COMPACTIFICATION OF TAME DELIGNE–MUMFORD STACKS

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Abstract. The main result of this paper is that separated Deligne–Mumford stacks in characteristic zero can be compactified. In arbitrary characteristic, we give a necessary and sufficient condition for a tame Deligne–Mumford stack to have a tame Deligne–Mumford compactification. The main tool is a new class of stacky modifications — *tame stacky blow-ups* — and a new étalification result which in it simplest form asserts that a tamely ramified finite flat cover becomes étale after a tame stacky blow-up. This should be compared to Raynaud–Gruson's flatification theorem which we extend to stacks.

Preliminary draft!

Introduction

A fundamental result in algebraic geometry is Nagata's compactification theorem which asserts that any variety can be embedded into a complete variety [Nag62]. More generally, if $f\colon X \to Y$ is a separated morphism of finite type between noetherian schemes, then there exists a compactification of f, that is, there is a proper morphism $\overline{f}\colon \overline{X} \to Y$ such that f is the restriction of $\overline{f}$ to an open subset $X \subseteq \overline{X}$ [Nag63].

The main theorem of this paper is an extension of Nagata's result to separated morphisms of finite type between quasi-compact *Deligne–Mumford stacks of characteristic zero* and more generally to tame Deligne–Mumford stacks. The compactification result relies on several different techniques which all are of independent interest.

- Tame stacky blow-ups — We introduce a class of stacky modifications, i.e., proper birational morphisms of stacks, and show that this class has similar properties as the class of blow-ups. A tame stacky blow-up is a composition of root stacks and ordinary blow-ups.
- Flatification — We generalize M. Raynaud and L. Gruson's flatification theorem by blow-ups [RG71] to stacks with quasi-finite diagonals.
- Étalification — We show that a generically étale and "tamely ramified" morphism $f\colon X \to Y$ of Deligne–Mumford stacks becomes étale after *tame stacky blow-ups* on X and Y. This is the étale

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analogue of the flatification theorem and useful even for morphisms of schemes.
- Riemann–Zariski spaces — Riemann–Zariski spaces was introduced [Zar40, Zar44] by Zariski in the 1940's. Classically, Riemann–Zariski spaces have been used in the context of resolution of singularities but have also been used in rigid geometry [FK06, Tem00, Tem04, CT09], for compactification problems [Nag62, Tem08] and in complex dynamics [FJ04].

Precise statements of the above results are given in Section 1. A crucial tool in this paper is the étale dévissage method described in [Ryd11]. The étale dévissage and stacky blow-ups are used to construct a compactification of $f\colon X \to Y$ after the compactification has been accomplished étale-locally on a suitable compactification of Y. An important consequence of Raynaud–Gruson's flatification theorem is that blow-ups are cofinal among modifications. Similarly, it follows from the étalification theorem that tame stacky blow-ups are cofinal among tame stacky modifications. This is the salient point in the proof of the compactification theorem.

Other compactification results for stacks. It is possible that the compactification result extends to *non-tame* stacks with finite automorphisms groups. For example, any normal Deligne–Mumford stack that is a global quotient can be compactified by a Deligne–Mumford stack that is a global quotient (proven by D. Edidin) although in this case we do not have any control of the stabilizers on the boundary (e.g., starting with a tame stack the compactification may be non-tame).

Another result in this vein is that *quasi-projective* DM-stacks in *characteristic zero* can be compactified by projective DM-stacks [Kre09]. These results suggest that any Deligne–Mumford stack can be compactified by a Deligne–Mumford stack. In a subsequent article we will prove that this is indeed the case, at least in equal characteristic.

A general compactification result for Deligne–Mumford stacks was announced in [Mat03]. However, the proof is severely flawed as the morphism $[V/G] \to \mathcal{X}$ in [Mat03, Cor. 2.2] is not representable as asserted by the author.

Pushouts. The actual construction of the compactification is accomplished through three different kinds of *push-outs*. The first is the push-out of an open immersion and an étale morphism and is used in the étale dévissage [Ryd11]. The second is the push-out of a closed immersion and a finite morphism [Fer03, Rao74b, Art70, Kol09, CLO09], cf. Appendix A, which is used to prove the compactification theorem for non-normal algebraic spaces. The third is the push-out of a closed immersion and an open immersion used in the context of Riemann–Zariski spaces. The last kind is the most subtle and these push-outs are, except in trivial cases, non-noetherian.

