots, x_n$ be an n-tuple of elements in an A-algebra R, and define the ring extension $\mathscr{A} \subseteq \mathscr{R}$ as in (2.3). Then we have that the elements $\varphi_n(x_1), \ldots, \varphi_n(x_n)$ in the R-algebra $\mathscr{R}$ form an $\mathscr{A}$-module basis.*

Proof. We obtain a map $\mathscr{R} \longrightarrow \mathrm{T}^n_A R[\alpha(x)^{-1}]$ by localizing the canonical map $j \colon \mathrm{TS}^{n-1,1}_A R \longrightarrow \mathrm{T}^n_A R$ of (2.3.1). Linear independence follows from Lemma (2.2). It remains to see that every tensor $y \in \mathrm{TS}^{n-1,1}_A R$ is in the $\mathscr{A}$-linear span of the elements $\varphi_n(x_1), \ldots, \varphi_n(x_n)$. From Proposition (1.7) we obtain the identity

$$\alpha^2(x)y = \sum_{i=1}^n (-1)^{n-i}\alpha(x)\alpha(x_{[i]}y)\varphi_n(x_i).$$

The product of two alternating tensors is a symmetric tensor, hence when inverting $\alpha^2(x)$ then y is in the $\mathscr{A}$-linear span of $\varphi_n(x_1), \dots, \varphi_n(x_n)$. □

3. Universal properties of $\mathscr{A} \longrightarrow \mathscr{R}$

In this section we will show that the extension $\mathscr{A} \subseteq \mathscr{R}$ constructed in the previous section, has a universal property. We will show that the pair represents the functor of pairs of étale extensions $B \longrightarrow E$, where E is an R-algebra and where the images of $x_1, \dots, x_n$ in E form a B-module basis.

3.1. **Trace map.** Recall that the extension $\mathscr{A} \subseteq \mathscr{R}$ we have constructed fits into the commutative diagram of A-algebras and A-algebra homomorphisms

$$\begin{array}{ccccc} R & \longrightarrow & R \otimes_A \mathscr{A} & \longrightarrow & \mathscr{R} \\ \uparrow & & \uparrow & & \\ A & \longrightarrow & \mathscr{A}. & & \end{array}$$

The composition of the two upper horizontal morphisms in the diagram above is $\varphi_n \colon R \longrightarrow \mathscr{R}$. As $\mathscr{R}$ is free of rank n as an $\mathscr{A}$-module, we have the $\mathscr{A}$-linear trace map $\mathscr{R} \longrightarrow \mathscr{A}$.

3.2. **Discriminant.** Let $B \longrightarrow E$ be a homomorphism of rings, where E is free of rank n as a B-module. To a given B-module basis $e_1, \dots, e_n$ of E we define the discriminant $d_E \in B$ as the determinant of the $(n \times n)$-matrix whose coefficient $c_{i,j}$ is the trace of the B-linear map $z \mapsto z \cdot e_i e_j$. It is easily seen that the discriminant d_E depends on the choice of basis of E, however the ideal $D_{E/B} \subseteq B$ generated by the discriminant does not. If $B \longrightarrow E$ is *locally free* it is thus clear that there is a locally principal ideal $D_{E/B} \subseteq B$, locally given by the discriminant.

Lemma 3.3. *For any element $z \in R$ we have that the trace of the $\mathscr{A}$-linear endomorphism $e \mapsto \varphi_n(z)e$ on $\mathscr{R}$ is $\varphi_1(z) + \cdots + \varphi_n(z)$.*

Proof. The elements $\varphi_n(x_1), \dots, \varphi_n(x_n)$ form, by Theorem (2.4), an $\mathscr{A}$-module basis of $\mathscr{R}$. The action of $\varphi_n(z)$ on the fixed basis element $\varphi_n(x_k)$ is $\varphi_n(zx_k)$. Using Corollary (1.9) we have that $\varphi_n(zx_k)$ is expressed as the sum

$$\sum_{i=1}^{n} \frac{\alpha(x_1, \dots, x_{i-1}, zx_k, x_{i+1}, \dots, x_n)\alpha(x)}{\alpha^2(x)} \varphi_n(x_i).$$

In particular we see that $\alpha(x_1,\dots,x_{k-1},zx_k,x_{k+1},\dots,x_n)\alpha(x)^{-1}$ is the k^{th} component of $\varphi_n(zx_k)$, and consequently that the trace of the endomorphism $e\mapsto\varphi_n(z)e$ is

$$\sum_{k=1}^{n}\frac{\alpha(x_1,\dots,x_{k-1},zx_k,x_{k+1},\dots,x_n)}{\alpha(x)}. \tag{3.3.1}$$

Now using the fact that the map α is $\mathrm{TS}^n_A R$-linear (1.3) we get that

$$\alpha(x)\sum_{k=1}^{n}\varphi_k(z)=\sum_{k=1}^{n}\alpha(x_1,\dots,x_{k-1},zx_k,x_{k+1},\dots,x_n).$$

Applying that identity to the sum (3.3.1) proves the lemma. □

3.4. **Polarized power sums.** For any element $z\in R$ the symmetric tensor

$$\mathfrak{p}(z):=\varphi_1(z)+\cdots+\varphi_n(z)\in\mathrm{TS}^n_A R,$$

is often referred to as polarized power sum [Wey39]. We will need the following observation.

Lemma 3.5. *Let $x=x_1,\dots,x_n$ and $y=y_1,\dots,y_n$ be two n-tuples of elements in R. Then we have that $\alpha(x)\alpha(y)=\det(\mathfrak{p}(x_iy_j))$.*

Proof. Let X be the $(n\times n)$-matrix with coefficients $X_{p,q}=\varphi_p(x_q)$, where $\varphi_p\colon R\longrightarrow\mathrm{T}^n_A R$ is the p^{th} co-projection. Then $\alpha(x)=\det(X)$. If Y is the matrix with coefficients $\varphi_p(y_q)$ we get that

$$\alpha(x)\alpha(y)=\det(X^T\cdot Y),$$

where X^T is the transpose of X. The coefficients of the product matrix X^TY are $\mathfrak{p}(x_py_q)$, proving the lemma. □

Proposition 3.6. *The discriminant of the extension $\mathscr{A}\longrightarrow\mathscr{R}$ is $\alpha^2(x)$. In particular we have that $\mathscr{A}\longrightarrow\mathscr{R}$ is étale.*

Proof. We have by Lemma (3.5) that $\alpha^2(x)$ is the determinant of the $(n\times n)$-matrix whose coefficient (i,j) equals $\mathfrak{p}(x_ix_j)$. By definition we have that $\mathfrak{p}(x_ix_j)=\varphi_1(x_ix_j)+\cdots+\varphi_n(x_ix_j)$, which by Lemma (3.3) equals the trace of the endomorphism $e\mapsto\varphi_n(x_ix_j)\cdot e$. As $\varphi_n(x_1),\dots,\varphi_n(x_n)$ is an $\mathscr{A}$-module basis of $\mathscr{R}$ (Theorem 2.4), we have that $\alpha^2(x)$ is the discriminant of the extension $\mathscr{A}\longrightarrow\mathscr{R}$. It then follows [$\mathrm{EGA}_{\mathrm{IV}}$, 18.2] that the extension is étale. □

3.7. **Algebra of polarized powers.** We let $P\subseteq\mathrm{TS}^n_A R$ be the A-subalgebra generated by the polarized power sums;

$$P=A[\mathfrak{p}(z)]_{z\in R}.$$

By Lemma (3.5) we have that $\alpha^2(x)\in P$.

Proposition 3.8. *The inclusion of A-algebras $P\subseteq\mathrm{TS}^n_A R$ becomes an isomorphism after localization with respect to $\alpha^2(x)$.*

Proof. Let $x = x_1 \otimes \cdots \otimes x_n$. If $y \in \mathrm{TS}^n_A R$ is a symmetric tensor we have by Proposition (1.3)

$$\alpha^2(x) \cdot y = \alpha(x) \cdot \alpha(xy),$$

and therefore $y = \alpha(x)\alpha(xy)/\alpha^2(x)$ in $\mathrm{TS}^n_A R[\alpha^2(x)^{-1}] = \mathscr{A}$. The symmetric tensor y is a sum of tensors $\sum_\gamma y_\gamma$, where each summand y_γ is of the form $y_{\gamma_1} \otimes \cdots \otimes y_{\gamma_n}$. Consequently $\alpha(xy)$ is the sum $\sum_\gamma \alpha(xy_\gamma)$. From Lemma (3.5) it follows that $\alpha(x)\alpha(xy)$ is a sum of products of polarized powers. In other words, the symmetric tensor $y \in P[\alpha^2(x)^{-1}]$. □

Remark 3.9. When the base ring A contains the field of rationals then we have that the A-algebra of symmetric tensors $\mathrm{TS}^n_A R$ is generated by its polarized power sums (see [Wey39]). It is moreover known that the polarized power sums do not always generate $\mathrm{TS}^n_A R$ in positive characteristic. We will later see (6.13) that the support of $\alpha^2(x)$ in $\mathrm{Spec}(\mathrm{TS}^n_A R)$, when running through all n-tuples x, is precisely the diagonals. What our proposition above therefore says is that the polarized power sums always generate the ring of invariants — as long as we stay away from the diagonals.

