THE HILBERT STACK

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ABSTRACT. Let $\pi\colon X \to S$ be a morphism of algebraic stacks that is locally of finite presentation with affine stabilizers. We prove that there is an algebraic S-stack—the Hilbert stack—parameterizing proper algebraic stacks mapping quasi-finitely to X. This was previously unknown, even for a morphism of schemes.

INTRODUCTION

Let $\pi\colon X \to S$ be a morphism of algebraic stacks. Define the *Hilbert stack*, $\underline{\mathrm{HS}}_{X/S}$, to be the S-stack that sends an S-scheme T to the groupoid of quasi-finite and representable morphisms $(Z \xrightarrow{s} X \times_S T)$, such that the composition $Z \xrightarrow{s} X \times_S T \xrightarrow{\pi_T} T$ is proper, flat, and of finite presentation.

Let $\underline{\mathrm{HS}}^{\mathrm{mono}}_{X/S} \subseteq \underline{\mathrm{HS}}_{X/S}$ be the S-substack whose objects are those $(Z \xrightarrow{s} X \times_S T)$ such that s is a monomorphism. The main results of this paper are as follows.

Theorem 1. *Let $\pi\colon X \to S$ be a non-separated morphism of noetherian algebraic stacks. Then $\underline{\mathrm{HS}}^{\mathrm{mono}}_{X/S}$ is never an algebraic stack.*

Theorem 2. *Let $\pi\colon X \to S$ be a morphism of algebraic stacks that is locally of finite presentation, with quasi-compact and separated diagonal, and affine stabilizers. Then $\underline{\mathrm{HS}}_{X/S}$ is an algebraic stack, locally of finite presentation over S, with quasi-affine diagonal over S.*

Theorem 3. *Let $X \to S$ be a morphism of algebraic stacks that is locally of finite presentation, with quasi-finite and separated diagonal. Let $Z \to S$ be a morphism of algebraic stacks that is proper, flat, and of finite presentation with finite diagonal. Then the S-stack $T \mapsto \mathrm{HOM}_T(Z \times_S T, X \times_S T)$ is algebraic, locally of finite presentation over S, with quasi-affine diagonal over S.*

Let $f\colon Y \to Z$ and $p\colon Z \to W$ be morphisms of stacks. Define the fibered category p_*Y, the *restriction of scalars of Y along p*, by $(p_*Y)(T) = Y(T \times_W Z)$.

Theorem 4. *Let $f\colon Y \to Z$ and $p\colon Z \to W$ be morphisms of algebraic stacks. Assume that p is proper, flat, and of finite presentation with finite diagonal and that f is locally of finite presentation with quasi-finite and separated diagonal. Then the restriction of scalars p_*Y is an algebraic stack, locally of finite presentation over W, with quasi-affine diagonal over W.*

Theorem 1 is similar to the main conclusion of [LS08], and is included for completeness. In the case that the morphism $\pi\colon X \to S$ is separated, the Hilbert stack, $\underline{\mathrm{HS}}_{X/S}$, is equivalent to the stack of properly supported algebras on X, which was shown to be

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algebraic in [Lie06]. Thus the new content of this paper is in the removal of separatedness assumptions from similar theorems in the existing literature. The statement of Theorem 2 for algebraic spaces appeared in [Art74, Appendix 1], but was left unproved due to a lack of foundational results. It is important to note that Theorems 2, 3, and 4 are completely new, even for schemes and algebraic spaces.

We wish to point out that if X is an algebraic S-stack with affine stabilizer groups, and $X^0 \subseteq X$ denotes the open locus where the inertia stack $I_{X/S}$ is quasi-finite, then there is an isomorphism of S-stacks $\underline{\mathrm{HS}}_{X_0/S} \to \underline{\mathrm{HS}}_{X/S}$. In particular, Theorem 2 is really about algebraic stacks with quasi-finite diagonals.

Theorems 3 and 4 generalize [Ols06b, Thm. 1.1 & 1.5] and [Aok06a, Aok06b] to the non-separated setting, and follow easily from Theorem 2. In the case where π is not flat, as was remarked in [Hal12b], Artin's Criterion is difficult to apply. Thus, to prove Theorem 2 we use the algebraicity criterion [*op. cit.*, Thm. A]. The results of [*op. cit.*, §9] show that it is sufficient to understand how infinitesimal deformations can be extended to global deformations (i.e. the effectivity of formal deformations).

The difficulty in extending infinitesimal deformations of the Hilbert stack lies in the dearth of "formal GAGA" type results—in the spirit of [EGA, III.5]—for non-separated schemes, algebraic spaces, and algebraic stacks. In this paper, we will prove a generalization of formal GAGA to non-separated morphisms of algebraic stacks. The proof of our version of non-separated formal GAGA requires the development of a number of foundational results on non-separated spaces, and forms the bulk of the paper.

0.1. **Background.** The most fundamental moduli problem in algebraic geometry is the Hilbert moduli problem for $\mathbb{P}^N_{\mathbb{Z}}$: find a scheme that parameterizes flat families of closed subschemes of $\mathbb{P}^N_{\mathbb{Z}}$. It was proven by Grothendieck [FGA, IV.3.1] that this moduli problem has a solution which is a disjoint union of projective schemes.

In general, given a morphism of schemes $X \to S$, one may consider the *Hilbert moduli problem*: find a scheme $\underline{\mathrm{Hilb}}_{X/S}$ parameterizing flat families of closed subschemes of X. It is more precisely described by its functor of points: for any scheme T, a map of schemes $T \to \underline{\mathrm{Hilb}}_{X/S}$ is equivalent to a diagram:

$$\begin{array}{ccc} Z & \hookrightarrow & X \times_S T \\ & \searrow & \downarrow \\ & & T, \end{array}$$

where the morphism $Z \to X \times_S T$ is a closed immersion, and the composition $Z \to T$ is proper, flat, and of finite presentation. Grothendieck, using projective methods, constructed the scheme $\underline{\mathrm{Hilb}}_{\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}}$.

In [Art69], M. Artin developed a new approach to constructing moduli spaces. It was proved, by M. Artin in [Art69, Cor. 6.2] and [Art74, Appendix], that the functor $\underline{\mathrm{Hilb}}_{X/S}$ had the structure of an algebraic space for any separated and locally finitely presented morphism of algebraic spaces $X \to S$. The algebraic space $\underline{\mathrm{Hilb}}_{X/S}$ is not, in general, a scheme—even if $X \to S$ is a proper morphism of smooth complex varieties. In more recent work, Olsson–Starr [OS03] and Olsson [Ols05] showed that the functor $\underline{\mathrm{Hilb}}_{X/S}$ is an algebraic space in the case of a separated and locally finitely presented morphism of algebraic stacks $X \to S$.

A separatedness assumption on a scheme is rarely restrictive to an algebraic geometer. Indeed, most schemes algebraic geometers are interested in are quasi-projective or proper. Let us examine some spaces that arise in the theory of moduli.

Example 0.1 (Picard Schemes). Let $C \to \mathbb{A}^1$ be the family of curves corresponding to a conic degenerating to a node. Then the Picard scheme $\mathrm{Pic}_{C/\mathbb{A}^1}$, which parameterizes families of line bundles on $C/\mathbb{A}^1$ modulo pullbacks from the base, is not separated. This is worked out in detail in [FGI$^+$05, Ex. 9.4.14].

Example 0.2 (Curves)**.** Let $\mathcal{U}$ be the stack of all curves. That is, a morphism $T \to \mathcal{U}$ from a scheme T, is equivalent to a morphism of algebraic spaces $C \to T$ that is proper, flat, finitely presented, and with one-dimensional fibers. In particular, $\mathcal{U}$ parameterizes all singular curves, which could be non-reduced and have many irreducible and connected components. In [Smy13, Appendix B], it was shown that $\mathcal{U}$ is an algebraic stack, locally of finite presentation over $\mathbb{Z}$. The stack $\mathcal{U}$ is interesting, as Hassett [Has03], Schubert [Sch91], and Smyth [Smy13] have constructed modular compactifications of $\mathcal{M}_g$, different from the classical Deligne–Mumford compactification [DM69], that are open substacks of $\mathcal{U}$. The algebraic stack $\mathcal{U}$ is not separated.

Unlike schemes, the norm for interesting moduli spaces is that they are non-separated. Indeed, families of interesting geometric objects tend not to have unique limits. This is precisely the reason why compactifying moduli spaces is an active, and very difficult, area of research.

Lundkvist and Skjelnes showed in [LS08] that for a non-separated morphism of noetherian algebraic spaces $X \to S$, the functor $\underline{\mathrm{Hilb}}_{X/S}$ is never an algebraic space. We will provide an illustrative example of this phenomenon.

Example 0.3. Consider the simplest non-separated scheme: the line with the doubled origin. Let $\Bbbk$ be a field and set $S = \operatorname{Spec} \Bbbk$. Let $X = \mathbb{A}^1_s \amalg_{\mathbb{A}^1_{s=t}-(0)} \mathbb{A}^1_t$, which we view as an S-scheme. Now, for a y-line, $\mathbb{A}^1_y$, we have a $D := \operatorname{Spec} \Bbbk[\![x]\!]$-morphism $T_y \colon D \to \mathbb{A}^1_y \times_S D$ given by

$$1 \otimes x \mapsto x, \quad y \otimes 1 \mapsto x.$$

Thus, we have an induced D-morphism:

$$i \colon D \xrightarrow{T_s} (\mathbb{A}^1_s) \times_S D \to X \times_S D.$$

Now, the fiber i_n of i over $D_n := \operatorname{Spec} \Bbbk[\![x]\!]/(x^{n+1})$ is topologically the inclusion of one of the two origins, which is a closed immersion. Note, however, that the map i is not a closed immersion. The closed immersions $i_n \colon D_n \to X \times_S D_n$ induce compatible S-morphisms $D_n \to \underline{\mathrm{Hilb}}_{X/S}$. If $\underline{\mathrm{Hilb}}_{X/S}$ is an algebraic space, then this data induces a unique S-morphism $D \to \underline{\mathrm{Hilb}}_{X/S}$. That is, there exists a closed immersion $j \colon Z \to X \times_S D$ whose fiber over D_n is i_n. One immediately deduces that j is isomorphic to i. But i is not a closed immersion, thus we have a contradiction.

Note that if $X \to S$ is separated, any monomorphism $Z \to X \times_S T$, such that $Z \to T$ is proper, is automatically a closed immersion. Thus, for a separated morphism $X \to S$, the stack $\underline{\mathrm{HS}}^{\mathrm{mono}}_{X/S}$ is equivalent to the Hilbert functor $\underline{\mathrm{Hilb}}_{X/S}$. In the case that the morphism $X \to S$ is non-separated, they are different. Note that in Example 0.3, the deformed object was still a monomorphism, so will not prove Theorem 1 for the line with the doubled-origin. Let us consider another example.

Example 0.4. Consider the line with doubled-origin again, and retain the notation and conventions of Example 0.3. Thus, we have an induced map over D:

$$D \amalg D \xrightarrow{T_s \amalg T_t} (\mathbb{A}^1_s \amalg \mathbb{A}^1_t) \times_S D \to X \times_S D.$$

Where $x = 0$, this becomes the inclusion of the doubled point, which is a closed immersion. Where $x \neq 0$, this becomes non-monomorphic.

The proof of [LS08, Thm. 2.6] is based upon Example 0.3. Arguing similarly, but with the last example, one readily obtains Theorem 1. Thus, for a non-separated morphism of schemes, the obstruction to the existence of a Hilbert scheme is that a monomorphism

$Z \hookrightarrow X$ can deform to a non-monomorphism. So, one is forced to parameterize non-monomorphic maps $Z \to X$. Such variants of the Hilbert moduli problem have been considered previously in the literature. We now list those variants that the authors are aware of at the time of publication and that are known to be algebraic.

- Vistoli's Hilbert stack, which parameterizes families of finite and unramified morphisms to a separated stack [Vis91].
- The stack of coherent algebras on a separated algebraic stack [Lie06].
- The stack of branchvarieties, which parameterizes geometrically reduced algebraic stacks mapping finitely to a separated algebraic stack. It has proper components when the target stack has projective coarse moduli space or admits a proper flat cover by a quasi-projective scheme [AK10, Lie06].
- There is a proper algebraic space parameterizing Cohen–Macaulay curves with fixed Hilbert polynomial mapping finitely, and birationally onto its image, to projective space [Høn04].
- The Hilbert stack of points for any morphism of algebraic stacks [Ryd11].

To subsume the variants of the Hilbert moduli problem listed above, we parameterize the quasi-finite morphisms $Z \to X$.

Example 0.5. Closed immersions, quasi-compact open immersions, quasi-compact unramified morphisms, and finite morphisms are all examples of quasi-finite morphisms. By Zariski's Main Theorem [EGA, IV.18.12.13], any quasi-finite and separated map of schemes $Z \to X$ factors as $Z \to \overline{Z} \to X$ where $Z \to \overline{Z}$ is an open immersion and $\overline{Z} \to X$ is finite.

We now define the *Generalized Hilbert moduli problem*: for a morphism of algebraic stacks $\pi\colon X \to S$, find an algebraic stack $\underline{\mathrm{HS}}_{X/S}$ such that a map $T \to \underline{\mathrm{HS}}_{X/S}$ is equivalent to the data of a quasi-finite and representable map $Z \to X \times_S T$, with the composition $Z \to T$ proper, flat, and of finite presentation. This is the fibered category that appears in Theorem 2. The main result in this paper, Theorem 2, is that this stack is algebraic.

Example 0.6. Let $X \to S$ be a separated morphism of algebraic stacks. Then every quasi-finite and separated morphism $Z \to X$, such that $Z \to S$ proper, is finite. Hence, $\underline{\mathrm{HS}}_{X/S}$ is the stack of properly supported coherent algebras on X. Lieblich [Lie06] showed that $\underline{\mathrm{HS}}_{X/S}$ is algebraic whenever $X \to S$ is locally of finite presentation and separated.

0.2. **Outline.** In §5, we will prove Theorem 2 using the algebraicity criterion [Hal12b, Thm. A]. To apply this criterion, like Artin's Criterion [Art74, Thm. 5.3], it is necessary to know that formally versal deformations of objects in the Hilbert stack can be effectivized. Note that effectivity results for moduli problems related to separated objects usually follow from the formal GAGA results of [EGA, III.5], and the relevant generalizations to algebraic stacks [OS03, Ols05]. Since we are concerned with non-separated objects, no previously published effectivity result applies.

In §4, we prove a generalization of formal GAGA for non-separated algebraic stacks. This is the main technical result of this paper. To be precise, let R be a noetherian ring that is separated and complete for the topology defined by an ideal $I \subseteq R$ (i.e., R is I-adic [EGA, 0_{I}.7.1.9]). Set $S = \operatorname{Spec} R$ and for each $n \geq 0$ let $S_n = \operatorname{Spec}(R/I^{n+1})$. Now let $\pi\colon X \to S$ be a morphism of algebraic stacks that is locally of finite type, with quasi-compact and separated diagonal, and affine stabilizers. For each $n \geq 0$ let $\pi_n\colon X_n \to S_n$ denote the pullback of the map π along the closed immersion $S_n \hookrightarrow S$. Suppose that for each $n \geq 0$, we have compatible quasi-finite S_n-morphisms $s_n\colon Z_n \to X_n$ such that the composition $\pi_n \circ s_n\colon Z_n \to S_n$ is proper. We show (Theorem 4.3) that there exists a unique, quasi-finite S-morphism $s\colon Z \to X$, such that the composition $\pi \circ s\colon Z \to S$ is proper, and that there are compatible X_n-isomorphisms $Z \times_X X_n \to Z_n$.

In §§1-3, we will develop techniques to prove the afforementioned effectivity result in §4. To motivate these techniques, it is instructive to explain part of Grothendieck's proof of formal GAGA for properly supported coherent sheaves [EGA, III.5.1.4].

So, let R be an I-adic noetherian ring, let $S = \operatorname{Spec} R$, and for each $n \geq 0$ let $S_n = \operatorname{Spec}(R/I^{n+1})$. For a morphism of schemes $f\colon Y \to S$ that is locally of finite type and separated, let $f_n\colon Y_n \to S_n$ denote the pullback of the morphism f along the closed immersion $S_n \hookrightarrow S$. Suppose that for each $n \geq 0$, we have a coherent Y_n-sheaf $\mathcal{F}_n$, properly supported over S_n, together with isomorphisms $\mathcal{F}_{n+1}|_{Y_n} \cong \mathcal{F}_n$. Then Grothendieck's formal GAGA [EGA, III.5.1.4] states that there is a unique coherent Y-sheaf $\mathcal{F}$, with support proper over S, such that $\mathcal{F}|_{Y_n} \cong \mathcal{F}_n$.

For various reasons, it is better to think of adic systems of coherent sheaves $\{\mathcal{F}_n\}$ as a coherent sheaf $\mathfrak{F}$ on the formal scheme $\widehat{Y}$ [EGA, I.10.11.3]. We say that a coherent sheaf $\mathfrak{F}$ on the formal scheme $\widehat{Y}$ is *effectivizable*, if there exists a coherent sheaf $\mathcal{F}$ on the scheme Y and an isomorphism of coherent sheaves $\widehat{\mathcal{F}} \cong \mathfrak{F}$ on the formal scheme $\widehat{Y}$. The effectivity problem is thus recast as: any coherent sheaf $\mathfrak{F}$ on the formal scheme $\widehat{Y}$, with support proper over $\widehat{S}$, is effectivizable. This is proven using the method of dévissage on the abelian category of coherent sheaves with proper support, $\mathbf{Coh}_{\mathrm{p}/S}(Y)$, on the scheme Y. The proof consists of the following sequence of observations.

(1) Given coherent sheaves $\mathcal{H}$ and $\mathcal{H}'$ on Y, with $\mathcal{H}'$ properly supported over S, the natural map of R-modules:

$$\operatorname{Ext}^i_{\mathcal{O}_Y}(\mathcal{H}, \mathcal{H}') \to \operatorname{Ext}^i_{\mathcal{O}_{\widehat{Y}}}(\widehat{\mathcal{H}}, \widehat{\mathcal{H}}')$$

is an isomorphism for all $i \geq 0$. In particular, the "$i = 0$" statement shows that the functor $\mathbf{Coh}_{\mathrm{p}/S}(Y) \to \mathbf{Coh}_{\mathrm{p}/S}(\widehat{Y})$ is fully faithful.

(2) By (1), it remains to prove that the functor $\mathbf{Coh}_{\mathrm{p}/S}(Y) \to \mathbf{Coh}_{\mathrm{p}/S}(\widehat{Y})$ is essentially surjective. It is sufficient to prove this essential surjectivity when Y is quasi-compact. By noetherian induction on Y, we may, in addition, assume that if $\mathfrak{F} \in \mathbf{Coh}_{\mathrm{p}/S}(\widehat{Y})$ is annihilated by some coherent ideal $J \subseteq \mathcal{O}_Y$ with $|\operatorname{supp}(\mathcal{O}_Y/J)| \subsetneq |Y|$, then $\mathfrak{F}$ is effectivizable.

(3) If we have an exact sequence of $\widehat{S}$-properly supported coherent sheaves on $\widehat{Y}$:

$$0 \longrightarrow \mathfrak{F}' \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{F}'' \longrightarrow 0$$

and two of $\mathfrak{F}'$, $\mathfrak{F}$, $\mathfrak{F}''$ are effectivizable, then the third is. This follows from the exactness of completion and the $i = 0{,}1$ statements of (1).

(4) Combine (2) and (3) to deduce that if $\alpha\colon \mathfrak{F} \to \mathfrak{F}'$ is a morphism in $\mathbf{Coh}_{\mathrm{p}/S}(\widehat{Y})$ such that $\mathfrak{F}'$ is effectivizable and $\ker \alpha$ and $\operatorname{coker} \alpha$ are annihilated by some coherent ideal $J \subseteq \mathcal{O}_Y$ with $|\operatorname{supp}(\mathcal{O}_Y/J)| \subsetneq |Y|$, then $\mathfrak{F}$ is effectivizable.

