ONE POSITIVE AND TWO NEGATIVE RESULTS FOR DERIVED CATEGORIES OF ALGEBRAIC STACKS

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Abstract. Let X be a quasi-compact algebraic stack with affine diagonal and let $\Psi_X \colon \mathsf{D}(\mathrm{QCoh}(X)) \to \mathsf{D}_{\mathrm{QCoh}}(X)$ be the natural functor. We prove that Ψ_X is an equivalence of categories if either $\mathsf{D}(\mathrm{QCoh}(X))$ or $\mathsf{D}_{\mathrm{QCoh}}(X)$ is compactly generated.

We also define a large class of algebraic stacks, the *poorly stabilized* stacks, and prove that if X is poorly stabilized with affine diagonal, then Ψ_X is not an equivalence. For poorly stabilized stacks X, the derived category $\mathsf{D}_{\mathrm{QCoh}}(X)$ is not compactly generated. None of these subtleties arise for schemes or algebraic spaces.

1. Introduction

Recall that if X is a quasi-compact scheme with affine diagonal, then the functor $\Psi_X \colon \mathsf{D}(\mathrm{QCoh}(X)) \to \mathsf{D}_{\mathrm{QCoh}}(X)$ is an equivalence of triangulated categories—see [BN93, Cor. 5.5] for the separated case (the argument adapts trivially to the case of affine diagonal) and [Stacks, 08H1] in the setting of algebraic spaces.

Recently, Krishna [Kri09, Cor. 3.7] extended the equivalence Ψ_X to a class of tame Deligne–Mumford stacks that satisfy the resolution property. We give a vast extension of this result.

Theorem 1.1. *Let X be an algebraic stack that is quasi-compact with affine diagonal. If either $\mathsf{D}_{\mathrm{QCoh}}(X)$ or $\mathsf{D}(\mathrm{QCoh}(X))$ is compactly generated, then the functor $\Psi_X \colon \mathsf{D}(\mathrm{QCoh}(X)) \to \mathsf{D}_{\mathrm{QCoh}}(X)$ is an equivalence of categories.*

In particular, Ψ_X is an equivalence for any Deligne–Mumford stack with affine diagonal (tame or not) [HR12]. It is natural to ask whether Ψ_X is always an equivalence of categories. On the positive side, recall that the restricted functor $\Psi_X^+ \colon \mathsf{D}^+(\mathrm{QCoh}(X)) \to \mathsf{D}^+_{\mathrm{QCoh}}(X)$ is always an equivalence of triangulated categories if X is a quasi-compact algebraic stack with affine diagonal [Lur04, Thm. 3.8] (also see [SGA6, Prop. II.3.5]).

In this note we introduce *poorly stabilized* algebraic stacks (see §4). This is a broad class of algebraic stacks in positive characteristic, which includes $B\mathbb{G}_a$ and $B\mathrm{GL}_n$ for $n > 1$. We prove

Theorem 1.2. *Let X be an algebraic stack that is quasi-compact with affine diagonal. If X is poorly stabilized, then the functor $\Psi_X \colon \mathsf{D}(\mathrm{QCoh}(X)) \to \mathsf{D}_{\mathrm{QCoh}}(X)$ is neither full nor faithful.*

Theorem 1.3. *Let X be an algebraic stack that is quasi-compact and quasi-separated. If X is poorly stabilized, then the triangulated category $\mathsf{D}_{\mathrm{QCoh}}(X)$ is not compactly generated.*

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Note that in the case where X has affine diagonal, Theorem 1.3 is a trivial consequence of Theorems 1.1 and 1.2. For stacks with non-affine stabilizer groups the situation is even worse: if $X = BE$, where E is an elliptic curve over $\mathbb{C}$, then the functor $\Psi^b_X \colon \mathsf{D}^b(\mathrm{Coh}(X)) \to \mathsf{D}^b_{\mathrm{Coh}}(X)$ is neither essentially surjective nor full.

We feel that Theorem 1.3 is somewhat surprising. Indeed, let X be a quasi-compact and quasi-separated scheme, then it is well-known that $\mathsf{D}_{\mathrm{QCoh}}(X)$ is compactly generated [BB03, Thm. 3.1.1(b)]. Now let X be a quasi-compact and quasi-separated algebraic stack. Recent work of Krishna [Kri09, Lem. 4.8], Ben-Zvi–Francis–Nadler [BZFN10, §3.3], Töen [Toë12, Cor. 5.2], and the first and third authors [HR12], has shown that frequently the unbounded derived category $\mathsf{D}_{\mathrm{QCoh}}(X)$ is also compactly generated.

Left-completeness. In the course of proving Theorem 1.2, we will prove that the triangulated category $\mathsf{D}(\mathrm{QCoh}(X))$ is not left-complete whenever X is poorly stabilized with affine diagonal. This generalizes an example of Neeman [Nee11] and amplifies some observations of Drinfeld–Gaitsgory [DG11, Rem. 1.2.10].

In Appendix B, we will prove that $\mathsf{D}_{\mathrm{QCoh}}(X)$ is left-complete for all algebraic stacks X. An analogous assertion in the context of derived algebraic geometry has been addressed by Drinfeld–Gaitsgory [DG11, Lem. 1.2.8]. In the Stacks Project [Stacks, 08IY] a similar result has been proved, albeit in a different context.

A requirement for a triangulated category to be left-complete is that it admits countable products. We were unable to locate a proof in the literature that $\mathsf{D}_{\mathrm{QCoh}}(X)$ admits countable products, however. Thus we also address this in Appendix B. By [Nee01b, Cor. 1.18], it suffices to prove that $\mathsf{D}_{\mathrm{QCoh}}(X)$ is well generated.

In Appendix A we show that if $\mathcal{M} \subseteq \mathcal{A}$ is an exact, coproduct-preserving inclusion of Grothendieck abelian categories, and $\mathcal{M} \subseteq \mathcal{A}$ is closed under extensions, then $\mathsf{D}_{\mathcal{M}}(\mathcal{A})$ is well generated—a result we expect to be of independent interest. Observe that the inclusion $\mathrm{QCoh}(X) \subseteq \mathrm{Mod}(X)$ is an exact, coproduct-preserving inclusion of Grothendieck abelian categories. We wish to point out that while [KS06, Prop. 14.2.4] is quite general, it does not apply in our situation. Indeed, they require that the embedding $\mathcal{M} \subseteq \mathcal{A}$ is closed under $\mathcal{A}$-subquotients, which is not the case for $\mathrm{QCoh}(X) \subseteq \mathrm{Mod}(X)$.