Compactification of schemes and algebraic spaces. Nagata's original proof of the compactification theorem for schemes uses Riemann–Zariski spaces and valuation theory. More modern and transparent proofs have subsequently been given by P. Deligne and B. Conrad [Con07] and W. Lütkebohmert [Lüt93]. There is also a modern proof by M. Temkin [Tem08] using Riemann–Zariski spaces.

Nagata's theorem has also been extended to a separated morphism $f\colon X \to Y$ of finite type between *algebraic spaces*. This is essentially due to J.-C. Raoult, who treated the case where either X is normal and Y is a scheme [Rao71] or Y is the spectrum of an excellent ring [Rao74a, Rao74b]. A proof for general X and Y has been announced by K. Fujiwara and F. Kato [FK06, 5.7], using rigid geometry and Riemann–Zariski techniques. The general case is also treated, without RZ-spaces, in an upcoming paper [CLO09]. We give an independent and simple proof in §6.

Blow-ups. All proofs of Nagata's theorem for schemes relies heavily on the usage of blow-ups and, in particular, on the first and the last of the following crucial properties of blow-ups:

(i) A blow-up on an open subset $U \subseteq X$ can be extended to a blow-up on X (extend by taking the closure of the center).
(ii) A blow-up on an closed subscheme $X_0 \subseteq X$ can be extended to a blow-up on X (extend by taking the same center).
(iii) Given an étale morphism $X' \to X$, then a blow-up on X' can be dominated by the pull-back of a blow-up on X.
(iv) Any proper birational representable morphism $X \to Y$ has a section after a blow-up on Y, cf. Corollary (5.1).

The first two properties are trivial, the third is proven via étale dévissage, cf. Proposition (4.14), and the last property follows from Raynaud–Gruson's flatification theorem. The compactification theorem for tame Deligne–Mumford stacks relies on analogous properties for tame stacky blow-ups. Here the first two properties are again trivial, the third property follows from a rather involved étale dévissage argument and the last property follows from the étalification theorem.

It should also be noted that the third property is used in the proof of the flatification and étalification results for algebraic spaces and stacks. Non-tame étalification and compactification of Deligne–Mumford stacks follow if we find a suitable "stackier" class of blow-ups having properties (i), (iii), (iv).

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Contents

1. Terminology and statement of results

1.1. **Terminology.** All algebraic stacks and algebraic spaces are assumed to be quasi-separated, that is, that the diagonal and the double-diagonal are quasi-compact. We do not require that the diagonal of an algebraic stack is separated but it is permissible for the reader to assume this. The main results on compactifications are valid for Deligne–Mumford stacks, or equivalently, stacks with unramified diagonal. Some of the results, notably flatification results are also valid for stacks with quasi-finite diagonals. To facilitate we use the terminology *quasi-Deligne–Mumford stacks* as in [SP]. To be able to use étale dévissage, we impose that quasi-Deligne–Mumford stacks have locally separated diagonals:

Definition (1.1). An algebraic stack X is *quasi-Deligne–Mumford* if Δ_X is quasi-finite and locally separated.

A morphism $f\colon X \to Y$ of algebraic stacks is *étale* (resp. unramified) if f is formally étale and locally of finite presentation (resp. formally unramified) and locally of finite type. A useful characterization is the following:

$$\begin{aligned} f \text{ is étale} &\iff f,\ \Delta_f \text{ and } \Delta_{\Delta_f} \text{ are flat and loc. of finite pres.}\\ f \text{ is unramified} &\iff f \text{ is loc. of finite type and } \Delta_f \text{ is étale.} \end{aligned}$$

For a thorough discussion of étale and unramified morphisms of stacks, see [Ryd09a, App. B]. A morphism $f\colon X \to Y$ of algebraic stacks is *quasi-finite* if f is of finite type, every fiber of f is discrete and every fiber of the diagonal Δ_f is discrete.

Definition (1.2). Let $f\colon X \to S$ be a morphism of Deligne–Mumford stacks and let $\psi\colon I_{X/S} \to X$ be the inertia group scheme. We say that f is *tame* if for every point $x\colon \mathrm{Spec}(k) \to X$, the group scheme $\psi^{-1}(x) \to \mathrm{Spec}(k)$ has rank prime to the characteristic of k. We say that f is *strictly tame* if for every point $x\colon \mathrm{Spec}(k) \to X$ and specialization $y\colon \mathrm{Spec}(k') \to S$ of $f \circ x$, the group scheme $\psi^{-1}(x) \to \mathrm{Spec}(k)$ has rank prime to the characteristic of k'.

Note that in characteristic zero, every morphism of Deligne–Mumford stacks is strictly tame. If S is equicharacteristic, then $f\colon X \to S$ is tame if and only if f is strictly tame.