Proposition 3.10. *Assume that the elements $x_1, \dots, x_n$ in R form an A-module basis. Then the discriminant $d_R \in A$ is mapped to $\alpha^2(x)$ by the structure map $A \longrightarrow \mathscr{A}$. The induced map $A_{d_R} \longrightarrow \mathscr{A}$ is an isomorphism, and the A-algebra homomorphism $\varphi_n \colon R \otimes_A A_{d_R} \longrightarrow \mathscr{R}$ is an isomorphism.*

Proof. We note that the canonical A-algebra morphism $\mathscr{A} \otimes_A R \longrightarrow \mathscr{R}$ is an isomorphism. Indeed, bijectivity follows as both $\mathscr{A}$-modules have the same basis: The elements $1 \otimes x_1, \dots, 1 \otimes x_n$ is a basis of $\mathscr{A} \otimes_A R$ by assumption, and these elements are mapped to $\varphi_n(x_1), \dots, \varphi_n(x_n)$ which form a basis of $\mathscr{R}$ by Theorem (2.4). In particular we have, for any $z \in R$, the following identity of traces:

$$\mathrm{Tr}_A(e \mapsto ze) \otimes 1 = \mathrm{Tr}_{\mathscr{A}}(e \mapsto \varphi_n(z)e). \tag{3.10.1}$$

By Lemma (3.3) we then have that $\mathfrak{p}(z) = \mathrm{Tr}_A(e \mapsto ze) \otimes 1$, and by Proposition (3.6) we have that the discriminant $d_R \in A$ of the extension $A \longrightarrow R$ is mapped to $\alpha^2(x) \in \mathscr{A}$ by the structure map $A \longrightarrow \mathscr{A}$. It now follows from (3.10.1) that the induced map $A_{d_R} \longrightarrow \mathscr{A}$ is surjective. Indeed, by Proposition (3.8) the A-algebra $\mathscr{A}$ is generated by the polarized powers $\mathfrak{p}(z)$ and the inverse of $\alpha^2(x)$.

It remains to the see injectivity. As our morphism $A_{d_R} \longrightarrow \mathscr{A}$ is the localization of the map $A \longrightarrow \mathrm{TS}^n_A R$, it suffices to see that the map $A \longrightarrow \mathrm{T}^n_A R$ is injective. Since the algebra R is a free A-module, it follows that $\mathrm{T}^n_A R$ is free, hence $A \longrightarrow \mathrm{T}^n_A R$ is injective. □

3.11. **A commutative diagram.** Let $B \longrightarrow E$ be a homomorphism of A-algebras. If $f\colon R \longrightarrow E$ is an A-algebra homomorphism, we obtain a natural A-algebra homomorphism of tensor products $\mathrm{T}^n_A R \longrightarrow \mathrm{T}^n_B E$, taking $z_1 \otimes \cdots \otimes z_n$ to $f(z_1) \otimes \cdots \otimes f(z_n)$. Consequently we get induced A-algebra homomorphisms $f_n\colon \mathrm{TS}^n_A R \longrightarrow \mathrm{TS}^n_B E$ and $f_{n-1,1}\colon \mathrm{TS}^{n-1,1}_A R \longrightarrow \mathrm{TS}^{n-1,1}_B E$ such that the following diagram of A-algebras

$$\begin{array}{ccc} \mathrm{TS}^n_A R & \xrightarrow{f_n} & \mathrm{TS}^n_B E \\ \downarrow & & \downarrow \\ \mathrm{TS}^{n-1,1}_A R & \xrightarrow{f_{n-1,1}} & \mathrm{TS}^{n-1,1}_B E \end{array} \tag{3.11.1}$$

commutes.

Theorem 3.12. *Let $B \longrightarrow E$ be an étale extension of A-algebras, and let $f\colon R \longrightarrow E$ be an A-algebra homomorphism. Assume that the elements $f(x_1), \dots, f(x_n)$ form a B-module basis of E. Then there is a unique A-algebra homomorphism*

$$\mathfrak{n}_{E/B}\colon \mathscr{A} \longrightarrow B,$$

such that $\mathscr{R} \otimes_{\mathscr{A}} B = E$ as quotients of $R \otimes_A B$. The homomorphism $\mathfrak{n}_{E/B}$ is determined by sending the polarized power sum $\mathfrak{p}(z)$ to $\mathrm{Tr}_B(e \mapsto f(z)e)$, the trace of the multiplication map by $f(z)$, for every $z \in R$.

Proof. We have the symmetric tensor $\alpha^2(f(x_1), \dots, f(x_n)) = \alpha^2(f(x))$ in $\mathrm{TS}^n_B E$. Let $\mathscr{A}_E$ and $\mathscr{R}_E$ denote the localization with respect to $\alpha^2(f(x))$ in $\mathrm{TS}^n_B E$ and of $\mathrm{TS}^{n-1,1}_B E$, respectively. The natural morphism $f_n\colon \mathrm{TS}^n_A R \longrightarrow \mathrm{TS}^n_B E$ takes $\alpha^2(x)$ to $\alpha^2(f(x))$. Then by localizing the commutative diagram (3.11.1) with respect to $\alpha^2(x)$ we obtain an $\mathscr{A}$-algebra homomorphism

$$\mathscr{R} \otimes_{\mathscr{A}} \mathscr{A}_E \longrightarrow \mathscr{R}_E. \tag{3.12.1}$$

The morphism (3.12.1) takes the basis $\varphi_n(x_1) \otimes 1, \dots, \varphi_n(x_n) \otimes 1$ of $\mathscr{R} \otimes_{\mathscr{A}} \mathscr{A}_E$ to the the elements $\varphi_n(f(x_1)), \dots, \varphi_n(f(x_n))$. By Theorem (2.4) we have that $\varphi_n(f(x_1)), \dots, \varphi_n(f(x_n))$ form an $\mathscr{A}_E$-module basis of $\mathscr{R}_E$. Consequently the $\mathscr{A}$-algebra homomorphism (3.12.1) is a surjective map of free rank n $\mathscr{A}_E$-modules, hence an isomorphism.

By Proposition (3.10) we have a natural identification $B = \mathscr{A}_E$, and $E = \mathscr{R}_E$, and consequently we have obtained an A-algebra homomorphism $\mathfrak{n}_{E/B}\colon \mathscr{A} \longrightarrow B$ such that $\mathscr{R} \otimes_{\mathscr{A}} B = E$.

It remains to see that the morphism $\mathfrak{n}_{E/B}$ is unique, and that it maps each polarized power sum $\mathfrak{p}(z)$ to the trace of the multiplication map of $f(z)$ on E. Note that since $\mathscr{R} \otimes_{\mathscr{A}} B = E$ we have the identity of traces $\mathrm{Tr}_{\mathscr{A}}(e \mapsto \varphi_n(z)e) \otimes 1 = \mathrm{Tr}_B(e \mapsto f(z)e)$, for any element $z \in R$. Combining that with Lemma (3.3) yield the identities

$$\mathfrak{n}_{E/B}(\mathfrak{p}(z)) = \mathrm{Tr}_{\mathscr{A}}(e \mapsto \varphi_n(z)e) \otimes 1 = \mathrm{Tr}_B(e \mapsto f(z)e).$$

From this we conclude that the morphism $\mathfrak{n}_{E/B}\colon \mathscr{A} \longrightarrow B$ has the announced action on $\mathfrak{p}(z)$, and since $\mathfrak{p}(z)$ generates $\mathscr{A}$ (Proposition 3.8) the morphism is unique. □

3.13. **The universal étale family, local description.** For each n-tuple of elements $x = x_1, \dots, x_n$ in R we have the open immersion of affine schemes $\operatorname{Spec}(\mathscr{A}(\alpha^2(x)) \subseteq \operatorname{Spec}(\operatorname{TS}^n_A R)$. Let $\mathbf{x}$ denote the set of all n-tuples of elements in R, and consider the union

$$\mathscr{U}_R = \bigcup_{x \in \mathbf{x}} \operatorname{Spec}(\mathscr{A}(\alpha^2(x)) \subseteq \operatorname{Spec}(\operatorname{TS}^n_A R).$$

By construction we have a map $\mathscr{Z}_R = \cup_{x\in\mathbf{x}} \operatorname{Spec}(\mathscr{R}(\alpha^2(x)) \longrightarrow \mathscr{U}_R$.

Corollary 3.14. *The family $\mathscr{Z}_R \longrightarrow \mathscr{U}_R$ is étale of rank n, and the natural map $\mathscr{Z}_R \longrightarrow \operatorname{Spec}(R) \times_{\operatorname{Spec}(A)} \mathscr{U}_R$ is a closed immersion. Furthermore, let B be an A-algebra, and let E be an algebra quotient of $R \otimes_A B$ which is étale of rank n as a B-algebra. Then there exists a unique morphism of schemes $\mathfrak{n}\colon \operatorname{Spec}(B) \longrightarrow \mathscr{U}_R$ such that the pull-back $\mathfrak{n}^*\mathscr{Z}_R = \operatorname{Spec}(E)$ as closed subschemes of $\operatorname{Spec}(R \otimes_A B)$.*

Proof. It is clear from the discussion in this section that the family $\mathscr{Z}_R \longrightarrow \mathscr{U}_R$ is a closed subscheme of $\operatorname{Spec}(R) \times_{\operatorname{Spec}(A)} \mathscr{U}_R$ and étale of rank n over $\mathscr{U}_R$. We will show its universal properties, and we begin with uniqueness.