2. Preliminaries

Let $\phi \colon X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic stacks. The functor $\phi_* \colon \mathrm{Mod}(X) \to \mathrm{Mod}(Y)$ restricts to a functor $\phi_{\mathrm{QCoh},*} \colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ and the categories $\mathrm{Mod}(X)$ and $\mathrm{QCoh}(X)$ are Grothendieck abelian [Stacks, 0781]. Thus, the functors ϕ_* and $\phi_{\mathrm{QCoh},*}$ both admit right derived functors on their respected unbounded derived categories [Stacks, 079P & 070K], which we denote as $\mathbf{R}\phi_*$ and $\mathbf{R}\phi_{\mathrm{QCoh},*}$. By [Ols07, Lem. 6.20], the restriction of $\mathbf{R}\phi_*$ to $\mathsf{D}^+_{\mathrm{QCoh}}(X)$ factors uniquely through $\mathsf{D}^+_{\mathrm{QCoh}}(Y)$ and if, in addition, ϕ is concentrated (e.g., representable), then the restriction of $\mathbf{R}\phi_*$ to $\mathsf{D}_{\mathrm{QCoh}}(X)$ factors through $\mathsf{D}_{\mathrm{QCoh}}(Y)$ [HR12, Thm. 2.3].

For an algebraic stack W let $\Psi_W \colon \mathsf{D}(\mathrm{QCoh}(W)) \to \mathsf{D}_{\mathrm{QCoh}}(W)$ denote the natural functor. The universal properties of right-derived functors provides a diagram:

$$\begin{array}{ccc} \mathsf{D}(\mathrm{QCoh}(X)) & \xrightarrow{\mathbf{R}\phi_{\mathrm{QCoh},*}} & \mathsf{D}(\mathrm{QCoh}(Y)) \\ \downarrow{\scriptstyle \Psi_X} & & \downarrow{\scriptstyle \Psi_Y} \\ \mathsf{D}_{\mathrm{QCoh}}(X) & \xrightarrow{\mathbf{R}\phi_*} & \mathsf{D}_{\mathrm{QCoh}}(Y), \end{array}$$

together with a natural transformation of functors:

$$\epsilon_\phi\colon \Psi_Y \circ \mathbf{R}\phi_{\mathrm{QCoh},*} \Rightarrow \mathbf{R}\phi_* \circ \Psi_X. \tag{2.1}$$

The following result, for schemes, is well-known [TT90, B.8].

Proposition 2.1. *Let $\phi\colon X \to Y$ be a morphism of algebraic stacks that is quasi-compact and quasi-separated. Suppose that both X and Y are quasi-compact with affine diagonal. If $M \in \mathsf{D}^+(\mathrm{QCoh}(X))$, then the morphism induced by (2.1):*

$$\epsilon_\phi(M)\colon \Psi_Y \circ \mathbf{R}\phi_{\mathrm{QCoh},*}(M) \to \mathbf{R}\phi_* \circ \Psi_X(M)$$

is an isomorphism. In particular, since $\Psi_Y^+\colon \mathsf{D}^+(\mathrm{QCoh}(Y)) \to \mathsf{D}^+_{\mathrm{QCoh}}(Y)$ is an equivalence [Lur04, Thm. 3.8]*, it follows that there is a natural isomorphism for each $M \in \mathsf{D}^+(\mathrm{QCoh}(X))$:*

$$\mathbf{R}\phi_{\mathrm{QCoh},*}(M) \to (\Psi_Y^+)^{-1} \circ \mathbf{R}\phi_* \circ \Psi_X^+(M).$$

Proof. The functors $\phi_{\mathrm{QCoh},*}$ and ϕ_* are left-exact, thus the functors $\mathbf{R}\phi_{\mathrm{QCoh},*}$ and $\mathbf{R}\phi_*$ are bounded below. Via standard "way-out" arguments, one readily reduces to proving the isomorphism above in the case $M \simeq N[0]$, where $N \in \mathrm{QCoh}(X)$. The isomorphism, in this case, reduces to proving that if $N \in \mathrm{QCoh}(X)$, then the natural morphism $\mathbf{R}^i\phi_{\mathrm{QCoh},*}N \to \mathbf{R}^i\phi_*N$ is an isomorphism for all integers $i \geq 0$, where $\mathbf{R}^i\phi_{\mathrm{QCoh},*}$ (resp. $\mathbf{R}^i\phi_*$) denotes the ith right-derived functor of $\phi_{\mathrm{QCoh},*}$ (resp. ϕ_*).

If X is an affine scheme, then the morphism ϕ is affine. Thus, by flat base change, the functor $\phi_{\mathrm{QCoh},*}$ is exact and $\mathbf{R}^i\phi_*N = 0$ for all $i > 0$ and all $N \in \mathrm{QCoh}(X)$. In particular, it follows that the result has been proven when X is an affine scheme. For the general case, the result now follows from the arguments of [TT90, B.8]. □

Corollary 2.2. *Let $\phi\colon X \to Y$ be a concentrated morphism of algebraic stacks. If X and Y are quasi-compact with affine diagonal, then there exists an integer $r \geq 0$ such that for all $M \in \mathsf{D}(\mathrm{QCoh}(X))$ and integers n the natural map:*

$$\tau^{\geq n}\mathbf{R}\phi_{\mathrm{QCoh},*}M \to \tau^{\geq n}\mathbf{R}\phi_{\mathrm{QCoh},*}\tau^{\geq n-r}M$$

is a quasi-isomorphism. It follows that

1. *$\mathbf{R}\phi_{\mathrm{QCoh},*}$ preserves small coproducts,*
2. *for all $M \in \mathsf{D}(\mathrm{QCoh}(X))$ the natural morphism induced by (2.1):*

$$\epsilon_\phi(M)\colon \Psi_Y \circ \mathbf{R}\phi_{\mathrm{QCoh},*}M \to \mathbf{R}\phi_* \circ \Psi_X(M)$$

   is an isomorphism, and
3. *if ϕ is quasi-affine, then $\mathbf{R}\phi_{\mathrm{QCoh},*}$ is conservative.*

Proof. The claims (1)–(3) are all simple consequences of the main claim and Proposition 2.1. Since ϕ is a concentrated morphism, there exists an integer $r \geq 0$ such that if $N \in \mathrm{QCoh}(X)$, then $\mathbf{R}^i\phi_*N = 0$ for all $i > r$. By Proposition 2.1 it follows that $\mathbf{R}^i\phi_{\mathrm{QCoh},*}N = 0$ for all $i > r$ too. The result now follows from [Stacks, 07K7]. □

Corollary 2.3. *Let X be a noetherian algebraic stack with affine diagonal. If C is a compact object of either $\mathsf{D}(\mathrm{QCoh}(X))$ or $\mathsf{D}_{\mathrm{QCoh}}(X)$, then C is quasi-isomorphic to a bounded complex of coherent sheaves on X.*

Proof. Let C be a compact object of $\mathsf{D}_{\mathrm{QCoh}}(X)$. By [HR12, Ex. 4.9], C is a perfect complex, thus belongs to $\mathsf{D}^b_{\mathrm{QCoh}}(X) \subseteq \mathsf{D}^+_{\mathrm{QCoh}}(X)$. By [Lur04, Thm. 3.8], it follows that $C \simeq \Psi_X(\tilde{C})$ for some $\tilde{C} \in \mathsf{D}(\mathrm{QCoh}(X))$. Note that $\tilde{C}$ even belongs to $\mathsf{D}^b_{\mathrm{Coh}(X)}(\mathrm{QCoh}(X))$. Combining [LMB, Prop. 15.4] with [SGA6, II.2.2], we deduce that C belongs to the image of $\mathsf{D}(\mathrm{Coh}(X)) \to \mathsf{D}_{\mathrm{QCoh}}(X)$.