(5) The result is true for quasi-projective morphisms $Y \to S$. This is proved via a direct argument.

(6) The Chow Lemma [EGA, II.5.6.1] gives a quasi-projective S-scheme Y' and a projective S-morphism $p\colon Y' \to Y$ that is an isomorphism over a dense open subset U of Y.

(7) Use (5) for the quasi-projective morphism $Y' \to S$ to show that $\widehat{p}^*\mathfrak{F} \cong \widehat{\mathcal{G}}$ for some S-properly supported coherent sheaf $\mathcal{G}$ on Y'.

(8) Use the Theorem on Formal Functions [EGA, III.4.1.5] to show that $(p_*\mathcal{G})^\wedge \cong \widehat{p}_*\widehat{p}^*\mathfrak{F}$. Thus $\widehat{p}_*\widehat{p}^*\mathfrak{F}$ is effectivizable with $\widehat{S}$-proper support.

(9) Note that we have an adjunction morphism $\eta\colon \mathfrak{F} \to \widehat{p}_*\widehat{p}^*\mathfrak{F}$. Pick a coherent ideal $J \subseteq \mathcal{O}_Y$ defining the complement of U. By [EGA, III.5.3.4], we may choose J so that it annihilates the kernel and cokernel of η. By (4), $\mathfrak{F}$ is effectivizable, and the result follows.

The proof of the non-separated effectivity result in §4 will be very similar to the technique outlined above, once the steps are appropriately reinterpreted. For a non-separated morphism of schemes $X \to S$, instead of the abelian category $\mathbf{Coh}\,(X)$ (resp. $\mathbf{Coh}_{\mathrm{p}/S}(X)$), we consider the non-abelian category $\mathbf{QF}_{\mathrm{s}}(X)$ (resp. $\mathbf{QF}_{\mathrm{p}/S}(X)$) that consists of quasi-finite and separated morphisms $Z \to X$ (resp. quasi-finite and separated morphism $Z \to X$ such that $Z \to S$ is proper). Instead of extensions, which do not make sense in a non-abelian category, we will use finite colimits.

In §1, we will show that $\mathbf{QF}_{\mathrm{s}}(X)$ is closed under finite colimits along finite morphisms. In §1 we also reinterpret (1) and (3) in terms of the preservation of these finite colimits under completion. The analogue of quasi-projective morphisms $Y \to S$ in (5) and (7) are morphisms that factor as $Y \to Y' \to S$, where $Y \to Y'$ is étale, and $Y' \to S$ is projective—note that it is essential that we do not assume that $Y \to Y'$ is separated. The analogue of the Chow Lemma used in (6), is a generalization, for non-separated schemes and algebraic spaces, due to Raynaud–Gruson [RG71, Cor. 5.7.13]. In §3, for a proper morphism $q\colon Y' \to Y$, we construct an adjoint pair $(q_!, q^*)\colon \mathbf{QF}_{\mathrm{s}}(Y') \leftrightarrows \mathbf{QF}_{\mathrm{s}}(Y)$ which takes the role of the adjoint pair $(q^*, q_*)\colon \mathbf{Coh}\,(Y) \leftrightarrows \mathbf{Coh}\,(Y')$. We also show that the adjoint pair can be constructed for locally noetherian formal schemes, and prove an analogue of (8) in this setting. In §4, we will combine the results of §§1 and 3 to obtain analogues of (4) and [EGA, III.5.3.4] in order to complete the analogue of step (9).

In §2, we introduce the generalized Stein factorization: every separated morphism of finite type with proper fibers factors canonically as a proper Stein morphism followed by a quasi-finite morphism. The adjoint pair of §3 is an immediate consequence of the existence of the generalized Stein factorization.

0.3. **Notation.** We introduce some notation here that will be used throughout the paper. For a category $\mathcal{C}$ and $X \in \mathbf{Obj}\,\mathcal{C}$, we have the *slice* category $\mathcal{C}/X$, with objects the morphisms $V \to X$ in $\mathcal{C}$, and morphisms commuting diagrams over X, which are called X-morphisms. If the category $\mathcal{C}$ has finite limits and $f\colon Y \to X$ is a morphism in $\mathcal{C}$, then for $(V \to X) \in \mathbf{Obj}(\mathcal{C}/X)$, define $V_Y := V \times_X Y$. Given a morphism $p\colon V' \to V$ in $\mathcal{C}/X$ there is an induced morphism $p_Y\colon V'_Y \to V_Y$ in $\mathcal{C}/Y$. There is an induced functor $f^*\colon \mathcal{C}/X \to \mathcal{C}/Y$ given by $(V \to X) \mapsto (V_Y \to Y)$.

Given a ringed space $U := (|U|, \mathcal{O}_U)$, a sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_U$, and a morphism of ringed spaces $g\colon V \to U$, we define the *pulled back ideal* $\mathcal{I}_V = \mathrm{im}(g^*\mathcal{I} \to \mathcal{O}_V) \subseteq \mathcal{O}_V$.

Let S be a scheme. An algebraic S-space is a sheaf F on the big étale site $(\mathbf{Sch}/S)_{\text{Ét}}$, such that the diagonal morphism $\Delta_F\colon F \to F \times_S F$ is represented by schemes, and there is a smooth surjection $U \to F$ from an S-scheme U. An algebraic S-stack is a stack H on $(\mathbf{Sch}/S)_{\text{Ét}}$, such that the diagonal morphism $\Delta_H\colon H \to H \times_S H$ is represented by algebraic S-spaces and there is a smooth surjection $U \to H$ from an algebraic S-space U. A priori, we make no separation assumptions on our algebraic stacks. We do show, however, that all algebraic stacks figuring in this paper possess quasi-compact and separated diagonals. Thus all of the results of [LMB] apply. We denote the $(2,1)$-category of algebraic stacks by $\mathbf{AlgStk}$.

1. Finite colimits of quasi-finite morphisms

In this section we will prove that the colimit of a finite diagram of finite morphisms between quasi-finite objects over an algebraic stack exists (Theorem 1.10). We will also prove that these colimits are compatible with completions (Theorem 1.18 and Corollary 1.20).

Let X be an algebraic stack. Denote the 2-category of algebraic stacks over X by $\mathbf{AlgStk}/X$. Define the full 2-subcategory $\mathbf{RSch}(X) \subseteq \mathbf{AlgStk}/X$ to have those objects $Y \xrightarrow{s} X$, where the morphism s is schematic. We now make three simple observations.

(1) The 1-morphisms in $\mathbf{RSch}(X)$ are schematic.

(2) Automorphisms of 1-morphisms in $\mathbf{RSch}(X)$ are trivial. Thus $\mathbf{RSch}(X)$ is naturally 2-equivalent to a 1-category.
(3) If the algebraic stack X is a scheme, then the natural functor $\mathbf{Sch}/X \to \mathbf{RSch}(X)$ is an equivalence of categories.

Definition 1.1. Let X be an algebraic stack. Define the following full subcategories of $\mathbf{RSch}(X)$:

(1) $\mathbf{RAff}(X)$ has objects the affine morphisms to X;
(2) $\mathbf{QF}_{\mathrm{s}}(X)$ has objects the quasi-finite and separated maps to X.

We also let $\mathbf{QF}_{\mathrm{s}}^{\mathrm{fin}}(X) \subseteq \mathbf{QF}_{\mathrm{s}}(X)$ denote the subcategory that has the same objects as $\mathbf{QF}_{\mathrm{s}}(X)$ but only has the morphisms $f\colon (Z \xrightarrow{s} X) \to (Z' \xrightarrow{s'} X)$ such that $Z \to Z'$ is finite. This is not a full subcategory.

Recall that quasi-finite, separated, and representable morphisms of algebraic stacks are schematic [LMB, Thm. A.2]. The subcategories $\mathbf{QF}_{\mathrm{s}}(X)$, $\mathbf{RAff}(X)$ and $\mathbf{RSch}(X)$ have all finite limits and these coincide with the limits in $\mathbf{AlgStk}/X$. The subcategory $\mathbf{QF}_{\mathrm{s}}^{\mathrm{fin}}(X)$ has fiber products and these coincide with the fiber products in $\mathbf{AlgStk}/X$. However, there is not a final object in $\mathbf{QF}_{\mathrm{s}}^{\mathrm{fin}}(X)$, except when every quasi-finite morphism to X is finite.

1.1. **Algebraic Stacks.** For the moment, we will be primarily concerned with the existence of colimits in the category $\mathbf{QF}_{\mathrm{s}}(X)$, where X is a locally noetherian algebraic stack. Colimits in algebraic geometry are usually subtle, so we restrict our attention to finite colimits in $\mathbf{QF}_{\mathrm{s}}^{\mathrm{fin}}(X)$. This includes the two types that will be useful in §4—pushouts of two finite morphisms and coequalizers of two finite morphisms. We will, however, pay attention to some more general types of colimits in the 2-category of algebraic stacks, as the added flexibility will simplify the exposition.

Remark 1.2. Note that because we have a fully faithful embedding of categories $\mathbf{QF}_{\mathrm{s}}(X) \subseteq \mathbf{RSch}(X)$, a categorical colimit in $\mathbf{RSch}(X)$, that lies in $\mathbf{QF}_{\mathrm{s}}(X)$, is automatically a categorical colimit in $\mathbf{QF}_{\mathrm{s}}(X)$. This will happen frequently in this section. Moreover, if a finite diagram $\{Z_i\}_{i\in I}$ in $\mathbf{QF}_{\mathrm{s}}^{\mathrm{fin}}(X)$ has a categorical colimit Z in $\mathbf{QF}_{\mathrm{s}}(X)$ and $Z_i \to Z$ is finite for every $i \in I$, then Z is also a categorical colimit in $\mathbf{QF}_{\mathrm{s}}^{\mathrm{fin}}(X)$. Indeed, it is readily seen that $\coprod_{i\in I} Z_i \to Z$ is surjective so any morphism $(Z \to W) \in \mathbf{QF}_{\mathrm{s}}(X)$, such that $Z_i \to Z \to W$ is finite for every $i \in I$, is finite.

Remark 1.3. It is important to observe that a categorical colimit in $\mathbf{AlgStk}/X$ is, in general, different from a categorical colimit in $\mathbf{RSch}(X)$. Indeed, let $X = \operatorname{Spec} \Bbbk$, and let G be a finite group. Then the X-stack $\mathrm{B}G$ is the colimit in $\mathbf{AlgStk}/X$ of the diagram $[G_X \rightrightarrows X]$. The colimit of this diagram in the category $\mathbf{RSch}(X)$ is just X.

The definitions that follow are closely related to those given in [Ryd13, Def. 2.2]. Recall that a map of topological spaces $g\colon U \to V$ is *submersive* if a subset $Z \subseteq |V|$ is open if and only if $g^{-1}(Z)$ is an open subset of $|U|$.

Definition 1.4. Consider a diagram of algebraic stacks $\{Z_i\}_{i\in I}$, an algebraic stack Z, and suppose that we have compatible maps $\phi_i\colon Z_i \to Z$ for all $i \in I$. Then we say that the data $(Z, \{\phi_i\}_{i\in I})$ is a

(1) *Zariski colimit* if the induced map on topological spaces $\phi\colon \varinjlim_i |Z_i| \to |Z|$ is a homeomorphism (equivalently, the map $\amalg_{i\in I}\phi_i\colon \amalg_{i\in I} Z_i \to Z$ is submersive and the map ϕ is a bijection of sets);
(2) *weak geometric colimit* if it is a Zariski colimit of the diagram $\{Z_i\}_{i\in I}$, and the canonical map of lisse-étale sheaves of rings $\mathcal{O}_Z \to \varprojlim_i (\phi_i)_*\mathcal{O}_{Z_i}$ is an isomorphism;

(3) *universal Zariski colimit* if for any algebraic Z-stack Y, the data $(Y, \{(\phi_i)_Y\}_{i\in I})$ is a Zariski colimit of the diagram $\{(Z_i)_Y\}_{i\in I}$;
(4) *geometric colimit* if it is a universal Zariski and weak geometric colimit of the diagram $\{Z_i\}_{i\in I}$;
(5) *uniform geometric colimit* if for any flat, algebraic Z-stack Y, the data $(Y, \{(\phi_i)_Y\}_{i\in I})$ is a geometric colimit of the diagram $\{(Z_i)_Y\}_{i\in I}$.

The exactness of flat pullback of sheaves and flat base change easily prove

Lemma 1.5. *Suppose that we have a finite diagram of algebraic stacks $\{Z_i\}_{i\in I}$. Then a geometric colimit $(Z, \{\phi_i\}_{i\in I})$ is a uniform geometric colimit.*

Also, note that weak geometric colimits in the category of algebraic stacks are not unique and thus are not categorical colimits (Remark 1.3). In the setting of schemes, however, we have the following Lemma.

Lemma 1.6. *Let $\{Z_i\}_{i\in I}$ be a diagram of schemes. Then every weak geometric colimit $(Z, \{\phi_i\}_{i\in I})$ is a colimit in the category of locally ringed spaces and, consequently, a colimit in the category of schemes.*

Proof. Fix a ringed space $W = (|W|, \mathcal{O}_W)$, together with compatible morphisms of ringed spaces $\psi_i\colon Z_i \to W$. We will produce a unique map of ringed spaces $\psi\colon Z \to W$ satisfying $\psi\phi_i = \psi_i$ for all $i \in I$. Since $(Z, \{\phi_i\}_{i\in I})$ is a Zariski colimit, it follows that there exists a unique, continuous, map of topological spaces $\psi\colon |Z| \to |W|$ such that $\psi\phi_i = \psi_i$. On the lisse-étale site, we also know that the map $\mathcal{O}_Z \to \varprojlim_i (\phi_i)_*\mathcal{O}_{Z_i}$ is an isomorphism. The natural functor from the lisse-étale site of Z to the Zariski site of Z also preserves limits. In particular, we deduce that as sheaves of rings on the topological space $|Z|$, the natural map $\mathcal{O}_Z \to \varprojlim_i (\phi_i)_*\mathcal{O}_{Z_i}$ is an isomorphism. The functor ψ_* preserves limits, thus the maps $\mathcal{O}_W \to (\psi_i)_*\mathcal{O}_{Z_i}$ induce a unique map to $\psi_*\mathcal{O}_Z$. Hence, we obtain a unique morphism of ringed spaces $\psi\colon Z \to W$ that is compatible with the data.

To complete the proof it remains to show that if, in addition, $W = (|W|, \mathcal{O}_W)$ is a locally ringed space and the morphisms ψ_i are morphisms of locally ringed spaces, then ψ is a morphism of locally ringed spaces. Let $z \in |Z|$. We must prove that the induced morphism $\mathcal{O}_{W,\psi(z)} \to \mathcal{O}_{Z,z}$ is local. Since $\amalg_{i\in I}|Z_i| \to |Z|$ is surjective, there exists an $i \in I$ and $z_i \in |Z_i|$ such that $\phi_i(z_i) = z$. By hypothesis, the morphism $\mathcal{O}_{W,\psi(z)} \to \mathcal{O}_{Z_i,z_i}$ is local. Also, $Z_i \to Z$ is a morphism of schemes so $\mathcal{O}_{Z,z} \to \mathcal{O}_{Z_i,z_i}$ is local. We immediately deduce that the morphism $\mathcal{O}_{W,\psi(z)} \to \mathcal{O}_{Z,z}$ is local, and the result is proven. □

The following criterion will be useful for verifying when a colimit is a universal Zariski colimit.

Lemma 1.7. *Consider a diagram of algebraic stacks $\{Z_i\}_{i\in I}$, an algebraic stack Z, and suppose that we have compatible maps $\phi_i\colon Z_i \to Z$ for all $i \in I$. If the map $\amalg_i\phi_i\colon \amalg_i Z_i \to Z$ is surjective and universally submersive and also*

(1) *for any geometric point $\operatorname{Spec} K \to Z$, the map $\phi_K\colon \varinjlim_i |(Z_i)_K| \to |\operatorname{Spec} K|$ is an injection of sets; or*
(2) *there is an algebraic stack X such that $Z, Z_i \in \mathbf{QF}_{\mathrm{s}}(X)$ for all $i \in I$, the maps ϕ_i are X-maps, and for any geometric point $\operatorname{Spec} L \to X$, the map $\phi_L\colon \varinjlim_i |(Z_i)_L| \to |Z_L|$ is an injection of sets,*

then $(Z, \{\phi_i\}_{i\in I})$ is a universal Zariski colimit of the diagram $\{Z_i\}_{i\in I}$.

Proof. To show (1), we observe that the universal submersiveness hypothesis on the map $\amalg_i\phi_i\colon \amalg_i Z_i \to Z$ reduces the statement to showing that for any morphism of algebraic stacks $Y \to Z$, the map $\phi_Y\colon \varinjlim_i |(Z_i)_Y| \to |Y|$ is a bijection of sets, which will follow if the map $\phi_K\colon \varinjlim_i |(Z_i)_K| \to |\operatorname{Spec} K|$ is bijective for any geometric point $\operatorname{Spec} K \to Y$.

By assumption, we know that ϕ_K is injective. For the surjectivity, we note that we have a commutative diagram of sets:

$$\begin{array}{ccc} \coprod_i |(Z_i)_K| & \xrightarrow{\alpha_K} & \varinjlim_i |(Z_i)_K| \, , \\ & \searrow^{\amalg_i \phi_i} & \downarrow{\scriptstyle \phi_K} \\ & & |\operatorname{Spec} K| \end{array}$$

where α_K and $\amalg_i \phi_i$ are surjective, thus ϕ_K is also surjective. For (2), given a geometric point $\operatorname{Spec} L \to X$, then $Z(\operatorname{Spec} L) = |Z_L|$ and $Z_i(\operatorname{Spec} L) = |(Z_i)_L|$, thus we may apply the criterion of (1) to obtain the claim. □

To obtain similarly useful results for stacks, we will need some relative notions, unlike the previous definitions which were all absolute.

Definition 1.8. Fix an algebraic stack X and a diagram $\{Z_i\}_{i\in I}$ in $\mathbf{RSch}(X)$. Suppose that $Z \in \mathbf{RSch}(X)$, and that we have compatible X-morphisms $\phi_i \colon Z_i \to Z$ for all $i \in I$. We say that the data $(Z, \{\phi_i\}_{i\in I})$ is a *uniform categorical colimit in* $\mathbf{RSch}(X)$ if for any flat, algebraic X-stack Y, the data $(Z_Y, \{(\phi_i)_Y\}_{i\in I})$ is the categorical colimit of the diagram $\{(Z_i)_Y\}_{i\in I}$ in $\mathbf{RSch}(Y)$.

Combining Lemmas 1.5 and 1.6 we obtain

Proposition 1.9. *Let X be an algebraic stack. If $\{Z_i\}_{i\in I}$ is a finite diagram in $\mathbf{RSch}(X)$, then a geometric colimit $(Z, \{\phi_i\}_{i\in I})$ in $\mathbf{RSch}(X)$ is also a uniform geometric and uniform categorical colimit in $\mathbf{RSch}(X)$.*

Proof. By Lemma 1.5, it is sufficient to prove that $(Z, \{\phi_i\}_{i\in I})$ is a categorical colimit in $\mathbf{RSch}(X)$. This follows from Lemma 1.6 and flat descent as we now explain.