1.2. **Statement of main results.** Our first result is the generalization of Raynaud–Gruson's flatification by blow-up to quasi-Deligne–Mumford stacks.

Theorem A (Flatification by blow-ups)**.** *Let S be a quasi-compact quasi-Deligne–Mumford stack and let $U \subseteq S$ be a quasi-compact open substack. Let $f\colon X \to S$ be a morphism of finite type of algebraic stacks such that $f|_{g^{-1}(U)}$ is flat and of finite presentation. Then there exists a U-admissible blow-up $\widetilde{S} \to S$ such that the strict transform $\widetilde{f}\colon \widetilde{X} \to \widetilde{S}$ is flat and of finite presentation.*

Proof. See Section 4. The idea is to use étale dévissage.

Theorem B (Compactification of representable morphisms)**.** *Let S be a quasi-compact quasi-Deligne–Mumford stack and let $f\colon X \to S$ be a representable and separated morphism of finite type. Then there exists a factorization $f = \overline{f} \circ j$ where j is an open immersion and $\overline{f}$ is proper and representable.*

Proof. See Section 6. The proof follows Raoult's ideas, using Nagata's compactification theorem for schemes and Raynaud–Gruson flatification by blow-ups. Étale dévissage is used to pass to the case where S is a scheme.

Theorem C (Tame étalification)**.** *Let $f\colon X \to S$ be a morphism of finite type between quasi-compact Deligne–Mumford stacks. Let $U \subseteq S$ be a quasi-compact open substack such that $f|_U$ is étale. Assume that f is* tamely ramified outside U*, e.g., that X is of characteristic zero. Then there exists a U-admissible tame stacky blow-up $\widetilde{S} \to S$ and an $f^{-1}(U)$-admissible blow-up $\widetilde{X} \to X \times_S \widetilde{S}$ such that $\widetilde{X} \to \widetilde{S}$ is étale.*

Proof. See Section 11. The proof uses Riemann–Zariski spaces and valuation theory. Tame stacky blow-ups are defined in Section 9.

Corollary D (Extensions of finite étale covers)**.** *Let X be a quasi-compact Deligne–Mumford stack and let $U \subseteq X$ be a quasi-compact open substack. Let $E_U \to U$ be a finite étale cover which is tamely ramified outside U (e.g.,* *assume that E_U/U has degree prime to p). Then there exists a U-admissible tame stacky blow-up $\widetilde{X} \to X$ such that $E_U \to U$ extends to a finite étale cover $\widetilde{E} \to \widetilde{X}$.*

Proof. See Section 12.

Corollary D is closely related to the generalized Abhyankar lemma. Indeed, the corollary follows from the generalized Abhyankar's lemma when X is regular and $X \setminus U$ is a simple normal crossings divisor. Riemann–Zariski spaces and non-discrete valuation theory is used to tackle the non-regular case. A partial generalization of Corollary D in the regular case has been given by M. Olsson [Ols09]. In *loc. cit.*, E_U/U is a G-torsor for a finite tame non-étale group scheme G.

Corollary E (Cofinality of tame stacky blow-ups among tame stacky modifications)**.** *Let S be a quasi-compact Deligne–Mumford stack and let $f\colon X \to S$ be a proper tame morphism of stacks which is an isomorphism over an open quasi-compact subset $U \subseteq S$. Then there exists a U-admissible tame stacky blow-up $X' \to X$ such that $X' \to X \to S$ is a U-admissible tame stacky blow-up.*

Proof. See Section 12.

Theorem F (Compactification of strictly tame morphisms)**.** *Let $f\colon X \to S$ be a separated morphism of finite type between quasi-compact Deligne–Mumford stacks. The following are equivalent*

(i) *f is strictly tame.*
(ii) *There exists a proper tame morphism $\overline{f}\colon \overline{X} \to S$ of Deligne–Mumford stacks and an open immersion $X \hookrightarrow \overline{X}$ over S.*

Moreover, it is possible to choose $\overline{X}$ such that for any $x \in |\overline{X}|$, we have that $\operatorname{stab}_{\overline{X}/S}(x) = A \times \prod_{i=1}^{n} \operatorname{stab}_{X/S}(x_i)$ where A is an abelian group and $x_1, x_2, \dots, x_n \in |X|$. Here $\operatorname{stab}(x)$ and $\operatorname{stab}(x_i)$ denote the geometric stabilizer groups.

Proof. See Section 13. The key ingredients in the proof are Riemann–Zariski spaces and the usage of tame stacky blow-ups together with Corollaries D and E.