Now let C be a compact object of $\mathsf{D}(\mathrm{QCoh}(X))$. Let $p\colon U \to X$ be a smooth surjection from an affine scheme U. The functor $p^*\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(U)$ is exact and gives rise to a derived functor $\mathbf{L}p^*_{\mathrm{QCoh}}\colon \mathsf{D}(\mathrm{QCoh}(X)) \to \mathsf{D}(\mathrm{QCoh}(U))$, which is left adjoint to $\mathbf{R}p_{\mathrm{QCoh},*}$. Corollary 2.2 implies that $\mathbf{R}p_{\mathrm{QCoh},*}$ preserves small coproducts, thus $\mathbf{L}p^*_{\mathrm{QCoh}}C \in \mathsf{D}(\mathrm{QCoh}(U))$ is compact. Since $U = \operatorname{Spec} A$ is affine, it follows that $\mathrm{QCoh}(U) \cong \mathrm{Mod}(A)$ and so $\mathbf{L}p^*_{\mathrm{QCoh}}C$ is a perfect complex [Stacks, 07LT]. In particular, $C \in \mathsf{D}^b_{\mathrm{Coh}(X)}(\mathrm{QCoh}(X))$. Arguing as before, we deduce that C belongs to the image of $\mathsf{D}(\mathrm{Coh}(X)) \to \mathsf{D}(\mathrm{QCoh}(X))$. □

Corollary 2.4. *Let X be a quasi-compact algebraic stack with affine diagonal. Every compact object of either $\mathsf{D}(\mathrm{QCoh}(X))$ or $\mathsf{D}_{\mathrm{QCoh}}(X)$ is perfect. Let $P \in \mathsf{D}(\mathrm{QCoh}(X))$ be a perfect complex. The following are equivalent:*

(1a) *P is compact.*
(1b) *$\Psi(P)$ is compact.*
(2a) *There exists an integer $r \geq 0$ such that the natural map*
$$\tau^{\geq n}\mathbf{R}\mathrm{Hom}(P, M) \to \tau^{\geq n}\mathbf{R}\mathrm{Hom}(P, \tau^{\geq n-r}M)$$
is a quasi-isomorphism for all $M \in \mathsf{D}(\mathrm{QCoh}(X))$ and integers n.
(2b) *There exists an integer $r \geq 0$ such that the natural map*
$$\tau^{\geq n}\mathbf{R}\mathrm{Hom}(\Psi(P), M) \to \tau^{\geq n}\mathbf{R}\mathrm{Hom}(\Psi(P), \tau^{\geq n-r}M)$$
is a quasi-isomorphism for all $M \in \mathsf{D}_{\mathrm{QCoh}}(X)$ and integers n.
(3) *There exists an integer $r \geq 0$ such that $\mathrm{Ext}^n(P, N) = 0$ for all $N \in \mathrm{QCoh}(X)$ and all $n \geq r$.*

Moreover, a set of compact objects $\{P_i\}$ of $\mathsf{D}(\mathrm{QCoh}(X))$ is generating if and only if $\{\Psi(P_i)\}$ is generating.

Proof. The first statement follows from the proof of Corollary 2.3.

If (3) holds, then the natural map $\mathrm{Ext}^p(P, M) \to \mathrm{Ext}^p(P, \tau_{\geq n-r}M)$ is an isomorphism for all $M \in \mathsf{D}(\mathrm{QCoh}(X))$ and $p \geq n$. This follows from [Stacks, 07K7] applied to the functor $\mathrm{Hom}(P, -)\colon \mathrm{QCoh}(X) \to \mathcal{A}b$.

Moreover, if (3) holds, then $\mathrm{Ext}^p(\Psi(P), M) \to \mathrm{Ext}^p(\Psi(P), \tau_{\geq n-r}M)$ is an isomorphism for all $M \in \mathsf{D}_{\mathrm{QCoh}}(X)$ and $p \geq n$. This follows using Spaltenstein resolutions exactly as in the proof of [LO08, Lem. 2.1.10] with ϵ_* replaced by
$$\mathrm{Hom}(P, -)\colon \mathrm{Mod}(X_{\text{lissét}}) \to \mathcal{A}b$$
(which is a right adjoint and hence commutes with limits). Thus (3) implies (2a) and (2b).

That (2a) implies (1a) follows from the fact that $H^n\big(P, \tau_{\geq n-r}(-)\big)$ preserves small coproducts. Similarly, (2b) implies (1b).

Now, suppose that (3) does not hold. Then there are sequences of sheaves $N_1, N_2, \dots \in \mathrm{QCoh}(X)$ and a strictly increasing sequence $d_1 < d_2 < \dots$ such that $\mathrm{Ext}^{d_i}(P, N_i) \neq 0$ and $\mathrm{Ext}^k(P, N_i) = 0$ for all $k > d_i$. It follows that
$$\mathrm{Hom}\Big(P, \bigoplus_i N_i[d_i]\Big) = \mathrm{Hom}\Big(P, \prod_i N_i[d_i]\Big) = \prod_i \mathrm{Ext}^{d_i}(P, N_i)$$
but
$$\bigoplus_i \mathrm{Hom}(P, N_i[d_i]) = \bigoplus_i \mathrm{Ext}^{d_i}(P, N_i) \neq \prod_i \mathrm{Ext}^{d_i}(P, N_i)$$
so P is not compact. An identical calculation for $\Psi(P)$ shows that $\Psi(P)$ is not compact.

The final claim follows immediately from (2a), (2b) and the t-exactness of Ψ since Ψ^+ is an equivalence of categories. □

Proof of Theorem 1.1. By assumption, either $\mathsf{D}(\mathrm{QCoh}(X))$ or $\mathsf{D}_{\mathrm{QCoh}}(X)$ is compactly generated. Corollary 2.4 then tells us that both are compactly generated and that Ψ sends compact objects to compact objects. Thus, Ψ admits a right adjoint $\mathbf{R}Q\colon \mathsf{D}_{\mathrm{QCoh}}(X) \to \mathsf{D}(\mathrm{QCoh}(X))$ that preserves small coproducts [Nee96, Thm. 5.1].

