REMARKS ON "THE RESOLUTION PROPERTY FOR SCHEMES AND STACKS" BY B. TOTARO

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1. Corrections and complements

There are two minor mistakes in [Tot04]. The first is in the last paragraph of the proof of Theorem 1.1 (last paragraph in Section 5 on p. 11). The total space of the frame bundle associated to $E_1 \oplus E_2$ is not a bundle over Y with fiber $\mathrm{GL}(n_1+n_2)/\big(\mathrm{GL}(n_1)\times\mathrm{GL}(n_2)\big)$. Here Y is the fiber product of the total spaces of the frame bundles associated to E_1 and E_2. Rather, it is a bundle over $\mathrm{GL}(n_1+n_2)/\big(\mathrm{GL}(n_1)\times\mathrm{GL}(n_2)\big)$ with fiber Y, cf. Lemma (1.1) below.

The second mistake is in the last paragraph of the proof of Proposition 9.1 (last paragraph on p. 20). The error is that Lemma 9.2 as stated does not apply. Indeed, there are no non-trivial $\mathbb{Z}/2\times T$-invariant neighborhoods of the singular points of Y. However, Lemma 9.2 is valid for not necessarily T-invariant S_i's as long as there is a T-invariant vector bundle which surjects onto $C=\bigoplus I_{S_i}$. There is also another general argument: if $Y\to X$ is finite and étale and Y has the resolution property, then so has X, cf. Lemma (1.3).

Lemma (1.1). *Let X be an algebraic stack and let E_1, E_2 be two vector bundles on X. Let F_1, F_2 and F denote the total spaces of the frame bundles corresponding to E_1, E_2 and $E_1\oplus E_2$. If $F_1\times_X F_2$ is quasi-affine then so is F. In particular, if F_1 or F_2 is quasi-affine then so is F.*

Proof. We can assume that E_1 and E_2 have constant ranks n_1 and n_2. Then $F=\mathrm{GL}_{n_1+n_2}\times^{\mathrm{GL}_{n_1}\times\mathrm{GL}_{n_2}}(F_1\times_X F_2)$. The induced map $F\to\mathrm{GL}_{n_1+n_2}/(\mathrm{GL}_{n_1}\times\mathrm{GL}_{n_2})$ is an fppf-fibration with fiber the quasi-affine scheme $F_1\times_X F_2$. Since $\mathrm{GL}_{n_1+n_2}/(\mathrm{GL}_{n_1}\times\mathrm{GL}_{n_2})$ is affine (a "Stiefel manifold"), it follows that F is quasi-affine. The last remark follows from the observation that $F_1\to X$ and $F_2\to X$ are affine. $\square$

Lemma (1.2). *Let $f\colon X\to Y$ be a finite fppf morphism of algebraic spaces. If X is affine (resp. quasi-affine) then so is Y.*

Proof. Every fiber of f has an affine open neighborhood by [$\mathrm{EGA}_{\mathrm{II}}$, Cor. 4.5.4]. Hence Y is an affine scheme (resp. a scheme) [SGA_3, Exp. V, Thm. 4.1]. If X is quasi-affine then so is Y by [$\mathrm{EGA}_{\mathrm{II}}$, Cor. 6.6.3]. $\square$

Lemma (1.3). *Let $f\colon X\to Y$ be a finite étale and surjective morphism of algebraic stacks. If X has the resolution property, then so has Y.*

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Proof. Let $\mathcal{E}$ be a locally free sheaf on X such that the total space of the corresponding frame bundle is quasi-affine[1]. To $\mathcal{E}$ we associate the vector bundle $V = \mathbb{V}(\mathcal{E}) = \operatorname{Spec}(\operatorname{Sym}(\mathcal{E}))$ and the frame bundle $\mathbf{Isom}(\mathcal{E}, \mathcal{O}_X^{\oplus n}) = \mathbf{Isom}(\mathcal{O}_X^{\oplus n}, \mathcal{E}^\vee) \hookrightarrow \mathbb{V}(\mathcal{E}^{\oplus n})$ (if $\mathcal{E}$ has constant rank n).

We will show that the total space of the frame bundle of $f_*\mathcal{E}$ is quasi-affine. First note that the total space of the frame bundle of $f_*\mathcal{E}$ is an algebraic space. Indeed, this follows from the observation that the total space is an algebraic space if and only if for every geometric point $y\colon \operatorname{Spec}(\overline{k}) \to Y$ the stabilizer action on the fiber $(f_*\mathcal{E})_y$ is faithful [EHKV01].

By Lemma (1.2) it is now enough to show that the total space of the frame bundle of $f^*f_*\mathcal{E}$ is quasi-affine. Since f is *étale*, we have that $f^*f_*\mathcal{E} \to \mathcal{E}$ is split. Indeed, we have that $X \times_Y X = \Delta_{X/Y}(X) \amalg Z$ and thus $f^*f_*\mathcal{E} = (\pi_2)_*(\pi_1)^*\mathcal{E} = \mathcal{E} \oplus (\pi_2|_Z)_*\big((\pi_1)^*\mathcal{E}|_Z\big)$. Since the total space of the frame bundle of $\mathcal{E}$ is quasi-affine, so is the total space of $f^*f_*\mathcal{E}$ by Lemma (1.1). $\square$

In particular, Lemma (1.3) shows that the resolution property holds for stacks and algebraic spaces of the form $[X/G]$ with X a smooth scheme and G a finite étale group. There are probably many examples of smooth algebraic spaces which are not of this form (e.g., simply connected smooth proper algebraic spaces).

Remark **(1.4).** It would be interesting to know whether Lemma (1.3) is valid if $f\colon X \to Y$ is finite fppf (this is a far more general situation). To show this, it is enough to show that the frame bundle of $f^*f_*\mathcal{E}$ has quasi-affine total space. I do not know if this is reasonable to expect since the obvious fibration implies that the total space is quasi-affine over a flag manifold (hence quasi-projective). This at least implies that the frame bundle of $f_*\mathcal{E}$ has quasi-projective total space and thus Y has the resolution property if Y is *normal*.

If Y has a quasi-projective (or at least divisorial) coarse moduli space, then it also follows that Y has the resolution property. Indeed, since X is a global quotient stack so is Y. Thus $Y = [V/\mathrm{GL}_n]$ where $V \to Y_{\mathrm{cms}}$ is affine so that V has a GL_n-equivariant ample sheaf (or ample family).

References

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[1] For X not normal, the existence is a recent (unpublished) result due to Philipp Gross.