Milnor excision of stacks, pushouts and (étale) nbhds

smooth

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§ Milnor squares

Fix a cartesian square of commutative rings s.th.

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[d] & A_0 \arrow[d] \\ B \arrow[r, two heads] & B_0 \end{tikzcd}$$

(i) horizontal maps surjective

(ii) cartesian square

$$I$$
$$\shortparallel$$
$(\Leftrightarrow \ker(A \to A_0) \xrightarrow{\cong} \ker(B \to B_0))$

not derived

$\Rightarrow B_0 \cong B \otimes_A A_0$

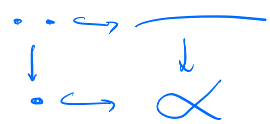
Special cases

(1) I nilpotent (def. thy)

(2) $A \to B$ also surjective "clutchings"

(3) B finite A-module "pinchings" e.g. $B = \tilde{A}$ normalization, I conductor $\subset A$, i.e. $I \cong IB$

Warning: In general A is rarely noetherian.



§ Milnor excision

Def: A functor $\mathcal{C}$: Rings $\to$ Cat$_\infty$ satisfies Milnor excision if

$$\begin{tikzcd} \mathcal{C}(A) \arrow[r] \arrow[d] & \mathcal{C}(A_0) \arrow[d] \\ \mathcal{C}(B) \arrow[r] & \mathcal{C}(B_0) \end{tikzcd}$$

homotopy cartesian

$A_0 = A/I$

$A_n = A/I^{n+1}$

pro-excision: repl $\mathcal{C}(A_0)$ with $\lim_n \mathcal{C}(A_n)$

Examples

- Milnor, Bass, Murthy '68 excision for Proj = VB f.g. proj modules and $K_{\leq 1}$ (false in general Swan?)
- Weibel '89 excision for KH.
- Kerz–Strunk–Tamme '18 pro-excision for K.
- Land–Tamme '19 pro-excision for localizing invariants
- Elmanto–Hoyois–Iwasa–Kelly '20 criterion for excision for cdh sheaves.
- ——"—— excision for SH.
- Bhatt–Mathew '18 excision for arc-sheaves (eg. étale coh, étale sheaves of sets)
- Bachmann–Khan–Ravi–Sosnilo (in prep) categorified version of Land–Tamme.

Milnor pro-excision for finite $A \to B$ + derived blowups $\rightsquigarrow$ abstract blowups

§ Milnor excision for stacks

Def: A Milnor square of algebraic (n-)stacks is a square s.th.:

$$\begin{tikzcd} X_0 \arrow[r, hook, "i"] \arrow[d, "f_0"'] & X \arrow[d, "f"] \\ Y_0 \arrow[r, hook, "j"'] & Y \end{tikzcd}$$

(i) vertical maps are affine
(ii) horizontal maps are closed immersion
(iii) $\mathcal{O}_Y \xrightarrow{\cong} f_*\mathcal{O}_X \times_{f_*i_*\mathcal{O}_{X_0}} j_*\mathcal{O}_{Y_0}$

$\iff$ flat/smooth-locally on Y it is a Milnor squares of affine schemes.

Rmks:
- Milnor squares are cartesian (but not derived cartesian)
- On "underlying" top spaces, they are cocartesian

[joint w/ J. Alper, J. Hall, D. Halpern-Leistner] ($\exists$ draft)

Thm 1 (AHHLR) Milnor squares satisfy excision for quasi-separated algebraic 2-stacks, i.e., Y is the pushout in the category of algebraic 2-stacks.

Rmks: Question is local on Y so WLOG Y affine.

Ex: $dBr \cong B^2\mathbb{G}_m \times B^1\mathbb{Z}$ $\quad dBr(X) =$ invertible quasi-coherent stacks $/X \overset{loc}{=} QCoh(dAz)$

Rmks: Easy cases

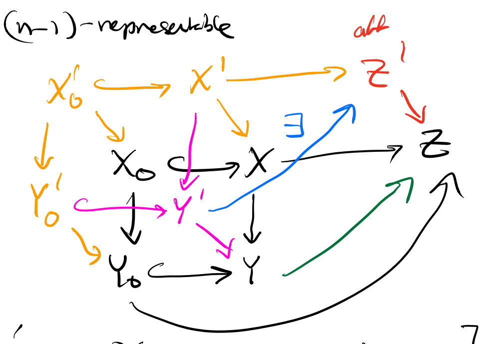
$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X \arrow[r, red] \arrow[d] & Z \\ Y_0 \arrow[r, hook] \arrow[urr, red, bend right] & Y \arrow[ur, red, dashed, "\exists ?"] & \end{tikzcd}$$

Z algebraic n-stack

(0) Z affine.
(1) $Z = BGL_n = VB_n$
(2) $Z = [\text{affine}/GL_n]$
(3) $Z = [\text{alg space}/GL_n]$ "quotient stack" $\rightsquigarrow$ reduces $Z =$ algebraic space

Strategy of proof:

1. Pick a smooth presentation $\underbrace{Z'}_{\text{affine}} \to Z$ ($n-1$)-representable
2. Pull-back to $Y_0 \leftarrow X_0 \to X$.
3. Glue to $Y' \to Y$ smooth surj such that I get a new Milnor square $(-)'$
4. Since Z' affine $\rightsquigarrow$ $Y' \to Z'$.
5. By descent: $Y \to Z$.