Consider the unit $\eta_M\colon M \to \mathbf{R}Q(\Psi(M))$ and the counit $\epsilon_M\colon \Psi(\mathbf{R}Q(M) \to M$ of the adjunction. Since Ψ^+ is an equivalence, we have that η_P and ϵ_P are isomorphisms for every compact object P. Since η and ϵ are triangulated functors that preserve small coproducts and $\mathsf{D}_{\mathrm{QCoh}}(X)$ or $\mathsf{D}(\mathrm{QCoh}(X))$ are compactly generated, it follows that η and ϵ are equivalences. We conclude that Ψ is an equivalence. □

3. The case of $B_k\mathbb{G}_a$ in positive characteristic

Throughout this section we let k denote a field of characteristic $p > 0$. Let $B_k\mathbb{G}_a$ be the algebraic stack classifying $\mathbb{G}_a$-torsors over k. We remind ourselves that the category of quasi-coherent sheaves on $B_k\mathbb{G}_a$ is the category of $\mathbb{G}_a$-modules, which is equivalent to the category of locally small modules over a certain ring R. In fact R is the ring

$$R = \frac{k[x_1, x_2, x_3, \ldots]}{(x_1^p, x_2^p, x_3^p, \ldots)}$$

and a module is locally small if every element is annihilated by all but finitely many x_i. Let us write $\mathsf{D}(R^{\mathrm{ls}})$ for the derived category of the category of locally small R–modules, and observe that $\mathsf{D}(R^{\mathrm{ls}}) \cong \mathsf{D}(\mathrm{QCoh}(B_k\mathbb{G}_a))$.

Proposition 3.1. *The only compact objects, in either* $\mathsf{D}(\mathrm{QCoh}(B_k\mathbb{G}_a))$ *or* $\mathsf{D}_{\mathrm{QCoh}}(B_k\mathbb{G}_a)$, *are the zero objects.*

Proof. The algebraic stack $B_k\mathbb{G}_a$ is noetherian with affine diagonal and so, by Corollary 2.3, every compact object is the image of a bounded complex of coherent sheaves. Let C be a compact object; we need to show that C vanishes.

Our compact object C is the image of a finite complex of finitely generated modules in $\mathsf{D}(R^{\mathrm{ls}})$. In particular, there exists an integer $n > 1$ such that x_i annihilates C for all $i \geq n$. Let us put this slightly differently: consider the ring homomorphisms $S \xrightarrow{\alpha} T \xrightarrow{\beta} R \xrightarrow{\gamma} T$ where

$$S = k[x_n]/(x_n^p), \qquad T = \frac{k[x_1, x_2, \ldots, x_{n-1}, x_n]}{(x_1^p, x_2^p, \ldots, x_{n-1}^p, x_n^p)}$$

where the maps $S \xrightarrow{\alpha} T \xrightarrow{\beta} R$ are the natural inclusions, and where $\gamma\colon R \to T$ is defined by

$$\gamma(x_i) = \begin{cases} x_i & \text{if } i \leq n \\ 0 & \text{if } i > n. \end{cases}$$

Note that $\gamma\beta = \mathrm{id}$. Restriction of scalars gives induced maps of derived categories, which we write as $\mathsf{D}(T) \xrightarrow{\gamma_*} \mathsf{D}(R^{\mathrm{ls}}) \xrightarrow{\beta_*} \mathsf{D}(T) \xrightarrow{\alpha_*} \mathsf{D}(S)$, and $\beta_*\gamma_* = \mathrm{id}$. Our complex C, which is a bounded complex annihilated by x_i for all $i \geq n$, is of the form γ_*B where $B \in \mathsf{D}^b(T)$ is a bounded complex of finite T–modules. And the fact that x_n annihilates C translates to saying that α_*B is a complex of modules annihilated by x_n, that is a complex of k–vector spaces. We wish to show that $C = 0$ or, equivalently, that α_*B is acyclic. We will show that if C is non-zero, then this gives rise to a contradiction.

Thus, assume that the cohomology of α_*B is non-trivial: in $\mathsf{D}(S)$ the complex α_*B is isomorphic to a non-zero sum of suspensions $k[\ell]$ of k. Then there are infinitely many integers m and non-zero maps in $\mathsf{D}(S)$ of the form $\alpha_*B \to k[m]$. Indeed, $\mathrm{Ext}^m_S(k,k) \neq 0$ for all $m \geq 0$. But α_* has a right adjoint $\alpha^\times = \mathrm{RHom}_S(T,-)$, and we deduce infinitely many non-zero maps in $\mathsf{D}(T)$ of the form $B \to \alpha^\times k[m] = \mathrm{Hom}_S(T,k)[m]$. Since $\mathsf{D}(T)$ is left-complete, these combine to a map in $\mathsf{D}(T)$

$$B \xrightarrow{\Psi} \prod_m \mathrm{Hom}_S(T,k)[m] \cong \coprod_m \mathrm{Hom}_S(T,k)[m]$$

for which the composites

$$B \xrightarrow{\Psi} \coprod_m \mathrm{Hom}_S(T,k)[m] \xrightarrow{\pi_m} \mathrm{Hom}_S(T,k)[m]$$

are non-zero. Applying γ_*, which preserves coproducts, we deduce maps

$$\gamma_* B \xrightarrow{\gamma_*\Psi} \coprod_m \gamma_*\mathrm{Hom}_S(T,k)[m] \xrightarrow{\gamma_*\pi_m} \gamma_*\mathrm{Hom}_S(T,k)[m]$$

whose composites cannot vanish in $\mathsf{D}(R^{\mathrm{ls}})$, since β_* takes them to non-zero maps. The equivalence $\mathsf{D}(R^{\mathrm{ls}}) \cong \mathsf{D}(\mathrm{QCoh}(B_k\mathbb{G}_a))$ gives us that the composites in $\mathsf{D}(\mathrm{QCoh}(B_k\mathbb{G}_a))$ do not vanish. Furthermore, the composites lie in $\mathsf{D}^+(\mathrm{QCoh}(B_k\mathbb{G}_a)) \subseteq \mathsf{D}(\mathrm{QCoh}(B_k\mathbb{G}_a))$, and on $\mathsf{D}^+(\mathrm{QCoh}(B_k\mathbb{G}_a))$ the map to $\mathsf{D}_{\mathrm{QCoh}}(B_k\mathbb{G}_a)$ is fully faithful [Lur04, Thm. 3.8]. Hence the images of the composites are non-zero in $\mathsf{D}_{\mathrm{QCoh}}(B_k\mathbb{G}_a)$ as well. But this contradicts the compactness of $C = \gamma_* B$. □

4. The general case

In this section we extend the results of the previous section and show that the presence of $\mathbb{G}_a$ in the stabilizer groups of an algebraic stack X is an obstruction to compact generation in positive characteristic. The existence of finite unipotent subgroups such as $\mathbb{Z}/p\mathbb{Z}$ and $\boldsymbol{\alpha}_p$ is an obstruction to the compactness of the structure sheaf $\mathcal{O}_X$ but does not rule out compact generation [HR13]. The only connected groups in characteristic p without unipotent subgroups are the groups of multiplicative type. The following well-known lemma characterizes the groups without $\mathbb{G}_a$'s.