When S is Deligne–Mumford, then Theorem B follows from Theorem F as we can replace a compactification $\overline{X}$ by its relative coarse moduli space. However, Theorem B is an essential ingredient in the proof of Theorem F.

2. Étale dévissage

The étale dévissage method reduces questions about general étale morphisms to étale morphisms of two basic types. The first type is finite étale coverings. The second is *étale neighborhoods* or equivalently pushouts of étale morphisms and open immersions — the étale analogue of open coverings consisting of two open subsets. This method was, to the author's best knowledge, introduced by Raynaud and Gruson in [RG71, 5.7] to reduce statements about algebraic spaces to statements for schemes. It is applied several times in this paper as well as in the paper [Ryd09b]. For the reader's convenience, we summarize the main results of [Ryd11] in this section. Twice, in Propositions (4.14) and (7.9), we will also use the dévissage method in a form that is not directly covered by Theorem (2.5).

Definition (2.1). Let X be an algebraic stack and let $Z \hookrightarrow |X|$ be a closed subset. An étale morphism $p\colon X' \to X$ is an *étale neighborhood of* Z if $p|_{Z_{\mathrm{red}}}$ is an isomorphism.

Theorem (2.2) ([Ryd11, Thm. A])**.** *Let X be an algebraic stack and let $U \subseteq X$ be an open substack. Let $f\colon X' \to X$ be an étale neighborhood of $X \setminus U$ and let $U' = f^{-1}(U)$. The natural functor*

$$(|_U, f^*)\colon \mathbf{QCoh}(X) \to \mathbf{QCoh}(U) \times_{\mathbf{QCoh}(U')} \mathbf{QCoh}(X')$$

is an equivalence of categories.

Theorem (2.3) ([Ryd11, Thm. B])**.** *Let X be an algebraic stack and let $j\colon U \to X$ be an open immersion. Let $p\colon X' \to X$ be an étale neighborhood of $X \setminus U$ and let $j'\colon U' \to X'$ be the pull-back of j. Then X is the pushout in the category of algebraic stacks of $p|_U$ and j'.*

Theorem (2.4) ([Ryd11, Thm. C])**.** *Let X' be a quasi-compact algebraic stack, let $j'\colon U' \to X'$ be a quasi-compact open immersion and let $p_U\colon U' \to U$ be a quasi-compact étale morphism. Then the pushout X of j' and p_U exists in the category of quasi-compact algebraic stacks. The resulting co-cartesian diagram*

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d, "p_U"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "p"] \\ U \arrow[r, "j"'] & X \end{tikzcd}$$

is also cartesian, j is a quasi-compact open immersion and p is an étale neighborhood of $X \setminus U$.

Theorem (2.5) ([Ryd11, Thm. D])**.** *Let X be a quasi-compact algebraic stack and let $\mathbf{E}$ be the 2-category of quasi-compact étale morphisms $Y \to X$. Let $\mathbf{D} \subseteq \mathbf{E}$ be a full subcategory such that*

(D1) *if $Y \in \mathbf{D}$ and $(Y' \to Y) \in \mathbf{E}$ then $Y' \in \mathbf{D}$,*
(D2) *if $Y' \in \mathbf{D}$ and $Y' \to Y$ is finite, surjective and étale, then $Y \in \mathbf{D}$, and*
(D3) *if $j\colon U \to Y$ and $f\colon Y' \to Y$ are morphisms in $\mathbf{E}$ such that j is an open immersion and f is an étale neighborhood of $Y \setminus U$, then $Y \in \mathbf{D}$ if $U, Y' \in \mathbf{D}$.*

Then if $(Y' \to Y) \in \mathbf{E}$ is representable and surjective and $Y' \in \mathbf{D}$, we have that $Y \in \mathbf{D}$. In particular, if there exists a representable and surjective morphism $Y \to X$ in $\mathbf{E}$ with $Y \in \mathbf{D}$ then $\mathbf{D} = \mathbf{E}$.

Note that the morphisms in $\mathbf{E}$ are not necessarily representable nor separated. In Theorem (2.4), even if X' and U have separated diagonals, the pushout X need not unless p_U is representable. We are thus naturally led to include algebraic stacks with non-separated diagonals.

The étale dévissage is sufficient for Deligne–Mumford stacks. For quasi-Deligne–Mumford stacks we will use a combination of étale dévissage and the following structure result.