Let $p \colon U \to X$ be a smooth surjection, where U is a scheme. Let $R = U \times_X U$, which is an algebraic space. Let $\tilde{R} \to R$ be an étale surjection, where $\tilde{R}$ is a scheme. If W_1 and W_2 belong to $\mathbf{RSch}(X)$, then fppf descent implies that the following sequence of sets:

$$\operatorname{Hom}_X(W_1, W_2) \longrightarrow \operatorname{Hom}_U((W_1)_U, (W_2)_U) \rightrightarrows \operatorname{Hom}_{\tilde{R}}((W_1)_{\tilde{R}}, (W_2)_{\tilde{R}})$$

is equalizing. Let W in $\mathbf{RSch}(X)$. Since $\varprojlim_{i\in I}$ preserves equalizers, it follows that both rows in the following commutative diagram are equalizing:

$$\begin{array}{ccccc} \operatorname{Hom}_X(Z, W) & \longrightarrow & \operatorname{Hom}_U(Z_U, W_U) & \rightrightarrows & \operatorname{Hom}_{\tilde{R}}(Z_{\tilde{R}}, Z_{\tilde{R}}) \\ \downarrow & & \downarrow & & \downarrow \\ \varprojlim_i \operatorname{Hom}_X(Z_i, W) & \longrightarrow & \varprojlim_i \operatorname{Hom}_U((Z_i)_U, W_U) & \rightrightarrows & \varprojlim_i \operatorname{Hom}_{\tilde{R}}((Z_i)_{\tilde{R}}, W_{\tilde{R}}). \end{array}$$

Observe that $(Z_U, \{(\phi_i)_U\})$ and $(Z_{\tilde{R}}, \{(\phi_i)_{\tilde{R}}\})$ are geometric colimits and so, by Lemma 1.6, also categorical colimits in $\mathbf{RSch}(U)$ and $\mathbf{RSch}(\tilde{R})$ respectively. Thus, the two vertical morphisms on the right in the above diagram are bijective. It follows that the left vertical arrow is bijective and that $(Z, \{\phi_i\}_{i\in I})$ is a categorical colimit in $\mathbf{RSch}(X)$. □

The following result is a generalization of [Kol11, Lem. 17], which treats the case of a finite equivalence relation of schemes.

Theorem 1.10. *Let X be a locally noetherian algebraic stack. Let $\{Z_i\}_{i\in I}$ be a finite diagram in $\mathbf{QF}^{\mathrm{fin}}_{\mathrm{s}}(X)$. Then a colimit $(Z, \{\phi_i\}_{i\in I})$ exists in $\mathbf{QF}^{\mathrm{fin}}_{\mathrm{s}}(X)$. Moreover, this colimit is a uniform categorical colimit in $\mathbf{RSch}(X)$ and a uniform geometric colimit.*

We will need some lemmas to prove Theorem 1.10. We first treat the affine case, then the finite case and finally the quasi-finite case.

Lemma 1.11. *Let X be an algebraic stack. Let $\{Z_i\}_{i\in I}$ be a finite diagram in $\mathbf{RAff}(X)$. Then this diagram has a categorical colimit in $\mathbf{RAff}(X)$, whose formation commutes with flat base change on X.*

Proof. By [LMB, Prop. 14.2.4], there is an anti-equivalence of categories between $\mathbf{RAff}(X)$ and the category of quasi-coherent sheaves of $\mathcal{O}_X$-algebras, which commutes with arbitrary change of base, and is given by $(Z \xrightarrow{s} X) \mapsto (\mathcal{O}_X \xrightarrow{s^\sharp} s_*\mathcal{O}_Z)$. Since the category of quasi-coherent $\mathcal{O}_X$-algebras has finite limits, it follows that if $s_i\colon Z_i \to X$ denotes the structure map of Z_i, then the categorical colimit is $Z = \operatorname{Spec}_X \varprojlim_i (s_i)_*\mathcal{O}_{Z_i}$. Since flat pullback of sheaves is exact, the formation of this colimit commutes with flat base change on X. □

Lemma 1.12. *Let X be a locally noetherian algebraic stack. Let $\{Z_i\}_{i\in I}$ be a finite diagram in $\mathbf{RSch}(X)$ such that $Z_i \to X$ is finite for every $i \in I$. Then this diagram has a uniform categorical and uniform geometric colimit Z in $\mathbf{RSch}(X)$ which is finite over X.*

Proof. By Lemma 1.11 the diagram has a categorical colimit $(Z, \{\phi_i\}_{i\in I})$ in $\mathbf{RAff}(X)$. If $s_i\colon Z_i \to X$ and $s\colon Z \to X$ denote the structure maps, then $s_*\mathcal{O}_Z \subseteq \prod (s_i)_*\mathcal{O}_{Z_i}$. Since X is locally noetherian, it follows that Z is finite over X. By Proposition 1.9, it remains to show that Z is a geometric colimit. Note that since $\coprod_{i\in I} Z_i \to Z$ is dominant and finite, it is surjective, universally closed and thus universally submersive.

We will now show that Z is a universal Zariski colimit using the criterion of Lemma 1.7(2). To apply this criterion we will employ a generalization of the arguments given in [Kol11, Lem. 17]. Let $\overline{x}\colon \operatorname{Spec} \Bbbk \to X$ be a geometric point. By [EGA, 0_{III}.10.3.1] this map factors as $\operatorname{Spec} \Bbbk \xrightarrow{\overline{x}^1} X^1 \xrightarrow{p} X$, where p is flat and X^1 is the spectrum of a maximal-adically complete, local noetherian ring with residue field $\Bbbk$. Since the algebraic stack Z is a uniform categorical colimit in $\mathbf{RAff}(X)$, we may replace X by X^1, and we denote the unique closed point of X by x.

For a finite X-scheme U, let $\pi_0(U)$ be its set of connected components. The scheme X is henselian [EGA, IV.18.5.14], so there is a unique universal homeomorphism $h_U\colon U_x \to \pi_0(U) \times \{x\} = \amalg_{m\in\pi_0(U)}\{x\}$ which is functorial with respect to U. In particular, there is a unique factorization $U \xrightarrow{s_U} \pi_0(U) \times X \to X$ such that $(s_U)_x = h_U$.

Let $W_i = \pi_0(Z_i) \times X$. Since $\pi_0(-)$ is a functor, we obtain a diagram $\{W_i\}_{i\in I}$ in $\mathbf{RAff}(X)$ and we let W be the categorical colimit of this diagram in $\mathbf{RAff}(X)$. It is readily seen that $W = \bigl(\varinjlim \pi_0(Z_i)\bigr) \times X$, so there is a bijection of sets $\varinjlim_i \pi_0(Z_i) \to \pi_0(W)$. Since Z is a categorical colimit, there is a canonical map $\mu\colon Z \to W$. In particular, the bijection $\nu_x\colon \varinjlim_i |(Z_i)_x| \to |W_x|$ factors as $\varinjlim_i |(Z_i)_x| \xrightarrow{\psi_x} |Z_x| \xrightarrow{\mu_x} |W_x|$ and thus $\psi_x\colon \varinjlim_i |(Z_i)_x| \to |Z_x|$ is injective. Hence, we have shown that Z is a universal Zariski colimit and it remains to show that Z has the correct functions.

There is a canonical morphism of sheaves of $\mathcal{O}_X$-algebras $\epsilon\colon \mathcal{O}_Z \to \varprojlim_i (\phi_i)_*\mathcal{O}_{Z_i}$, which we have to show is an isomorphism. By functoriality, we have an induced morphism of sheaves of $\mathcal{O}_X$-algebras:

$$\epsilon_2\colon s_*\mathcal{O}_Z \xrightarrow{s_*\epsilon} s_*\bigl(\varprojlim_i (\phi_i)_*\mathcal{O}_{Z_i}\bigr) \xrightarrow{\epsilon_1} \varprojlim_i s_*(\phi_i)_*\mathcal{O}_{Z_i}.$$

Since the functor s_* is left exact, ϵ_1 is an isomorphism; by construction of Z the map ϵ_2 is an isomorphism and so $s_*\epsilon$ is an isomorphism. Since s is affine, the functor s_* is faithfully exact and we conclude that the map ϵ is an isomorphism of sheaves. □

Proof of Theorem 1.10. By hypothesis, the structure morphisms $s_i\colon Z_i \to X$ are quasi-finite, separated, and representable. By Zariski's Main Theorem [LMB, Thm. 16.5(ii)], there are finite X-morphisms $\overline{s_i}\colon \overline{Z_i} \to X$ and open immersions $u_i\colon Z_i \hookrightarrow \overline{Z_i}$.

Let $W_i = \prod_{i\to j} \overline{Z_j}$ where the product is fibered over X and let $\pi_g\colon W_i \to \overline{Z_j}$ denote the projection onto the factor corresponding to $g\colon i \to j$. Further, let $v_i\colon Z_i \to W_i$ be the morphism such that $\pi_g \circ v_i = u_j \circ g_*\colon Z_i \xrightarrow{g_*} Z_j \xrightarrow{u_j} \overline{Z_j}$. We also have a natural morphism $g_*\colon W_i \to W_j$ such that $\pi_h \circ g_* = \pi_{hg}$ for every arrow $h\colon j \to k$. This gives a diagram $\{W_i\}_{i\in I}$ such that the morphisms $v_i\colon Z_i \to W_i$ induce a morphism of diagrams.

Since $u_i\colon Z_i \to \overline{Z_i}$ is an immersion, so is $v_i\colon Z_i \to W_i$. Now, replace W_i with the schematic image of $Z_i \to W_i$. Then $Z_i \to W_i$ is an open immersion with dense image and $\{W_i\}_{i\in I}$ is a diagram with finite structure morphisms $W_i \to X$. Thus the diagram $\{W_i\}_{i\in I}$ has a uniform geometric and uniform categorical colimit $W \in \mathbf{RSch}(X)$ and $W \to X$ is finite (Lemma 1.12).

We will now show that for any arrow $i \to j$, the open substack $Z_j \subseteq W_j$ is pulled back to the open substack $Z_i \subseteq W_i$. We have canonical maps $Z_i \xrightarrow{\alpha} Z_j \times_{W_j} W_i \xrightarrow{\beta} W_i$ and since the maps $\beta\circ\alpha$ and β are open immersions, so is $\alpha\colon Z_i \to Z_j \times_{W_j} W_i$. Similarly, we have $Z_i \xrightarrow{\alpha} Z_j \times_{W_j} W_i \xrightarrow{\gamma} Z_j$ where $\gamma\circ\alpha$ and γ are finite morphisms. Thus $\alpha\colon Z_i \to Z_j \times_{W_j} W_i$ is an open and closed immersion. Since $u_i = \beta\circ\alpha\colon Z_i \to W_i$ has dense image, it follows that α is an isomorphism.

Let $|Z|$ be the set-theoretic image of $\coprod_i |Z_i|$ in $|W|$. Since $|W|$ is a colimit of $\{|W_i|\}_{i\in I}$ in the category of sets, and $|Z_i| = |Z_j| \times_{|W_j|} |W_i|$ for all $i \to j$, it follows that $|Z_i| = |Z| \times_{|W|} |W_i|$. Moreover, since $|W|$ is a Zariski colimit, it has the quotient topology, so $|Z| \subseteq |W|$ is open.

We let $Z \subseteq W$ be the open substack with underlying topological space $|Z|$. As we noted above, the diagram $\{Z_i\}_{i\in I}$ is obtained as the pull-back of the diagram $\{W_i\}_{i\in I}$ along the open immersion $Z \to W$. Since W is a uniform geometric colimit, the pull-back Z is a uniform geometric colimit. It is a uniform categorical colimit in $\mathbf{RSch}(X)$ since $Z \in \mathbf{RSch}(X)$ (Proposition 1.9). Finally, note that $Z_i \to Z$ is finite since it is a pull-back of the finite morphism $W_i \to W$, so Z is a colimit in $\mathbf{QF}^{\mathrm{fin}}_{\mathrm{s}}(X)$ (Remark 1.2). ☐

1.2. **Completions of schemes.** Here we will show that the colimits constructed in Theorem 1.10 remain colimits after completing along a closed subset. Denote the category of formal schemes by $\mathbf{FSch}$. We require some more definitions that are analogous to those given in §1.1.

Definition 1.13. Consider a diagram of formal schemes $\{\mathfrak{Z}_i\}_{i\in I}$, a formal scheme $\mathfrak{Z}$, and suppose that we have compatible maps $\varphi_i\colon \mathfrak{Z}_i \to \mathfrak{Z}$ for every $i \in I$. Then we say that the data $(\mathfrak{Z}, \{\varphi_i\}_{i\in I})$ is a

(1) *formal Zariski colimit* if the induced map on topological spaces $\varinjlim_i |\mathfrak{Z}_i| \to |\mathfrak{Z}|$ is a homeomorphism;
(2) *formal weak geometric colimit* if it is a formal Zariski colimit and the canonical map of sheaves of rings $\mathcal{O}_{\mathfrak{Z}} \to \varprojlim_i (\varphi_i)_*\mathcal{O}_{\mathfrak{Z}_i}$ is a topological isomorphism, where we give the latter sheaf of rings the limit topology (this is nothing other than $\mathfrak{Z}$ being the colimit in the category of topologically ringed spaces).

If, in addition, the formal scheme $\mathfrak{Z}$ is locally noetherian, and the maps $\varphi_i\colon \mathfrak{Z}_i \to \mathfrak{Z}$ are topologically of finite type, then we say that the data $(\mathfrak{Z}, \{\varphi_i\}_{i\in I})$ is a

(3) *universal formal Zariski colimit* if for any adic formal $\mathfrak{Z}$-scheme $\mathfrak{Y}$, the data $(\mathfrak{Y}, \{(\varphi_i)_{\mathfrak{Y}}\}_{i\in I})$ is the formal Zariski colimit of the diagram $\{(\mathfrak{Z}_i)_{\mathfrak{Y}}\}_{i\in I}$;
(4) *formal geometric colimit* if it is a universal formal Zariski and a formal weak geometric colimit of the diagram $\{\mathfrak{Z}_i\}_{i\in I}$;
(5) *uniform formal geometric colimit* if for any adic flat formal $\mathfrak{Z}$-scheme $\mathfrak{Y}$, the data $(\mathfrak{Y}, \{(\varphi_i)_{\mathfrak{Y}}\}_{i\in I})$ is a formal geometric colimit of the diagram $\{(\mathfrak{Z}_i)_{\mathfrak{Y}}\}_{i\in I}$.

We have two lemmas which are the analogues of Lemmas 1.5 and 1.6 for formal schemes.

Lemma 1.14. *Let $\{\mathfrak{Z}_i\}_{i\in I}$ be a diagram of formal schemes. Then every formal weak geometric colimit $(\mathfrak{Z}, \{\varphi_i\}_{i\in I})$ is a colimit in the category of topologically locally ringed spaces and, consequently, a colimit in the category* $\mathbf{FSch}$.

Lemma 1.15. *Consider a finite diagram of locally noetherian formal schemes* $\{\mathfrak{Z}_i\}_{i\in I}$*, and a locally noetherian formal scheme* $\mathfrak{Z}$*, together with finite morphisms* $\varphi_i\colon \mathfrak{Z}_i \to \mathfrak{Z}$*. If the data* $(\mathfrak{Z}, \{\varphi_i\}_{i\in I})$ *is a formal geometric colimit of the diagram* $\{\mathfrak{Z}_i\}_{i\in I}$*, then it is a uniform formal geometric colimit.*

Proof. Let $\varpi\colon \mathfrak{Y} \to \mathfrak{Z}$ be an adic flat morphism of locally noetherian formal schemes. It suffices to show that the canonical map $\vartheta_{\mathfrak{Y}}\colon \mathcal{O}_{\mathfrak{Y}} \to \varprojlim_i [(\varphi_i)_{\mathfrak{Y}}]_*\mathcal{O}_{(\mathfrak{Z}_i)_{\mathfrak{Y}}}$ is a topological isomorphism. Since φ_i is finite for all i, and we are taking a finite limit, we conclude that it suffices to show that $\vartheta_{\mathfrak{Y}}$ is an isomorphism of coherent $\mathcal{O}_{\mathfrak{Y}}$-modules. By hypothesis, the map $\vartheta_{\mathfrak{Z}}\colon \mathcal{O}_{\mathfrak{Z}} \to \varprojlim_i (\varphi_i)_*\mathcal{O}_{\mathfrak{Z}_i}$ is an isomorphism, and since ϖ is adic flat, ϖ^* is an exact functor from coherent $\mathcal{O}_{\mathfrak{Z}}$-modules to coherent $\mathcal{O}_{\mathfrak{Y}}$-modules, thus commutes with finite limits. Hence, we see that $\vartheta_{\mathfrak{Y}}$ factors as the following sequence of isomorphisms:

$$\mathcal{O}_{\mathfrak{Y}} \cong \varpi^*\mathcal{O}_{\mathfrak{Z}} \cong \varprojlim_i \varpi^*[(\varphi_i)_*\mathcal{O}_{\mathfrak{Z}_i}] \cong \varprojlim_i [(\varphi_i)_{\mathfrak{Y}}]_*\mathcal{O}_{(\mathfrak{Z}_i)_{\mathfrak{Y}}}. \qquad \square$$

Definition 1.16. Fix a locally noetherian formal scheme $\mathfrak{X}$ and let $\{\mathfrak{Z}_i\}_{i\in I}$ be a diagram in $\mathbf{FSch}/\mathfrak{X}$, where each formal scheme $\mathfrak{Z}_i$ is locally noetherian. Let $\mathfrak{Z} \in \mathbf{FSch}/\mathfrak{X}$ also be locally noetherian, and suppose that we have compatible $\mathfrak{X}$-morphisms $\varphi_i\colon \mathfrak{Z}_i \to \mathfrak{Z}$ that are topologically of finite type. Then we say that the data $(\mathfrak{Z}, \{\varphi_i\}_{i\in I})$ is a *uniform categorical colimit in* $\mathbf{FSch}/\mathfrak{X}$ if for any adic flat formal $\mathfrak{X}$-scheme $\mathfrak{Y}$, the data $(\mathfrak{Z}_{\mathfrak{Y}}, \{(\varphi_i)_{\mathfrak{Y}}\}_{i\in I})$ is the categorical colimit of the diagram $\{(\mathfrak{Z}_i)_{\mathfrak{Y}}\}_{i\in I}$ in $\mathbf{FSch}/\mathfrak{Y}$.

Definition 1.17. Let $\mathfrak{X}$ be a formal scheme. Define $\mathbf{QF}_{\mathrm{s}}(\mathfrak{X})$ to be the category whose objects are adic, quasi-finite, and separated maps $(\mathfrak{Z} \xrightarrow{\sigma} \mathfrak{X})$. A morphism in $\mathbf{QF}_{\mathrm{s}}(\mathfrak{X})$ is an $\mathfrak{X}$-morphism $\mathfrak{f}\colon (\mathfrak{Z} \xrightarrow{\sigma} \mathfrak{X}) \to (\mathfrak{Z}' \xrightarrow{\sigma'} \mathfrak{X})$.

For a scheme X, and a closed subset $|V| \subseteq |X|$, we define the *completion functor*

$$c_{X,|V|}\colon \mathbf{Sch}/X \to \mathbf{FSch}/\widehat{X}_{/|V|}, \quad (Z \xrightarrow{s} X) \mapsto \big(\widehat{Z}_{/s^{-1}|V|} \to \widehat{X}_{/|V|}\big).$$

Note that restricting $c_{X,|V|}$ to $\mathbf{QF}_{\mathrm{s}}(X)$ has essential image contained in $\mathbf{QF}_{\mathrm{s}}(\widehat{X}_{/|V|})$.