Lemma 4.1. *Let G be a group scheme of finite type over an algebraically closed field k. Then the following are equivalent:*

(1) *G^0_{red} is semiabelian, that is, a torus or the extension of an abelian variety by a torus;*
(2) *there is no subgroup $\mathbb{G}_a \hookrightarrow G$.*

Proof. By Chevalley's Theorem [Con02, Thm. 1.1] there is an extension $1 \to H \to G^0_{\mathrm{red}} \to A \to 1$ where H is smooth, affine and connected and A is an abelian variety. A subgroup $\mathbb{G}_a \hookrightarrow G$ would have to be contained in H which implies that H is not a torus. Conversely, recall that $H(k)$ is generated by its semi-simple and unipotent elements by the Jordan Decomposition Theorem [Bor91, Thm. 4.4]. If H is not a torus, then there exist non-trivial unipotent elements in $H(k)$. But any non-trivial unipotent element of $H(k)$ lies in a subgroup $\mathbb{G}_a \hookrightarrow G$. The result follows. □

If k is of positive characteristic, then we say that G is poor if G^0_{red} is not semiabelian. We say that an algebraic stack X is *poorly stabilized* if there exists a geometric point x of X whose residue field $\kappa(x)$ is of characteristic $p > 0$ and stabilizer group scheme G_x is poor. In particular, the algebraic stacks $B_k\mathbb{G}_a$ and $B_k\mathrm{GL}_n$ for $n > 1$ are poorly stabilized in positive characteristic. The following characterization of poorly stabilized algebraic stacks will be useful.

Lemma 4.2. *Let X be a quasi-separated algebraic stack.*

(1) *The stack X is poorly stabilized if and only if there exists a field k of characteristic $p > 0$ and a representable morphism $\phi\colon B_k\mathbb{G}_a \to X$.*

(2) *If X has affine stabilizers, then every representable morphism $\phi\colon B_k\mathbb{G}_a \to X$ is quasi-affine.*

Proof. Let k be an algebraically closed field and let $x\colon \operatorname{Spec}(k) \to X$ be a geometric point with stabilizer group scheme G. This induces a representable morphism $BG \to X$. If X is poorly stabilized, then there exists a point x such that G^0_{red} is not semiabelian. By the previous lemma, there is a subgroup $\mathbb{G}_a \hookrightarrow G$ and hence a representable morphism $B\mathbb{G}_a \to BG$.

Conversely, given a representable morphism $\phi\colon B_k\mathbb{G}_a \to X$, there is an induced representable morphism $\psi\colon B_k\mathbb{G}_a \to B_kG$. The morphism ψ is induced by some subgroup $\mathbb{G}_a \hookrightarrow G$ (unique up to conjugation) so X is poorly stabilized.

The structure morphism $\iota_x\colon \mathcal{G}_x \hookrightarrow X$ of the residual gerbe $\mathcal{G}_x$ at x is quasi-affine [Ryd11, Thm. B.2] and $\phi = \iota_x \circ \rho \circ \psi$ where $\rho\colon B_kG \to \mathcal{G}_x$ is affine. If X has affine stabilizers, then G is affine and it follows that the quotient $G/\mathbb{G}_a$ is quasi-affine since $\mathbb{G}_a$ is unipotent [Ros61, Thm. 3]. We conclude that the morphism $\psi\colon B_k\mathbb{G}_a \to B_kG$, as well as ϕ, is quasi-affine. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 4.2, there exists a field of characteristic $p > 0$ and a quasi-affine morphism $\phi\colon B_k\mathbb{G}_a \to X$. By Corollary 2.2, there exists an integer $n \geq 1$ such that if $N \in \operatorname{QCoh}(B_k\mathbb{G}_a)$, then $\mathbf{R}\phi_{\mathrm{QCoh},*}N \in \mathsf{D}^{[0,n]}(\operatorname{QCoh}(X))$. By [Nee11, Thm. 1.1], there exists $M \in \operatorname{QCoh}(B_k\mathbb{G}_a)$ such that the natural map in $\mathsf{D}(\operatorname{QCoh}(B_k\mathbb{G}_a))$:

$$\bigoplus_{i\geq 0} M[in] \to \prod_{i\geq 0} M[in]$$

is not a quasi-isomorphism—note that while [Nee11, Thm. 1.1] only proves the above assertion in the case where $n = 1$, a simple argument by induction on n gives the claim above. Corollary 2.2 now implies that the natural map:

$$\bigoplus_{i\geq 0} \mathbf{R}\phi_{\mathrm{QCoh},*}M[in] \to \prod_{i\geq 0} \mathbf{R}\phi_{\mathrm{QCoh},*}M[in]$$

is not a quasi-isomorphism. Since $\mathbf{R}\phi_{\mathrm{QCoh},*}M \in \mathsf{D}^{[0,n]}(\operatorname{QCoh}(X))$, it follows that $\mathsf{D}(\operatorname{QCoh}(X))$ is not left-complete.

To see that the functor Ψ_X is neither full nor faithful, let $L = \mathbf{R}\phi_{\mathrm{QCoh},*}M$, $S = \oplus_{i\geq 0} L[in]$, and $P = \prod_{i\geq 0} L[in]$. Since $\prod_{i\geq 0} M[in]$ is not bounded above [Nee11, Rem. 1.2] and ϕ is concentrated, it follows that P is not bounded above (Corollary 2.2). Note that the functor Ψ_X preserves small coproducts and is t-exact. Thus, because $\mathsf{D}_{\mathrm{QCoh}}(X)$ is left-complete (Theorem B.1), we have natural isomorphisms in $\mathsf{D}_{\mathrm{QCoh}}(X)$:

$$\Psi_X(S) \simeq \bigoplus_{i\geq 0} \Psi_X(L)[in] \simeq \prod_{i\geq 0} \Psi_X(L)[in].$$

Now, for any $K \in \mathsf{D}(\operatorname{QCoh}(X))$, the following diagram commutes:

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathsf{D}(\operatorname{QCoh}(X))}(K, S) & \longrightarrow & \operatorname{Hom}_{\mathsf{D}_{\mathrm{QCoh}}(X)}(\Psi_X(K), \Psi_X(S)) \\
\downarrow & & \downarrow \\
\operatorname{Hom}_{\mathsf{D}(\operatorname{QCoh}(X))}(K, P) & & \operatorname{Hom}_{\mathsf{D}_{\mathrm{QCoh}}(X)}(\Psi_X(K), \prod_{i\geq 0} \Psi_X(L)[in]) \\
\downarrow & & \downarrow \\
\prod_{i\geq 0} \operatorname{Hom}_{\mathsf{D}(\operatorname{QCoh}(X))}(K, L[in]) & \longrightarrow & \prod_{i\geq 0} \operatorname{Hom}_{\mathsf{D}_{\mathrm{QCoh}}(X)}(\Psi_X(K), \Psi_X(L)[in]).
\end{array}$$

The horizontal map on the bottom is an isomorphism by [Lur04, Thm. 3.8] and because $L[in] \in \mathsf{D}^+(\mathrm{QCoh}(X))$. Both lower vertical maps are also isomorphisms by the definition of the product in the respective categories. We have already shown above that the upper vertical map on the right is an isomorphism.

