Mayer-Vietoris squares and descent of étale sheaves

11/12 - 2015

I MV-squares

a) motivation

b) definition

c) examples

d) theorems (FR/MB + main thm)

II Application 1

push-out, gluing of alg spaces.

III Proof of main thm

a) Recollement

b) Gabber rigidity

IV Application 2

Descent for univ subm. of étale sheaves

I Mayer-Vietoris squares

$X = U \cup V \qquad j_1 : U \to X \quad j_2 : V \to X$

$\mathcal{F}$ sheaf $\Rightarrow \quad 0 \to \mathcal{F}(X) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V)$

(exact for flasque) $\Rightarrow$ LES $\cdots \to H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F}) \times H^i(V, \mathcal{F}) \to H^i(U \cap V, \mathcal{F}) \to \cdots$

triangle $\qquad \mathcal{F} \to Rj_{1*} j_1^* \mathcal{F} \oplus Rj_{2*} j_2^* \mathcal{F} \to Rh_* h^* \mathcal{F} \to \mathcal{F}[1]$

Gluing of sheaves: $QCoh(X) \xrightarrow{\cong} QCoh(U) \times_{QCoh(U \cap V)} QCoh(V)$

$$\mathcal{F} \longmapsto \left(\mathcal{F}|_U, \mathcal{F}|_V, (\mathcal{F}|_U)|_{U \cap V} \overset{\varphi}{\cong} (\mathcal{F}|_V)|_{U \cap V} \right)$$

$$\acute{E}t(X) \cong \acute{E}t(U) \times_{\acute{E}t(U \cap V)} \acute{E}t(V) \qquad \text{etc.}$$

$$\begin{tikzcd} U \cap V \arrow[r, hook] \arrow[d, hook] & V \arrow[d, hook] \\ U \arrow[r, hook, "j"] & X \end{tikzcd}$$

$\leftarrow$ open nbhd of $X \setminus U$.

General square:

$$\begin{tikzcd} U' \arrow[r, hook, "j'"] \arrow[d, "f_U"] & X' \arrow[d, "f"] \\ U \arrow[r, hook, "j"'] & X \end{tikzcd}$$

$i : Z \hookrightarrow X$ some complement of U.

"f nbhd of Z"

Ex: a) f étale neighborhood of Z: f étale, $f|_Z$ iso.

Nisnevich square, upper distinguished, Morel-Voevodsky elem.

(K-theory, motives, $\mathbb{A}^1$-homotopy)

b) tubular neighborhoods

$$\begin{tikzcd} U' \arrow[r, hook] \arrow[d] & N_{Z/X} \arrow[r] \arrow[d, "f"] & Z \arrow[l, bend right, "0\text{-sect.}"'] \arrow[d, equal] \\ U \arrow[r, hook] & X & Z \arrow[l, hook'] \end{tikzcd}$$

f diffeo. in nbhd of Z.

c) formal neighborhoods $X' = X^{\wedge}_Z$ $\quad f|_{Z^{(n)}}$ iso $\forall n$.

Def: The square is MV if

(0) j qc open, i fin pres closed im

(1) $f|_Z$ isomorphism

(2) $\mathrm{Tor}^i_{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Z) = 0 \quad \forall i > 0$

$\Big\} \Leftrightarrow Lf^*\mathcal{O}_Z \simeq \mathcal{O}_Z \Leftrightarrow X' \overset{L}{\underset{X}{\times}} Z \cong Z$

$f|_Z$ derived iso

The square is weak MV if

(0) + (1') $f|_Z$ iso $\forall Z$, $|Z| = |X \setminus U|$

Lemma: $\left.\begin{array}{l} f \text{ flat along } f^{-1}(Z) \\ f|_Z \text{ iso} \end{array}\right\} \Rightarrow f$ MV $\Rightarrow f$ weak MV

and converse holds if X, X' noetherian.

pf: converse:

$$\begin{tikzcd} X'^{\wedge}_Z \arrow[r, "\text{flat}"] \arrow[d, equal] & X' \arrow[d] \\ X^{\wedge}_Z \arrow[r, "\text{flat}"'] & X \end{tikzcd}$$

Ex: a) $X = \operatorname{Spec} A$, $Z = \operatorname{Spec} A/I$, $U = X \setminus Z$

$X' = \operatorname{Spec}(A^\wedge_I)$ $\quad A^\wedge_I = \varprojlim_n A/I^n$

I f.gen $\Rightarrow$ weak MV

A noeth $\Rightarrow$ MV (f is flat)

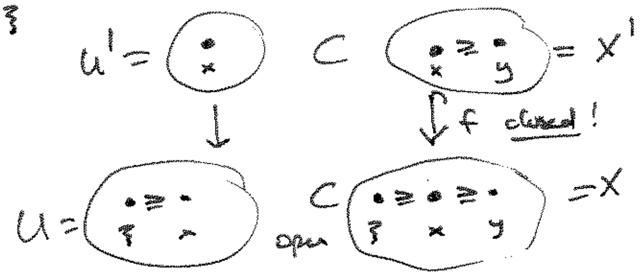
b) Etale nbhds.

c) V valuation-ring of finite dim > 1 (non-noeth!)