Assume that there are two morphisms $\mathfrak{n}_1, \mathfrak{n}_2\colon \operatorname{Spec}(B) \longrightarrow \mathscr{U}_R$ such that the two pull-backs $\mathfrak{n}_1^*\mathscr{Z}_R = \mathfrak{n}_2^*\mathscr{Z}_R$ coincide with $\operatorname{Spec}(E)$. For any n-tuple x of elements in R, write $\mathscr{U}_{R,x} = \operatorname{Spec}\big(\mathscr{A}(\alpha^2(x))\big)$. We define the open subsets $T_{x,y} = \mathfrak{n}_1^{-1}(\mathscr{U}_{R,x}) \cap \mathfrak{n}_2^{-1}(\mathscr{U}_{R,y})$, for any two n-tuples $x, y \in \mathbf{x}$ of elements in R. Then $\{T_{x,y}\}_{x,y\in\mathbf{x}}$ is an open cover of $\operatorname{Spec}(B)$. Moreover $T_{x,y}$ is affine as $\mathscr{U}_{R,x}$ is affine. We let $T_{x,y} = \operatorname{Spec}(B_{x,y})$. The images in $E_{x,y} = E \otimes_B B_{x,y}$ of either the elements $x = x_1, \dots, x_n$ or the elements $y = y_1, \dots, y_n$ form a $B_{x,y}$-module basis.

Let $\mathfrak{n}_x\colon \mathscr{A}(\alpha^2(x)) \longrightarrow B_{x,y}$ and $\mathfrak{n}_y\colon \mathscr{A}(\alpha^2(y)) \longrightarrow B_{x,y}$ be the homomorphisms corresponding to the morphisms of schemes $\mathfrak{n}_1|_{T_{x,y}}$ and $\mathfrak{n}_2|_{T_{x,y}}$. By Theorem (3.12) $\mathfrak{n}_x$ and $\mathfrak{n}_y$ send polarized power sums onto traces. Thus, by Lemma (3.5), both $\mathfrak{n}_x$ and $\mathfrak{n}_y$ send $\alpha^2(x)$ and $\alpha^2(y)$ onto a generator of the discriminant ideal $D_{E_{x,y}} \subseteq B_{x,y}$. By the étaleness assumption of E we thus have that both $\mathfrak{n}_x$ and $\mathfrak{n}_y$ factor through $\operatorname{TS}^n_A R[\alpha^2(x)^{-1}\alpha^2(y)^{-1}]$. By the uniqueness of $\mathfrak{n}_x$ and $\mathfrak{n}_y$ it follows that these factorizations are equal. Thus $\mathfrak{n}_1 = \mathfrak{n}_2$.

Now, let B and E be as in the corollary. Then there is an affine open covering $\{\operatorname{Spec}(B_\gamma)\}_\gamma$ of $\operatorname{Spec}(B)$ such that for each index γ there are n elements $x_\gamma = x_{\gamma_1}, \dots, x_{\gamma_n}$ in R whose images in $E_\gamma = E \otimes_B B_\gamma$ form a B_γ-module basis. By Theorem (3.12) we get, for each γ, a morphism

$$\mathfrak{n}_\gamma\colon \operatorname{Spec}(B_\gamma) \longrightarrow \mathscr{U}_{R,x_\gamma} \subseteq \mathscr{U}_R$$

such that the family $\mathscr{Z}_R$ is pulled back to $\operatorname{Spec}(E_\gamma)$. The restrictions of $\mathfrak{n}_\gamma$ and $\mathfrak{n}_{\gamma'}$ to the intersection $\operatorname{Spec}(B_\gamma) \cap \operatorname{Spec}(B_{\gamma'})$ coincide by the uniqueness shown above. Thus the maps $\{\mathfrak{n}_\gamma\}$ glue to a morphism $\mathfrak{n}\colon \operatorname{Spec}(B) \longrightarrow \mathscr{U}_R$ with the required property. □

4. Generically étale families

By some minor modifications of the results in the previous section we will obtain a representing pair $\mathscr{A}_+ \longrightarrow \mathscr{R}_+$ for the functor of generically étale families. We use the notation introduced in the preceding sections, and in particular we have the n-tuple $x = x_1, \dots, x_n$ of elements in R fixed.

4.1. **A canonical ideal.** We have (Proposition 1.3) that the alternator map $\alpha\colon \mathrm{T}^n_A R \longrightarrow \mathrm{T}^n_A R$ is $\mathrm{TS}^n_A R$-linear, and since $\mathrm{T}^n_A R$ has a product structure we obtain a $\mathrm{TS}^n_A R$-module map

$$\alpha \times \alpha\colon\ \mathrm{T}^n_A R \otimes_{\mathrm{TS}^n_A R} \mathrm{T}^n_A R \longrightarrow \mathrm{T}^n_A R,$$

taking two elements y and z of $\mathrm{T}^n_A R$ to $\alpha(y)\alpha(z)$. From (2.1) we have that $\alpha(y)\alpha(z)$ is invariant under the $\mathfrak{S}_n$-action. Consequently the $\mathrm{TS}^n_A R$-module given as the image of $\alpha \times \alpha$ is an ideal of $\mathrm{TS}^n_A R$, and this ideal we will denote by I_R. We will refer to it as the *canonical ideal.*

Remark 4.2. There is a natural homomorphism of rings from the divided powers ring $\Gamma^n_A R$ to the invariant ring $\mathrm{TS}^n_A R$. Under this map the canonical ideal I_R is the image of the norm ideal $I \subseteq \Gamma^n A_R$ defined in ([ES04]).

4.3. **The subscheme defined by the canonical ideal.** It is clear from the definition of the canonical ideal $I_R \subseteq \mathrm{TS}^n_A R$ that it is generated by elements of the form $\alpha(y)\alpha(z)$, with n-tuples $y = y_1, \dots, y_n$ and $z = z_1, \dots, z_n$ of elements in R. Furthermore, if we let $D(s) \subseteq \operatorname{Spec}(\mathrm{TS}^n_A R)$ denote the open subset where $s \in \mathrm{TS}^n_A R$ does not vanish, then we have

$$D(\alpha(y)\alpha(z)) = D(\alpha^2(y)) \cap D(\alpha^2(z)) \quad \text{in} \quad \operatorname{Spec}(\mathrm{TS}^n_A R).$$

Let $\Delta \subseteq \operatorname{Spec}(\mathrm{TS}^n_A R)$ denote the closed subscheme defined by I_R. We then have, using the notation of the preceding section, that

$$\mathscr{U}_R = \bigcup_{x \in \mathbf{x}} \operatorname{Spec}(\mathscr{A}(\alpha^2(x))) = \operatorname{Spec}(\mathrm{TS}^n_A R) \setminus \Delta, \tag{4.3.1}$$

where $\mathbf{x}$ is the set of all n-tuples of elements in R. Let

$$\psi\colon\ \operatorname{Spec}(\mathrm{TS}^{n-1,1}_A R) \longrightarrow \operatorname{Spec}(\mathrm{TS}^n_A R)$$

denote the morphism corresponding to the canonical ring homomorphism $i\colon \mathrm{TS}^n_A R \longrightarrow \mathrm{TS}^{n-1,1}_A R$ (2.3.1). We then have, with the notation of Corollary (3.14), that the family $\mathscr{Z}_R = \psi^{-1}(\mathscr{U}_R)$.

4.4. **The Rees algebras.** The Rees algebra $\oplus_{m\geq 0} I_R^m$ of the canonical ideal $I_R \subseteq \mathrm{TS}_A^n R$ is a graded ring, and we let

$$\mathscr{A}_+ = \mathscr{A}_+(\alpha^2(x)) = \big(\oplus_{m\geq 0} I_R^m \big)_{(\alpha^2(x))}$$

denote the degree zero part of the localization at the degree one element $\alpha^2(x) \in I_R$ of the Rees algebra $\oplus_{m\geq 0} I_R^m$. The natural inclusion of rings $\mathrm{TS}_A^n R \longrightarrow \mathrm{TS}_A^{n-1,1} R$ (2.3.1) induces a graded ring homomorphism between their Rees algebras, and then an induced ring homomorphism

$$\mathscr{A}_+ \longrightarrow \mathscr{R}_+ := \big(\oplus_{m\geq 0} I_R^m \,\mathrm{TS}_A^{n-1,1} R \big)_{(\alpha^2(x))}. \tag{4.4.1}$$

The A-algebra homomorphism $\varphi_n \colon R \longrightarrow \mathrm{TS}_A^{n-1,1} R$ gives an R-algebra structure on $\mathscr{R}_+$.

Proposition 4.5. *The $\mathscr{A}_+$-algebra $\mathscr{R}_+$ is free as an $\mathscr{A}_+$-module, and the elements $\varphi_n(x_1), \dots, \varphi_n(x_n)$ form a basis. In particular we have that the A-algebra homomorphism $R \otimes_A \mathscr{A}_+ \longrightarrow \mathscr{R}_+$ is surjective. Furthermore, the discriminant of the extension $\mathscr{A}_+ \longrightarrow \mathscr{R}_+$ is a non-zero divisor in $\mathscr{A}_+$.*

Proof. Linear independence of $\varphi_n(x_1), \dots, \varphi_n(x_n)$ follows from Theorem (2.4) as the localization of $\mathscr{A}_+$ in the element $\alpha^2(x) \in \mathrm{TS}_A^n R$ is injective, and we have $(\mathscr{A}_+)_{\alpha^2(x)} = \mathscr{A}$. From Proposition (1.7) elements of the form $y \in \mathrm{TS}_A^{n-1,1} R$ are in the linear span of $\varphi_n(x_1), \dots, \varphi_n(x_n)$, and consequently they form a basis.