If $K = P$, then the upper vertical map on the left is not surjective. Indeed, if there was a morphism $P \to S$ such that the composition with $S \to P$ was $\mathrm{id}\colon P \to P$, then P would be a direct summand of S. Since S is bounded above and P is not bounded above, this is a contradiction, and there is no such morphism. It follows that the horizontal map on the top is not surjective and so Ψ_X is not full.

If K is a cocone of $S \to P$, that is, $K \to S \to P \to K[1]$ is a triangle, then the upper vertical map on the left is not injective and it follows that Ψ_X is not faithful. □

Finally, we prove Theorem 1.3.

Proof of Theorem 1.3. By Lemma 4.2, there exists a field of characteristic $p > 0$ and a representable morphism $\phi\colon B_k\mathbb{G}_a \to X$. Note that $\mathbf{R}\phi_*\mathcal{O}_{B_k\mathbb{G}_a} \in \mathsf{D}_{\mathrm{QCoh}}(X)$ and is non-zero. If $\mathsf{D}_{\mathrm{QCoh}}(X)$ is compactly generated, then there is a compact object $M \in \mathsf{D}_{\mathrm{QCoh}}(X)$ and a non-zero map $M \to \mathbf{R}\phi_*\mathcal{O}_{B_k\mathbb{G}_a}$. By adjunction, there is a non-zero map $\mathbf{L}\phi^*M \to \mathcal{O}_{B_k\mathbb{G}_a}$. But the functor $\mathbf{L}\phi^*$ sends compact objects of $\mathsf{D}_{\mathrm{QCoh}}(X)$ to compact objects of $\mathsf{D}_{\mathrm{QCoh}}(B_k\mathbb{G}_a)$ [HR12, Ex. 4.8 & Thm. 2.3]. By Lemma 3.1, it follows that $\mathbf{L}\phi^*M \simeq 0$ and we have a contradiction. Hence $\mathsf{D}_{\mathrm{QCoh}}(X)$ is not compactly generated. □

Appendix A. $\mathsf{D}_{\mathcal{M}}(\mathcal{A})$ is well generated

Throughout this section let $\mathcal{A}$ be a Grothendieck abelian category, and let $\mathcal{M} \subseteq \mathcal{A}$ be a Grothendieck abelian subcategory. The embedding $\mathcal{M} \to \mathcal{A}$ is assumed to be fully faithful, exact and coproduct-preserving, and $\mathcal{M}$ is assumed to be closed under extensions in $\mathcal{A}$. But $\mathcal{M}$ is not assumed to contain $\mathcal{A}$–subobjects or quotient objects of its objects. The example we have in mind is where X is an algebraic stack, $\mathcal{A}$ is the category of sheaves of $\mathcal{O}_X$–modules on X, and $\mathcal{M}$ is the subcategory of quasi-coherent sheaves. The main theorem of this section is

Theorem A.1. *Let $\mathsf{D}_{\mathcal{M}}(\mathcal{A}) \subseteq \mathsf{D}(\mathcal{A})$ be the full subcategory of the (unbounded) derived category $\mathsf{D}(\mathcal{A})$ of $\mathcal{A}$, whose objects are chain complexes in $\mathcal{A}$ with cohomology in $\mathcal{M}$. Then $\mathsf{D}_{\mathcal{M}}(\mathcal{A})$ is well generated.*

We will prove the theorem by a sequence of lemmas. We begin with

Construction A.2. Let g' be a generator for the abelian category $\mathcal{A}$ and let g'' be a generator for the abelian subcategory $\mathcal{M}$. Then $g = g' \oplus g''$ certainly generates the abelian category $\mathcal{A}$. For this choice of g we construct a regular cardinal μ as in [Nee13, Defn. 1.11] and a subcategory $\mathcal{B} \subseteq \mathcal{A}$ as in [Nee13, Defn. 1.13]. It follows that our $\mathcal{B}$ satisfies the statements of [Nee13, Lem. 1.14, Prop. 1.15, Rem. 1.16, and Prop. 1.18].

In addition we will use the little Lemma

Lemma A.3. *Every object of $\mathcal{A}$ is the μ–filtered colimit of its subobjects belonging to $\mathcal{B}$, and every object of $\mathcal{M}$ is the μ–filtered colimit of its subobjects belonging to $\mathcal{B} \cap \mathcal{M}$.*

Proof. Let X be an object in $\mathcal{A}$. By [Nee13, Prop. 1.15(i)] the coproduct of fewer than μ objects in $\mathcal{B}$ belongs to $\mathcal{B}$. If $\{X_\lambda,\ \lambda \in \Lambda\}$ is a set of fewer than μ subobjects of X, all belonging to $\mathcal{B}$, then the map $\Phi\colon \coprod_{\lambda\in\Lambda} X_\lambda \to X$ is a morphism from an object in $\mathcal{B}$ to X. The image of Φ is a subobject of X containing all the X_λ, and belongs to $\mathcal{B}$ because it is a quotient of an object in $\mathcal{B}$, see [Nee13, Prop. 1.15(ii)]. Hence the partially ordered set of subobjects of X belonging to $\mathcal{B}$ is μ–filtered.

If X belongs to $\mathcal{M}$ and the X_λ belong to $\mathcal{B} \cap \mathcal{M}$ then the map Φ from before is a morphism in $\mathcal{M}$ and its image belongs to $\mathcal{M}$; by the paragraph above it also belongs to $\mathcal{B}$, and therefore the partially ordered set of subobjects of X belonging to $\mathcal{B} \cap \mathcal{M}$ is μ–filtered.

It remains to show that the colimits of these partially ordered sets of subobjects are X. Now g is a generator for $\mathcal{A}$ and there is a surjection $g^\lambda \to X$, hence X is the colimit of the images of subcoproducts $g^A \subseteq g^\lambda$ where the cardinality of A is $< \mu$. And if X belongs to $\mathcal{M}$ then there is a surjection $\{g''\}^\lambda \to X$, and X is the colimit of the images of subcoproducts $\{g''\}^A$ where the cardinality of A is $< \mu$. The images belong to $\mathcal{B}$ by [Nee13, Prop. 1.15(i) and (ii)], and if X belongs to $\mathcal{M}$ then the map $\{g''\}^A \to X$ is a morphism in $\mathcal{M}$ whose image lies in $\mathcal{M}$. □

Lemma A.4. *Let $f\colon X \to Y$ be a surjective morphism in $\mathcal{A}$, and let $M \subseteq Y$ be a subobject belonging to $\mathcal{B}$. Then there exists a subobject $L \subseteq X$, with $L \in \mathcal{B}$ and $M = \mathrm{Im}(L \to Y)$.*