Proposition (2.6). *Let S be a quasi-compact quasi-Deligne–Mumford stack. Then there exists:*

(i) *a representable, étale and surjective morphism $S' \to S$.*
(ii) *a finite, faithfully flat and finitely presented morphism $U' \to S'$ where U' is affine.*

In particular, S' admits an affine coarse moduli space S'_{cms}. If S is of finite type over a noetherian scheme S_0, then S'_{cms} is of finite type over S_0 and $S' \to S'_{\mathrm{cms}}$ is quasi-finite and proper.

Proof. The existence of $U' \to S' \to S$ with U' quasi-affine is [Ryd11, Thm. 7.2]. The existence of a quasi-affine coarse moduli space S'_{cms} with the ascribed finiteness property is then classical, cf. [Ryd07, §4]. After Zariski-localization of S'_{cms} we may assume that U' and S'_{cms} are affine. $\square$

Part 1. Modifications, blow-ups, flatification and compactification of algebraic spaces

3. Stackpairs and modifications

In this section we give a simple framework for (stacky) U-admissible modifications.

Definition (3.1). A *stackpair* (X, U) is an open immersion of quasi-compact algebraic stacks $j\colon U \to X$. We say that (X, U) is *strict* if U is schematically dense, i.e., if the adjunction map $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective. A morphism of stackpairs $(X', U') \to (X, U)$ is a morphism $f\colon X' \to X$ of stacks such that $U' \subseteq f^{-1}(U)$. If $f, g\colon (X', U') \to (X, U)$ are morphisms, then a 2-morphism $\tau\colon f \Rightarrow g$ is a 2-morphism of the underlying morphisms $f, g\colon X' \to X$. We let **StackP** denote the 2-category of stackpairs.

We say that a morphism $f\colon (S', U') \to (S, U)$ is flat, étale, proper, representable, etc., if the underlying morphism $f\colon S' \to S$ has this property. We say that $f\colon (S', U') \to (S, U)$ is *cartesian* if $f^{-1}(U) = U'$.

The 2-category **StackP** has 2-fiber products. Explicitly, if $p\colon (X, W) \to (S, U)$ and $f\colon (S', U') \to (S, U)$ are morphism of stackpairs and we let $X' = X \times_S S'$ and $W' = W \times_U U'$, then

$$\begin{array}{ccc} (X', W') & \longrightarrow & (X, W) \\ \downarrow & & \downarrow{\scriptstyle p} \\ (S', U') & \xrightarrow{f} & (S, U) \end{array}$$

is 2-cartesian. As usual we say that $(X', W') \to (S', U')$ is the pull-back of p along f.

Given a stackpair (X, U) we let $(X, U)_{\mathrm{strict}} = (\overline{U}, U)$ where $\overline{U}$ is the schematic closure of U in X. Then $(-,-)_{\mathrm{strict}}$ is a right adjoint to the inclusion functor of strict stackpairs into all stackpairs. Thus, if p and f are morphisms of strict stackpairs, then the square

$$\begin{tikzcd} (X',W')_{\mathrm{strict}} \arrow[r] \arrow[d] & (X,W) \arrow[d, "p"] \\ (S',U') \arrow[r, "f"] & (S,U) \end{tikzcd}$$

is 2-cartesian in the category of strict stackpairs. Even if the stackpairs are not strict, the above square makes sense and we say that $(X',W')_{\mathrm{strict}} \to (S',U')$ is the *strict transform* of p. Note that if (X,W) is strict and if f is flat and cartesian then the strict transform of p coincides with the pull-back of p.

If (S,U) is a stackpair and $\mathcal{F}$ is a quasi-coherent sheaf on S, then we let the strictification $\mathcal{F}_{\mathrm{strict}}$ be the image of $\mathcal{F} \to j_*j^*\mathcal{F}$ where $j\colon U \to S$ is the canonical open immersion. The strict transform of $\mathcal{F}$ along $f\colon (S',U') \to (S,U)$ is the quasi-coherent sheaf $(f^*\mathcal{F})_{\mathrm{strict}}$.

Definition (3.2). Let (S,U) be a stackpair. A *modification* of (S,U) or a *U-admissible modification* of S is a proper representable morphism $\pi\colon (X,U) \to (S,U)$ such that (X,U) is strict and $\pi^{-1}(U) \to U$ is an isomorphism.

We say that S has *the completeness property* if every quasi-coherent sheaf of $\mathcal{O}_S$-modules is the direct limit of finitely presented $\mathcal{O}_S$-modules [Ryd09b, Def. 4.2].

Definition (3.3) ([Ryd09b, Def. 4.6])**.** We say that an algebraic stack S is *pseudo-noetherian* if S is quasi-compact (and quasi-separated) and for every finitely presented morphism $S' \to S$ of algebraic stacks, the stack S' has the completeness property.

