AN EXAMPLE WHERE STACK STRUCTURE CANNOT BE EXTENDED

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Abstract. We give an example of a normal singular threefold X and smooth Deligne-Mumfords stacks $\mathscr{U}$ with coarse moduli space the smooth locus $U \subseteq X$ of the threefold such that the stack $\mathscr{U}$ cannot be extended to a stack $\mathscr{X}$ having coarse moduli space X. The singularity is a terminal singularity and there exists a stack $\mathscr{X}'$ with coarse moduli space a resolution of the singularity. As a consequence, the contraction theorem for schemes in the minimal model program does not seem to generalize to Deligne-Mumford stacks. On the other hand, there exists an Artin stack with *good moduli space* X.

1. Finite étale covers

Recall the following definition of B. Noohi [Noo04]:

Definition (1.1). Let $\mathscr{X}$ be a Deligne-Mumford stack. We say that $\mathscr{X}$ is *uniformizable* if there exists a finite étale cover $U \to \mathscr{X}$ such that U is an algebraic space. We say that $\mathscr{X}$ is (algebraically) *simply connected* if every finite étale cover $\mathscr{X}' \to \mathscr{X}$ is trivial, i.e., every connected component of $\mathscr{X}'$ is isomorphic to its image.

For a definition of root stacks and root gerbes we refer to C. Cadman's paper [Cad07].

Proposition (1.2). *Let X be a scheme.*

(i) *Let $D \hookrightarrow X$ be an effective Cartier divisor such that $\mathcal{O}(D)$ is trivial. Then the root stack $X(\sqrt[r]{D})$ is uniformizable.*

(ii) *Let $\mathcal{L}$ be an invertible sheaf, then the root gerbe $X(\sqrt[r]{\mathcal{L}^r})$ is neutral and hence uniformizable.*

Proof. If $\mathcal{O}(D)$ is trivial, we can construct a cyclic covering $X' \to X$ of degree r ramified along D. Explicitly $X' = \operatorname{Spec}_X(\mathcal{O}_X[t]/t^r - s)$ where s is a global section of $\mathcal{O}(D) = \mathcal{O}_X$ defining D. The induced morphism $X' \to X(\sqrt[r]{D})$ is finite and étale. The second statement is obvious from the definition. □

For a converse statement we have:

Proposition (1.3). *Let k be a field (resp. a separable closed field). Let $P \in \mathbb{P}^1_k$ be a k-rational point and let $r \geq 2$ be an integer. Then the stacks $\mathbb{P}^1(\sqrt[r]{P})$ and $\mathbb{P}^1(\sqrt[r]{\mathcal{O}(1)})$ are not uniformizable (resp. simply connected).*

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Proof. There is a finite birational flat morphism $\mathbb{P}^1(\sqrt[r]{P}) \to \mathbb{P}^1(\sqrt[r]{\mathcal{O}(1)})$ (in particular representable) so it is enough to show that the first stack is not uniformizable (resp. is simply connected). We can also assume that k is separably closed and it is enough to show that the first stack is simply connected in this case. But $\mathbb{P}^1(\sqrt[r]{P})$ equals the weighted projective line $\mathbb{P}(1,r)$ which is simply connected. For an explicit argument, let $X \to \mathbb{P}^1(\sqrt[r]{P})$ be the $\mathbb{G}_m$-torsor corresponding to $\mathcal{O}(P^{1/r})$. Explicitly, we have

$$\mathbb{P}^1 = \operatorname{Spec}(k[x,y]) \setminus \{0\}/\mathbb{G}_m, \quad D = V(f), f \in k < x,y >$$
$$X = \operatorname{Spec}(A[x,y,z]/z^r - f) \setminus \{0\}$$

where $\mathbb{G}_m$ acts on X with weight r on x and y and weight 1 on z. The scheme X sits inside $\mathbb{P}^2$ with complementary codimension 2 so X is simply connected. Since $X \to \mathbb{P}^1(\sqrt[r]{P})$ has connected fibers it follows that $\mathbb{P}^1(\sqrt[r]{P})$ is simply connected. $\square$

More generally for any $n \geq 1$, $r \geq 2$, $D \hookrightarrow \mathbb{P}^n_k$ smooth of degree d relatively prime to r the stacks $\mathbb{P}^n(\sqrt[r]{D})$ and $\mathbb{P}^n(\sqrt[r]{\mathcal{O}(d)})$ are (probably) not uniformizable and this could perhaps be extended to any simply connected (smooth) base scheme and (smooth) divisor D such that $\mathcal{O}(D)$ is not trivial and has no r^{th} root.

Example (1.4). A non-simply connected example with D torsion in the Picard group. Let $X' = \mathbb{A}^2 = \operatorname{Spec}(k[x,y])$, let $X = X'/\mathfrak{S}_2 = \operatorname{Spec}(k[x^2,xy,y^2])$, let $X_0 = X \setminus \{0\}$ be the smooth locus of X and let $X'_0 = X' \setminus \{0\}$ be the inverse image of X_0. The Weil divisor $D = \mathrm{V}(x^2,xy) \hookrightarrow X$ is $\mathbb{Q}$-Cartier. Let $D_0 = D \cap X_0$ be the restriction to the smooth locus and let $D'_0 = \mathrm{V}(x=0)$ be the inverse image in X'_0. Let $r \geq 2$ be an integer. There is a finite étale morphism

$$X'_0\left(\sqrt[r]{D'_0}\right) \to X_0\left(\sqrt[r]{D_0}\right)$$

The stack $X'_0(\sqrt[r]{D'_0})$ is uniformizable and hence so is $X_0(\sqrt[r]{D_0})$. Not that $\mathcal{O}(D_0)$ is non-trivial since X is normal and D is not Cartier. On the other hand $2D = \mathrm{V}(x^2)$ is Cartier and $\mathcal{O}(2D)$ is trivial.