Proof. Let N be the inverse image of $M \subseteq Y$ under the epimorphism $X \to Y$. By Lemma A.3 the object N is the μ–filtered colimit of its subobjects $N_\lambda \in \mathcal{B}$. Hence M is the μ–filtered colimit of its subobjects $\mathrm{Im}(N_\lambda \to M)$, but $M \in \mathcal{B}$ is μ–presentable by [Nee13, Prop. 1.18]. Therefore the identity map $\mathrm{id}\colon M \to M$ factors through the image of some N_λ, which means we may choose a λ so that $L = N_\lambda \to M$ is surjective. □

Lemma A.5. *Let $X \subseteq Y \subseteq Z$ be objects in $\mathcal{A}$ with $Y/X \in \mathcal{M}$. Suppose N is a subobject of Z belonging to $\mathcal{B}$. Then we may find an object $N' \in \mathcal{B}$, with $N \subseteq N' \subseteq Z$, and such that*

(1) $\frac{N' \cap Y}{N' \cap X} \in \mathcal{B} \cap \mathcal{M}$, *and*
(2) *the natural map* $\frac{N}{N \cap Y} \to \frac{N'}{N' \cap Y}$ *is an isomorphism.*

Proof. The object $N \cap Y$ is a subobject of $N \in \mathcal{B}$ and belongs to $\mathcal{B}$ by [Nee13, Prop. 1.15(ii)]. Hence the composite $N \cap Y \to Y \to Y/X$ is a morphism from $N \cap Y \in \mathcal{B}$ to the object $Y/X \in \mathcal{M}$. By Lemma A.3 the object Y/X is the μ–filtered colimit of its subobjects belonging to $\mathcal{B} \cap \mathcal{M}$, and $N \cap Y \in \mathcal{B}$ is μ–presentable by [Nee13, Prop. 1.18]. Therefore the map $N \cap Y \to Y/X$ factors through a subobject $M \subseteq Y/X$, with $M \in \mathcal{B} \cap \mathcal{M}$.

By Lemma A.4, applied to the epimorphism $Y \to Y/X$ and the object $M \subseteq Y/X$, we may choose an subobject $L \subseteq Y$, with L belonging to $\mathcal{B}$, and such that the image of L in Y/X is M. Let N' be the image of the natural map $L \oplus N \to Z$. Because N' is a quotient of $N \oplus L \in \mathcal{B}$ it must belong to $\mathcal{B}$, and we leave it to the reader to check that (i) and (ii) of the Lemma are satisfied. □

Lemma A.6. *Given any non-zero object $Z \in \mathsf{D}_{\mathcal{M}}(\mathcal{A})$, there is an object $N \in \mathsf{D}^-_{\mathcal{B}\cap\mathcal{M}}(\mathcal{B})$ and a non-zero map $N \to Z$.*

Proof. If Z is the chain complex

$$\cdots \longrightarrow Z^{i-1} \xrightarrow{\partial} Z^i \xrightarrow{\partial} Z^{i+1} \longrightarrow \cdots,$$

we let $Y^i \subseteq Z^i$ be the cycles, in other words the kernel of $\partial\colon Z^i \to Z^{i+1}$, and $X^i \subseteq Y^i$ be the boundaries, that is the image of $\partial\colon Z^{i-1} \to Z^i$. We are assuming that $Z \in \mathsf{D}_{\mathcal{M}}(\mathcal{A})$ is non-zero, meaning its cohomology is not all zero; without loss of generality we may assume $H^0(Z) \neq 0$. Thus Y^0/X^0 is a non-zero object of $\mathcal{M}$.

By Lemma A.3 the object $Y^0/X^0 \in \mathcal{M}$ is the colimit of its subobjects belonging to $\mathcal{B} \cap \mathcal{M}$; since $Y^0/X^0 \neq 0$ we may choose a subobject $M \subseteq Y^0/X^0$, with $M \in$

$\mathcal{B} \cap \mathcal{M}$ and $M \neq 0$. By Lemma A.4, applied to the surjection $Y^0 \to Y^0/X^0$ and the subobject $M \subseteq Y^0/X^0$, we may choose a subobject $N^0 \subseteq Y^0$ belonging to $\mathcal{B}$ and such that the image of N^0 in Y^0/X^0 is M. Since Y^0 is the kernel of $Z^0 \to Z^1$ this gives us a commutative square

$$\begin{array}{ccc} N^0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ Z^0 & \longrightarrow & Z^1 \end{array}$$

where the vertical maps are monomorphisms and such that the image of the map $N^0 \to Y^0/X^0 = H^0(Z)$ is non-zero and belongs to $\mathcal{B} \cap \mathcal{M}$.

We propose to inductively extend this to the left. We will define a commutative diagram

$$\begin{array}{ccccccccccccccccc} & & & & N^i & \longrightarrow & N^{i+1} & \longrightarrow & \cdots & \longrightarrow & N^{-1} & \longrightarrow & N^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & Z^{i-1} & \longrightarrow & Z^i & \longrightarrow & Z^{i+1} & \longrightarrow & \cdots & \longrightarrow & Z^{-1} & \longrightarrow & Z^0 & \longrightarrow & Z^1 & \longrightarrow & \cdots \end{array}$$

where

(1) The subobjects $N^j \subseteq Z^j$ belong to $\mathcal{B}$.
(2) For $j > i$ the cohomology of $N^{j-1} \to N^j \to N^{j+1}$ belongs to $\mathcal{B} \cap \mathcal{M}$.
(3) Let K^i be the kernel of the map $N^i \to N^{i+1}$. Then the image of the natural map $K^i \to H^i(Z)$ belongs to $\mathcal{B} \cap \mathcal{M}$.

Since we have constructed N^0 we only need to prove the inductive step. Let us therefore suppose we have constructed the diagram as far as i; we need to extend it to $i-1$. By Lemma A.4, applied to the surjection $Z^{i-1} \to X^i$ and to the subobject $K^i \cap X^i \subseteq X^i$, we may choose an object $L \subseteq Z^{i-1}$, with $L \in \mathcal{B}$ and where the image of L in X^i is $K^i \cap X^i$. By Lemma A.5, applied to the inclusions $X^{i-1} \subseteq Y^{i-1} \subseteq Z^{i-1}$ and the subobject $L \subseteq Z^{i-1}$, we may find a subobject $N^{i-1} \subseteq Z^{i-1}$ belonging to $\mathcal{B}$ such that

(4) $\frac{N^{i-1} \cap Y^{i-1}}{N^{i-1} \cap X^{i-1}}$ belongs to $\mathcal{B} \cap \mathcal{M}$.
(5) $\frac{L}{L \cap Y^{i-1}} \cong \frac{N^{i-1}}{N^{i-1} \cap Y^{i-1}}$.