Theorem 1.18. *Let X be a locally noetherian scheme, let $|V| \subseteq |X|$ be a closed subset, and let $\{Z_i\}_{i\in I}$ be a finite diagram in $\mathbf{QF}^{\mathrm{fin}}_{\mathrm{s}}(X)$. If the scheme Z denotes the categorical colimit of the diagram in $\mathbf{Sch}/X$, which exists by Theorem 1.10, then $c_{X,|V|}(Z)$ is a uniform categorical and uniform formal geometric colimit of the diagram $\{c_{X,|V|}(Z_i)\}_{i\in I}$ in $\mathbf{FSch}/\widehat{X}_{/|V|}$.*

Proof. Let $\mathfrak{X} = \widehat{X}_{/|V|}$, $\mathfrak{Z} = c_{X,|V|}(Z)$, and $\mathfrak{Z}_{\mathfrak{i}} = c_{X,|V|}(Z_i)$. By Lemmas 1.14 and 1.15, it suffices to show that $\mathfrak{Z}$ is a formal geometric colimit of the diagram $\{\mathfrak{Z}_{\mathfrak{i}}\}_{i\in I}$. We first show that $\mathfrak{Z}$ is a universal formal Zariski colimit. Let $\varpi\colon \mathfrak{Y} \to \mathfrak{X}$ be an adic morphism of locally noetherian formal schemes. Let $\mathcal{I}$ be a coherent sheaf of radical ideals defining $|V| \subseteq |X|$ so that $(\varpi^{-1}\mathcal{I})\mathcal{O}_{\mathfrak{Y}}$ is an ideal of definition of $\mathfrak{Y}$. Let $X_0 = V(\mathcal{I}) \subseteq X$ and $Y_0 = \mathfrak{Y} \times_{\mathfrak{X}} X_0$. Then by Theorem 1.10 the map of topological spaces $\varinjlim_{i\in I} |Z_i \times_X Y_0| \to |Z \times_X Y_0|$ is a homeomorphism. Noting that $|\mathfrak{Z}_{\mathfrak{Y}}| = |Z \times_X Y_0|$ and $|\mathfrak{Z}_{\mathfrak{i}\mathfrak{Y}}| = |Z_i \times_X Y_0|$, we conclude that $\mathfrak{Z}$ is a universal formal Zariski colimit and it remains to show that $\mathfrak{Z}$ has the correct functions.

Let $\phi_i\colon Z_i \to Z$ denote the canonical morphisms. By Theorem 1.10 we have an isomorphism of sheaves of rings $\epsilon\colon \mathcal{O}_Z \to \varprojlim_i (\phi_i)_*\mathcal{O}_{Z_i}$. Hence, since ϕ_i is finite, and completion is exact on coherent modules [EGA, I.10.8.9], by [EGA, I.10.8.8(i)] we have an isomorphism of coherent $\mathcal{O}_{\mathfrak{Z}}$-algebras:

$$\mathcal{O}_{\mathfrak{Z}} \xrightarrow{\epsilon^\wedge} (\varprojlim_i \phi_{i*}\mathcal{O}_{Z_i})^\wedge \cong \varprojlim_i (\phi_{i*}\mathcal{O}_{Z_i})^\wedge \cong \varprojlim_i \widehat{\phi}_{i*}\mathcal{O}_{\mathfrak{Z}_{\mathfrak{i}}}.$$

It remains to show that this isomorphism is topological (where we endow the right side with the limit topology). A general fact here is that the topology on the right is the subspace topology of the product topology on $\prod_{i\in I}\widehat{\phi}_{i*}\mathcal{O}_{\mathfrak{Z}_i}$. Since ϕ_i is finite, Krull's Theorem [EGA, 0_{I}.7.3.2] implies that the limit and adic topologies coincide. The result follows. □

1.3. **Completions of algebraic stacks.** Here we observe that the colimits constructed in Theorem 1.10 remain colimits after completing along a closed subset. To avoid developing the theory of formal algebraic stacks, we will work with adic systems of algebraic stacks. This has the advantage of being elementary as well as sufficient for our purposes.

Let X be a locally noetherian algebraic stack and suppose that $|V| \subseteq |X|$ is a closed subset, defined by a coherent $\mathcal{O}_X$-ideal I. For each $n \geq 0$ let $X_n = V(I^{n+1})$ and define:

$$\mathbf{QF}_{\mathrm{s}}(\widehat{X}_{/|V|}) := \varprojlim_n \mathbf{QF}_{\mathrm{s}}(X_n).$$

There is also a completion functor:

$$c_{X,|V|}\colon \mathbf{QF}_{\mathrm{s}}(X) \to \mathbf{QF}_{\mathrm{s}}(\widehat{X}_{/|V|}), \quad (Z \to X) \mapsto (Z \times_X X_n \to X_n)_{n\geq 0}.$$

Remark 1.19. If X is a locally noetherian scheme and $|V| \subseteq |X|$ is a closed subset, then the completion of X along $|V|$, which we also denote as $\widehat{X}_{/|V|}$, is a locally noetherian formal scheme. In particular, the category $\mathbf{QF}_{\mathrm{s}}(\widehat{X}_{/|V|})$ and the functor $c_{X,|V|}$ are currently defined twice—as above and in the previous subsection. Note that the definition of an adic morphism of locally noetherian formal schemes implies that these two categories are naturally equivalent. Moreover, the natural equivalence respects the two definitions of $c_{X,|V|}$. These are abuses of notation and for that we apologize. We firmly believe, however, that this notation will be sufficiently convenient to outweigh any potential confusion that may arise.

We conclude this section with a corollary, which is an immediate consequence of Theorem 1.10, smooth descent, and Theorem 1.18.

Corollary 1.20. *Let X be a locally noetherian algebraic stack. Suppose that $|V| \subseteq |X|$ is a closed subset. Let $\{Z_i\}_{i\in I}$ be a diagram in $\mathbf{QF}^{\mathrm{fin}}_{\mathrm{s}}(X)$ and let $Z \in \mathbf{QF}_{\mathrm{s}}(X)$ be the categorical colimit, which exists by Theorem 1.10. Then $c_{X,|V|}(Z)$ is a categorical colimit of the diagram $\{c_{X,|V|}(Z_i)\}_{i\in I}$ in $\mathbf{QF}_{\mathrm{s}}(\widehat{X}_{/|V|})$, and remains so after flat and locally noetherian base change on X.*

2. Generalized Stein factorizations

A morphism of schemes $f\colon X \to Y$ is *Stein* if the morphism $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism. If f is a proper morphism of locally noetherian schemes that is Stein, then Zariski's Connectedness Theorem [EGA, III.4.3.2] implies that f is surjective with geometrically connected fibers. Note that if f is quasi-compact and quasi-separated then it factors as:

$$X \xrightarrow{r} \widetilde{X} \xrightarrow{\tilde{f}} Y,$$

where r is Stein and $\tilde{f}$ is affine. Indeed, we simply take $\widetilde{X} = \operatorname{Spec}_Y(f_*\mathcal{O}_X)$. In general, this factorization says little about f. In the case where Y is locally noetherian and f is proper, however, one obtains the well-known *Stein factorization* of f [EGA, III.4.3.1]. In this case, $\tilde{f}$ is finite and r is proper and surjective with geometrically connected fibers. Note that the Stein factorization is also unique and compatible with flat base change on Y. We would like to generalize these properties of the Stein factorization to non-proper morphisms.

We also wish to point out that the above discussion is perfectly valid for locally noetherian formal schemes. That is, if $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ is a proper morphism of locally noetherian formal schemes, then it admits a Stein factorization: $\mathfrak{X} \xrightarrow{\rho} \widetilde{\mathfrak{X}} \xrightarrow{\tilde{\varphi}} \mathfrak{Y}$, where $\tilde{\varphi}$ is finite and ρ is proper and Stein. The Stein factorization is compatible with (not necessarily adic) flat base change on $\mathfrak{Y}$ (Proposition A.1) and ρ has geometrically connected fibers (Corollary A.2).

In this section we will address the following question: suppose that $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ is an adic morphism of locally noetherian formal schemes that is locally of finite type. Is there a factorization of φ as $\mathfrak{X} \xrightarrow{\rho} \widetilde{\mathfrak{X}} \xrightarrow{\tilde{\varphi}} \mathfrak{Y}$, where $\tilde{\varphi}$ is locally quasi-finite and ρ is proper and Stein? If such a factorization exists, we call it a *generalized Stein factorization*. A desirable property of a generalized Stein factorization will be that it is compatible with flat base change on $\mathfrak{Y}$.

Note that a necessary condition for $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ to admit a generalized Stein factorization is that every $w \in |\mathfrak{X}|$ lies in a proper connected component of the fiber $\mathfrak{X}_{\varphi(w)}$. We will call such morphisms *locally quasi-proper*. If φ is separated (resp. quasi-compact), then so is $\tilde{\varphi}$ in any generalized Stein factorization since ρ is proper and surjective. In §2.4 we prove

Theorem 2.1. *Let $\mathfrak{Y}$ be a locally noetherian formal scheme. Suppose that $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ is locally quasi-proper and separated. Then φ admits a generalized Stein factorization $\mathfrak{X} \xrightarrow{\rho} \widetilde{\mathfrak{X}} \xrightarrow{\tilde{\varphi}} \mathfrak{Y}$ which is unique and compatible with flat base change on $\mathfrak{Y}$. Moreover, $\tilde{\varphi}$ is separated and if φ is quasi-compact, so is $\tilde{\varphi}$.*

In future work, we will prove Theorem 2.1 for algebraic stacks in order to compare our generalized Stein factorizations with the connected component fibrations of [LMB, §6.8] and [Rom11, Thm. 2.5.2]. We wish to point out that Theorem 2.1 should extend to a reasonable theory of formal algebraic spaces or stacks. Unfortunately, the only treatise on formal algebraic spaces that we are aware of is D. Knutson's [Knu71, V], and this requires everything to be separated [Knu71, V.2.1]—an assumption we definitely do not wish to impose. We are not aware of any written account of a theory of formal algebraic stacks.

Remark 2.2. We will construct the generalized Stein factorization by working étale-locally on $\mathfrak{Y}$. If $f\colon X \to Y$ is a separated and locally quasi-proper morphism of schemes that is quasi-compact, then there is a more explicit construction of the generalized Stein factorization. If $\overline{Y}$ denotes the integral closure of Y in $f_*\mathcal{O}_X$, then the image of $X \to \overline{Y}$ is open and equals $\widetilde{X}$. This follows from Theorem 2.1 and [EGA, IV.8.12.3]. Proving directly that the image is open, that $X \to \widetilde{X}$ is proper and that $\widetilde{X} \to Y$ is quasi-finite is possible but not trivial. This construction also has the deficiency that we were unable to easily adapt it to locally noetherian formal schemes.

2.1. **Uniqueness and base change of generalized Stein factorizations.** In this subsection we address the uniqueness and base change assertions in Theorem 2.1. Both assertions will also be important for the proof of the existence of generalized Stein factorizations in Theorem 2.1.

Lemma 2.3. *Let $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of locally noetherian formal schemes that admits a factorization $\mathfrak{X} \xrightarrow{\rho} \widetilde{\mathfrak{X}} \xrightarrow{\tilde{\varphi}} \mathfrak{Y}$ where ρ is proper and $\tilde{\varphi}$ is locally quasi-finite and separated. Then ρ is Stein if and only if the natural map:*

$$\operatorname{Hom}_{\mathfrak{Y}}(\widetilde{\mathfrak{X}}, \mathfrak{Z}) \to \operatorname{Hom}_{\mathfrak{Y}}(\mathfrak{X}, \mathfrak{Z})$$

is bijective for every locally quasi-finite and separated morphism $s\colon \mathfrak{Z} \to \mathfrak{Y}$. In particular, for locally quasi-proper and separated morphisms, the generalized Stein factorization, if it exists, is unique.

Proof. A $\mathfrak{Y}$-morphism $\widetilde{\mathfrak{X}} \to \mathfrak{Z}$ is equivalent to a section of the projection $\mathfrak{Z} \times_{\mathfrak{Y}} \widetilde{\mathfrak{X}} \to \widetilde{\mathfrak{X}}$. In particular, we may now replace $\mathfrak{Y}$ with $\widetilde{\mathfrak{X}}$ and $\mathfrak{Z}$ with $\mathfrak{Z} \times_{\mathfrak{Y}} \widetilde{\mathfrak{X}}$. Thus we have reduced the result to the situation where φ is proper and $\widetilde{\mathfrak{X}} = \mathfrak{Y}$.

Now let $\mathfrak{X} \xrightarrow{\beta'} \operatorname{St}_{\mathfrak{Y}}(\mathfrak{X}) \xrightarrow{b} \mathfrak{Y}$ be the Stein factorization of φ (which exists because φ is proper). Fix a $\mathfrak{Y}$-morphism $\alpha\colon \mathfrak{X} \to \mathfrak{Z}$. Since $\mathfrak{Z}$ is separated over $\mathfrak{Y}$ and φ is proper, it follows that α is proper. Thus, there is a Stein factorization of $\alpha\colon \mathfrak{X} \xrightarrow{\alpha'} \mathrm{St}_{\mathfrak{Z}}(\mathfrak{X}) \xrightarrow{a} \mathfrak{Z}$. Note, however, that $\gamma\colon \mathrm{St}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathfrak{Y}$ is quasi-finite and proper, thus finite [EGA, III.4.8.11]. Next observe that we have natural isomorphisms of coherent sheaves of $\mathcal{O}_{\mathfrak{Y}}$-algebras:

$$b_*\mathcal{O}_{\mathrm{St}_{\mathfrak{Y}}(\mathfrak{X})} \cong b_*\beta'_*\mathcal{O}_{\mathfrak{X}} \cong \gamma_*\alpha'_*\mathcal{O}_{\mathfrak{X}} \cong \gamma_*\mathcal{O}_{\mathrm{St}_{\mathfrak{Z}}(\mathfrak{X})}.$$

Hence, there exists a unique $\mathfrak{Y}$-isomorphism $\delta\colon \mathrm{St}_{\mathfrak{Y}}(\mathfrak{X}) \to \mathrm{St}_{\mathfrak{Z}}(\mathfrak{X})$ that is compatible with the data. If φ is Stein, then $\mathrm{St}_{\mathfrak{Y}}(\mathfrak{X}) = \mathfrak{X}$ and $b = \mathrm{id}_{\mathfrak{X}}$. We deduce that there is a uniquely induced $\mathfrak{Y}$-morphism $\mathfrak{Y} \xrightarrow{\delta} \mathrm{St}_{\mathfrak{Z}}(\mathfrak{X}) \xrightarrow{a} \mathfrak{Z}$. Conversely, we have a Stein factorization $\mathfrak{X} \to \mathrm{St}_{\widetilde{\mathfrak{X}}}(\mathfrak{X}) \to \widetilde{\mathfrak{X}}$. By the Stein case already considered, we see that $\mathrm{St}_{\widetilde{\mathfrak{X}}}(\mathfrak{X})$ and $\widetilde{\mathfrak{X}}$ both satisfy the same universal property for locally quasi-finite and separated $\mathfrak{Y}$-schemes. The result follows. $\square$

Fix a locally noetherian formal scheme $\mathfrak{Y}$ and let $\mathbf{LQP}_{\mathrm{s}}(\mathfrak{Y})$ (resp. $\mathbf{LQF}_{\mathrm{s}}(\mathfrak{Y})$) denote the category of locally quasi-proper (resp. locally quasi-finite) and separated morphisms $\mathfrak{X} \to \mathfrak{Y}$. Lemma 2.3 allows us to reinterpret Theorem 2.1 as the existence of a left adjoint:

$$\mathrm{St}_{\mathfrak{Y}}\colon \mathbf{LQP}_{\mathrm{s}}(\mathfrak{Y}) \to \mathbf{LQF}_{\mathrm{s}}(\mathfrak{Y})$$

to the inclusion $\mathbf{LQF}_{\mathrm{s}}(\mathfrak{Y}) \subseteq \mathbf{LQP}_{\mathrm{s}}(\mathfrak{Y})$ such that if $\mathfrak{Z} \in \mathbf{LQP}_{\mathrm{s}}(\mathfrak{Y})$, then:

(1) the natural morphism $\mathfrak{Z} \to \mathrm{St}_{\mathfrak{Y}}(\mathfrak{Z})$ is proper;
(2) if $\mathfrak{Y}' \to \mathfrak{Y}$ is flat, then we have a natural isomorphism:

$$\mathrm{St}_{\mathfrak{Y}'}(\mathfrak{Z} \times_{\mathfrak{Y}} \mathfrak{Y}') \cong \mathrm{St}_{\mathfrak{Y}}(\mathfrak{Z}) \times_{\mathfrak{Y}} \mathfrak{Y}'.$$

Indeed, the proper morphism $\mathfrak{Z} \to \mathrm{St}_{\mathfrak{Y}}(\mathfrak{Z})$ is Stein by Lemma 2.3 and it is unique by adjunction. The remainder of this section is devoted to the proof of the existence of an adjoint $\mathrm{St}_{\mathfrak{Y}}$ with properties (1) and (2). The properness of $\mathfrak{Z} \to \mathrm{St}_{\mathfrak{Y}}(\mathfrak{Z})$ is not automatic, but a consequence of the construction. We can, however, address property (2) immediately.

Lemma 2.4. *Fix a cartesian diagram of locally noetherian formal schemes:*

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{p'} & \mathfrak{X} \\ {\scriptstyle\varphi'}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathfrak{Y}' & \xrightarrow{p} & \mathfrak{Y}. \end{array}$$

Suppose that φ is separated and admits a generalized Stein factorization $\mathfrak{X} \to \mathrm{St}_{\mathfrak{Y}}(\mathfrak{X}) \to \mathfrak{Y}$. Then φ' admits a generalized Stein factorization $\mathfrak{X}' \to \mathrm{St}_{\mathfrak{Y}'}(\mathfrak{X}') \to \mathfrak{Y}'$ and there is a naturally induced $\mathfrak{Y}'$-morphism:

$$h\colon \mathrm{St}_{\mathfrak{Y}'}(\mathfrak{X}') \to \mathrm{St}_{\mathfrak{Y}}(\mathfrak{X}) \times_{\mathfrak{Y}} \mathfrak{Y}',$$

which is a finite universal homeomorphism. In addition, if p is flat, then the morphism h is an isomorphism.

Proof. The morphism φ' factors as $\mathfrak{X}' \xrightarrow{\gamma} \mathrm{St}_{\mathfrak{Y}}(\mathfrak{X}) \times_{\mathfrak{Y}} \mathfrak{Y}' \xrightarrow{\sigma} \mathfrak{Y}'$ where σ is locally quasi-finite and separated and γ is proper. Next observe that the Stein factorization of γ is the generalized Stein factorization of φ'. Thus γ factors as:

$$\mathfrak{X}' \to \mathrm{St}_{\mathfrak{Y}'}(\mathfrak{X}') \xrightarrow{h} \mathrm{St}_{\mathfrak{Y}}(\mathfrak{X}) \times_{\mathfrak{Y}} \mathfrak{Y}'.$$

By Zariski's Connectedness Theorem (Corollary A.2) $\mathfrak{X}' \to \mathrm{St}_{\mathfrak{Y}'}(\mathfrak{X}')$ and γ have geometrically connected fibers. It follows that h is a finite universal homeomorphism (it is finite with geometrically connected fibers). In addition, if p is flat, then since cohomology commutes with flat base change (Proposition A.1), γ is already Stein, and so h is an isomorphism. $\square$

2.2. **Local decompositions.** Before we prove the existence of the adjoint $\mathrm{St}_{\mathfrak{Y}}$ it will be necessary to understand the étale local structure of locally quasi-proper morphisms. We accomplish this by generalizing the well-known structure results that are available for locally quasi-finite and separated morphisms [EGA, IV.18.5.11c, IV.18.12.1].