It remains to see that the discriminant of the extension $\mathscr{A}_+ \longrightarrow \mathscr{R}_+$ is a non-zero divisor. As the localization map $\mathscr{A}_+ \longrightarrow \mathscr{A}$ is injective, and $\mathscr{R}_+ \otimes_{\mathscr{A}_+} \mathscr{A} = \mathscr{R}$, we need only to see that the discriminant of $\mathscr{A} \longrightarrow \mathscr{R}$ is a non-zero divisor in $\mathscr{A}_+$. By Proposition (3.6) the discriminant is $\alpha^2(x)$, which by definition is a unit in $\mathscr{A}$. □

4.6. **The Grothendieck–Deligne norm map.** Let $B \longrightarrow E$ be an extension of rings, where E is free of rank n as a B-module. The norm induced by the determinant map $\det \colon E \longrightarrow B$ corresponds to a B-algebra homomorphism $\sigma_E \colon \mathrm{TS}_B^n E \longrightarrow B$ ([Rob80], [Del73, 6.3, p. 180], and [Ive70]). The algebra homomorphism σ_E takes $y \otimes \cdots \otimes y$ to $\det(e \mapsto ye)$.

Lemma 4.7. *Let $B \longrightarrow E$ be an extension of A-algebras, and let $f \colon R \longrightarrow E$ be an A-algebra homomorphism. Assume that the elements $f(x) = f(x_1), \dots, f(x_n)$ form a B-module basis of E, and let $d_E \in B$ denote the discriminant of the the extension.*

(i) *The extension of the canonical ideal I_R by the induced map $\mathrm{TS}_A^n R \longrightarrow \mathrm{TS}_B^n E$ coincides with the canonical ideal I_E and the ideal generated by $\alpha^2(f(x_1), \dots, f(x_n))$.*
(ii) *The element $\alpha^2(f(x))$ in $\mathrm{TS}_B^n E$ is mapped to the discriminant d_E by the homomorphism $\sigma_E \colon \mathrm{TS}_B^n E \longrightarrow B$.*

(iii) *We have the following commutative diagram of A-algebras and A-algebra homomorphisms*

$$\begin{array}{ccccc} \mathrm{TS}^n_A R & \xrightarrow{\mathrm{can}} & \mathrm{TS}^n_B E & \xrightarrow{\sigma_E} & B \\ \downarrow{\scriptstyle \mathrm{can}} & & & & \downarrow{\scriptstyle \mathrm{can}} \\ \mathscr{A} & & \xrightarrow{\mathfrak{n}_{E/B}} & & B_{d_E}, \end{array} \tag{4.7.1}$$

where $\mathfrak{n}_{E/B}$ is the morphism of Theorem (3.12).

Proof. As $f(x_1), \dots, f(x_n)$ is a B-module basis of E it follows from (1.8) that any alternating tensor $\alpha(z_1, \dots, z_n)$ is in the B-module spanned by $\alpha(f(x))$. In particular we see that $I_R \mathrm{TS}^n_B E = I_E = (\alpha^2(f(x)))$, and the first statement of the lemma follows.

Let z be an element of the free B-module E. We have (cf. [Ive70] p. 9, Section 2.5) that the homomorphism σ_E sends the polarized power sum $\mathfrak{p}(z)$ to the trace of the multiplication map $e \mapsto ze$. Thus, the map σ_E and $\mathfrak{n}_{E/B}$ have the same action on polarized powers. Furthermore, as $\alpha^2(f(x)) = \det(\mathfrak{p}(f(x_i x_j)))$ by Lemma (3.5) and we have that $f(x_1), \dots, f(x_n)$ form a B-module basis of E it follows that $\alpha^2(f(x))$ is mapped to the discriminant d_E of $B \longrightarrow E$. We have then proved the second statement of the lemma. We have also that the localization of the composite map $\sigma_E \circ \mathrm{can}\colon \mathrm{TS}^n_A R \longrightarrow B$ with respect to the element $\alpha^2(x)$ is the morphism $\mathfrak{n}_{E/B}$, proving the third statement. □

4.8. **Induced map of Rees algebras.** Let $B \longrightarrow E$ be an extension as in Lemma (4.7). From the A-algebra homomorphism $\mathrm{TS}^n_A R \longrightarrow B$ appearing as the top horizontal row in (4.7.1), we obtain a graded homomorphism of Rees algebras

$$\oplus_{m \geq 0} I_R^m \longrightarrow \oplus_{m \geq 0} I_R^m B = \oplus_{m \geq 0} (d_E)^m. \tag{4.8.1}$$

We have furthermore by Lemma (4.7) that $\alpha^2(x)$ in $\mathrm{TS}^n_A R$ is mapped to the discriminant d_E by the map (4.8.1). If we let B_+ denote the degree zero part of the localization of $\oplus_{m \geq 0} I_R^m B$ with respect to the degree one element d_E, we obtain from (4.8.1) an induced A-algebra homomorphism

$$\mathfrak{n}^+_{E/B}\colon \mathscr{A}_+ \longrightarrow B_+. \tag{4.8.2}$$

Let $E_+ := E \otimes_B B_+$.

We will in the sequel use the notation $\mathrm{Ker}(d_E) \subseteq B$ for the kernel of the localization map $B \longrightarrow B_{d_E}$. With this notation we have

$$B_+ = B / \mathrm{Ker}(d_E), \tag{4.8.3}$$

as $I_R B = (d_E)$.

Lemma 4.9. *Let $A \longrightarrow R$ be a homomorphism of rings, and assume that $x_1,\dots,x_n$ form an A-module basis for R. Then the induced A-algebra homomorphism $\mathfrak{n}^+_{R/A}\colon \mathscr{A}_+ \longrightarrow A_+$ (4.8.2) is an isomorphism, and the induced map $\mathscr{R}_+ \longrightarrow R_+$ is an isomorphism.*

Proof. We have that $A_+ = A/\operatorname{Ker}(d_R)$, where d_R is the discriminant of $A \longrightarrow R$. Since $\mathfrak{n}^+_{R/A}\colon \mathscr{A}_+ \longrightarrow A_+$ is an A-algebra homomorphism it is necessarily surjective. Injectivity we prove in the following way. From Lemma (4.7) we have the following commutative diagram of A-algebras

(4.9.1)
$$\begin{array}{ccc} \mathscr{A}_+ & \xrightarrow{\mathfrak{n}^+_{R/A}} & A_+ \\ \downarrow & & \downarrow \\ \mathscr{A} & \xrightarrow{\mathfrak{n}_{R/A}} & A_{d_R}. \end{array}$$

The vertical arrows are injective being the localizations in their respective discriminants, and the bottom horizontal map $\mathfrak{n}_{R/A}$ is an isomorphism by Proposition (3.10). Injectivity of $\mathfrak{n}^+_{R/A}$ now follows from the commutative diagram (4.9.1).

As $\mathscr{R}_+$ is both an $\mathscr{A}_+$-algebra and an R-algebra we have an induced A-algebra homomorphism

(4.9.2)
$$R \otimes_A \mathscr{A}_+ \longrightarrow \mathscr{R}_+.$$

The map (4.9.2) sends the $\mathscr{A}_+$-module basis $x_1 \otimes 1,\dots,x_n \otimes 1$ to $\varphi_n(x_1),\dots,\varphi_n(x_n)$. By Proposition (4.5) it follows that the map (4.9.2) is an isomorphism of $\mathscr{A}_+$-modules, hence an isomorphism of algebras. As R_+ is by definition $R\otimes_A A_+$, and we have $A_+ = \mathscr{A}_+$ we obtain from (4.9.2) the isomorphism $\mathscr{R}_+ \longrightarrow R_+$. □

Theorem 4.10. *Let $A \longrightarrow R$ be a homomorphism of algebras, and let $x_1,\dots,x_n$ be elements of R. Let $B \longrightarrow E$ be an extension of A-algebras, and assume that $f\colon R \longrightarrow E$ is an A-algebra homomorphism such that the elements $f(x_1),\dots,f(x_n)$ form a B-module basis of E. Then the A-algebra homomorphism $\mathfrak{n}^+_{E/B}\colon \mathscr{A}_+ \longrightarrow B_+$ (4.8.2) is the unique A-algebra homomorphism such that*

$$\mathscr{R}_+ \otimes_{\mathscr{A}_+} B_+ = E_+$$

as quotients of $R\otimes_A B_+$.