By (5) the images of L and N^{i-1} under the map $Z^{i-1} \to Z^i$ are the same, but L was chosen to map onto $K^i \cap X^i$. This means that the cohomology of $N^{i-1} \to N^i \to N^{i+1}$ is $\frac{K^i}{K^i \cap X^i}$ and belongs to $\mathcal{B} \cap \mathcal{M}$ by part (iii) of the inductive hypothesis. The kernel K^{i-1} of the map $N^{i-1} \to N^i$ is $N^{i-1} \cap Y^{i-1}$, and the map to $H^{i-1}(Z)$ takes it to the object $\frac{N^{i-1} \cap Y^{i-1}}{N^{i-1} \cap X^{i-1}}$, which belongs to $\mathcal{B} \cap \mathcal{M}$ by (4). □

Proof of Theorem A.1. Since $\mathcal{A}$ is a Grothendieck abelian category [Nee01a, Thm. 0.2] tells us that $\mathsf{D}(\mathcal{A})$ is well generated. Now consider the subcategory $\mathsf{D}^-_{\mathcal{B}\cap\mathcal{M}}(\mathcal{B})$: it is essentially small, and [Nee01b, Prop. 8.4.2] says that $\mathsf{D}^-_{\mathcal{B}\cap\mathcal{M}}(\mathcal{B})$ must be contained in $\{\mathsf{D}(\mathcal{A})\}^\alpha$ for some regular cardinal α. If $\mathcal{T} = \mathrm{Loc}\big(\mathsf{D}^-_{\mathcal{B}\cap\mathcal{M}}(\mathcal{B})\big)$ is the localizing subcategory generated by $\mathsf{D}^-_{\mathcal{B}\cap\mathcal{M}}(\mathcal{B})$ then [Nee01b, Thm. 4.4.9] informs us that $\mathcal{T}$ is well generated. Since $\mathsf{D}^-_{\mathcal{B}\cap\mathcal{M}}(\mathcal{B}) \subset \mathsf{D}_{\mathcal{M}}(\mathcal{A})$ and $\mathsf{D}_{\mathcal{M}}(\mathcal{A})$ is localizing it follows that $\mathcal{T} \subset \mathsf{D}_{\mathcal{M}}(\mathcal{A})$.

We know that $\mathcal{T}$ is well generated; to finish the proof it suffices to show that the inclusion $\mathcal{T} \subset \mathsf{D}_{\mathcal{M}}(\mathcal{A})$ is an equality. In any case the inclusion is a coproduct-preserving functor from the well generated category $\mathcal{T}$ and must have a right adjoint. For every object $Y \in \mathsf{D}_{\mathcal{M}}(\mathcal{A})$ there is a triangle in $\mathsf{D}_{\mathcal{M}}(\mathcal{A})$

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

with $X \in \mathcal{T}$ and $Z \in \mathcal{T}^{\perp}$. It suffices to prove that $Z = 0$, which comes from Lemma A.6: if Z were non-zero there would be a non-zero map $N \to Z$ with $N \in \mathsf{D}^{-}_{\mathcal{B}\cap\mathcal{M}}(\mathcal{B}) \subset \mathcal{T}$, contradicting $Z \in \mathcal{T}^{\perp}$. □

Appendix B. $\mathsf{D}_{\mathrm{QCoh}}(X)$ is left-complete

In this section we prove the following Theorem.

Theorem B.1. *If X is an algebraic stack, then $\mathsf{D}_{\mathrm{QCoh}}(X)$ is well generated. In particular, it admits small products. Moreover, $\mathsf{D}_{\mathrm{QCoh}}(X)$ is left-complete.*

Proof. The inclusion $\mathrm{QCoh}(X) \subseteq \mathrm{Mod}(X_{\text{lissét}})$ is exact, stable under extensions, and coproduct preserving. Since $\mathrm{QCoh}(X)$ and $\mathrm{Mod}(X_{\text{lissét}})$ are Grothendieck abelian categories [Stacks, 07A5 & 0781], it follows that $\mathsf{D}_{\mathrm{QCoh}}(X)$ is well generated (Theorem A.1). By [Nee01b, Cor. 1.18], $\mathsf{D}_{\mathrm{QCoh}}(X)$ admits small products.

It remains to prove that $\mathsf{D}_{\mathrm{QCoh}}(X)$ is left-complete. Let $p\colon U \to X$ be a smooth surjection from an algebraic space U. Let $U^{+}_{\bullet,\text{ét}}$ denote the resulting strictly simplicial algebraic space [Ols07, §4.1]. By [LO08, Ex. 2.2.5], there is an equivalence of triangulated categories $\mathsf{D}_{\mathrm{QCoh}}(X) \simeq \mathsf{D}_{\mathrm{QCoh}}(U^{+}_{\bullet,\text{ét}})$. The inclusion $\mathrm{QCoh}(U^{+}_{\bullet,\text{ét}}) \subseteq \mathrm{Mod}(U^{+}_{\bullet,\text{ét}})$ is exact, stable under extensions, and coproduct preserving. It follows that the functor $\omega\colon \mathsf{D}_{\mathrm{QCoh}}(U^{+}_{\bullet,\text{ét}}) \to \mathsf{D}(U^{+}_{\bullet,\text{ét}})$ is exact and coproduct preserving. As we already have seen, the category $\mathsf{D}_{\mathrm{QCoh}}(X) \simeq \mathsf{D}_{\mathrm{QCoh}}(U^{+}_{\bullet,\text{ét}})$ is well generated. Thus the functor ω admits a right adjoint λ [Nee01b, Prop. 1.20]. Because the functor ω is fully faithful, the adjunction $\mathrm{id} \Rightarrow \lambda \circ \omega$ is an isomorphism of functors.

Note that because λ is a right adjoint, it preserves products. In particular, it remains to prove that if $K \in \mathsf{D}_{\mathrm{QCoh}}(U^{+}_{\bullet,\text{ét}})$, then there exists a distinguished triangle in $\mathsf{D}(U^{+}_{\bullet,\text{ét}})$ (where we also take the products in $\mathsf{D}(U^{+}_{\bullet,\text{ét}})$):

$$\omega(K) \longrightarrow \prod_{n\geq 0} \tau^{\geq -n}\omega(K) \xrightarrow{1-\text{shift}} \prod_{n\geq 0} \tau^{\geq -n}\omega(K) \longrightarrow \omega(K)[1].$$

Indeed, this follows from the observation that $\tau^{\geq -n}\omega(K) \simeq \omega(\tau^{\geq -n}K)$ for all integers n and $K \to \lambda \circ \omega(K)$ is an isomorphism.

Let $(W \to U_n)$ be an object of $U^{+}_{\bullet,\text{ét}}$. The resulting slice $U^{+}_{\bullet,\text{ét}}/(W \to U_n)$ is equivalent to the small étale site on W. In particular, it follows that if W is an affine scheme and $M \in \mathrm{QCoh}(U^{+}_{\bullet,\text{ét}}) \cong \mathrm{QCoh}(X)$, then $H^p(U^{+}_{\bullet,\text{ét}}/(W \to U_n), M) = 0$ for all $p > 0$ [Stacks, 01XB & 0756]. Now let $\mathcal{B} \subseteq U^{+}_{\bullet,\text{ét}}$ denote the full subcategory consisting of those objects $(W \to U_n)$, where W is an affine scheme. It follows that $\mathcal{B}$ satisfies the requirements of [Stacks, 08U3] and we deduce the result. □

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