Stacky blow-ups, étalification and compactification of stacks

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KTH Mathematics

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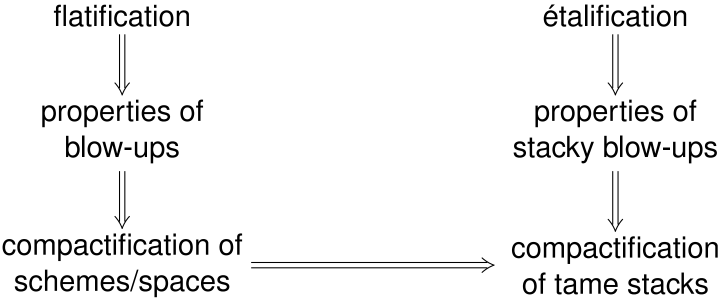
Usually, one wants $X \subseteq \overline{X}$ where $\overline{X}$ is smooth and $\overline{X} \setminus X$ is snc. This is accomplished by first choosing **any** compactification $X \subseteq \widetilde{X}$ and then taking a resolution of singularities $\overline{X} \to \widetilde{X}$.

Outline


flatification
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compactification of schemes/spaces
étalification
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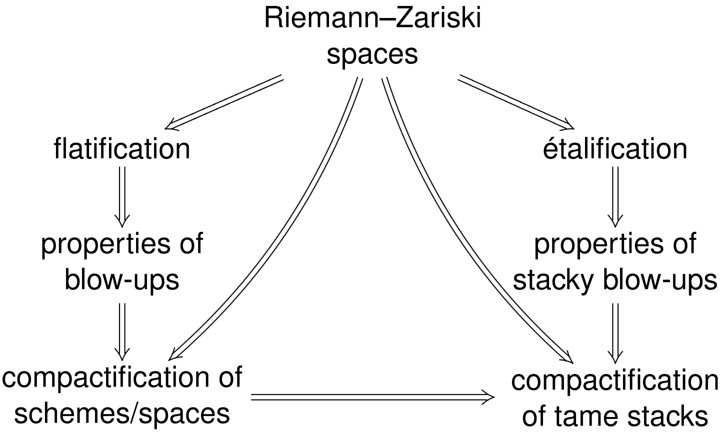
Riemann–Zariski spaces
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Contents

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Theorem (Nagata '62)

Every separated variety X/k admits an open embedding into a complete (=compact, proper) variety $\overline{X}/k$.

There is also a generalization for schemes:

Theorem (Nagata '63, Deligne, Lütkebohmert '93, Conrad '07)

Let $f\colon X \to Y$ be separated and of finite type. Then $f = \overline{f} \circ j$ where $j\colon X \to \overline{X}$ is an open embedding and $\overline{f}\colon \overline{X} \to Y$ is proper.

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Lurking in the background:

- Riemann–Zariski space of valuations of $K(X)$.
- Raynaud–Gruson's flatification theorem.

Compactification of algebraic spaces

There is an analogue for algebraic spaces:

Theorem (Raoult '74)

Every separated ***algebraic space*** *X/k of finite type admits an open embedding into a proper algebraic space $\overline{X}/k$.*

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For X **normal**, use that $X = Z/G$ where Z is a scheme and G is a finite group acting (not necessarily freely) on Z.
For general X, use **push-outs** to pass from a compactification of the normalization $\widetilde{X}$ to a compactification of X. □

Compactification of Deligne–Mumford stacks

Theorem (R.)

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Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism of DM-stacks. Then

$$f \text{ is finite type, separated and strictly tame} \iff f = (\text{proper and tame}) \circ (\text{open embedding})$$

Tameness

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A DM-stack $\mathscr{X}$ is **tame** if $\forall x \in |\mathscr{X}|$, char $k(x) \nmid |\mathrm{stab}(x)|$.

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$f\colon \mathscr{X} \to \mathscr{Y}$ is **strictly tame** if $\forall y_\xi \in |\mathscr{Y}|$, $y_0 \in \overline{\{y_\xi\}}$, $x \in f^{-1}(y_\xi)$

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Remedy for 1 — étale devissage (arXiv:1005.2171).
Remedy for 2 — stacky blow-ups.

Main lemma

Lemma

Let

$$\begin{array}{ccc} \mathscr{X}' & \longrightarrow & \mathscr{X} \\ \downarrow f' & \square & \downarrow f \\ \mathscr{Y}' & \xrightarrow{g} & \mathscr{Y} \end{array}$$

be a cartesian diagram of DM-stacks such that

- $f\colon \mathscr{X} \to \mathscr{Y}$ *is separated, of finite type and strictly tame.*
- $g\colon \mathscr{Y}' \to \mathscr{Y}$ *is representable, étale and surjective.*

Then f *is tamely compactifiable if and only if* f' *is so.*

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Proof.