Examples of pseudo-noetherian stacks are noetherian stacks [LMB00, Prop. 15.4] and quasi-Deligne–Mumford stacks [Ryd09b, Thm. A]. We say that a stackpair (S,U) is pseudo-noetherian if S is pseudo-noetherian. Then U is pseudo-noetherian as well. Essentially all stackpairs will be pseudo-noetherian but it is crucial to allow non-noetherian stackpairs as we will use stackpairs (S,U) where S is the spectrum of a non-discrete valuation ring.

4. Blow-ups and flatification for stacks

In this section we extend Raynaud–Gruson's flatification theorem [RG71, Thm. 5.2.2] to quasi-Deligne–Mumford stacks. We also give a more general version of the theorem involving an unramified morphism. Some of the results are valid for more general stacks, e.g., finite morphisms of arbitrary stacks can be flatified via Fitting ideals. We begin with the fundamental properties of blow-ups and then use étale dévissage in Proposition (4.14) to prove that blow-ups can be "quasi-extended" along étale (and unramified) morphisms.

Let (S,U) be a stackpair. Recall that a morphism $f\colon X \to S$ is a *blow-up* if $X = \mathrm{Bl}_Z X = \mathrm{Proj}(\bigoplus_{d\geq 0} \mathcal{I}^d)$ for some ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_S$ of *finite type* so that f is projective. Here $Z \hookrightarrow X$ denotes the finitely presented closed substack defined by the ideal $\mathcal{I}$.

A blow-up is U*-admissible* if $\mathcal{I}$ can be chosen such that $\mathcal{I}|_U = \mathcal{O}_U$. Then $f|_U$ is an isomorphism. If in addition U is schematically dense in $S \setminus V(\mathcal{I})$, or equivalently, if (X, U) is a strict stackpair, then we say that the blow-up is *strictly* U*-admissible*. We say that $f\colon (X, U) \to (S, U)$ is a blow-up if $f\colon X \to S$ is a strictly U-admissible blow-up. In particular, every blow-up of stackpairs is a modification of stackpairs.

Warning **(4.1).** There are two fine points here to be observed.

(i) If $f\colon X \to S$ is a blow-up then f may be a U-admissible modification without being a U-admissible blow-up. This happens if the intersection of U and the center of the blow-up is a non-empty Cartier divisor. This issue is settled by Corollary (4.11).

(ii) If (S, U) is not strict, then a U-admissible blow-up need not be strictly U-admissible. This annoying detail is handled by Lemma (4.4).

We begin with two lemmas that are obvious for noetherian stacks.

Lemma (4.2). *Let S be a pseudo-noetherian stack and let $S' \to S$ be a blow-up. Then S' is a pseudo-noetherian stack.*

Proof. Let $\mathcal{I}$ be a finite type ideal such that S' is the blow-up of S in $\mathcal{I}$. Choose a finitely presented sheaf $\mathcal{F}$ with a surjection $\mathcal{F} \twoheadrightarrow \mathcal{I}$. Then $S' = \mathrm{Proj}(\bigoplus_{d\geq 0} \mathcal{I}^d)$ is a closed subscheme of $T = \mathrm{Proj}(\bigoplus_{d\geq 0} \mathcal{F}^d)$. As $T \to S$ is of finite presentation, we have that T is pseudo-noetherian. As $S' \hookrightarrow T$ is affine, it follows that S' is pseudo-noetherian [Ryd09b, Prop. 4.5]. $\square$

Lemma (4.3). *Let (S, U) be a pseudo-noetherian stackpair.*

(i) *There exists a finitely presented closed substack $Z \hookrightarrow S$ such that $|Z| = |S| \setminus |U|$.*

(ii) *If $Z_U \hookrightarrow U$ is a finitely presented closed substack, then there is a finitely presented closed substack $Z \hookrightarrow S$ such that $Z \cap U = Z_U$.*

(iii) *If $S' \hookrightarrow S$ is a closed substack and $Z' \hookrightarrow S'$ is a finitely presented closed substack such that $Z' \cap U = \emptyset$, then there is a finitely presented closed substack $Z \hookrightarrow S$ such that $Z \cap S' = Z'$ and $Z \cap U = \emptyset$.*

Proof. (i) is a special case of [Ryd09b, Prop. 8.12]. In (ii), let $Z_1 = \overline{Z_U}$ be the closure of Z_U in S and in (iii) let $Z_1 = Z'$ seen as a closed substack of S. As S is pseudo-noetherian we can write $Z_1 = \varprojlim_\lambda Z_\lambda$ as a limit of finitely presented closed substacks of S. For sufficiently large λ we have that $Z_\lambda \cap U = Z_U$ in (ii) and that $Z_\lambda \cap S' = Z'$ and $Z_\lambda \cap U = \emptyset$ in (iii). We can thus take $Z = Z_\lambda$ for sufficiently large λ. $\square$