Proposition 2.5. *Let $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ be an adic morphism of locally noetherian formal schemes that is locally of finite type and separated. Let $y\colon \mathrm{Spec}(k) \to \mathfrak{Y}$ be a point and suppose that Z is a connected component of the fiber $\mathfrak{X}_y$. If Z is proper, then there exists an adic étale neighborhood $(\mathfrak{Y}', y') \to (\mathfrak{Y}, y)$ and an open and closed immersion $i\colon \mathfrak{X}_Z \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}'$ where $\mathfrak{X}_Z \to \mathfrak{Y}'$ is proper and $(\mathfrak{X}_Z)_{y'} \cong Z$.*

Proof. We first prove the result in the setting of a morphism of locally noetherian schemes $f\colon X \to Y$. We now immediately reduce to the case where Y is affine and f is quasi-compact. Let $k_0 \subseteq k$ denote the separable closure of the residue field $k(y)$ in k. Then the resulting morphism $X_y \to X_{k_0}$ induces a bijection on connected components [EGA, IV.4.3.2]. So, we may replace k with k_0 and Z by the corresponding connected component in X_{k_0}. By standard limit and smearing out arguments [EGA, IV.8] we further reduce to the case where Y is local and henselian with closed point y and residue field k.

By Chow's Lemma [EGA, II.5.6.1], there exists a quasi-projective Y-scheme W and a proper and surjective Y-morphism $p\colon W \to X$. If the result has been proved for $W \to Y$, then we claim that this implies the result for $f\colon X \to Y$. Indeed, each connected component of $p^{-1}(Z)$ is proper, thus we may write $W = W_{p^{-1}(Z)} \amalg W'$ where $W_{p^{-1}(Z)} \to Y$ is proper, and the connected components of $W_{p^{-1}(Z)}$ coincide with the connected components of $(W_{p^{-1}(Z)})_y = p^{-1}(Z)$. Now take $X_Z = p(W_{p^{-1}(Z)})$ and $X' = p(W')$. Then X_Z is proper over Y and it remains to show that $X_Z \cap X' = \emptyset$. Note that $X_Z \cap X'$ is a closed subset of X_Z. If it is non-empty, then it must have non-empty intersection with Z. But this would imply that $Z \cap X' \neq \emptyset$, hence $p^{-1}(Z) \cap W' \neq \emptyset$—a contradiction. We deduce that it remains to prove the result when X is also quasi-projective over Y.

In this case, there is an open and dense immersion $i\colon X \hookrightarrow \overline{X}$ where $\overline{X}$ is a projective Y-scheme. Note that if V is a connected component of $\overline{X}_y$ and $V \cap Z \neq \emptyset$, then $V \cap Z = V$ since Z is proper. By Corollary B.4, there is also a bijection between the set of connected components of $\overline{X}_y$ and $\overline{X}$. Set X_Z to be the union of those connected components of $\overline{X}$ that meet Z. Observe that $X_Z \cap X$ is an open subset of X_Z which contains $(X_Z)_y = Z$. Since $X_Z \to Y$ is proper and Y is a local scheme, we have that $X_Z \cap X = X_Z$ and we conclude that $X_Z \subseteq X$. In particular, we readily deduce that X_Z is an open and closed subset of X and we have the claimed decomposition $X = X_Z \amalg X'$ with $(X_Z)_y = Z$.

Now let $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ be as in the Proposition and let $\mathfrak{I}$ be an ideal of definition for $\mathfrak{Y}$. Consider the locally noetherian scheme $Y = V(\mathfrak{I}) \subseteq \mathfrak{Y}$. Let X be the fiber product $\mathfrak{X} \times_{\mathfrak{Y}} Y$ and let $f\colon X \to Y$ denote the induced morphism. Since f is adic, X is a locally noetherian scheme and f is locally of finite type and separated. By what we have proven for schemes, there is an étale neighborhood $(Y', y') \to (Y, y)$ and an open and closed immersion $i\colon X_Z \to X \times_Y Y'$, where $X_Z \to Y'$ is proper and $(X_Z)_{y'} \cong Z$. By [EGA, IV.18.1.2], there is a unique lift of the étale neighborhood $(Y', y') \to (Y, y)$ to an adic étale neighborhood $(\mathfrak{Y}', y') \to (\mathfrak{Y}, y)$ such that the pullback along $(Y, y) \to (\mathfrak{Y}, y)$ coincides with $(Y', y') \to (Y, y)$. The result follows. □

Remark 2.6. In future work we will prove Proposition 2.5 for non-separated morphisms of non-noetherian algebraic stacks. We do not treat this here as the necessary reformulations in the non-locally-separated case would take us too far afield.

We conclude this subsection with a trivial, but important, corollary.

Corollary 2.7. *Let $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of locally noetherian formal schemes that is separated and locally quasi-proper. Then there exists an adic étale morphism $\mathfrak{Y}' \to \mathfrak{Y}$, and an open and closed immersion $i\colon \mathfrak{U} \hookrightarrow \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}'$ such that the composition $\mathfrak{U} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}' \to \mathfrak{X}$ is surjective and the composition $\mathfrak{U} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}' \to \mathfrak{Y}'$ is proper.*

Thus Corollary 2.7 implies that any separated and locally quasi-proper morphism $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ admits an étale slice that has a generalized Stein factorization. Thus in order to construct the generalized Stein factorization of φ, we can use étale descent. To describe the relevant descent data, we will address a more general problem in §2.3: for a proper and Stein morphism, which étale morphisms are pulled back from the base?

2.3. **Pullbacks of étale morphisms.** Let $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ be a proper and Stein morphism of locally noetherian formal schemes. In this subsection we will identify those étale morphisms to $\mathfrak{X}$ that are pulled back from $\mathfrak{Y}$.

So, let $\mathfrak{S}$ be a locally noetherian formal scheme. Define $\mathbf{Et}_{\mathrm{s}}(\mathfrak{S})$ to be the category of morphisms $(\mathfrak{V} \to \mathfrak{S})$ that are adic étale and separated. Also, define $\mathbf{OC}(\mathfrak{S})$ to be the set of open and closed immersions into $\mathfrak{S}$. Let $\mathfrak{I}$ be an ideal of definition of $\mathfrak{S}$. If we let $S = V(\mathfrak{I})$, then S is a locally noetherian scheme and the natural map $\mathbf{OC}(\mathfrak{S}) \to \mathbf{OC}(S)$ is bijective. More generally, an easy consequence of [EGA, IV.18.1.2] is that the functor $\mathbf{Et}_{\mathrm{s}}(\mathfrak{S}) \to \mathbf{Et}_{\mathrm{s}}(S)$ is an equivalence of categories.

Let $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ be an adic morphism of locally noetherian formal schemes. There is an induced functor

$$\varphi^*\colon \mathbf{Et}_{\mathrm{s}}(\mathfrak{Y}) \to \mathbf{Et}_{\mathrm{s}}(\mathfrak{X}), \quad (\mathfrak{V} \to \mathfrak{Y}) \mapsto (\mathfrak{V} \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{X}).$$

Let $\mathbf{Et}_{\mathrm{s,t}/\mathfrak{Y}}(\mathfrak{X})$ be the full subcategory of $\mathbf{Et}_{\mathrm{s}}(\mathfrak{X})$ with objects those $\mathfrak{W} \to \mathfrak{X}$ such that for any geometric point $\overline{y}$ of $\mathfrak{Y}$, and any connected component Z of $\mathfrak{W}_{\overline{y}}$, the induced morphism $Z \to \mathfrak{X}_{\overline{y}}$ is an open and closed immersion. Note that $\mathbf{OC}(\mathfrak{X}) \subseteq \mathbf{Et}_{\mathrm{s,t}/\mathfrak{Y}}(\mathfrak{X})$. In the following Lemma, we characterize the essential image of φ^*.

Lemma 2.8. *Let $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ be a proper and Stein morphism of locally noetherian formal schemes.*

(1) *The functor $\varphi^*\colon \mathbf{Et}_{\mathrm{s}}(\mathfrak{Y}) \to \mathbf{Et}_{\mathrm{s}}(\mathfrak{X})$ is fully faithful with image $\mathbf{Et}_{\mathrm{s,t}/\mathfrak{Y}}(\mathfrak{X})$.*
(2) *The induced map $\mathbf{OC}(\mathfrak{Y}) \to \mathbf{OC}(\mathfrak{X})$ is bijective.*
(3) *If $\mathfrak{W} \in \mathbf{Et}_{\mathrm{s,t}/\mathfrak{Y}}(\mathfrak{X})$, then it admits a generalized Stein factorization $\mathfrak{W} \to \mathrm{St}_{\mathfrak{Y}}(\mathfrak{W}) \to \mathfrak{Y}$, with $\mathrm{St}_{\mathfrak{Y}}(\mathfrak{W}) \to \mathfrak{Y}$ étale and separated. Moreover, $\mathfrak{W} = \mathfrak{Y} \times_{\mathfrak{X}} \mathrm{St}_{\mathfrak{Y}}(\mathfrak{W})$.*

Proof. For (1) and (2), by passing to the underlying morphism of schemes we obtain, via Zariski's Connected Theorem (Corollary A.2), a proper and surjective morphism of locally noetherian schemes $f\colon X \to Y$ with geometrically connected fibers. By Proposition B.1 and Remark B.2, the functor $f^*\colon \mathbf{Et}_{\mathrm{s}}(Y) \to \mathbf{Et}_{\mathrm{s}}(X)$ is fully faithful with image $\mathbf{Et}_{\mathrm{s,t}/Y}(X)$ and the induced map $\mathbf{OC}(\mathfrak{Y}) \to \mathbf{OC}(\mathfrak{X})$ is bijective. The claim (3) follows from (1) and Lemma 2.4. □

2.4. **Existence of generalized Stein factorizations.** In this subsection we will prove Theorem 2.1. Before we can accomplish this we require one more Lemma. This Lemma is well-known for schemes, though requires some care for formal schemes.

Lemma 2.9. *Let $\mathfrak{X}$ be a locally noetherian formal scheme and let $\mathfrak{U} \to \mathfrak{X}$ be an adic, locally quasi-finite, and separated morphism. Suppose that $[\mathfrak{R} \rightrightarrows \mathfrak{U}]$ is an adic flat equivalence relation over $\mathfrak{X}$ such that $\mathfrak{R} \to \mathfrak{U} \times_{\mathfrak{X}} \mathfrak{U}$ is a closed immersion. Then this equivalence relation has a formal geometric quotient $\mathfrak{Z}$ that is adic, locally quasi-finite, and separated over $\mathfrak{X}$. Moreover, taking the quotient commutes with arbitrary (not necessarily adic) base change, the quotient map $\mathfrak{U} \to \mathfrak{Z}$ is adic and faithfully flat, and $\mathfrak{R} = \mathfrak{U} \times_{\mathfrak{Z}} \mathfrak{U}$.*

Proof. Let $\mathfrak{I}$ be an ideal of definition for $\mathfrak{X}$ and denote $X_{\mathfrak{I}} = V(\mathfrak{I})$, $U_{\mathfrak{I}} = \mathfrak{U} \times_{\mathfrak{X}} X_{\mathfrak{I}}$, and $R_{\mathfrak{I}} = \mathfrak{R} \times_{\mathfrak{X}} X_{\mathfrak{I}}$. Then, the hypotheses ensure that $[R_{\mathfrak{I}} \rightrightarrows U_{\mathfrak{I}}]$ is an fppf equivalence relation over $X_{\mathfrak{I}}$. We observe that the quotient of this fppf equivalence relation in the category of algebraic spaces is a scheme $Z_{\mathfrak{I}}$, as it is locally quasi-finite and separated over $X_{\mathfrak{I}}$ [Knu71, II.6.16]. Note that for any other ideal of definition $\mathfrak{J} \supset \mathfrak{I}$ there is a natural isomorphism $Z_{\mathfrak{I}} \times_{\mathfrak{X}} X_{\mathfrak{J}} \cong Z_{\mathfrak{J}}$. Hence, the directed system $\{Z_{\mathfrak{I}}\}_{\mathfrak{I}}$ is adic over $\mathfrak{X}$. Taking the colimit of this system in the category of topologically ringed spaces produces a locally noetherian formal scheme $\mathfrak{Z}$, which is adic, locally quasi-finite, and separated over $\mathfrak{X}$. It is now immediate that $\mathfrak{Z}$ is a formal geometric quotient of the given equivalence relation. That $\mathfrak{Z}$ has the stated properties is an easy consequence of the corresponding results for its truncations $Z_{\mathfrak{J}}$. □

We now proceed to the proof of Theorem 2.1.

Proof of Theorem 2.1. The uniqueness and stability under base change of generalized Stein factorizations are consequences of Lemmas 2.3 and 2.4. Thus it remains to address the existence of a generalized Stein factorization $\mathfrak{X} \xrightarrow{\rho} \widetilde{\mathfrak{X}} \xrightarrow{\tilde{\varphi}} \mathfrak{Y}$ and show that $\tilde{\varphi}$ has the prescribed properties.

Note that if $\mathfrak{Y} \to \mathfrak{T}$ is locally quasi-finite and separated, then $\mathrm{St}_{\mathfrak{Y}}(\mathfrak{X}) \cong \mathrm{St}_{\mathfrak{T}}(\mathfrak{X})$ over $\mathfrak{T}$. Moreover, $\mathbf{Et}_{\mathrm{s,t}/\mathfrak{Y}}(\mathfrak{X}) \simeq \mathbf{Et}_{\mathrm{s,t}/\mathfrak{T}}(\mathfrak{X})$. Thus if $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ factors as $\mathfrak{X} \to \mathfrak{Y}' \to \mathfrak{Y}$, where $\mathfrak{X} \to \mathfrak{Y}'$ is proper and $\mathfrak{Y}' \to \mathfrak{Y}$ is locally quasi-finite and separated, then the Stein factorization of $\mathfrak{X} \to \mathfrak{Y}'$ gives the generalized Stein factorization of $\mathfrak{X} \to \mathfrak{Y}$.

So, we fix a locally quasi-proper and separated morphism $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$. By Corollary 2.7 there is an adic, étale, and separated morphism $\mathfrak{Y}' \to \mathfrak{Y}$, together with an open and closed immersion $i\colon \mathfrak{U} \hookrightarrow \mathfrak{X}' := \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}'$, such that the induced morphism $\mathfrak{U} \to \mathfrak{Y}'$ is proper, and the induced morphism $\mathfrak{U} \to \mathfrak{X}$ is adic, étale, separated, and surjective. By the remarks above we obtain a generalized Stein factorization $\mathfrak{U} \to \mathrm{St}_{\mathfrak{Y}}(\mathfrak{U}) \to \mathfrak{Y}$ (note that $\mathrm{St}_{\mathfrak{Y}}(\mathfrak{U}) = \mathrm{St}_{\mathfrak{Y}'}(\mathfrak{U})$).

Let $\mathfrak{Y}'' = \mathfrak{Y}' \times_{\mathfrak{Y}} \mathfrak{Y}'$ and take $s_1, s_2\colon \mathfrak{Y}'' \to \mathfrak{Y}'$ to denote the two projections. Let $\mathfrak{X}'' = \mathfrak{X}' \times_{\mathfrak{X}} \mathfrak{X}'$ and denote by t_1 and t_2 the two projections. For $j = 1$ and 2 let $\mathfrak{R}_j$ denote the pullback of $\mathfrak{U}$ along t_j. Note that the morphisms $i_j\colon \mathfrak{R}_j \to \mathfrak{X}''$ are open and closed immersions. Let $\mathfrak{R} = \mathfrak{R}_1 \cap \mathfrak{R}_2 = \mathfrak{U} \times_{\mathfrak{X}} \mathfrak{U}$. Then we obtain an adic étale equivalence relation $[\mathfrak{R} \rightrightarrows \mathfrak{U}]$ with quotient $\mathfrak{X}$.

Note that both morphisms $\mathfrak{R} \to \mathfrak{U}$ belong to $\mathbf{Et}_{\mathrm{s,t}/\mathfrak{Y}}(\mathfrak{U})$. The morphism $\mathfrak{U} \to \mathrm{St}_{\mathfrak{Y}}(\mathfrak{U})$ is also proper and Stein. By Lemma 2.8(1) we see that the functor $\mathbf{Et}_{\mathrm{s}}(\mathrm{St}_{\mathfrak{Y}}(\mathfrak{U})) \to \mathbf{Et}_{\mathrm{s,t}/\mathfrak{Y}}(\mathfrak{U})$ is an equivalence of categories (note that $\mathbf{Et}_{\mathrm{s,t}/\mathfrak{Y}}(\mathfrak{U}) \simeq \mathbf{Et}_{\mathrm{s,t}/\mathrm{St}_{\mathfrak{Y}}(\mathfrak{U})}(\mathfrak{U})$). Functoriality, together with Lemma 2.8(3), produces an adic étale groupoid $[\mathrm{St}_{\mathfrak{Y}}(\mathfrak{R}) \rightrightarrows \mathrm{St}_{\mathfrak{Y}}(\mathfrak{U})]$ in $\mathbf{LQF}_{\mathrm{s}}(\mathfrak{Y})$ which pulls back to the equivalence relation $[\mathfrak{R} \rightrightarrows \mathfrak{U}]$. We will now verify that $\mathrm{St}_{\mathfrak{Y}}(\mathfrak{R}) \to \mathrm{St}_{\mathfrak{Y}}(\mathfrak{U}) \times_{\mathfrak{Y}} \mathrm{St}_{\mathfrak{Y}}(\mathfrak{U})$ is a closed immersion.

Note that for $j = 1$ and 2 we have that $\mathfrak{R} \in \mathbf{OC}(\mathfrak{R}_j)$. By Lemma 2.8(2,3) we thus see that $\mathrm{St}_{\mathfrak{Y}}(\mathfrak{R}) \in \mathbf{OC}(\mathrm{St}_{\mathfrak{Y}}(\mathfrak{R}_j))$. We also know that $\mathrm{St}_{\mathfrak{Y}}(\mathfrak{R}_j)$ is the pullback of $\mathrm{St}_{\mathfrak{Y}}(\mathfrak{U}) \to \mathfrak{Y}'$ along $s_j\colon \mathfrak{Y}'' \to \mathfrak{Y}'$ (Lemma 2.4). Thus we obtain a $\mathfrak{Y}$-isomorphism:

$$\mathrm{St}_{\mathfrak{Y}}(\mathfrak{R}_1) \times_{\mathfrak{Y}''} \mathrm{St}_{\mathfrak{Y}}(\mathfrak{R}_2) \to \mathrm{St}_{\mathfrak{Y}}(\mathfrak{U}) \times_{\mathfrak{Y}} \mathrm{St}_{\mathfrak{Y}}(\mathfrak{U}).$$

We have already seen, however, that the map $\mathrm{St}_{\mathfrak{Y}}(\mathfrak{R}) \to \mathrm{St}_{\mathfrak{Y}}(\mathfrak{R}_1)$ is an open and closed immersion. Since everything is separated, it follows that the natural map $\mathrm{St}_{\mathfrak{Y}}(\mathfrak{R}) \to \mathrm{St}_{\mathfrak{Y}}(\mathfrak{R}_1) \times_{\mathfrak{Y}''} \mathrm{St}_{\mathfrak{Y}}(\mathfrak{R}_2) = \mathrm{St}_{\mathfrak{Y}}(\mathfrak{U}) \times_{\mathfrak{Y}} \mathrm{St}_{\mathfrak{Y}}(\mathfrak{U})$ is a closed immersion. Thus, we have an adic étale equivalence relation $[\mathrm{St}_{\mathfrak{Y}}(\mathfrak{R}) \rightrightarrows \mathrm{St}_{\mathfrak{Y}}(\mathfrak{U})]$ in $\mathbf{LQF}_{\mathrm{s}}(\mathfrak{Y})$ satisfying the hypotheses of Lemma 2.9.