$P \subset V$ prime ideal, $\operatorname{Spec} V = \{ \underset{\xi}{(0)} \geq \{p\} \geq \{m\} \}$ $\qquad \xi \geq x \geq y$

$U = \operatorname{Spec}(V_p)$ $\qquad X' = \operatorname{Spec}(V/p)$

$X = \operatorname{Spec}(V)$



Lemma: A weak MV square is a push-out in cat of top spaces:

$$|X| = |X'| \underset{|U'|}{\times} |U|$$

Thm (Ferrand-Raynaud '70, Moret-Bailly '96)

$$\text{MV-square} \Rightarrow \mathrm{QCoh}_{f\text{-flat}}(X) \xrightarrow{\simeq} \mathrm{QCoh}(X') \times_{\mathrm{QCoh}(U')} \mathrm{QCoh}_{f_U\text{-flat}}(U)$$

($\mathcal{F} \in \mathrm{QCoh}(X)$ f-flat if $Lf^*\mathcal{F} \xrightarrow{\simeq} f^*\mathcal{F}$, e.g. $\mathcal{F}$ flat or f flat)

Main Thm (Hall-Rydh '15)

$$\text{weak MV-square} \Rightarrow \acute{E}t(X) \xrightarrow{\simeq} \acute{E}t(X') \times_{\acute{E}t(U')} \acute{E}t(U)$$

II Application 1

Thm (HR'15) MV-square, X excellent, f flat.

$$\Psi : \mathrm{AlgSp}(X) \xrightarrow{\simeq} \mathrm{AlgSp}(X') \times_{\mathrm{AlgSp}(U')} \mathrm{AlgSp}(U)$$

pf: WLOG f regular ($X' = X^\wedge$)
~~Popescu~~: f limit of smooth $\Rightarrow$ have descent of (f.p.) alg sp along f.

Need to construct descent data for $X' \times_X X' \rightrightarrows X' \longrightarrow X$.

$$\begin{tikzcd} U' \arrow[d, "\Delta_{f_U}"'] & \subset & X' \arrow[d, "\Delta_f"] \\ U' \times_U U' & \subset & X' \times_X X' \end{tikzcd}$$

MV-square (highly non-noeth, sim. to valuation example)

Descent data follows from:

Thm (HR'15) ~~weak~~ MV-square $\Rightarrow$ push-out in cat. of alg sp.

($\Leftrightarrow \Psi$ fully faithful)

pf: Use gluing of étale to reduce to push-out in cat of affine schemes.

and $\Gamma(X, \mathcal{O}_X) = \Gamma(X', \mathcal{O}_{X'}) \times_{\Gamma(U', \mathcal{O}_{U'})} \Gamma(U, \mathcal{O}_U)$

Cor: MV-square, X exc, f flat $\to$ p-o in cat of alg. stacks.

pf: Use gluing of alg sp..

↑
used in Tannaka duality.

III Étale sheaves: pf of main thm

<u>Recollement</u>: $\acute{E}t(X) \simeq \left(\acute{E}t(Z),\ \acute{E}t(U),\ \underset{\overset{\parallel}{i'^*f^*j_*}}{i^*j_*}\right)$

$$\acute{E}t(X') \simeq \left(\acute{E}t(Z),\ \acute{E}t(U'),\ i'^*j'_*\right)$$

$$\acute{E}t(X') \times_{\acute{E}t(U')} \acute{E}t(U) \simeq \left(\acute{E}t(Z),\ \acute{E}t(U),\ i'^*j'_*f_U^*\right)$$

Natural thm $f^*j_* \xrightarrow[\lambda]{} j'_*f_U^*$. Enough to prove λ iso.

WLOG X, X' henselian.

<u>Thm</u> (Gabber rigidity) weak MV, X, X' henselian, then

$$\begin{array}{ccc} \Gamma(U, \mathcal{F}) & \xrightarrow{\ \cong\ } & \Gamma(U', f_U^*\mathcal{F}) \\ \parallel & & \parallel \\ \Gamma(X', f^*j_*\mathcal{F}) & & \Gamma(X', j'_*f_U^*\mathcal{F}) \end{array} \qquad \forall \mathcal{F} \in \acute{E}t(U)$$

<u>pf</u>: Reduce to proving $Oc(U) \xrightarrow{\simeq} Oc(U')$.

Replace X w/ hu in Z $\Rightarrow$ MV-square.

Replace X w/ norm of X in U $\Rightarrow$ X' norm in U' (FR/MB)

$$\begin{tikzcd} Oc(Z) & Oc(X) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d] & Oc(U) \arrow[d] \\ Oc(Z) \arrow[u, equal] & Oc(X') \arrow[l, "\cong"] \arrow[r, "\simeq"] & Oc(U) \end{tikzcd}$$

by henselity (proper bc) by normality

IV Application 2

$$f : X' \longrightarrow X \qquad \begin{matrix} X' \times_X X' \\ \shortparallel \end{matrix} \qquad X''' \;\substack{\longrightarrow\\ \longrightarrow \\ \longrightarrow}\; X'' \;\underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}}\; X' \longrightarrow X$$

$$\Phi_f : \text{Ét}(X) \longrightarrow \text{Ét}(X' \to X)$$

$$= \left\{ E' \in \text{Ét}(X'),\ \varphi : \pi_1^* E' \xrightarrow{\sim} \pi_2^* E' \quad \begin{matrix} \\ \varphi \text{ satisfies cocycle condition on } X''' \end{matrix} \right\}$$

SGA1/SGA4

- f univ subm $\Rightarrow \Phi_f$ fully faithful
- f étale surj $\Rightarrow \Phi$ equiv (general topos fact)
- f proper surj $\Rightarrow$ —"— (proper base change)
- f f. flat + l.f.p. $\Rightarrow$ —"— (reduce to qff, fin surj + ét)

Thm (R'07) f univ <u>subtrusive</u>, l.f.p. $\Rightarrow \Phi$ equiv.

Thm (HR'15) f univ <u>submersive</u>, qc. $\Rightarrow \Phi_{cons}$ equiv.

Rmk. X noeth.: subm $\Leftrightarrow$ subt.