Lemma (4.4) (Strictification)**.** *Let (S, U) be a pseudo-noetherian stackpair. Then there is a blow-up $(S', U) \to (S, U)$ where, by definition, (S', U) is strict.*

Proof. Let S' be the blow-up of S in a center Z such that $|Z| = |S| \setminus |U|$. $\square$

Let $g\colon (S', U') \to (S, U)$ be a morphism of *strict* stackpairs. If $f\colon (X, U) \to (S, U)$ is a blow-up with center $Z \hookrightarrow S$, then the strict transform $f'\colon (X', U') \to (S', U')$ is a blow-up with center $g^{-1}(Z) \hookrightarrow S'$.

Lemma (4.5). *Let (S, U) be a pseudo-noetherian strict stackpair and let $j\colon S' \hookrightarrow S$ be a closed substack. Then every blow-up of $(S', U \cap S')$ extends to a blow-up of (S, U), that is, if $f'\colon (X', U \cap S') \to (S', U \cap S')$ is a blow-up, then there exists a blow-up $f\colon (X, U) \to (S, U)$ such that the strict transform of f along j is f'.*

Proof. This is an immediate consequence of Lemma (4.3). □

The proofs of the following three lemmas are essentially identical to the proofs in the case of schemes (replace quasi-compact and quasi-separated scheme with pseudo-noetherian stack). The only difference is that we require the blow-ups to be strictly U-admissible.

Lemma (4.6) ([RG71, Lem. 5.1.4] and [Con07, Lem. 1.2])**.** *Let (S, U) be a pseudo-noetherian stackpair. If $f\colon (S', U) \to (S, U)$ and $g\colon (S'', U) \to (S', U)$ are blow-ups, then so is $f \circ g$.*

Using Lemma (4.4) and Lemma (4.6) we will often replace non-strict stackpairs with strict stackpairs.

Lemma (4.7) ([RG71, Lem. 5.1.5])**.** *Let (S, U) be a pseudo-noetherian stackpair. Let $U = \coprod_{i=1}^{n} U_i$ be a partition of U in open and closed substacks. Then there exists a blow-up $(S', U) \to (S, U)$ such that S' admits a partition $S' = \coprod_{i=1}^{n} S'_i$ with $S'_i \cap U = U_i$.*

Proof. After replacing (S, U) with a blow-up we can assume that it is a strict stackpair. Then proceed as in [RG71, Lem. 5.1.5]. □

Lemma (4.8) ([RG71, Lem. 5.3.1])**.** *Let (S, U) be a pseudo-noetherian stackpair and let $V \subseteq S$ be a quasi-compact open substack. Then any blow-up of $(V, U \cap V)$ extends to a blow-up of (S, U).*

Proof. After replacing $(S, U \cup V)$ with a blow-up we can assume that it is a strict stackpair. The lemma then follows from Lemma (4.3). □

Lemma (4.9). *Let (S, U) be a pseudo-noetherian stackpair. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_S$-module such that $\mathcal{F}$ is locally generated by r elements over U. Then there exists a blow-up $(S', U) \to (S, U)$ such that the strict transform of $\mathcal{F}$ is locally generated by r elements.*

Proof. After replacing (S, U) with a blow-up we can assume that (S, U) is strict. The construction of Fitting ideals commutes with arbitrary base change and in particular makes sense for ideals of finite type on arbitrary stacks. As $\mathcal{F}$ is of finite presentation, the Fitting ideals of $\mathcal{F}$ are of finite type. The lemma is satisfied by taking the blow-up of S in the r^{th} Fitting ideal of $\mathcal{F}$, cf. [RG71, Lem. 5.4.2]. □

Proposition (4.10) (Flatification of modules)**.** *Let (S, U) be a pseudo-noetherian stackpair. Let $\mathcal{F}$ be an $\mathcal{O}_S$-module of finite type which is flat and of finite presentation over U. Then there exists a blow-up $\pi\colon (S', U) \to (S, U)$ such that the strict transform of $\mathcal{F}$ along π is flat and finitely presented.*

Proof. As $\mathcal{F}|_U$ is flat and of finite presentation, it is locally free (smooth-locally) and the rank of $\mathcal{F}$ is locally constant over U. Let $U = \coprod_i U_i$ be a partition of U in open and closed subsets such that $\mathcal{F}|_{U_i}$ has constant rank i. After replacing (S,U) with a blow-up, this partition of U extends to a partition $S=\coprod_i S_i$ by Lemma (4.7). After replacing (S,U) with (S_i,U_i) we can thus assume that $\mathcal{F}|_U$ has constant rank r.