Let $\widetilde{\mathfrak{X}} \in \mathbf{LQF}_{\mathrm{s}}(\mathfrak{Y})$ be the quotient of this equivalence relation. Since $\mathrm{St}_{\mathfrak{Y}}(\mathfrak{U}) \to \widetilde{\mathfrak{X}}$ is adic étale and $\mathrm{St}_{\mathfrak{Y}}(\mathfrak{R}) = \mathrm{St}_{\mathfrak{Y}}(\mathfrak{U}) \times_{\widetilde{\mathfrak{X}}} \mathrm{St}_{\mathfrak{Y}}(\mathfrak{U})$, the proper and Stein morphism $\mathfrak{U} \to \mathrm{St}_{\mathfrak{Y}}(\mathfrak{U})$, with descent data given by $\mathfrak{R} \to \mathrm{St}_{\mathfrak{Y}}(\mathfrak{R})$, descends to a proper and Stein morphism $\mathfrak{X} \to \widetilde{\mathfrak{X}}$. This is the generalized Stein factorization of φ by Lemma 2.3. □

3. ADJUNCTIONS FOR QUASI-FINITE SPACES

If $p\colon X' \to X$ is a morphism of algebraic stacks, then there is a pullback functor $p^*\colon \mathbf{QF}_{\mathrm{s}}(X) \to \mathbf{QF}_{\mathrm{s}}(X')$ given by $(Z \to X) \mapsto (Z \times_X X' \to X')$. More generally, given a closed subset $|V| \subseteq |X|$, set $|V'| = p^{-1}|V|$. Then there is an induced pullback functor $\widehat{p}^*_{/|V|}\colon \mathbf{QF}_{\mathrm{s}}(\widehat{X}_{/|V|}) \to \mathbf{QF}_{\mathrm{s}}(\widehat{X}'_{/|V'|})$. In this short section we will use generalized Stein factorizations to construct left adjoints to these functors when p is proper and schematic (i.e., those proper morphisms that are representable by schemes). This is certainly true in greater generality, though we limit ourselves to this situation for simplicity.

Definition 3.1. Suppose that $p\colon X' \to X$ is a proper morphism of locally noetherian algebraic stacks. Let $|V| \subseteq |X|$ be a closed subset and let $|V'| = p^{-1}|V|$. Let $\mathbf{QF}_{\mathrm{s,qp}/X}(\widehat{X}'_{/|V'|})$ denote the full subcategory of $\mathbf{QF}_{\mathrm{s}}(\widehat{X}'_{/|V'|})$ consisting of those adic systems $(Z_n \to X'_n)_{n\geq 0}$ such that the composition $Z_0 \to X'_0 \to X_0$ has proper fibers.

We now have the following Theorem.

Theorem 3.2. *Suppose that $p\colon X' \to X$ is a proper and schematic morphism of locally noetherian algebraic stacks. Let $|V| \subseteq |X|$ be a closed subset and let $|V'| = p^{-1}|V|$. Then the functor $\widehat{p}^*_{/|V|}\colon \mathbf{QF}_{\mathrm{s}}(\widehat{X}_{/|V|}) \to \mathbf{QF}_{\mathrm{s}}(\widehat{X}'_{/|V'|})$ factors through $\mathbf{QF}_{\mathrm{s,qp}/X}(\widehat{X}'_{/|V'|})$ and admits a left adjoint:*

$$(\widehat{p}_{/|V|})_!\colon \mathbf{QF}_{\mathrm{s,qp}/X}(\widehat{X}'_{/|V'|}) \to \mathbf{QF}_{\mathrm{s}}(\widehat{X}_{/|V|}),$$

which is compatible with locally noetherian and flat base change on X. Moreover, for $(Z \to X') \in \mathbf{QF}_{\mathrm{s,qp}/X}(X')$, the induced natural map:

$$(\widehat{p}_{/|V|})_! \circ c_{X',|V'|}(Z \to X') \to c_{X,|V|} \circ p_!(Z \to X)$$

is an isomorphism.

Proof. We wish to point out that if $|V| = |X|$, then $\widehat{X}_{/|V|} = X$ and $\mathbf{QF}_{\mathrm{s}}(\widehat{X}_{/|V|}) = \mathbf{QF}_{\mathrm{s}}(X)$. Thus we denote $\widehat{p}^*_{/|V|}$ by p^* and the functor $p_!$ (which appears at the end of the statement) is a left adjoint to p^*.

Now the first claim about the factorization is trivial. For the existence and properties of $(\widehat{p}_{/|V|})_!$, by smooth descent, it is sufficient to prove the result in the case where $p\colon X' \to X$ is a morphism of locally noetherian schemes. Set $\mathfrak{X} = \widehat{X}_{/|V|}$ and $\mathfrak{X}' = \widehat{X}'_{/|V'|}$ and let $\pi\colon \mathfrak{X}' \to \mathfrak{X}$ be the induced morphism.

Let $(Z_n \to X'_n)_{n\geq 0} \in \mathbf{QF}_{\mathrm{s,qp}/\mathfrak{X}}(\mathfrak{X}')$ which we may also view as a quasi-finite and separated morphism of locally noetherian formal schemes $\mathfrak{Z} \to \mathfrak{X}'$ such that the composition $\mathfrak{Z} \to \mathfrak{X}' \to \mathfrak{X}$ is locally quasi-proper and separated. By Theorem 2.1, there is a generalized Stein factorization $\mathfrak{Z} \xrightarrow{\rho} \mathrm{St}_{\mathfrak{X}}(\mathfrak{Z}) \to \mathfrak{X}$ and $\mathrm{St}_{\mathfrak{X}}(\mathfrak{Z}) \to \mathfrak{X}$ is quasi-finite and separated. We set $\pi_!\mathfrak{Z} = \mathrm{St}_{\mathfrak{X}}(\mathfrak{Z})$ and note that $\pi_!$ is a left adjoint to π^* by Lemma 2.3. The compatibility with flat base change and completions is implied by the flat base change property of $\mathrm{St}_{\mathfrak{X}}$, together with the fact that $\mathfrak{X} \to X$ is a (non-adic) flat morphism [EGA, I.10.8.9]. □

4. The Existence Theorem

Definition 4.1. Fix a morphism of algebraic stacks $\pi\colon X \to Y$ and a closed subset $|Y_0| \subseteq |Y|$. Let $|X_0| = \pi^{-1}|Y_0|$. Define $\mathbf{QF}_{\mathrm{p}/Y}(\widehat{X}_{/|X_0|})$ to be the full subcategory of $\mathbf{QF}_{\mathrm{s,qp}/Y}(\widehat{X}_{/|X_0|})$ consisting of those $(Z_n \to X)_{n\geq 0}$ such that the composition $Z_0 \to X \to Y$ is proper.

Note that $\mathbf{QF}_{\mathrm{p}/Y}(\widehat{X}_{/|X_0|})$ is also a full subcategory of $\mathbf{QF}^{\mathrm{fin}}_{\mathrm{s}}(\widehat{X}_{/|X_0|})$. That is, every morphism in $\mathbf{QF}_{\mathrm{p}/Y}(\widehat{X}_{/|X_0|})$ is finite.

Throughout this section we fix a noetherian ring R and an ideal $I \subseteq R$, such that R is I-adic. Let $S = \operatorname{Spec} R$ and $S_n = \operatorname{Spec}(R/I^{n+1})$. Consider a morphism of algebraic stacks $\pi\colon X \to S$ that is locally of finite type. For $n \geq 0$ let $X_n = X \times_S S_n$. It will be convenient for us to abbreviate the symbol $\widehat{X}_{/|X_0|}$ to $\widehat{X}$. Consider the I-adic completion functor:

$$\Psi_{\pi,I}\colon \mathbf{QF}_{\mathrm{p}/S}(X) \longrightarrow \mathbf{QF}_{\mathrm{p}/S}(\widehat{X}), \quad (Z \to X) \mapsto (Z \times_X X_n \to X_n)_{n\geq 0}.$$

In this section we prove an Existence Theorem, which gives sufficient conditions on the morphism $\pi\colon X \to S$ for the functor $\Psi_{\pi,I}$ to be an equivalence of categories.

In the case that $\pi\colon X \to S$ is separated, the category $\mathbf{QF}_{\mathrm{p}/S}(X)$ is equivalent to the category of finite algebras over X with S-proper support. An immediate consequence of [Ols05, Thm. 1.4], with the amplification [Ols06b, Thm. A.1][1], is that the functor $\Psi_{\pi,I}$ is an equivalence in this situation. Note that [*loc. cit.*] also gives a straightforward proof that $\Psi_{\pi,I}$ is fully faithful when π is no longer assumed to be separated. Indeed, we have

Lemma 4.2. *Let $\pi\colon X \to S$ be a morphism of algebraic stacks that is locally of finite type with quasi-compact and separated diagonal. Then the functor $\Psi_{\pi,I}$ is fully faithful.*

Proof. We begin with the following general observation: let $f\colon V \to W$ be a representable morphism of algebraic S-stacks. Let $\mathrm{Sec}(V/W)$ denote the set of sections to f. Denote by $f_n\colon V_n \to W_n$ the pullback of f along $S_n \hookrightarrow S$. If W is a proper algebraic stack over S, and f is also separated and locally of finite type, then the natural map $\mathrm{Sec}(V/W) \to \varprojlim_n \mathrm{Sec}(V_n/W_n)$ is bijective. Indeed, a section $t\colon W \to V$ of f is equivalent to a closed immersion $W \to W \times_S V$ such that the composition with the first projection is the identity. The algebraic S-stack $W \times_S V$ is also separated and [Ols06b, Thm. A.1] now gives the claim. To see that the functor $\Psi_{\pi,I}$ is fully faithful, we simply observe that if $Z, \widetilde{Z} \in \mathbf{QF}_{\mathrm{p}/S}(X)$, then

$$\mathrm{Hom}_{\mathbf{QF}_{\mathrm{p}/S}(X)}(Z, \widetilde{Z}) = \mathrm{Sec}([Z \times_X \widetilde{Z}]/Z). \qquad \square$$

Proving that $\Psi_{\pi,I}$ is an equivalence when π is not necessarily separated is the main technical contribution of this paper. In this section, we will prove

Theorem 4.3. *Let $\pi\colon X \to S$ be a morphism of algebraic stacks that is locally of finite type with quasi-compact and separated diagonal and affine stabilizers. Then $\Psi_{\pi,I}$ is an equivalence.*

Some interesting special cases of stacks with affine stabilizers are:

(1) global quotient stacks,
(2) stacks of global type [Ryd09, Defn. 2.1],
(3) stacks with quasi-finite diagonal,
(4) algebraic spaces, and
(5) schemes.

We wish to point out that Theorem 4.3 is even new for schemes. As outlined in §0.2, we will prove Theorem 4.3 by a dévissage on the non-abelian category $\mathbf{QF}_{\mathrm{p}/S}(X)$. This dévissage is combined with the Raynaud–Gruson Chow Lemma [RG71, Cor. 5.7.13] to reduce to proving that $\Psi_{\pi,I}$ is essentially surjective for a very special class of morphisms of schemes $\pi\colon X \to S$. Before we get into our main lemmas to set up the dévissage, the following definitions will be useful.

Let $\pi\colon X \to S$ be a morphism of algebraic stacks that is locally of finite type with quasi-compact and separated diagonal. We say that $(Z_n \to X_n)_{n\geq 0} \in \mathbf{QF}_{\mathrm{p}/S}(\widehat{X})$ is *effectivizable* if it lies in the essential image of $\Psi_{\pi,I}$. If $(Z_n \to X_n)_{n\geq 0} \in \mathbf{QF}_{\mathrm{p}/S}(\widehat{X})$ is effectivizable, we say that an object $(Z \to X) \in \mathbf{QF}_{\mathrm{p}/S}(X)$ together with an isomorphism $e\colon \Psi_{\pi,I}(Z \to X) \to (Z_n \to X_n)_{n\geq 0}$ is an *effectivization* of $(Z_n \to X_n)_{n\geq 0}$. Note that given two effectivizations $((Z \to X), e)$, $((Z' \to X), e')$ of an effectivizable $(Z_n \to X_n)_{n\geq 0}$, then there exists a unique isomorphism $\alpha\colon (Z' \to X) \to (Z \to X)$ such that $e \circ \Psi_{\pi,I}(\alpha) = e'$ (Lemma 4.2).

Let $(\varphi_n)_{n\geq 0}\colon (Z'_n \to X_n)_{n\geq 0} \to (Z_n \to X_n)_{n\geq 0}$ be a morphism in $\mathbf{QF}^{\mathrm{fin}}_{\mathrm{s}}(\widehat{X})$. Let $J \subseteq \mathcal{O}_X$ be a coherent sheaf of ideals. If X is a scheme, then (φ_n) corresponds to a finite

[1]This amplification, from coherent sheaves on proper stacks to coherent sheaves with proper support on separated stacks, is implicit in [Ols05, Thm. 1.5].

morphism of locally noetherian formal schemes $\varphi\colon \mathfrak{Z}' \to \mathfrak{Z}$. Let $\varphi^\sharp\colon \mathcal{O}_{\mathfrak{Z}} \to \varphi_*\mathcal{O}_{\mathfrak{Z}'}$ denote the corresponding homomorphism. We say that φ is *J-admissible* if $J_{\mathfrak{Z}} \cap \ker\varphi^\sharp = 0$ and $\operatorname{coker}\varphi^\sharp$ is annihilated by $J_{\mathfrak{Z}}$. The last condition is equivalent to $\varphi_* J_{\mathfrak{Z}'} \subseteq \operatorname{im}\varphi^\sharp$. We say that φ is *strongly J-admissible* if $J_{\mathfrak{Z}} \cap \ker\varphi^\sharp = 0$ and $\varphi_* J_{\mathfrak{Z}'} = \varphi^\sharp(J_{\mathfrak{Z}})$. If X is any algebraic stack, locally of finite type over S, then we say that $(\varphi_n)_{n\geq 0}$ is J-admissible (resp. strongly J-admissible) if there exists a smooth surjection from a scheme $V \to X$ such that the adic system $(\varphi_n)_V : Z'_{V_n} \to Z_{V_n}$ induces a J-admissible (resp. strongly J-admissible) morphism $\varphi_V : \mathfrak{Z}'_V \to \mathfrak{Z}_V$ of formal schemes. This notion does not depend on $V \to X$.

Our dévissage methods hinge on the following two lemmas. Our first Lemma forms the analogue of Step (4) given in §0.2. It is here that we first utilize Corollary 1.20.

Lemma 4.4. *Let $\pi\colon X \to S$ be a morphism of algebraic stacks that is locally of finite type with quasi-compact and separated diagonal. Let $J \subseteq \mathcal{O}_X$ be a coherent ideal and let $(\varphi_n)_{n\geq 0}\colon (Z'_n \to X_n)_{n\geq 0} \to (Z_n \to X_n)_{n\geq 0}$ be a J-admissible morphism in $\mathbf{QF}_{\mathrm{p}/S}(\widehat{X})$. Suppose that $(Z'_n \to X_n)_{n\geq 0}$ and $(Z_n \times_X V(J) \to X_n)_{n\geq 0}$ are effectivizable. Then $(Z_n \to X_n)_{n\geq 0}$ is effectivizable.*

Proof. There exists a cocartesian diagram in $\mathbf{QF}_{\mathrm{s}}(\widehat{X})$ (Corollary 1.20):

$$\begin{array}{ccc} (Z'_n \times_X V(J) \to X_n)_{n\geq 0} & \hookrightarrow & (Z'_n \to X_n)_{n\geq 0} \\ \downarrow & & \downarrow{\scriptstyle (\varphi_n)_{n\geq 0}} \\ (Z_n \times_X V(J) \to X_n)_{n\geq 0} & \hookrightarrow & (W_n \to X_n)_{n\geq 0}, \end{array}$$

where $(W_n \to X_n)_{n\geq 0} \in \mathbf{QF}_{\mathrm{p}/S}(X)$ is effectivizable to $(W \to X)$. The universal properties show that there is a uniquely induced morphism $(b_n)_{n\geq 0}\colon (W_n \to X_n)_{n\geq 0} \to (Z_n \to X_n)_{n\geq 0}$ such that each b_n is finite. We will show that

(1) (b_n) is strongly J-admissible, and
(2) if (φ_n) is strongly J-admissible, then (b_n) is an isomorphism.

We may then apply the procedure twice, first for (φ_n), and then with (b_n) instead of (φ_n), and deduce that (Z_n) is the pushout of an effectivizable diagram and hence effectivizable.

That (b_n) is strongly J-admissible (resp. an isomorphism) can be verified smooth-locally on X. Thus after replacing X by an affine and noetherian scheme, we may pass from the adic systems above to noetherian formal schemes. That is, $(\varphi_n)_{n\geq 0}$ is given by a finite morphism of noetherian formal schemes $\varphi\colon (\mathfrak{Z}' \to \widehat{X}) \to (\mathfrak{Z} \to \widehat{X})$.

By assumption, φ is J-admissible (resp. strongly J-admissible), so we also know that $J_{\mathfrak{Z}} \cap \ker\varphi^\sharp = 0$ and that $\varphi_* J_{\mathfrak{Z}'} \subseteq \operatorname{im}\varphi^\sharp$ (resp. $\varphi_* J_{\mathfrak{Z}'} = \varphi^\sharp(J_{\mathfrak{Z}})$). Moreover, $(b_n)_{n\geq 0}$ is given by a finite morphism of noetherian formal schemes $\beta\colon \mathfrak{W} \to \mathfrak{Z}$.

That β is strongly J-admissible (resp. an isomorphism) is Zariski local on $\mathfrak{Z}$. Thus, we may assume that $\mathfrak{Z} = \operatorname{Spf}_{IA} A$, $\mathfrak{Z}' = \operatorname{Spf}_{IA'} A'$, and that φ is given by a finite morphism $f\colon A \to A'$. Let $B = A/JA \times_{A'/JA'} A'$. By Theorem 1.18, there is a natural isomorphism $\mathfrak{W} \cong \operatorname{Spf}_{IA} B$. It remains to prove that the natural map $A \to B$ is strongly J-admissible (resp. an isomorphism). To this end we form the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & JA & \xrightarrow{m\mapsto(m,0)} & A \times A' & \to & A/JA \times A' & \to & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_2} & & \\ 0 & \to & JA' & \to & A' & \to & A'/JA' & \to & 0, \end{array}$$

where $d_1(a, a') = f(a) - a'$ and $d_2(\overline{a}, a') = \overline{f}(\overline{a}) - \overline{a'}$. By the Snake Lemma, there is an exact sequence of A-modules:

$$0 \longrightarrow JA \cap \ker(f) \longrightarrow A \longrightarrow B \xrightarrow{\delta} JA'/f(JA) \longrightarrow 0.$$

Thus, $A \to B$ is an isomorphism if and only if $f\colon A \to A'$ is strongly J-admissible.

In general, we have that $JB \subseteq 0 \times JA'$ and thus inclusions $f(JA) \subseteq JB \subseteq JA'$. We also note that the composition $JB \to 0 \times JA' \to JA'/f(JA)$ equals $\delta|_{JB}$.

If f is J-admissible, then $JA \cap \ker(f) = 0$ and $JA' \subseteq f(A)$. The second condition implies that $J^2A' \subseteq Jf(A) = f(JA)$, so $\delta(JB) = 0$ and hence $f(JA) = JB$ as submodules of JA'. The first condition says that $JA = f(JA)$ and it follows that $A \to B$ is injective from the exact sequence. Putting this together, we have that $A \to B$ is injective and that $JA = f(JA) = JB$, so $A \to B$ is strongly J-admissible. □

In order to apply Lemma 4.4 we have our next Lemma. This is the analogue of [EGA, III.5.3.4] in our situation; forming part of Step (9) in the outline given in §0.2. To prove this Lemma, we must combine the main techniques of the paper developed thus far—specifically Corollary 1.20 and Theorem 2.1.