Use stacky blow-ups and étale devissage. Required properties of stacky blow-ups follow from étalification. □

Proof of compactification theorem for tame stacks

Theorem (R.)

Let $\mathscr{Y}$ be a DM-stack and let $f : \mathscr{X} \to \mathscr{Y}$ be separated, of finite type and strictly tame. Then f has a tame compactification.

Sketch of proof.

Use Main Lemma and Riemann–Zariski spaces.

Skip details

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2. Find a (stacky) X-admissible blow-up $\widetilde{Y} \to Y$ and an étale covering $U \to \widetilde{Y}$ such that $\mathscr{X} \times_Y U \to \widetilde{Y}$ has a compactification (this is relatively easy and uses Riemann–Zariski spaces).

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3. Conclude by Main Lemma. □

Outline

1. Compactification of varieties and stacks
2. Flatification and blow-ups
   - Blow-ups
   - Flatification via blow-ups (Raynaud–Gruson)
3. Étalification and stacky blow-ups
4. Relation with toric geometry

Blow-ups

Let $p \colon X \to Y$ be a morphism between schemes (or stacks).

Definition

- We say that p is a **modification** if p is proper and birational. If $p^{-1}(U) \to U$ is an isomorphism, then we say that p is ***U*-admissible**.

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- If in addition $U \subseteq Y$ is an open subscheme such that $\mathcal{I}|_U = \mathcal{O}_U$ (so that $p^{-1}(U) \to U$ is an isomorphism) then we say that p is ***U*-admissible**.

Flatification

Theorem (Raynaud–Gruson '71)

Let $f\colon X \to Y$ be a finite type morphism of schemes such that $f|_U$ is flat for some open $U \subseteq Y$. Then $\exists$ U-admissible blow-up $\widetilde{Y} \to Y$ such that the strict transform $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ is flat.

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Remark: $\widetilde{X}$ is the closure of $f^{-1}(U)$ in $X \times_Y \widetilde{Y}$.

Properties of blow-ups

1. (Open extension)

2. (Cofinality)

3. (Étale quasi-extension)

($\mathfrak{B}$ denotes a blow-up. Also U-admissible variants.)

Properties of blow-ups

1. (Open extension) For every diagram of solid arrows there exists a blow-up $\overline{X} \to \overline{Y}$ such that $\overline{X}|_Y \cong X$. [trivial]

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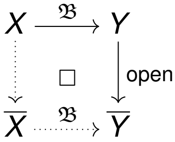
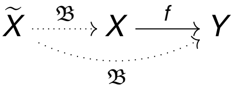


2. (Cofinality) Every modification $f\colon X \to Y$ is dominated by a blowup. [flatification]



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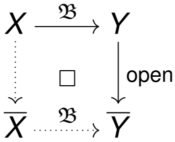

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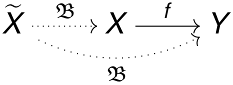
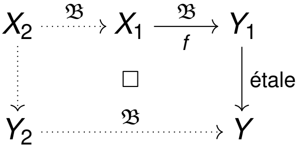


3. (Étale quasi-extension) For every diagram of solid arrows there exists blow-ups $X_2 \to X_1$ and $Y_2 \to Y$ as indicated. [étale devissage]



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Outline

Root stacks

Definition

Let $D \hookrightarrow X$ be a Cartier divisor and $r \geq 1$ an integer. The **root stack** $X_{D,r}$ is an X-stack roughly defined as

$$\mathrm{Hom}\left(T, X_{D,r}\right) = \left\{f\colon T \to X, E \in \mathrm{Div}(T) \mid f^{*}D = rE\right\}$$

(for precise definition, replace divisor with line bundle + section)

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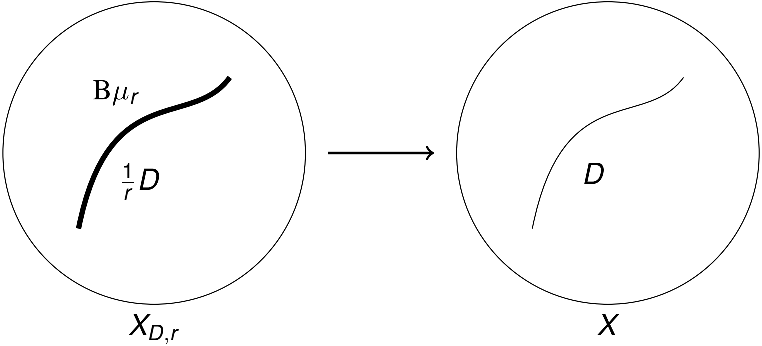
Facts

1. $X_{D,r}$ is a tame Artin stack and Deligne–Mumford if $p \nmid r$.
2. $\pi \colon X_{D,r} \to X$ is a **flat** $(X \setminus D)$-admissible modification.
3. $\left(X_{D,r}\right)_{\mathrm{cms}} = X$.
4. $\pi^{-1}(D)_{\mathrm{red}} \to D_{\mathrm{red}}$ is a μ_r-gerbe.
5. If $D = rE$ then $\left(X_{D,r}\right)^{\mathrm{norm}} = X^{\mathrm{norm}}$.