Proof. From the canonical map $\operatorname{TS}^n_A R \longrightarrow \operatorname{TS}^n_B E$ we obtain an induced A-algebra homomorphism of graded rings

$$\oplus_{m\geq 0} I_R^m \longrightarrow \oplus_{m\geq 0} I_R^m \operatorname{TS}^n_B E.$$

By Lemma (4.7) (i) we have that the extension $I_R \operatorname{TS}^n_B E$ is the canonical ideal I_E. We let $(\mathscr{A}_E)_+$ denote the degree zero part of the localization of the graded ring $\oplus_{m\geq 0} I_R^m \operatorname{TS}^n_B E = \oplus_{m\geq 0} I_E^m$ at the degree one element $\alpha^2(f(x))$. And similarly we let $(\mathscr{R}_E)_+$ denote the localization of the graded ring we obtain from the natural map $\mathrm{TS}^n_B E \longrightarrow \mathrm{TS}^{n-1,1}_B E$. From the commutative diagram of A-algebras in (3.11.1) we obtain the following commutative diagram

(4.10.1)
$$\begin{array}{ccc} \mathscr{A}_+ & \longrightarrow & (\mathscr{A}_E)_+ \\ \downarrow & & \downarrow \\ \mathscr{R}_+ & \longrightarrow & (\mathscr{R}_E)_+. \end{array}$$

Since $(\mathscr{A}_E)_+$ is the localization of the Rees algebra of the canonical ideal $I_E \subseteq \mathrm{TS}^n_B E$ we have by Proposition (4.5) that the elements $\varphi_n(f(x_1)), \dots, \varphi_n(f(x_n))$ form an $(\mathscr{A}_E)_+$-module basis for the algebra $(\mathscr{R}_E)_+$. The commutative diagram (4.10.1) induces a canonical $\mathscr{A}_+$-algebra homomorphism

(4.10.2)
$$\mathscr{R}_+ \otimes_{\mathscr{A}_+} (\mathscr{A}_E)_+ \longrightarrow (\mathscr{R}_E)_+.$$

The $\mathscr{A}_+$-algebra homomorphism (4.10.2) sends $\varphi_n(z) \otimes 1$ to $\varphi_n(f(z))$. Hence the homomorphism (4.10.2) identifies the $(\mathscr{A}_E)_+$-module basis $\varphi_n(x_1) \otimes 1, \dots, \varphi_n(x_n) \otimes 1$ with the basis $\varphi_n(f(x_1)), \dots, \varphi_n(f(x_n))$, and consequently (4.10.2) is an isomorphism of $\mathscr{A}_+$-algebras. Furthermore, by Lemma (4.9) we have a natural identification $(\mathscr{A}_E)_+ = B_+$ and $(\mathscr{R}_E)_+ = E_+$. Hence we have shown the existence of a morphism $\mathscr{A}_+ \longrightarrow B_+$ with the desired property.

For uniqueness we note that a morphism $\eta\colon \mathscr{A}_+ \longrightarrow B_+$ such that $\mathscr{R}_+ \otimes_{\mathscr{A}_+} B_+ = E_+$ would have to map the discriminant d^+ of $\mathscr{A}_+ \longrightarrow \mathscr{R}_+$ to the discriminant d_E of $B \longrightarrow E$. When we localize $\mathscr{A}_+$ in the discriminant d^+ we obtain $\mathscr{A}$, and consequently a commutative diagram of A-algebras

(4.10.3)
$$\begin{array}{ccc} \mathscr{A}_+ & \xrightarrow{\eta} & B_+ \\ \downarrow \scriptstyle{\mathrm{can}} & & \downarrow \scriptstyle{\mathrm{can}} \\ \mathscr{A} & \longrightarrow & B_{d_E}. \end{array}$$

As $B_{d_E} \longrightarrow E_{d_E}$ is étale it follows from Theorem (3.12) that the bottom horizontal row of the above (4.10.3) diagram is $\mathfrak{n}_{E_d/B_d}$. Consequently the morphisms η and $\mathfrak{n}^+_{E/B}$ coincide after localization with respect to the discriminant d^+. As the vertical arrows in the diagram above are injective, both discriminants being non-zero divisors, it follows that $\eta = \mathfrak{n}^+_{E/B}$. □

4.11. **Generically étale families.** Let $B \longrightarrow E$ be a finite and locally free homomorphism of rings. If the local generators of the discriminant ideal $D_{E/B}$ are non-zero divisors then we say that the extension $B \longrightarrow E$ is *generically étale*. Equivalently, the family is generically étale if the closed subscheme of $\mathrm{Spec}(B)$, corresponding to the ideal $D_{E/B}$, is an effective Cartier divisor. Another, perhaps more geometric, characterization is the following. The finite and locally free extension $B \longrightarrow E$ is generically étale if and only if the open subset $U \subseteq \operatorname{Spec}(B)$ of points where the fibers are étale is schematically dense in $\operatorname{Spec}(B)$.

4.12. **The universal generically étale family, local description.** In analogy with the end of Section 3 we give here a local treatment for the universal properties of the space of generically étale families. Let $\mathbf{x}$ denote the set of all n-tuples of elements $x_1, \dots, x_n$ in R. We let

$$\mathscr{U}_R^+ = \bigcup_{x \in \mathbf{x}} \operatorname{Spec}(\mathscr{A}_+(\alpha^2(x))) \subseteq \operatorname{Proj}(\oplus_{m \geq 0} I_R^m).$$

Using identities similar to those in (4.3) one sees that the open subset $\mathscr{U}_R^+$ equals the scheme $\operatorname{Proj}(\oplus_{m\geq 0} I_R^m)$. We note furthermore, that the family $\mathscr{Z}_R^+ = \cup_{x\in\mathbf{x}} \operatorname{Spec}(\mathscr{R}_+(\alpha^2(x))) \longrightarrow \mathscr{U}_R^+$ is generically étale, and is a closed subscheme of $\operatorname{Spec}(R) \times_{\operatorname{Spec}(A)} \mathscr{U}_R^+$.

Corollary 4.13. *Let B be an A-algebra, and let E be an algebra quotient of $R \otimes_A B$. Assume that the homomorphism of rings $B \longrightarrow E$ is a generically étale extension of rank n. Then there exist a unique morphism of schemes*

$$\mathfrak{n}^+ \colon \operatorname{Spec}(B) \longrightarrow \mathscr{U}_R^+,$$

such that the pull-back of $\mathscr{Z}_R^+$ along $\mathfrak{n}^+$ is $\operatorname{Spec}(E)$.

Proof. The existence and uniqueness of the homomorphism $\mathfrak{n}^+$ follows by Theorem (4.10) and arguments similar to those in the proof of Corollary (3.14). □

5. The space of étale families

In this section we show how to construct a space parameterizing étale families in a fixed separated algebraic space $X \longrightarrow S$ ([LMB00, 6.6]). We thereby obtain an abstract and global version of Corollary (3.14), but not the explicit statement about the basis as in Theorem (3.12).

5.1. **Disjoint sections.** Let $f \colon X \longrightarrow S$ be a morphism of algebraic spaces. A *section* of f is a morphism $s \colon S \longrightarrow X$ such that $f \circ s = \mathrm{id}_S$. Two sections $s, s' \colon S \longrightarrow X$ are called *disjoint* if $(s, s') \colon S \longrightarrow X \times_S X$ does not intersect the diagonal.

Remark 5.2. For a separated morphism $X \longrightarrow S$ we have that a section $s \colon S \longrightarrow X$ is equivalent with a closed subspace $Z \subseteq X$, with Z isomorphic to S. Note that two disjoint sections $s, s' \colon S \longrightarrow X$ determines a closed subspace Z isomorphic to the disjoint union of two copies of S.

For any S-space T we let $U_X^n(T)$ denote the set of *unordered* n-tuples of sections $s_1, \dots, s_n$ of the second projection $X \times_S T \longrightarrow T$, where the sections are pairwise disjoint. Clearly U_X^n describes a functor; the functor of n *unordered disjoint sections* of $X \longrightarrow S$.

5.3. **Diagonals and their complement.** Let $X \longrightarrow S$ be separated, and let $X^n_S := X \times_S \cdots \times_S X$ denote the n-fold product. We let V_X denote the open complement of the diagonals in X^n_S.

5.4. **Action of the symmetric group.** The permutation action $\mathfrak{S}_n$ on X^n_S induces an action $\rho\colon V_X \times \mathfrak{S}_n \longrightarrow V_X$. It is clear that the map $(\pi_1, \rho)\colon V_X \times \mathfrak{S}_n \longrightarrow V_X \times_S V_X$ is a monomorphism of algebraic spaces, where π_1 denotes the projection on the first factor. We say that the action ρ is *free*, and we note that (π_1, ρ) describes an étale equivalence relation on V_X.

Lemma 5.5. *Let $X \longrightarrow S$ be a separated algebraic space. Then U^n_X is the presheaf quotient of the equivalence relation $V_X \times \mathfrak{S}_n \underset{\rho}{\overset{\pi_1}{\rightrightarrows}} V_X$.*

Proof. Clearly X^n_S parameterizes n ordered sections as we have the canonical identification

$$\operatorname{Hom}_S(T, X^n_S) = \prod_{i=1}^{n} \operatorname{Hom}_T(T, X \times_S T),$$

for any S-space T. Similarly we get that V_X parameterizes n ordered disjoint sections of $X \longrightarrow S$. The action of $\mathfrak{S}_n$ on V_X corresponds to the action on $\operatorname{Hom}_S(T, X^n_S)$ given by permuting the sections, from where it follows that the presheaf quotient is U^n_X. □

5.6. **The space of étale families.** We let $\mathscr{U}^n_X$ denote the sheafification of the presheaf U^n_X in the étale topology. As a consequence of the above lemma we have that the sheaf $\mathscr{U}^n_X$ is the quotient sheaf of the equivalence relation $V_X \times \mathfrak{S}_n \underset{\rho}{\overset{\pi_1}{\rightrightarrows}} V_X$.

Proposition 5.7. *Let $X \longrightarrow S$ be a separated algebraic space. The sheaf $\mathscr{U}^n_X$ is an algebraic space and represents the functor of closed subspaces of X that are étale and of rank n over the base.*

Proof. As the algebraic space $V_X \longrightarrow S$ is separated, the map

$$(\pi_1, \rho)\colon V_X \times \mathfrak{S}_n \longrightarrow V_X \times_S V_X$$

is a closed immersion and hence quasi-affine. We then have that the quotient sheaf $\mathscr{U}^n_X$ of the étale equivalence relation

$$V_X \times \mathfrak{S}_n \underset{\rho}{\overset{\pi_1}{\rightrightarrows}} V_X$$

is an algebraic space [Knu71, II, Prop. 3.14].