Theorem (1.5). *Let S be the spectrum of a henselian local ring with closed point s and let $f\colon X \to S$ be a proper morphism of stacks with finite diagonal. The functor*

$$\{\textit{Finite étale covers of } X\} \longrightarrow \{\textit{Finite étale covers of } X_s\}$$
$$(E \to X) \longmapsto (E_s \to X_s)$$

is an equivalence of categories.

Proof. The theorem is well-known when X is a scheme (cf. [$\mathrm{EGA_{IV}}$, Thm. 18.3.4] for the case where S is noetherian and complete and [Art69, Thm. 3.1] or [$\mathrm{SGA_4}$, Exp. XII, Thm. 5.9 bis] for the henselian case). To prove the case where X is a stack, choose a finite (non-flat) surjection $Z \to X$ [Ryd09, Thm. B]. By descent along finite surjections, there is an equivalence between the category of finite étale covers of X (resp. X_s) and the category of finite étale of Z (resp. Z_s) together with a descent datum on $Z \times_X Z$ (resp. $(Z \times_X Z)_s$), cf. [$\mathrm{SGA_1}$, Exp. VI, Thm. 4.7] or [Ryd07]. From the case of schemes, the category of finite étale covers of Z equipped with a descent datum is equivalent to the category of finite étale covers of Z_s equipped with a descent datum. $\square$

Theorem (1.6). *Let k be a field and let $X \hookrightarrow \mathbb{A}^4$ be the hypersurface $\operatorname{Spec}(k[x,y,z,w]/xy-zw)$. Let X_0 be the smooth locus of X and let $D_0 \hookrightarrow X_0$ be the Cartier divisor given by $x = z = 0$. Let $\mathscr{X}_0$ be either the orbifold $X_0(\sqrt[r]{D_0})$ or the gerbe $X_0(\sqrt[r]{\mathcal{O}(D_0)})$ so that X_0 is the coarse moduli space of $\mathscr{X}_0$. Then there does not exists a stack $\mathscr{X}$ with finite diagonal and coarse moduli space X such that $\mathscr{X}|_{X_0} \cong \mathscr{X}_0$.*

Some introductory comments (which are not needed for the proof). The scheme X is a non-simplicial toric variety and hence is normal but the singular point is not a quotient singularity. The Picard group of X_0 is $\mathbb{Z}$ and is generated by D and $(X_0)_{\overline{k}}$ is simply connected.

Proof. We can assume that k is separably closed. Let W denote the spectrum of the (strict) henselization of X at the singular point $x = y = z = w = 0$. If a stack $\mathscr{X}$ as in the Theorem exists, then $\mathscr{X} \times_X W$, and *a fortiori* $\mathscr{X}_0 \times_X W$, are uniformizable. Let $X' \to X$ be the small resolution such that the closure of D_0 intersects the exceptional fiber in one point. Explicitly $X' = (\operatorname{Spec}(k[u_1,u_2,u_3,u_4]) \setminus \{u_1 = u_2 = 0\})/\mathbb{G}_m$ where $\mathbb{G}_m$ acts freely with weights $(1,1,-1,-1)$ and $x = u_1u_3$, $y = u_2u_4$, $z = u_1u_4$, $w = u_2u_3$. The closure D' of D_0 in X' is given by $u_1 = 0$, the exceptional fiber $E \cong \mathbb{P}^1 \hookrightarrow X'$ is defined by $u_3 = u_4 = 0$ and $D'|_E$ has degree one. Let $\mathscr{X}' = X(\sqrt[r]{D'})$.

Now, the finite étale covers of $\mathscr{X}_0 \times_X W$ coincide with the finite étale covers of $\mathscr{X}' \times_X W$ since $\mathscr{X}_0 \subseteq \mathscr{X}'$ has complementary codimension 2 and $\mathscr{X}'$ is regular (this is why we study the small resolution of X). Since $\mathscr{X}' \to X$ is proper and W is henselian, the finite étale covers of $\mathscr{X}' \times_X W$ are the same as the finite étale covers of $\mathscr{X}'|_0 \cong \mathbb{P}^1(\sqrt[r]{P}) \cong \mathbb{P}(1,r)$ by Theorem (1.5) but this stack is simply connected as we saw in Proposition (1.3). Thus, $\mathscr{X}_0 \times_X W$ is simply connected and hence not uniformizable. This contradicts the existence of $\mathscr{X}$. $\square$

Remark **(1.7).** The stack $\mathscr{X}_0$ of the theorem has an *Artin compactification*, i.e., there is a stack $\mathscr{X}$ with *good moduli space* X [Alp08]. This is the stack $[\mathbb{A}^4(\sqrt[r]{u_1 = 0})/\mathbb{G}_m]$. The moduli map $\mathscr{X} \to X$ is universally closed but not separated.

The scheme X has terminal singularities. This indicates that there is a smooth DM-stack such that when running the minimal model program on this stack we are supposed to do a contraction which exists on the level of coarse moduli spaces but not on the level of stacks, not even as a rational map. A solution to this could be to instead work in the category of Artin stacks. This has some twists to it, for example an open immersion of Artin stack could act as a contraction!

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