Root stacks (picture)



Locally a ramified μ_r-cover:

$$X = \mathrm{Spec}(A), \quad D = \{z = 0\}, \quad X_{D,r} = \left[\mathrm{Spec}(A[w]/w^r - z)/\mu_r\right]$$

Stacky blow-ups

Definition

Let X be a stack, let $Z \hookrightarrow X$ be a closed substack and $r \geq 1$ an integer. We let $\mathrm{Bl}_{Z,r}X = (\mathrm{Bl}_Z X)_{E,r}$ where $E \hookrightarrow \mathrm{Bl}_Z X$ is the exceptional divisor.

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Definition

Let $p\colon X \to Y$ be a morphism between stacks. We say that p is a **stacky blow-up** if there exists a (finitely presented) closed substack $Z \hookrightarrow X$ and an integer $r \geq 1$ such that $X = \mathrm{Bl}_{Z,r}Y$. If $Z \cap U = \emptyset$ for some U then p is ***U*-admissible**.

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Caution: A sequence of blow-ups is a blow-up (Raynaud–Gruson) but a sequence of stacky blow-ups is not a stacky blow-up.

Tame étalification

Theorem (R.)

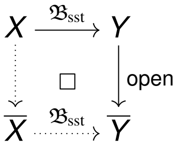
Let $f\colon X \to Y$ be a finite type morphism of stacks such that $f|_U$ is étale for some open $U \subseteq Y$ and f is tamely ramified. Then $\exists$ a commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\mathfrak{B}_{\mathrm{sst}}} & X \\ {\scriptstyle \widetilde{f}}\big\downarrow & \circ & \big\downarrow{\scriptstyle f} \\ \widetilde{Y} & \xrightarrow{\mathfrak{B}_{\mathrm{sst}}} & Y \end{array}$$

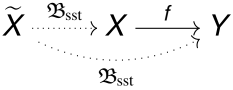
where $\widetilde{X} \to X$ and $\widetilde{Y} \to Y$ are sequences of stacky blow-ups and $\widetilde{f}$ is étale.

Properties of stacky blow-ups

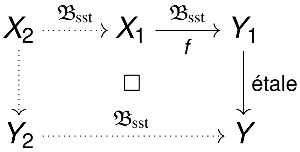
1. (Open extension) [trivial]



2. (Cofinality) Every **tame** stacky modification $f\colon X \to Y$ is dominated by a sequence of stacky blow-ups. [tame étalification]



3. (Étale quasi-extension) [étale devissage]

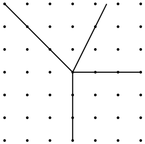


($\mathfrak{B}_{\mathrm{sst}}$ denotes a sequence of stacky blow-ups.)

Outline

Toric stacks

Let $N = \mathbb{Z}^d$ and let $\Sigma \subseteq N_{\mathbb{Q}}$ be a rational simplicial fan.

Σ 

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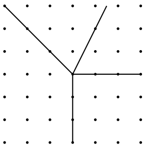
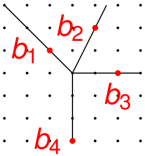


To Σ we associate the toric variety X_Σ. Let $\rho_1, \rho_2, \ldots, \rho_n$ be the rays in Σ and choose generators $b_i \in \rho_i \cap N$ of ρ_i.

$\overline{\Sigma}$



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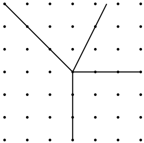
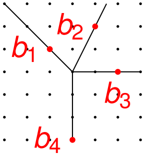
Let $N = \mathbb{Z}^d$ and let $\Sigma \subseteq N_{\mathbb{Q}}$ be a rational simplicial fan.

Σ



To Σ we associate the toric variety X_Σ. Let $\rho_1, \rho_2, \ldots, \rho_n$ be the rays in Σ and choose generators $b_i \in \rho_i \cap N$ of ρ_i.

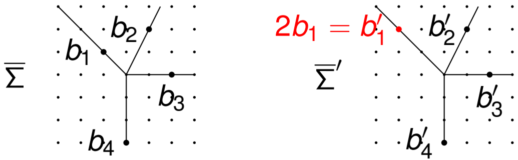
$\overline{\Sigma}$



To the **stacky fan** $\overline{\Sigma} = (\Sigma, \mathbf{b})$ we associate a **toric stack** $\mathscr{X}_{\overline{\Sigma}}$.
Toric stacks are always **regular** and tame.