What remains to show is that sections of $\mathscr{U}^n_X$ corresponds to closed subspaces that are étale of rank n over the base. By Lemma (5.5) a section over an S-space T of the presheaf U^n_X is n unordered disjoint sections of $X \times_S T \longrightarrow T$. Equivalently, as $X \longrightarrow S$ is separated, a section over T is a closed subspace $Z \subseteq X \times_S T$ such that Z is isomorphic to a disjoint union of n copies of T. Consequently, a section over T of the sheaf $\mathscr{U}_X^n$ is given by an étale covering $T' \longrightarrow T$ and a closed subspace $Z' \subseteq X \times_S T'$, isomorphic to a disjoint union of n copies of T'. The pull-back of the family $Z' \longrightarrow T'$ along the two different projection maps $T' \times_T T' \longrightarrow T'$ coincide. Therefore we have that the section over T of $\mathscr{U}_X^n$ is given by a unique closed subspace $Z \subseteq X \times_S T$ such that $Z \longrightarrow T$ is étale and finite of rank n.

Further, as every finite étale morphism $Z \longrightarrow T$ trivializes after an étale base change, the set of sections over T is the set of all closed subspaces $Z \subseteq X \times_S T$ that are finite, étale and of rank n. □

Remark 5.8. Let $\mathfrak{S}_{n-1}$ act by permuting the first $n-1$ factors of X_S^n, and consider the induced action on V_X. The induced map $V_X/\mathfrak{S}_{n-1} \longrightarrow \mathscr{U}_X^n$ will be étale of rank n, and one can check that this is in fact the universal family of étale, rank n closed subspaces of X. In Proposition (6.12) we will prove this fact using a different approach.

Remark 5.9. When $X = \operatorname{Spec}(R)$ is affine over base $S = \operatorname{Spec}(A)$ then we have that $\mathscr{U}_X^n$ is the open complement of the diagonals in $\operatorname{Spec}(\operatorname{TS}_A^n R)$. One can furthermore check that the support of the closed subscheme defined by the canonical ideal $I_R \subseteq \operatorname{TS}_A^n R$ is the diagonals. Hence Corollary (3.14) can be obtained as a consequence of Proposition (5.7). However, the more explicit description given by Theorem (3.12) is what will be important for us.

6. The space of generically étale families

6.1. **Finite group quotients.** When a finite group G acts on a separated algebraic space X, the geometric quotient X/G exists as an algebraic space; this is an unpublished result of Deligne [Knu71, p. 183]. When the base is locally Noetherian and $X \longrightarrow S$ is locally of finite type, proofs of this existence result are given in [KM97] and in [Kol97]. It is furthermore possible to extend the proof given by Kollár in [Kol97] to the general setting with any separated algebraic space $X \longrightarrow S$, for details we refer to [Ryd07].

We will be interested in the particular case with the symmetric group $\mathfrak{S}_n$ of n letters acting by permuting the factors of the n-fold product $X_S^n = X \times_S \cdots \times_S X$. For a separated algebraic space $X \longrightarrow S$ we will denote the quotient space with $\operatorname{Sym}_S^n X$.

6.2. **Symmetric spaces.** A well-known fact is that geometric quotients commute with flat base change [MFK94, p. 9] or, for finite groups [SGA$_1$, Exp. V, Prop. 1.9]. Thus if $T \longrightarrow S$ is flat, and X_T denotes

$X \times_S T$, then we have the cartesian diagram

(6.2.1)
$$\begin{array}{ccc} \mathrm{Sym}^n_T(X_T) & \longrightarrow & \mathrm{Sym}^n_S X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S. \end{array}$$

This fact will be useful when we want to reduce to the case with an affine base scheme S.

6.3. **Fixed-point reflecting morphisms.** We recall some properties of geometric quotients (see [Knu71], [Ryd07]). Let G be a finite group acting on an algebraic space Y. The stabilizer group G_y of a point $y \in Y$ is defined as the inverse image of the point (y, y) by the map $G \times_S Y \longrightarrow Y \times_S Y$. A G-equivariant morphism $f \colon Y \longrightarrow X$ is *fixed-point reflecting* in a point $y \in Y$ if the stabilizer group G_y equals the stabilizer group $G_{f(y)}$. The fixed-point reflecting set, with respect to a given étale separated map $Y \longrightarrow X$, we denote by $Y|_{\mathrm{fpr}}$. The set $Y|_{\mathrm{fpr}} \subseteq Y$ is an open G-invariant subset, and we denote the geometric quotient $Y|_{\mathrm{fpr}}/G$ by $Y/G|_{\mathrm{fpr}}$. We have furthermore a cartesian diagram

(6.3.1)
$$\begin{array}{ccc} Y|_{\mathrm{fpr}} & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y/G|_{\mathrm{fpr}} & \longrightarrow & X/G, \end{array}$$

where the horizontal maps are étale.

Proposition 6.4. *Let $X \longrightarrow S$ be a separated algebraic space, and let $Y \longrightarrow X$ be étale, with Y affine. Let $T \longrightarrow S$ be an S-space, and let $Z \subseteq Y \times_S T$ be a closed subspace, which is finite and locally free of rank n over T. Let $\sigma_Z \colon T \longrightarrow \mathrm{Sym}^n_S Y$ denote the morphism corresponding to the determinant map (4.6). Then the morphism $\sigma_Z \colon T \longrightarrow \mathrm{Sym}^n_S Y$ factors through the fixed-point reflecting set $\mathrm{Sym}^n_S Y|_{\mathrm{fpr}}$ if and only if the composite morphism $Z \subseteq Y \times_S T \longrightarrow X \times_S T$ is a closed immersion.*

Proof. Since the morphism $Z \longrightarrow T$ is finite we have that the morphism $i \colon Z \longrightarrow X \times_S T$ is a closed immersion if and only if the induced morphism over points is a closed immersion. Thus we can assume $T = \mathrm{Spec}(K)$, where K is an algebraically closed field. The support of the family Z is then a finite set of points $z_1, \dots, z_p$ in $Y \times_S \mathrm{Spec}(K)$. Let m_i denote the length of the local ring at z_i, $i = 1, \dots, p$. The determinant map $T \longrightarrow \mathrm{Sym}^n_T Z$ takes the family $Z \longrightarrow T$ to the cycle $m_1 \cdot z_1 + \cdots + m_p \cdot z_p$ [Ive70, Prop. 4.7]. The determinant map composed with the induced map $\mathrm{Sym}^n_T Z \longrightarrow \mathrm{Sym}^n_S Y$ is the morphism σ_Z. Let $z \in Y^n_S$ be the n-tuple of points where the m_1 first coordinates are z_1, the next m_2 coordinates are z_2 etc. The set of points in Y^n_S that we obtain by shuffling the order of the n-tuple z is the inverse image $q_n^{-1}(\sigma_Z(T))$ by the quotient map $q_n \colon Y^n_S \longrightarrow \mathrm{Sym}^n_S Y$.

Since $Z \subseteq Y \times_S T$ is a closed immersion and $Y \longrightarrow X$ an étale map, the composition $i\colon Z \longrightarrow X \times_S T$ is a closed immersion if and only if it is injective on points ([EGA$_{\mathrm{IV}}$, Prop. 17.2.6, Cor. 18.12.6]). The map i being injective is equivalent with the set $q_n^{-1}(\sigma_Z(T)) \subset Y^n \times_S T$ being a subset of the fixed-point reflecting set $(Y^n_S)|_{\mathrm{fpr}}$ with respect to the map $Y^n_S \longrightarrow X^n_S$. $\square$

6.5. **Covers of symmetric quotients with affine base.** Let the base space S be affine, and let $X \longrightarrow S$ be a separated algebraic space. We choose an étale covering $\coprod X_\gamma \longrightarrow X$, with X_γ affine for each γ, such that the induced map

$$\coprod_\gamma (X_\gamma)^n_S \longrightarrow X^n_S \tag{6.5.1}$$

is surjective. In other words, a covering such that any n-tuple of points in X lies in the image of some $X_\gamma \longrightarrow X$. Then $\coprod_\gamma \mathrm{Sym}^n_S(X_\gamma)|_{\mathrm{fpr}} \longrightarrow \mathrm{Sym}^n_S X$ is an étale cover. For two indices γ and γ' we define

$$(X_\gamma \times_X X_{\gamma'})^n|_{\mathrm{fpr}} := (X_\gamma)^n|_{\mathrm{fpr}} \times_{X^n_S} (X_{\gamma'})^n|_{\mathrm{fpr}},$$

where we have suppressed the base S in the notation, and where we have used the notation introduced in (6.3). We have the following commutative diagram
(6.5.2)

$$\begin{array}{ccccc}
\coprod_{\gamma,\gamma'}(X_\gamma \times_X X_{\gamma'})^n|_{\mathrm{fpr}} & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & \coprod_\gamma (X_\gamma)^n|_{\mathrm{fpr}} & \xrightarrow{p} & X^n_S \\
\downarrow q & & \downarrow q & & \downarrow q \\
\coprod_{\gamma,\gamma'} \mathrm{Sym}^n_S(X_\gamma \times_X X_{\gamma'})|_{\mathrm{fpr}} & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & \coprod_\gamma \mathrm{Sym}^n_S(X_\gamma)|_{\mathrm{fpr}} & \xrightarrow{\pi} & \mathrm{Sym}^n_S X,
\end{array}$$

where the q's are the quotient maps, p_1, p_2 and p are the natural maps, and where π_1, π_2 and π are the induced ones. By (6.3) we have that the three squares: $qp = \pi q$ and $qp_i = \pi_i q$ for $i = 1, 2$; are cartesian. Furthermore we have that π_1 and π_2 form an étale equivalence relation with quotient π.