(3) $\iff$ excision for $Sm_{n-1}: X \longmapsto \{X' \to X \text{ smooth } (n-1)\text{-stacks}\}$

Rmks:
- When Z DM 1-stack, $\exists Z' \to Z$ étale rep. Enough with excision for Et_0 [Bhatt–Mathew]
- Different approach to (3) in general: Work locally on Y $\Rightarrow$ $\exists$ section $Y_0 \to Y_0'$ $\Rightarrow$ $\exists$ section $X_0 \to X_0'$. Try to extend this section to $X \to X'$!

§ Pushouts

Thm 2 Excision holds for Sm_1.

More generally:

Thm 3 (existence of Ferrand pushouts) A Milnor (Ferrand) datum

$$\begin{tikzcd} X_0' \arrow[r, hook, "\text{closed imm}"] \arrow[d, "\text{affine}"'] & X' \ (\text{qsep}) \\ Y_0' \ (\text{qsep}) & \end{tikzcd}$$

of algebraic 1-stacks fits in a Milnor square. ($\Rightarrow$ pushout by $\underset{\text{for 1-stacks}}{\text{Thm 1}}$)

proof: We prove Thm 3

(a) first for 1-stacks with affine diagonal
(b) then for 1-stacks with affine 2^{nd} diagonal
(c) then for all 1-stacks (have iso 3^{rd} diag.)

$$\begin{tikzcd} Y_0'' \ (\text{aff}) \arrow[d, "\text{smooth}"'] & X_0'' \ (\text{aff}) \arrow[l] \arrow[r, hook] \arrow[d, "\text{smooth}"] \arrow[dl, phantom, "\urcorner"] \arrow[dr, phantom, "\ulcorner"] & X'' \ (\text{aff}) \arrow[d, "\text{smooth}"] \\ Y_0' & X_0' \arrow[l] \arrow[r, hook] & X' \end{tikzcd}$$

1. Pick smooth presentation $Y_0'' \longrightarrow Y_0'$.
2. Pullback to $X_0'' \longrightarrow X_0'$. ($X_0''$ affine because $X_0' \to Y_0'$ affine)
3. Extend to $X'' \longrightarrow X'$ (Thm 4 below)
4. Get Milnor square of affine schemes:

$$\begin{tikzcd} X_0'' \arrow[r, hook] \arrow[d] & X'' \arrow[d] \\ Y_0'' \arrow[r, hook] & Y'' = \operatorname{Spec}\left(\Gamma(X'', \mathcal{O}_{X''}) \underset{\Gamma(Y_0'', \mathcal{O}_{Y''})}{\times} \Gamma(X_0'', \mathcal{O}_{X_0''}) \right) \end{tikzcd}$$

5. Get another Milnor datum $Y_0'' \times_{Y_0'} Y_0'' \leftarrow X_0'' \times_{X_0'} X_0'' \hookrightarrow X'' \times_{X'} X''$ of algebraic spaces

   They are (a) affine (b) affine diag (c) affine second diagonal $\Rightarrow$ pushout exist.
6. Get smooth groupoid of Milnor squares. Taking quotients gives the result. $\square$

§ Smooth neighborhoods

Thm 4 ($\exists$ of smooth nbhds) Let $X_0' \hookrightarrow X'$ closed immersion of algebraic 1-stacks. Let $g_0 : X_0'' \to X_0'$ smooth (or Étale) with X_0'' affine. Then $\exists\, g : X'' \to X'$ smooth (Étale) with X'' affine extending g_0.

Rmk: $g_0 = id_{X_0'}$ (when X_0' affine) is interesting.

Ex: (X', X_0') with X_0' affine admits a henselization (X'^h, X_0') which is affine. (previously unknown even for X' scheme)

pf: (0) reduction to finite type $/\mathbb{Z}$. (abs noeth approx for 1-stacks)

(1) Deformation theory $\rightsquigarrow \exists\, g_n : X_n'' \to X_n'$ smooth extending g_0

(obstruction lies in $H^2(X_0'', \underbrace{L^\vee_{X_0''/X_0'}}_{\text{concentrated in coh. deg} \leq 0 \text{ b/c smooth}} \otimes I^n/I^{n+1}) = 0$ b/c X_0'' affine)

Let $X_n'' = \operatorname{Spec} A_n$. Then $\{A_n\}$ adic system $\rightsquigarrow A = \varprojlim_n A_n$ complete adic noetherian rings.
Define $\hat{X}'' := \operatorname{Spec} \hat{A}$.

(2) Tannaka duality $\rightsquigarrow \exists!\, \hat{g} : \hat{X}'' \to X'$ formally smooth.
extending $g_n : X_n'' \to X_n' \hookrightarrow X'$.

That is:

$$(*) \quad \operatorname{Mor}(\hat{X}'', X') \xrightarrow{\simeq} \varprojlim_n \operatorname{Mor}(X_n'', X') \quad \text{equiv of } \infty\text{-grpds}$$

$$\hat{g} \longmapsto \{g_n\}$$

That $(*)$ holds for 1-stacks follows from Tannaka duality: (only valid for 1-stacks)

$$\operatorname{Mor}(Z, X) \xrightarrow{\simeq} \operatorname{Hom}_{\otimes}^{r}(\operatorname{Coh}(X), \operatorname{Coh}(Z))$$

right-exact monoidal functors

using that $\operatorname{Coh}(\hat{X}'') = \varprojlim \operatorname{Coh}(X_n'')$.

(3) Artin algebraization $\rightsquigarrow \exists\, \underset{\text{aff.}}{X''} \to X$ smooth. $\square$

Rmks:

- If X_0 is a point, then A m-adic and $(*)$ holds for n-stacks X' th. and can be proven without Tannaka duality. Thus seems reasonable that $(*)$ holds for n-stacks in general.
- Alternative approach to step ③ in Thm 1 also uses $(*)$ and only proven for 1-stacks.