Toric stacks and root stacks

Let D_i be the toric divisor corresponding to the ray ρ_i. Taking the r^{th} root stack of D_i results in the toric stack with stacky fan $\overline{\Sigma}' = \{\Sigma', \mathbf{b}'\}$ where $\Sigma' = \Sigma$ and $b'_j = b_j$ for $j \neq i$ and $b'_i = rb_i$:



2^{d} root stack of D_1

Star subdivisions

In particular, any star subdivision is obtained by first taking some root stacks and then a blow-up in a smooth center:

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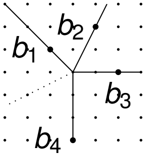

$\overline{\Sigma}$

$\mathscr{X}_{\overline{\Sigma}}$

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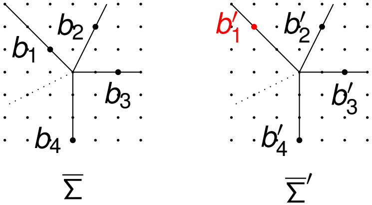

$$\mathscr{X}_{\overline{\Sigma}} \xleftarrow{\quad \mathrm{Bl}_{D_1,2} \quad} \mathscr{X}_{\overline{\Sigma}'}$$

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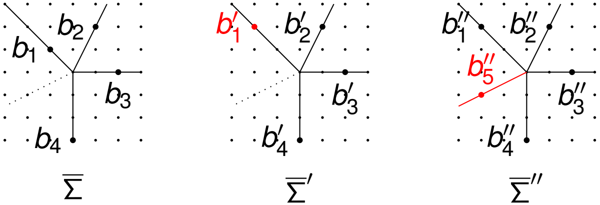

$$\mathscr{X}_{\overline{\Sigma}} \xleftarrow{\mathrm{Bl}_{D_1,2}} \mathscr{X}_{\overline{\Sigma}'} \xleftarrow{\mathrm{Bl}_{D'_1 \cap D'_4,1}} \mathscr{X}_{\overline{\Sigma}''}$$

Weak factorization of toric stacks

In the language of toric stacks and stacky blow-ups we have:

Theorem (Włodarczyk '98)

1. *A proper birational map $\mathscr{X}_{\overline{\Sigma}} \dashrightarrow \mathscr{X}_{\overline{\Sigma}'}$ between toric stacks factors as a sequence of stacky blow-ups and stacky blow-downs with smooth equivariant centers.*
2. *A proper birational map $X_{\Sigma} \dashrightarrow X_{\Sigma'}$ between regular toric varieties factors as a sequence of blow-ups and blow-downs with smooth equivariant centers.*

Weak factorization

Theorem (Abramovich–Karu–Matsuki–Włodarczyk ’02, W ’03)

A proper birational map $X \dashrightarrow Y$ between regular varieties over a field of characteristic zero, factors as a sequence of blow-ups and blow-downs with smooth centers.

Weak factorization

Theorem (Abramovich–Karu–Matsuki–Włodarczyk '02, W '03)

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Conjecture (R.)

A proper birational map $\mathscr{X} \dashrightarrow \mathscr{Y}$ between regular DM-stacks over a field of characteristic zero, factors as a sequence of stacky blow-ups and stacky blow-downs with smooth centers.

End of talk

The end

Quasi-projective varieties and stacks

Let X/k be a variety. The following are equivalent:

1. X is quasi-projective.
2. $\exists$ open embedding $X \subseteq \overline{X}$ with $\overline{X}$ projective.
3. $\exists$ embedding $X \hookrightarrow \mathbb{P}^n_k$.

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Definition (char. 0)

Let $\mathscr{X}/k$ be a separated DM-stack of finite type over a field k of characteristic zero. The stack $\mathscr{X}$ is **(quasi-)projective** if:

1. $\mathscr{X}$ is a **global quotient stack**, i.e., $\mathscr{X} = [U/\mathrm{GL}_n]$ for some algebraic space U.
2. The coarse moduli space $\mathscr{X}_{\mathrm{cms}}$ is (quasi-)projective.

Quasi-projective varieties and stacks (cont.)

Theorem (Kresch '09)

Let $\mathscr{X}/k$ be a DM-stack of characteristic zero. The following are equivalent:

1. *$\mathscr{X}$ is quasi-projective.*
2. *$\exists$ an open embedding $\mathscr{X} \subseteq \overline{\mathscr{X}}$ into a projective stack.*
3. *$\exists$ an embedding $\mathscr{X} \hookrightarrow \mathscr{P}$ where $\mathscr{P}$ is a smooth projective DM-stack.*

Moreover, every smooth DM-stack with (quasi-)projective cms is (quasi-)projective.