Lemma 4.5. *Let X be an algebraic stack of finite type over S with quasi-compact and separated diagonal. Let $(Z_n \to X_n)_{n\geq 0} \in \mathbf{QF}_{\mathrm{p}/S}(\widehat{X})$. Suppose that there is a proper and schematic morphism $p\colon X' \to X$ and an open substack $U \subseteq X$ such that the morphism $p^{-1}(U) \to U$ is finite and flat. Assume, in addition, that $\widehat{p}^*(Z_n \to X_n)_{n\geq 0} \in \mathbf{QF}_{\mathrm{p}/S}(\widehat{X}')$ is effectivizable. Then there exists a coherent ideal $J \subseteq \mathcal{O}_X$, with $|\operatorname{supp}(\mathcal{O}_X/J)|$ equal to the complement of U in X, and a J-admissible morphism $(\varphi_n)_{n\geq 0}\colon (\widetilde{Z}_n \to X_n)_{n\geq 0} \to (Z_n \to X_n)_{n\geq 0}$ with $(\widetilde{Z}_n \to X_n)_{n\geq 0}$ effectivizable.*

Proof. Let $X'' = X' \times_X X'$. For $j = 1$ and 2 let $s_j\colon X'' \to X'$ denote the jth projection. Let $q = p \circ s_1$ and observe that there is a 2-morphism $\alpha\colon q \Rightarrow p \circ s_2$. By hypothesis, $\widehat{p}^*(Z_n \to X_n)_{n\geq 0}$ admits an effectivization $(Z' \to X') \in \mathbf{QF}_{\mathrm{p}/S}(X')$. By Theorem 3.2, for $j = 1$ and 2, there exist natural morphisms $\eta_j\colon (s_j)_!(s_j)^*Z' \to Z'$ in $\mathbf{QF}_{\mathrm{p}/S}(X')$. By Theorem 3.2, there are also natural morphisms $p_!(\eta_j)\colon p_!(s_j)_!(s_j)^*Z' \to p_!Z'$ in $\mathbf{QF}_{\mathrm{p}/S}(X)$. Moreover, there is a natural isomorphism of functors $p_!(s_1)_! \Rightarrow q_!$. The 2-morphism α also induces a natural isomorphism of functors $p_!(s_2) \Rightarrow q_!$ and an isomorphism $\Psi_{\pi\circ q,I}(s_1^*Z_1') \cong \Psi_{\pi\circ q,I}(s_2^*Z_2')$ in $\mathbf{QF}_{\mathrm{p}/S}(\widehat{X}'')$, where $\pi\colon X \to S$ denotes the structure morphism of X. By Lemma 4.2, we obtain an isomorphism $\nu\colon s_1^*Z_1' \to s_2^*Z_2'$ in $\mathbf{QF}_{\mathrm{p}/S}(X'')$. Let $Z'' = s_1^*Z'$. Combining the afforementioned isomorphisms, we obtain for $j = 1$ and 2 natural morphisms $t_j\colon q_!Z'' \to p_!Z'$ in $\mathbf{QF}_{\mathrm{p}/S}(X)$. The morphisms t_j are finite, thus the coequalizer diagram $[q_!Z'' \rightrightarrows p_!Z']$ has a colimit, $\widetilde{Z}$ in $\mathbf{QF}_{\mathrm{p}/S}(X)$ (Theorem 1.10). By Theorem 3.2 there are natural isomorphisms:

$$\Psi_{\pi,I}(p_!Z') \cong \widehat{p}_!(Z' \times_{X'} X_n' \to X_n')_{n\geq 0} \cong \widehat{p}_!\widehat{p}^*(Z_n \to X_n)_{n\geq 0}$$
$$\Psi_{\pi,I}(q_!Z'') \cong \widehat{q}_!(Z'' \times_{X''} X_n'' \to X_n'')_{n\geq 0} \cong \widehat{q}_!\widehat{q}^*(Z_n \to X_n)_{n\geq 0}.$$

Moreover, the morphism $\widehat{p}_!\widehat{p}^*(Z_n \to X_n)_{n\geq 0} \to (Z_n \to X_n)_{n\geq 0}$ coequalizes the two morphisms $\widehat{q}_!\widehat{q}^*(Z_n \to X_n)_{n\geq 0} \rightrightarrows \widehat{p}_!\widehat{p}^*(Z_n \to X_n)_{n\geq 0}$. By Corollary 1.20, there is a uniquely induced morphism $(\varphi_n)_{n\geq 0}\colon (\widetilde{Z} \times_X X_n \to X_n)_{n\geq 0} \to (Z_n \to X_n)_{n\geq 0}$ in $\mathbf{QF}_{\mathrm{p}/S}(\widehat{X})$.

It remains to prove that $(\varphi_n)_{n\geq 0}$ is J-admissible for a suitable ideal J. First, let $J \subseteq \mathcal{O}_X$ be any coherent ideal defining the complement of U. We will show that $(\varphi_n)_{n\geq 0}$ is J^N-admissible for sufficiently large N. Note that the verification of this is local on X for the smooth topology. Indeed, by Corollary 1.20 and Theorem 3.2, the construction of $(\varphi_n)_{n\geq 0}$ is compatible with smooth base change on X. So, we may henceforth assume that $X = \operatorname{Spec} B$, where B is an R-algebra of finite type, and that $(Z_n \to X_n)_{n\geq 0}$ is given by a morphism of locally noetherian formal schemes $(\mathfrak{Z} \to \widehat{X})$. We next observe that we may work Zariski locally on $\mathfrak{Z}$. To see this, we note that the morphism $\widehat{p}_!\widehat{p}^*\mathfrak{Z} \to \mathfrak{Z}$ is just the Stein factorization of $\widehat{p}^*\mathfrak{Z} \to \mathfrak{Z}$ (Lemma 2.3) and this is compatible with flat base change on $\mathfrak{Z}$ (Lemma 2.4), and similarly for the morphisms $\widehat{q}_!\widehat{q}^*\mathfrak{Z} \rightrightarrows \widehat{p}_!\widehat{p}^*\mathfrak{Z}$. Also, let

$\widetilde{Z}^\wedge = \widetilde{\mathfrak{Z}}$ and let $(\varphi_n)_{n\geq 0}$ be given by the finite morphism $\varphi\colon \widetilde{\mathfrak{Z}} \to \mathfrak{Z}$. Then Theorem 1.18 implies that $\mathcal{O}_{\widetilde{\mathfrak{Z}}} = \ker(\mathcal{O}_{\widehat{p}_!\widehat{p}^*\mathfrak{Z}} \xrightarrow{\widehat{t}_1^\sharp - \widehat{t}_2^\sharp} \mathcal{O}_{\widehat{q}_!\widehat{q}^*\mathfrak{Z}})$. Thus the formation of $\widetilde{\mathfrak{Z}}$ is also compatible with flat base change on $\mathfrak{Z}$. Consequently, we may assume that $\mathfrak{Z} = \operatorname{Spf}_{IA} A$, for some B-algebra A that is IA-adically complete. Let $Z = \operatorname{Spec} A$. Then the morphism $\widehat{Z} \to \widehat{X}$ induces a morphism of schemes $Z \to X$.

Let $p_Z\colon Z' \to Z$ (resp. $q_Z\colon Z'' \to Z$) denote the pullback of p (resp. q) along $Z \to X$. Let $A' = \Gamma(Z', \mathcal{O}_{Z'})$ and $A'' = \Gamma(Z'', \mathcal{O}_{Z''})$ (which are both coherent A-modules because p_Z and q_Z are proper). Then there are two induced morphisms d_1, $d_2\colon A' \to A''$ and $\widetilde{A} := \Gamma(\widetilde{Z}, \mathcal{O}_{\widetilde{Z}}) = \ker(A' \xrightarrow{d_1 - d_2} A'')$. The morphism $\varphi\colon \widetilde{\mathfrak{Z}} \to \mathfrak{Z}$ induces a natural finite map $f\colon A \to \widetilde{A}$.

It remains to show that we can find an $N \gg 0$ such that $(J^N A) \cap \ker f = 0$ and $J^N A$ annihilates $\operatorname{coker} f$. Now, on the complement, U_Z, of the support of A/JA on $\operatorname{Spec} A$, the morphism $p_Z^{-1}(U_Z) \to U_Z$ is finite and flat. So, if $g \in JA$, then $A''_g = A'_g \otimes_{A_g} A'_g$. By finite flat descent, $\varphi_g\colon A_g \to \widetilde{A}_g$ is an isomorphism. Since A is noetherian, we deduce immediately that the kernel and cokernel of φ are annihilated by $J^N A$ for some $N \gg 0$. By the Artin–Rees Lemma [EGA, 0_{I}.7.3.2.1], and by possibly increasing N, we can ensure that $(J^N A) \cap \ker f = 0$. The result follows. □

We may finally prove Theorem 4.3.

Proof of Theorem 4.3. By Lemma 4.2, it remains to show that the functor $\Psi_{\pi,I}$ is essentially surjective. We divide the proof of this into a number of cases.
Basic case. Let $\pi\colon X \to S$ be a morphism of schemes that is of finite type and factors as $X \xrightarrow{f} P \xrightarrow{g} S$, where f is étale and g is projective. Let $(Z_n \to X_n)_{n\geq 0} \in \mathbf{QF}_{\mathrm{p}/S}(\widehat{X})$, which we may regard as a quasi-finite and separated morphism of locally noetherian formal schemes $(\mathfrak{Z} \to \widehat{X})$. Then the composition $\mathfrak{Z} \to \widehat{X} \to \widehat{P}$ is quasi-finite and proper, since P is separated and $\mathfrak{Z}$ is S-proper. Hence, the morphism $\mathfrak{Z} \to \widehat{P}$ is finite [EGA, III.4.8.11]. As $g\colon P \to S$ is projective, there is a finite P-scheme Z such that $(\widehat{Z} \to \widehat{P}) \cong (\mathfrak{Z} \to \widehat{P})$ in $\mathbf{QF}_{\mathrm{p}/S}(\widehat{P})$ [EGA, III.5.4.4]. The morphism f is étale, thus Corollary B.5 implies that there is a unique P-morphism $Z \to X$ lifting the given P_0-morphism $Z_0 \to X_0$. Thus we have obtained $(Z \to X) \in \mathbf{QF}_{\mathrm{p}/S}(X)$ with completion isomorphic to $\mathfrak{Z}$ over $\widehat{X}$. We conclude that the functor $\Psi_{\pi,I}$ is an equivalence of categories in this case.
Quasi-compact with quasi-finite and separated diagonal case. Let $\pi\colon X \to S$ be a morphism of algebraic stacks that is of finite type with quasi-finite and separated diagonal. We now prove that the functor $\Psi_{\pi,I}$ is an equivalence by noetherian induction on the closed subsets of $|X|$. Thus, for any closed immersion $i\colon V \hookrightarrow X$ with $|V| \subsetneq |X|$, we may assume that the functor $\Psi_{\pi\circ i,I}\colon \mathbf{QF}_{\mathrm{p}/S}(V) \to \mathbf{QF}_{\mathrm{p}/S}(\widehat{V})$ is an equivalence.

Let $(Z_n \to X_n)_{n\geq 0} \in \mathbf{QF}_{\mathrm{p}/S}(X)$. Then there exists a 2-commutative diagram:

$$\begin{array}{ccc} X & \xleftarrow{p} & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xleftarrow{g} & P, \end{array}$$

such that g is projective, f is étale, X' is a scheme, and p is proper, schematic, surjective, and over a dense open subset $U \subseteq X$ the morphism $p^{-1}(U) \to U$ is finite and flat [Ryd09, Thm. 8.9]. By the Basic Case considered above, $\widehat{p}^*(Z_n \to X_n)_{n\geq 0} \in \mathbf{QF}_{\mathrm{p}/S}(\widehat{X'})$ is effectivizable. By Lemma 4.5, there exists a coherent ideal $J \subseteq \mathcal{O}_X$, with $|\operatorname{supp}(\mathcal{O}_X/J)|$ equal to the complement of U in X, and a J-admissible morphism $(\varphi_n)_{n\geq 0}\colon (\widetilde{Z}_n \to X_n)_{n\geq 0} \to (Z_n \to X_n)_{n\geq 0}$ with $(\widetilde{Z}_n \to X_n)_{n\geq 0}$ effectivizable. By noetherian induction, $(Z_n \times_X V(J) \to X_n)_{n\geq 0} \in \mathbf{QF}_{\mathrm{p}/S}(\widehat{X})$ is effectivizable. By Lemma 4.4, $(Z_n \to X_n)_{n\geq 0}$ is effectivizable.

General case. We now assume that $\pi\colon X \to S$ is as in the statement of the Theorem. It remains to prove that the functor $\Psi_{\pi,I}$ is essentially surjective. To show this, we first reduce to the case where π is quasi-compact. Let $(Z_n \to X_n)_{n\geq 0} \in \mathbf{QF}_{\mathrm{p}/S}(\widehat{X})$. Let O_X denote the set of quasi-compact open substacks of X. The set O_X is ordered by inclusion, and we note that $\{|W \times_X Z_0|\}_{W\in O_X}$ is an open cover of the quasi-compact topological space $|Z_0|$. Thus, there is a $W \in O_X$ such that the canonical X-morphism $W \times_X Z_0 \to Z_0$ is an isomorphism. In particular, the map $Z_0 \to X$ factors uniquely as $Z_0 \to W \hookrightarrow X$. Since the open immersion $W \hookrightarrow X$ is étale, it follows that the maps $Z_n \to X$ also factor compatibly through W. We can thus replace X with W and assume henceforth that X is quasi-compact.

Next, we note that there is an open substack $X^{\mathrm{qf}} \subseteq X$ with the property that a point x of X belongs to X^{qf} if and only if the stabilizer at x is finite. Indeed, by Chevalley's Theorem [EGA, IV.13.1.4], the locus of points in the inertia stack $I_{X/S} \to X$ that are isolated in their fibers is an open subset of $I_{X/S}$. Intersecting this open set with the identity section $e\colon X \to I_{X/S}$ gives X^{qf} as $I_{X/S} \to X$ is a relative group stack (the dimension of a finite type group algebraic space over a field is its local dimension at the identity).

Arguing as before and applying the previous case, it now remains to show that the map $Z_0 \to X$ factors through X^{qf}. So we let $I_X \to X$ (resp. $I_{Z_0} \to Z_0$) denote the inertia stack of X (resp. Z_0) over S. The morphism $Z_0 \to X$ is separated and representable thus $I_{Z_0} \to I_X \times_X Z_0$ is a closed immersion over Z_0. By assumption, $I_X \to X$ has affine fibers and $I_{Z_0} \to Z_0$ is proper. Hence the fibers of $I_{Z_0} \to Z_0$ are proper and affine, thus finite. By Zariski's Main Theorem [LMB, Cor. A.2.1], $I_{Z_0} \to Z_0$ is finite, and so Z_0 has finite diagonal over S (because Z_0 is separated over S). We now form the 2-commutative diagram:

$$\begin{array}{ccc} Z_0 \times_X Z_0 & \longrightarrow & Z_0 \times_S Z_0 \\ & \searrow & \downarrow \\ & & Z_0. \end{array}$$

The morphism $Z_0 \to X$ is also quasi-finite, thus the morphism $Z_0 \times_X Z_0 \to Z_0$ is quasi-finite. The morphism $Z_0 \times_S Z_0 \to Z_0$ has finite diagonal, thus $Z_0 \times_X Z_0 \to Z_0 \times_S Z_0$ is quasi-finite. We may now conclude that if z is a point of Z_0, then its image in X has finite stabilizer. Thus Z_0 factors through X^{qf} and we deduce the result. □

Remark 4.6. It is possible that Theorem 4.3 extends to any stack X with locally quasi-finite diagonal, that is, without requiring that Δ_X is separated and quasi-compact. For such X, we consider the full subcategory $\mathbf{LQF}_{\mathrm{p}/S}(\widehat{X}_{/|X_0|}) \subseteq \mathbf{LQF}(\widehat{X}_{/|X_0|})$ consisting of systems of locally quasi-finite representable morphisms $(Z_n \to X)_{n\geq 0}$ such that the composition $Z_0 \to X \to S$ is proper.

The generalization of Lemma 4.2, that $\Psi_{\pi,I}\colon \mathbf{LQF}_{\mathrm{p}/S}(X) \longrightarrow \mathbf{LQF}_{\mathrm{p}/S}(\widehat{X})$ is fully faithful, follows as for stacks with separated diagonals, but using Theorem 4.3 instead of [Ols05, Thm. 1.4]. The corresponding generalizations of Lemmas 4.4 and 4.5 would follow if we extend the results of Sections 1 and 2 to include certain non-separated quasi-finite morphisms. This is probably not too difficult to accomplish, although one may have to develop a theory of non-separated formal algebraic spaces.

Finally, to obtain the generalization of Theorem 4.3 we would also need a suitable Chow lemma for stacks with locally quasi-finite diagonals. It would suffice to find a proper generically finite and flat morphism $p\colon X' \to X$ together with an étale, not necessarily representable, morphism $f\colon X' \to P$ with $P \to S$ projective. Whether such a proper covering exists is not clear to the authors.

5. Algebraicity of the Hilbert stack

Before we get to the proof Theorem 2, we will require an analysis of some spaces of sections. Let T be a scheme and let $s\colon V' \to V$ be a representable morphism of algebraic

T-stacks. Define $\underline{\mathrm{Sec}}_T(V'/V)$ to be the sheaf that takes a T-scheme W to the set of sections of the morphism $V' \times_T W \to V \times_T W$. We require an improvement of [Ols06b, Prop. 5.10] and [Lie06, Lem. 2.10], which we prove using [Hal12a, Thm. D].

Proposition 5.1. *Let T be a scheme and let $p\colon Z \to T$ be a morphism of algebraic stacks that is proper, flat, and of finite presentation. Let $s\colon Q \to Z$ be a quasi-finite, separated, finitely presented, and representable morphism. Then the T-sheaf $\underline{\mathrm{Sec}}_T(Q/Z) = p_*Q$ is represented by a quasi-affine T-scheme which is of finite presentation.*

Proof. By a standard limit argument [Ryd09, Prop. B.3], we can assume that T is noetherian. By Zariski's Main Theorem [LMB, Thm. 16.5(ii)], there is a finite morphism $\overline{Q} \to Z$ and an open immersion $Q \hookrightarrow \overline{Q}$ over Z. In particular, we see that there is a natural transformation of T-sheaves $\underline{\mathrm{Sec}}_T(Q/Z) \to \underline{\mathrm{Sec}}_T(\overline{Q}/Z)$ which is represented by open immersions. Hence, we may assume for the remainder that the morphism $s\colon Q \to Z$ is finite. Next, observe that $\underline{\mathrm{Sec}}_T(Q/Z)$ is a subfunctor of the T-presheaf $\underline{\mathrm{Hom}}_{\mathcal{O}_Z/T}(s_*\mathcal{O}_Q, \mathcal{O}_Z)$: the sections are the module homomorphisms that are also algebra homomorphisms. This is characterized by the vanishing of certain maps

$$s_*\mathcal{O}_Q \otimes_{\mathcal{O}_Z} s_*\mathcal{O}_Q \to \mathcal{O}_Z \quad \text{and} \quad \mathcal{O}_Z \to \mathcal{O}_Z$$

corresponding to the compatibility with the multiplication and the unit. By [Hal12a, Thm. D], the sheaf $\underline{\mathrm{Hom}}_{\mathcal{O}_Z/T}(s_*\mathcal{O}_Q, \mathcal{O}_Z)$ is represented by a scheme that is affine and of finite type over T. Similarly, $\underline{\mathrm{Hom}}_{\mathcal{O}_Z/T}(\mathcal{O}_Z, \mathcal{O}_Z)$ and $\underline{\mathrm{Hom}}_{\mathcal{O}_Z/T}(s_*\mathcal{O}_Q \otimes_{\mathcal{O}_Z} s_*\mathcal{O}_Q, \mathcal{O}_Z)$ are affine. It follows that $\underline{\mathrm{Sec}}_T(Q/Z) \hookrightarrow \underline{\mathrm{Hom}}_{\mathcal{O}_Z/T}(s_*\mathcal{O}_Q, \mathcal{O}_Z)$ is represented by closed immersions and the result follows. □

Corollary 5.2. *Let T be a scheme and let $X \to T$ be a morphism of algebraic stacks that is locally of finite presentation. Suppose that we have quasi-finite, separated and representable morphisms $s_i\colon Z_i \to X$ for $i = 1, 2$ such that Z_1 and Z_2 are proper, flat and of finite presentation over T. Then the functor $\mathrm{Hom}_X(Z_1, Z_2)$ on $\mathbf{Sch}/T$, given by $T' \mapsto \mathrm{Hom}_{X_{T'}}\big((Z_1)_{T'}, (Z_2)_{T'}\big)$, is represented by a scheme that is quasi-affine over T. In particular, the open subfunctor $\mathrm{Isom}_X(Z_1, Z_2) \subseteq \mathrm{Hom}_X(Z_1, Z_2)$ parameterizing isomorphisms is represented by a scheme that is quasi-affine over T.*

Proof. Note that $\mathrm{Hom}_X(Z_1, Z_2) = \underline{\mathrm{Sec}}_T((Z_1 \times_X Z_2)/Z_1)$. Thus, by Proposition 5.1 the functor $\mathrm{Hom}_X(Z_1, Z_2)$ has the asserted properties. □

Proof of Theorem 2. We may immediately reduce to the case where S is an affine scheme. Arguing as in the proof of the General case of Theorem 4.3, we may reduce to the situation where the morphism $X \to S$ is of finite presentation with quasi-finite and separated diagonal. Standard limit methods [Ryd09, Prop. B.2 & B.3] now permit us to further assume that S is the spectrum of an excellent noetherian ring R.