Lemma 6.6. *Let $A \longrightarrow A'$ be a homomorphism of rings, R be an A-algebra and $R' = R \otimes_A A'$. Then the extension of the canonical ideal I_R (4.1) by the homomorphism $\mathrm{TS}^n_A(R) \longrightarrow \mathrm{TS}^n_{A'}(R')$ equals the canonical ideal $I_{R'}$.*

Proof. Recall the alternator map $\alpha_R\colon \mathrm{T}^n_A R \longrightarrow \mathrm{T}^n_A R$ (1.2.1) and that the canonical ideal I_R is generated by $\alpha(x)\alpha(y)$ for $x, y \in \mathrm{T}^n_A R$. As $\alpha_{R'} = \alpha_R \otimes 1\colon \mathrm{T}^n_{A'} R' \longrightarrow \mathrm{T}^n_{A'} R'$ is A'-linear the lemma follows. $\square$

Lemma 6.7. *Let $X' \longrightarrow X$ be an étale morphism of S-schemes where $S = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(R')$ and $X = \mathrm{Spec}(R)$ are affine schemes. Let $\mathscr{I}$ and $\mathscr{I}'$ be the ideal sheaves corresponding to the canonical ideals* *of* $\mathrm{TS}^n_A(R)$ *and* $\mathrm{TS}^n_A(R')$ *respectively. Then the pull-back of* $\mathscr{I}$ *by the morphism*

$$\mathrm{Sym}^n_S(X')|_{\mathrm{fpr}} \longrightarrow \mathrm{Sym}^n_S(X)$$

equals the restriction of $\mathscr{I}'$.

Proof. The canonical ideal I_R of $\mathrm{TS}^n_A R$ is the image of the $\mathrm{TS}^n_A R$-linear map

$$\alpha \times \alpha \colon \mathrm{T}^n_A R \otimes_{\mathrm{TS}^n_A R} \mathrm{T}^n_A R \longrightarrow \mathrm{TS}^n_A R. \tag{6.7.1}$$

Let $f_\gamma \in \mathrm{TS}^n_A(R')$ be symmetric tensors such that the principal open sets $D(f_\gamma)$ cover $\mathrm{Sym}^n_S(X')|_{\mathrm{fpr}}$. From (6.3.1) we have that the diagram

$$\begin{array}{ccc} (\mathrm{T}^n_A R')_{f_\gamma} & \longleftarrow & \mathrm{T}^n_A R \\ \uparrow & & \uparrow \\ (\mathrm{TS}^n_A R')_{f_\gamma} & \longleftarrow & \mathrm{TS}^n_A R \end{array}$$

is co-cartesian for every γ. Therefore, by applying the change of basis $\mathrm{TS}^n_A R \longrightarrow (\mathrm{TS}^n_A R')_{f_\gamma}$ to the morphism (6.7.1) we obtain

$$\alpha \times \alpha \colon (\mathrm{T}^n_A R')_{f_\gamma} \otimes_{(\mathrm{TS}^n_A R')_{f_\gamma}} (\mathrm{T}^n_A R')_{f_\gamma} \longrightarrow (\mathrm{TS}^n_A R')_{f_\gamma}.$$

Thus the extension of the canonical ideal I_R in $(\mathrm{TS}^n_A R')_{f_\gamma}$ equals the localization of the canonical ideal $I_{R'}$ in f_γ. □

Proposition 6.8. *Let* $X \longrightarrow S$ *be a separated algebraic space. The canonical ideals* $I_R \subseteq \mathrm{TS}^n_A R$ *as defined in Section (4.1) for* A*-algebras* R*, glue to an ideal sheaf* $\mathscr{I}_X$ *on* $\mathrm{Sym}^n_S(X)$. *If* $Y \longrightarrow X$ *is an étale morphism then the pull-back of* $\mathscr{I}_X$ *along* $\mathrm{Sym}^n_S(Y)|_{\mathrm{fpr}} \longrightarrow \mathrm{Sym}^n_S(X)$ *is the restriction of the canonical ideal sheaf* $\mathscr{I}_Y$ *on* $\mathrm{Sym}^n_S(Y)$.

Proof. Let $\coprod_\beta S_\beta \longrightarrow S$ be an étale cover with S_β affine and let $X_\beta = X \times_S S_\beta$. For any β choose an étale cover $\coprod_\gamma X_{\beta,\gamma} \longrightarrow X_\beta$ as in (6.5.1). Then the canonical ideal sheaves on $\mathrm{Sym}_{S_\beta}(X_{\beta,\gamma})$ glue to an ideal sheaf $\mathscr{I}_{X_\beta}$ on $\mathrm{Sym}_{S_\beta}(X_\beta)$ using Lemma (6.7) and the étale equivalence relation of (6.5.2). It is further clear that $\mathscr{I}_{X_\beta}$ is independent on the choice of étale cover of X_β.

Using that symmetric products commute with flat base change (6.2.1) and Lemma (6.6) it follows that the sheaves $\mathscr{I}_{X_\beta}$ glue to an ideal sheaf $\mathscr{I}_X$ on $\mathrm{Sym}^n_S(X)$ which is independent on the choice of covering of S. The last statement is obvious from the construction. □

6.9. **The addition of points map.** Let $q_n \colon X^n_S \longrightarrow \mathrm{Sym}^n_S X$ be the quotient map. We let $\mathfrak{S}_{n-1}$ act on the first $n-1$ copies of X^n_S and denote the geometric quotient $X^n_S/\mathfrak{S}_{n-1}$ with $\mathrm{Sym}^{n-1,1}_S X$. Since the geometric quotient equals the categorical quotient in the category of separated algebraic spaces ([Kol97, Cor. 2.15], [Ryd07]), and as q_n is $\mathfrak{S}_{n-1}$-invariant it factors through $X^n_S \longrightarrow \mathrm{Sym}^{n-1,1}_S X$. We have a canonical morphism

$$\psi_X \colon \mathrm{Sym}^{n-1,1}_S X \longrightarrow \mathrm{Sym}^n_S X. \tag{6.9.1}$$

Lemma 6.10. *Let $Y \longrightarrow X$ be an étale map. Then the pull-back of the family ψ_X along the morphism $\mathrm{Sym}^n_S Y|_{\mathrm{fpr}} \longrightarrow \mathrm{Sym}^n_S X$ is canonically identified with the restriction of the family ψ_Y to the open subset $\mathrm{Sym}^n_S Y|_{\mathrm{fpr}} \subseteq \mathrm{Sym}^n_S Y$.*

Proof. Let $U = (Y^n_S)|_{\mathrm{fpr}}$ denote the $\mathfrak{S}_n$-invariant fixed-point reflecting set of $Y^n_S \longrightarrow X^n_S$. We then have that the induced map $U/\mathfrak{S}_{n-1} \longrightarrow U/\mathfrak{S}_n$ is the restriction of $\psi_Y \colon Y^n/\mathfrak{S}_{n-1} \longrightarrow Y^n/\mathfrak{S}_n$ to the open subset $U/\mathfrak{S}_n = \mathrm{Sym}^n_S(Y)|_{\mathrm{fpr}} \subseteq \mathrm{Sym}^n_S Y$. We need to see that the pull-back of the family $\psi_X \colon X^n/\mathfrak{S}_{n-1} \longrightarrow X^n/\mathfrak{S}_n$ along the map $U/\mathfrak{S}_n \longrightarrow X^n/\mathfrak{S}_n$ equals $U/\mathfrak{S}_{n-1} \longrightarrow U/\mathfrak{S}_n$. First note that U is the base change of X^n by the étale morphism $U/\mathfrak{S}_n \longrightarrow X^n/\mathfrak{S}_n$ as the diagram (6.3.1) is cartesian. As taking the quotient with $\mathfrak{S}_{n-1}$ commutes with flat base change (6.2) the result follows. □

Lemma 6.11. *Let $Z \longrightarrow S$ be a finite map of algebraic spaces, and let $W \longrightarrow Z$ be an étale cover. Then there exists an étale cover $S' \longrightarrow S$ such that $W \times_S S' \longrightarrow Z \times_S S'$ has a section. In particular, suppose that $Z \subseteq X$ is a closed immersion of algebraic spaces, with Z finite over the base. Then, for a separated and étale cover $Y \longrightarrow X$ there exists an étale cover $S' \longrightarrow S$ such that the closed immersion $Z \times_S S' \longrightarrow X \times_S S'$ lifts to a closed immersion $Z \times_S S' \longrightarrow Y \times_S S'$.*

Proof. Let $x \in S$ be a point, and let A^h_x denote the strictly local ring at x. Let E_x be the coordinate ring of the affine scheme $Z \times_S \mathrm{Spec}(A^h_x)$. Then E_x is a product of local Henselian rings with separably closed residue fields. For every closed point z_i of $\mathrm{Spec}(E_x)$ choose a point w_i of $W \times_S \mathrm{Spec}(A^h_x)$ above z_i. As $W \longrightarrow Z$ is étale there is a section $Z \times_S \mathrm{Spec}(A^h_x)) \longrightarrow W \times_S \mathrm{Spec}(A^h_x)$ mapping z_i to w_i. By a standard limit argument, this section extends to a section $Z \times_S U \longrightarrow W \times_S U$ where U is an étale neighborhood around the point $x \in S$, proving the first claim. □

Proposition 6.12. *Let $X \longrightarrow S$ be a separated algebraic space. Let $\Delta \subseteq \mathrm{Sym}^n_S X$ denote the closed subspace defined by the canonical ideal sheaf $\mathscr{I}_X$. We have a canonical identification $\mathscr{U}^n_X = \mathrm{Sym}^n_S X \setminus \Delta$, where $\mathscr{U}^n_X$ is the algebraic space of (5.6). Moreover, the family $\psi_X^{-1}(\mathscr{U}^n_X) \longrightarrow \mathscr{U}^n_X$ is the universal family of closed subspaces in $X \longrightarrow S$, that are étale of rank n over the base.*

Proof. Denote the open subspace $U = \mathrm{Sym}^n_S X \setminus \Delta$, and let $Z = \psi_X^{-1}(U)$. By Proposition (5.7) it suffices to show that the pair (U, Z) represents the functor parameterizing closed subspaces of X that are étale of rank n over the base.