We are now in a position to employ the algebraicity criterion [Hal12b, Thm. A] to prove that the stack $\underline{\mathrm{HS}}_{X/S}$ is algebraic and locally of finite presentation over S. Standard methods show that the S-groupoid $\underline{\mathrm{HS}}_{X/S}$ is a limit preserving stack for the étale topology. In [Hal12b, Lem. 9.3] it is proved that the S-groupoid $\underline{\mathrm{HS}}_{X/S}$ is $\mathbf{Aff}$-homogeneous. By Theorem 4.3, the S-groupoid is effective. Let T be an S-scheme, let $X_T = X \times_S T$, and let $f_T\colon X_T \to T$ be the projection. In [Hal12b, §9] it is shown, using the results of [Ols06a], that if $(Z \xrightarrow{g} X_T) \in \underline{\mathrm{HS}}_{X/S}(T)$, then for each quasi-coherent $\mathcal{O}_T$-module I there are natural isomorphisms:

$$\mathrm{Aut}_{\underline{\mathrm{HS}}_{X/S}/S}\big((Z \to X_T), I\big) \cong \mathrm{Hom}_{\mathcal{O}_Z}(L_{Z/X_T}, g^*f_T^*I)$$
$$\mathrm{Def}_{\underline{\mathrm{HS}}_{X/S}/S}\big((Z \to X_T), I\big) \cong \mathrm{Ext}^1_{\mathcal{O}_Z}(L_{Z/X_T}, g^*f_T^*I).$$

By [Hal12a, Thm. C], it follows that $\underline{\mathrm{HS}}_{X/S}$ satisfies the conditions on automorphisms and deformations required by [Hal12b, Thm. A]. In [Hal12b, §9], a 2-step obstruction theory for $\underline{\mathrm{HS}}_{X/S}$ at $(Z \xrightarrow{g} X_T)$ is also described:

$$\mathrm{o}^1\big((Z \xrightarrow{g} X_T), -\big) : \mathrm{Exal}_S(T, -) \Rightarrow \mathrm{Hom}_{\mathcal{O}_Z}\Big(g^*\mathcal{T}or_1^{S,\tau,f}(\mathcal{O}_T, \mathcal{O}_X), g^* f_T^*(-)\Big)$$
$$\mathrm{o}^2\big((Z \xrightarrow{g} X_T), -\big) : \ker \mathrm{o}^1\big((Z \xrightarrow{g} X_T), -\big) \Rightarrow \mathrm{Ext}^2_{\mathcal{O}_Z}\big(L_{Z/X_T}, g^* f_T^*(-)\big).$$

Again, by [Hal12a, Thm. C], it now follows that $\underline{\mathrm{HS}}_{X/S}$ satisfies the conditions on obstructions required by [Hal12b, Thm. A]. Having met all the hypotheses of [Hal12b, Thm. A] we conclude that the S-groupoid $\underline{\mathrm{HS}}_{X/S}$ is an algebraic stack that is locally of finite presentation over S. Finally, Corollary 5.2 implies that the diagonal of $\underline{\mathrm{HS}}_{X/S}$ is quasi-affine and we deduce the result. □

Appendix A. Coherent cohomology of formal schemes

In this appendix, we prove that cohomology commutes with flat base change of formal schemes. This is well-known for schemes, but we could not find a suitable reference for formal schemes. Note that in the adic case, this result follows from the more general result [AJL99, Prop. 7.2(b)].

Proposition A.1. *Consider a cartesian diagram of locally noetherian formal schemes:*

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{p'} & \mathfrak{X} \\ {\scriptstyle \varpi'}\downarrow & & \downarrow{\scriptstyle \varpi} \\ \mathfrak{Y}' & \xrightarrow{p} & \mathfrak{Y}, \end{array}$$

where ϖ is proper and p is flat. Let $\mathfrak{F}$ be a coherent $\mathcal{O}_{\mathfrak{X}}$-module, then for any $q \geq 0$ the base change morphism $p^* R^q\varpi_*\mathfrak{F} \to R^q\varpi'_* p'^*\mathfrak{F}$ is a topological isomorphism.

Proof. By [EGA, III.3.4.5.1], the statement is Zariski local on $\mathfrak{Y}$ and $\mathfrak{Y}'$ so we may assume that $\mathfrak{Y} = \mathrm{Spf}_I R$ for some I-adic noetherian ring R, $\mathfrak{Y}' = \mathrm{Spf}_{I'} R'$ for some I'-adic noetherian ring R', $IR' \subseteq I'$, and $R \to R'$ is a flat morphism of rings. It suffices to prove that the morphism $H^q(\mathfrak{X}, \mathfrak{F})\widehat{\otimes}_R R' \to H^q(\mathfrak{X}', p'^*\mathfrak{F})$ is a topological isomorphism. Note that p factors as $\mathrm{Spf}_{I'} R' \xrightarrow{r} \mathrm{Spf}_{IR'} R' \xrightarrow{q} \mathrm{Spf}_I R$. Since q is adic, we are reduced to proving the Proposition when p is also adic or when $R = R'$. Note that cohomology can be shown to commute with adic flat base change by minor modifications to the statements and arguments of [EGA, 0_{III}.13.7.8], combined with [EGA, 0_{III}.13.7.7] and [EGA, III.3.4.3] to deal with flat base change instead of localizations. It remains to prove the Proposition when $R = R'$.

For each k, $l \geq 0$, let $R_{k,l} = R/(I^{k+1}, I'^{l+1})$, $R_k = R/I^{k+1}$, and $R'_l = R/I'^{l+1}$. If $k \geq l$, then $R_{k,l} = R'_l$ (because $I \subseteq I'$). Also, for each fixed k, the noetherian ring R_k is I'-adically complete, and the natural map $R_k \to \varprojlim_l R_{k,l}$ is a topological isomorphism. Let $X_{k,l} = \mathfrak{X} \otimes_R R_{k,l}$, $X_k = \mathfrak{X} \otimes_R R_k$ (which we view as noetherian schemes), $F_{k,l} = \mathfrak{F} \otimes_R R_{k,l}$, and $F_k = \mathfrak{F} \otimes_R R_k$. By [EGA, III.4.1.7], for each $k \geq 0$, there are natural induced topological isomorphisms:

$$H^q(X_k, F_k)^\wedge \cong \varprojlim_l H^q(X_{k,l}, F_{k,l}),$$

where $H^q(X_k, F_k)^\wedge$ denotes the completion of the R_k-module $H^q(X_k, F_k)$ with respect to the I'-adic topology. Note, however, that $X_k \to \mathrm{Spec}\, R_k$ is proper, thus $H^q(X_k, F_k)$ is a coherent R_k-module [EGA, III.3.2.1]. Consequently, $H^q(X_k, F_k)$ is I'-adically complete [EGA, 0_{I}.7.3.6], and we have topological isomorphisms:

$$H^q(X_k, F_k) \cong \varprojlim_l H^q(X_{k,l}, F_{k,l}).$$

Let $X'_l = \mathfrak{X} \otimes_R R'_l$ and $F'_l = \mathfrak{F} \otimes_R R'_l$. Applying the functor $\varprojlim_k$ to both sides of the isomorphism above we obtain natural isomorphisms of R'-modules:

$$\varprojlim_k H^q(X_k, F_k) \cong \varprojlim_k \varprojlim_l H^q(X_{k,l}, F_{k,l}) \cong \varprojlim_l \varprojlim_k H^q(X_{k,l}, F_{k,l}) \cong \varprojlim_l H^q(X'_l, F'_l).$$

By [EGA, III.3.4.4], we have naturally induced topological isomorphisms:

$$H^q(\mathfrak{X}, \mathfrak{F}) \to \varprojlim_k H^q(X_k, F_k) \quad \text{and} \quad H^q(\mathfrak{X}', \mathfrak{F}') \to \varprojlim_l H^q(X'_l, F'_l),$$

and we deduce the result. □

As a Corollary, we can prove Zariski's Connectedness Theorem for formal schemes, using the same argument as [EGA, III.4.3.2 and 4.3.4].

Corollary A.2. *Let $\varpi\colon \mathfrak{X} \to \mathfrak{Y}$ be a proper and Stein morphism of locally noetherian formal schemes. Then ϖ has geometrically connected fibers.*

Proof. Use [EGA, 0_{III}.10.3.1] and Proposition A.1 to reduce to $\mathfrak{Y} = \operatorname{Spf}_{IR} R$ where R is a complete local ring with maximal ideal I and algebraically closed residue field R/I. Then $|\mathfrak{X}|$ equals the unique fiber of ϖ. Finally observe that $\varpi_*\mathcal{O}_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{Y}}$ is local so $|\mathfrak{X}|$ is connected. □

Appendix B. Henselian pairs

In this appendix, we address some simple results about étale morphisms that we could not locate in the literature. As an application, we strengthen a well-known result about the stability of henselian pairs under proper morphisms.

For an algebraic space S, let $\mathbf{Et}(S)$ denote the category of morphisms of algebraic spaces $V \to S$ that are étale. Let $S_{\text{ét}}$ denote the small étale site of S: this has underlying category $\mathbf{Et}(S)$ and the coverings are jointly surjective families of morphisms. It is well-known that the natural functor:

$$\mathbf{Et}(S) \to \mathbf{Sh}(S_{\text{ét}}), \quad (V \to S) \mapsto \operatorname{Hom}_S(-, V)$$

is an equivalence of categories [Mil80, V.1.5], where $\mathbf{Sh}(S_{\text{ét}})$ denotes category of $S_{\text{ét}}$-sheaves of sets (even if S is a scheme, we need to allow V to be an algebraic space). If $f\colon X \to S$ is a morphism of algebraic spaces, there are thus natural functors $f^*\colon \mathbf{Et}(S) \to \mathbf{Et}(X)$ and $f_*\colon \mathbf{Et}(X) \to \mathbf{Et}(S)$, with f^* left adjoint to f_*. We define $\mathbf{Et}_{t/S}(X)$ to be the full subcategory of $\mathbf{Et}(X)$ with objects $W \to X$ such that for every geometric point $\overline{s}$ of S and every connected component Z of $W_{\overline{s}}$, the induced morphism $Z \to X_{\overline{s}}$ is an open and closed immersion. We now have the main technical result of this appendix.

Proposition B.1. *Let $f\colon X \to S$ be a proper and surjective morphism of schemes with geometrically connected fibers. Then the functor $f^*\colon \mathbf{Et}(S) \to \mathbf{Et}(X)$ is fully faithful with image $\mathbf{Et}_{t/S}(X)$.*

Proof. To show that f^* is fully faithful, it is sufficient to prove that if $V \in \mathbf{Et}(S)$, then the natural map $V \to f_*f^*V$ is an isomorphism. This may be verified over the geometric points of $\overline{s}$ of S. Note, however, that the functor f^* is trivially compatible with arbitrary base change on S. Since f is proper, it is a basic case of the Proper Base Change Theorem [SGA4, XII.5.1(i)] that f_* is also compatible with arbitrary base change on S. Thus, it suffices to prove the result where $S = \operatorname{Spec} k$ with k an algebraically closed field. Since f is surjective, this is trivial.

To classify the image of f^*, it remains to show that if $W \in \mathbf{Et}_{t/S}(X)$, then the natural map $f^*f_*W \to W$ is an isomorphism. This may be verified at geometric points $\overline{x}$ of X. The remarks above about base change also apply here, so we are again reduced to the situation where $S = \operatorname{Spec} k$ with k an algebraically closed field. In this case, the connected components of W are all open and closed subschemes of X. Since X is geometrically connected, we conclude that $W = \amalg_{w\in\pi_0(W)} X$. It remains to show that the natural map $\amalg_{w\in\pi_0(W)} S \to f_*W$ is an isomorphism. This may be checked on global sections (since k is algebraically closed) and there we have a natural bijection $\pi_0(W) \to \mathrm{Hom}_X(X, W)$. □

Remark B.2. Let $f\colon X \to S$ be a proper morphism of schemes with geometrically connected fibers and let $\gamma\colon V_1 \to V_2$ be a morphism in $\mathbf{Et}_{\mathrm{t}/S}(X)$. Then γ is an open immersion (resp. an open and closed immersion, resp. separated) if and only if the induced morphism $f_*\gamma\colon f_*V_1 \to f_*V_2$ is such. The first is an easy consequence of the fact that open immersions are categorical monomorphisms in $\mathbf{Et}(X)$, and since f_* preserves products, it preserves monomorphisms. The other two cases are even easier.

For a scheme S, let $\mathbf{OC}(S)$ denote its set of open and closed subsets. Note that $\mathbf{OC}(X) \subseteq \mathbf{Et}_{\mathrm{t}/S}(X)$. A *henselian pair* (S, S_0) consists of a scheme S and a closed immersion $S_0 \hookrightarrow S$ such that for any finite morphism $g\colon S' \to S$, the natural map $\mathbf{OC}(S') \to \mathbf{OC}(S' \times_S S_0)$ is bijective [EGA, IV.18.5.5].

Example B.3. Note that if B is a noetherian ring, separated and complete for the topology defined by an ideal $I \subseteq B$, then $(\operatorname{Spec} B, \operatorname{Spec} B/I)$ is a henselian pair [EGA, IV.18.5.16(ii)].

We now obtain the following improvement of [SGA4, XII.5.5] and [EGA, IV.18.5.19], where it is proved for henselian pairs of the form $(\operatorname{Spec} A, \{\mathfrak{m}\})$, where A is a henselian local ring and $\mathfrak{m}$ is the maximal ideal of A.

Corollary B.4. *Let (S, S_0) be a henselian pair and let $f\colon X \to S$ be a proper morphism of noetherian schemes. Then $(X, X \times_S S_0)$ is a henselian pair.*

Proof. Let $X_0 = X \times_S S_0$ and take $f_0\colon X_0 \to S_0$ to be the induced morphism. It is sufficient to prove that $\mathbf{OC}(X) \to \mathbf{OC}(X_0)$ is bijective. If $X \to \widetilde{X} \to S$ denotes the Stein factorization of f, then Proposition B.1 and Remark B.2 implies that we have a bijection $\mathbf{OC}(\widetilde{X}) \to \mathbf{OC}(X)$. Since $\widetilde{X} \to S$ is finite, $(\widetilde{X}, \widetilde{X} \times_S S_0)$ is a henselian pair [EGA, IV.18.5.6]. A similar analysis applies to the Stein factorization of f_0, $X_0 \to \widetilde{X}_0 \xrightarrow{\tilde{f}_0} S_0$, where we also obtain a bijection $\mathbf{OC}(\widetilde{X}_0) \to \mathbf{OC}(X_0)$. Denote by $g\colon X_0 \to \widetilde{X} \times_S S_0$ and $\tilde{g}\colon \widetilde{X} \times_S S_0 \to S_0$ the induced morphisms. There are now natural maps of coherent $\mathcal{O}_{S_0}$-algebras:

$$\tilde{g}_*\mathcal{O}_{\widetilde{X}\times_S S_0} \to \tilde{g}_*g_*\mathcal{O}_{X_0} \cong (f_0)_*\mathcal{O}_{X_0} \cong (\tilde{f}_0)_*\mathcal{O}_{\widetilde{X}_0}.$$

Whence we obtain a finite S_0-morphism $h\colon \widetilde{X}_0 \to \widetilde{X} \times_S S_0$. But the morphisms $X_0 \to \widetilde{X}_0$ and $X_0 \to \widetilde{X} \times_S S_0$ are both surjective with geometrically connected fibers, thus h also is surjective with geometrically connected fibers. Consequently, h is a universal homeomorphism [EGA, IV.18.12.11] so that $\mathbf{OC}(\widetilde{X} \times_S S_0) \to \mathbf{OC}(\widetilde{X}_0)$ is bijective. The result follows. □

Note that one of the strengths of henselian pairs is that they enable the computation of sections of sheaves. Indeed, if (S, S_0) is a henselian pair, with S quasi-compact and quasi-separated, and $V \in \mathbf{Et}(S)$, then the natural map $\mathrm{Hom}_S(S, V) \to \mathrm{Hom}_{S_0}(S_0, V \times_S S_0)$ is bijective [SGA4, XII.6.5(i)].

Corollary B.5. *Let (S, S_0) be a henselian pair, where S is a noetherian scheme. Fix a scheme Y over S. Let Z and X be schemes over Y such that $X \to Y$ is étale and $Z \to S$ is proper. Then the natural map:*

$$\mathrm{Hom}_Y(Z, X) \to \mathrm{Hom}_Y(Z \times_S S_0, X)$$

is bijective.

Proof. The map in the statement is identified with the natural map:

$$\mathrm{Hom}_Z(Z, V) \to \mathrm{Hom}_{Z\times_S S_0}(Z \times_S S_0, V \times_S S_0)$$

where $V = X \times_Y Z$ is étale over Z. Since $(Z, Z\times_S S_0)$ is a henselian pair (Corollary B.4), the result follows. □

Remark B.6. Note that [SGA4, XII.6.5(i)] and [SGA4, XII.5.1(i)] are quite elementary. The first result follows from the following facts: (i) every sheaf is a direct limit of constructible sheaves [SGA4, IX.2.7.2], and (ii) every constructible sheaf embeds in a product of push-forwards of constant sheaves along finite morphisms [SGA4, IX.2.14]. To prove the second result, one reduces to the case where S is henselian, then to $X = \mathbb{P}^n_S$ using a suitable Chow lemma, and finally to S noetherian and henselian using approximation. Then one concludes using [SGA4, XII.5.1(i)] and the Stein factorization [SGA4, XII.5.8].

For completeness, let us mention some generalizations of the results in this section to algebraic spaces and stacks. The first result, [SGA4, XII.6.5(i)], is easily extended to noetherian stacks and to quasi-compact and quasi-separated Deligne–Mumford stacks. The second result, [SGA4, XII.5.1(i)], then follows for noetherian stacks using the Chow lemma [Ols05, Thm. 1.1] and for quasi-compact and quasi-separated Deligne–Mumford stacks using the Chow lemma [Ryd09, Thm. B]. Proposition B.1, Corollary B.4 and Corollary B.5 thus follow for such stacks. The noetherian assumption in Corollary B.4 can be removed for Deligne–Mumford stacks using the fact that there exists Stein factorizations for proper morphisms of non-noetherian stacks, although one gets an integral morphism instead of a finite morphism. However, this is not a problem as integral morphisms can be approximated by finite morphisms [Ryd09, Thm. A].

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