We can by (6.2) assume that the base $S = \mathrm{Spec}(A)$ is affine, and we let $\coprod_\gamma X_\gamma \longrightarrow X$ be an étale covering as given in the construction (6.5.1). Let U_γ denote the inverse image of U along $\mathrm{Sym}^n_S(X_\gamma)|_{\mathrm{fpr}} \longrightarrow \mathrm{Sym}^n_S X$. Let Z_γ be the pull-back of the family $Z \longrightarrow U$ to U_γ and define

$U_{\gamma,\gamma'}$ and $Z_{\gamma,\gamma'}$ in a similar way. We then have a cartesian diagram of étale equivalence relations

$$\begin{array}{ccccc} \coprod_{\gamma,\gamma'} Z_{\gamma,\gamma'} & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & \coprod_\gamma Z_\gamma & \xrightarrow{p} & Z \\ \downarrow & & \downarrow & & \downarrow \\ \coprod_{\gamma,\gamma'} U_{\gamma,\gamma'} & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & \coprod_\gamma U_\gamma & \xrightarrow{\pi} & U. \end{array} \tag{6.12.1}$$

By Proposition (6.8) we have that $U_\gamma = \mathrm{Sym}^n_S(X_\gamma)|_{\mathrm{fpr}} \cap D(\mathscr{I}_{X_\gamma})$, where $D(\mathscr{I}_{X_\gamma})$ is the open subset of $\mathrm{Sym}^n_S(X_\gamma)$ defined by the non-vanishing of the ideal sheaf $\mathscr{I}_{X_\gamma}$. By Lemma (6.10) the family $Z_\gamma \longrightarrow U_\gamma$ is simply the restriction of the family ψ_{X_γ} to U_γ. Now it follows from Corollary (3.14) and Proposition (6.4) that (U_γ, Z_γ) parameterizes closed subspaces $W \subseteq X_\gamma$ that are étale of rank n over the base, and which are also closed subspaces of X. In particular, we have that $Z \longrightarrow U$ is étale of rank n. The universal properties of (U, Z) then follows from Lemma (6.11) and the above diagram. □

Remark 6.13. It follows from the proposition that the support $|\Delta|$ of the space defined by the canonical sheaf of ideals $\mathscr{I}_X$, equals the diagonals. This can also be verified directly.

6.14. **The space of generically étale families.** We let $\mathscr{G}^n_X$ denote the blow-up of $\mathrm{Sym}^n_S X$ along the closed subspace $\Delta \subseteq \mathrm{Sym}^n_S X$ defined by canonical ideal $\mathscr{I}_X$. We furthermore let $\mathscr{Z}$ denote the blow-up of $\mathrm{Sym}^{n-1,1}_S(X)$ along $\psi_X^{-1}\Delta$, where ψ_X is the canonical morphism (6.9.1). The morphism ψ_X then induces a morphism

$$\mathscr{Z}_X \longrightarrow \mathscr{G}^n_X. \tag{6.14.1}$$

Remark 6.15. The property of being generically étale is not stable under base change, as one easily realizes by taking the fiber of a point in the discriminant locus of the family. However the property of being generically étale is stable under flat base change, and in particular under étale base change.

Theorem 6.16. *Let $X \longrightarrow S$ be a separated algebraic space, and let $\mathscr{Z}_X \longrightarrow \mathscr{G}^n_X$ be as in* (6.14.1)*. Then the family $\mathscr{Z}_X \longrightarrow \mathscr{G}^n_X$ is generically étale of rank n, and has the following universal property. For any S-space T, and any closed subspace $Z \subseteq X \times_S T$ such that the projection $Z \longrightarrow T$ is generically étale of rank n, there exists a unique morphism $f\colon T \longrightarrow \mathscr{G}^n_X$ such that the pull-back $f^*\mathscr{Z}_X = Z$, as subspaces of $X \times_S T$.*

Proof. Proceeding as in the proof of Proposition (6.12), replacing $Z \longrightarrow U$ with $\mathscr{Z}_X \longrightarrow \mathscr{G}^n_X$, we obtain a cartesian diagram similar to (6.12.1). In this diagram, the vertical arrows are the blow-ups of the canonical morphisms ψ_{X_γ} and $\psi_{X_{\gamma,\gamma'}}$, in the corresponding canonical ideals, restricted to the fixed-point reflecting loci. This is because blowing up commutes with flat base change. Arguing as in the proof of Proposition (6.12), it then follows that $\mathscr{Z}_X \longrightarrow \mathscr{G}^n_X$ is generically étale and has the ascribed universal property by (4.12), Corollary (4.13), Proposition (6.4) and Lemma (6.11). □

6.17. **Schematic closure.** Let $f\colon Y \longrightarrow X$ be a *quasi-compact* immersion of algebraic spaces. Then $f_*\mathscr{O}_Y$ is quasi-coherent, and in particular the kernel of $\mathscr{O}_X \longrightarrow f_*\mathscr{O}_Y$ is a quasi-coherent sheaf of ideals in $\mathscr{O}_X$ [Knu71, II, Prop. 4.6]. The closed subspace in X determined by this ideal sheaf is the *schematic closure* of $f\colon Y \longrightarrow X$.

Corollary 6.18. *Let $\mathscr{H}^n_X$ denote the Hilbert functor of flat, finite rank n families of closed subspaces in $X \longrightarrow S$. Then $\mathscr{U}^n_X$ is an open subspace of $\mathscr{H}^n_X$, and $\mathscr{G}^n_X$ equals the schematic closure of $\mathscr{U}^n_X$ in $\mathscr{H}^n_X$.*

Proof. As $X \longrightarrow S$ is separated the Hilbert functor is representable by an algebraic space (see e.g. [Ryd10], [ES04]). It is clear that $\mathscr{U}^n_X$ is open in $\mathscr{H}^n_X$ being the complement of the discriminant of the universal family. As the discriminant is a locally principal subspace we have that the open immersion $\mathscr{U}^n_X \subseteq \mathscr{H}^n_X$ is quasi-compact. Let $G \subseteq \mathscr{H}^n_X$ denote the schematic closure of $\mathscr{U}^n_X$. It is clear that the restriction of the universal family $\xi \longrightarrow \mathscr{H}^n_X$ to G satisfies the universal property for generically étale families. Consequently, by the theorem we have that $\mathscr{G}^n_X = G$. □

Remark 6.19. Let R be an A-algebra. There is a canonical A-algebra homomorphism $\tau\colon \Gamma^n_A(R) \longrightarrow \mathrm{TS}^n_A R$ from the divided powers algebra to the ring of symmetric tensors. The map τ is in general neither surjective nor injective [Lun08].

The situation can be globalized (see [Ryd08]) for a separated algebraic space $X \longrightarrow S$, giving a map $t\colon \mathrm{Sym}^n_S X \longrightarrow \Gamma^n_{X/S}$. In ([ES04]) they defined a closed subspace $\Delta' \subseteq \Gamma^n_{X/S}$ whose blow-up yields the good component of the Hilbert functor $\mathscr{H}^n_X$. The closed subspace $\Delta \subseteq \mathrm{Sym}^n_S X$ that we consider in this article is the inverse image $t^{-1}(\Delta')$. Even though the map $t\colon \mathrm{Sym}^n_S X \longrightarrow \Gamma^n_{X/S}$ is not an isomorphism we have that the two corresponding blow-ups of Δ' and $t^{-1}(\Delta')$ are isomorphic; indeed both blow-ups are identified with the schematic closure of $\mathscr{U}^n_X$ inside the Hilbert space $\mathscr{H}^n_X$.

Remark 6.20. The universal family $\mathscr{Z}_X$ we obtained by blowing up the closed subspace $\psi_X^{-1}(\Delta) \subseteq \mathrm{Sym}^{n-1,1}_S X$, or equivalently by taking the strict transform of ψ_X along $\mathscr{G}^n_X \longrightarrow \mathrm{Sym}^n_S X$. There is however a natural way to make a finite family flat. Let B denote the blow up of $\mathrm{Sym}^n_S X$ along the n^{th} Fitting ideal of the family $\psi_X\colon \mathrm{Sym}^{n-1,1}_S X \longrightarrow \mathrm{Sym}^n_S X$. Then the strict transform $E \longrightarrow \mathrm{B}$ of the family ψ_X is flat ([RG71, 5.4]). The Fitting ideal is in general different from the canonical ideal.

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