As S has the completeness property and $\mathcal{F}$ is of finite type and finitely presented over U, there exists a finitely presented $\mathcal{O}_S$-module $\mathcal{G}$ and a surjection $\mathcal{G}\twoheadrightarrow\mathcal{F}$ which is an isomorphism over U [Ryd09b, Rmk. 4.4]. As the strict transforms of $\mathcal{F}$ and $\mathcal{G}$ are equal, we can thus replace $\mathcal{F}$ with $\mathcal{G}$ and assume that $\mathcal{F}$ is finitely presented. The corollary then follows from Lemma (4.9), cf. [RG71, Lem. 5.4.3]. □

The following result shows that a blow-up $\pi\colon S'\to S$ that is an isomorphism over U is almost U-admissible. We will generalize this statement to arbitrary representable modifications in Corollary (5.1).

Corollary (4.11). *Let (S,U) be a pseudo-noetherian stackpair. Let $\mathcal{I}$ be a finitely generated ideal defining a blow-up $\pi\colon S'\to S$ such that $\pi|_U$ is an isomorphism (i.e., $\mathcal{I}$ is invertible over U). Then there is a blow-up $(S'',U)\to(S',U)$ such that the composition $(S'',U)\to(S',U)\to(S,U)$ is a blow-up.*

Proof. According to Proposition (4.10), there exists a blow-up $(S'',U)\to(S,U)$, given by some ideal $\mathcal{J}$, which flatifies $\mathcal{I}$. Then $\mathcal{I}\mathcal{O}_{S''}$ (which equals the strict transform of $\mathcal{I}$) is invertible and hence there is a unique morphism $S''\to S'$. The morphism $S''\to S'$ is the blow-up of $\mathcal{J}\mathcal{O}_{S'}$. □

Definition (4.12). Let S be an algebraic stack and let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two finite type ideal sheaves defining blow-ups $X_1\to S$ and $X_2\to S$. We say that $X_2\to S$ *dominates* the blow-up $X_1\to S$ if $\mathcal{I}_1\mathcal{O}_{X_2}$ is invertible, so that there is a unique morphism $X_2\to X_1$ over S.

Remark **(4.13).** Let $X_1\to S$ and $X_2\to S$ be two blow-ups and let $S'\to S$ be covering in the fpqc topology. Then X_2 dominates X_1 if and only if $X_2\times_S S'\to S'$ dominates $X_1\times_S S'\to S'$. Indeed, a sheaf on X_2 is invertible if and only if its inverse image is an invertible sheaf on $X_2\times_S S'$ [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 2.5.2]. Moreover, if $X_2\to S$ is the blow-up in $\mathcal{I}_2$ and $X_2\to S$ dominates $X_1\to S$ then $X_2\to X_1$ is the blow-up in $\mathcal{I}_2\mathcal{O}_{X_1}$.

Proposition (4.14). *Let $f\colon(S',U')\to(S,U)$ be a cartesian morphism of pseudo-noetherian stacks. Let $(X',U')\to(S',U')$ be a blow-up. Assume that one of the following holds:*

(a) *f is a closed immersion.*
(b) *f is representable and étale.*
(b′) *f is representable, étale and $f|_{X\setminus U}$ is finite of constant rank d.*
(b″) *f is representable, étale and $f|_{X\setminus U}$ is an isomorphism.*
(c) *f is representable and unramified and S is quasi-Deligne–Mumford.*

Then there exists a blow-up $(X,U)\to(S,U)$ such that $(X\times_S S',U')\to(S',U')$ dominates $(X',U')\to(S',U')$. If (S,U) is strict, and (a) *or* (b″) *holds, then it can further be arranged so that $X\times_S S'=X'$.*

Proof. Let $\mathcal{I}'$ be an ideal sheaf defining the blow-up $X' \to S'$. After replacing S and S' and X with the strict transforms along a blow-up as in Lemma (4.4), we can assume that (S, U) is strict. Case (a) is then Lemma (4.5). The reduction from case (b$''$) to case (b) uses the étale dévissage of §2 but does not immediately follow from Theorem (2.5).

Case (b$''$): As $\mathcal{I}'|_{U'}$ equals $\mathcal{O}_{U'}$, it follows from Theorem (2.2) that $\mathcal{I}'$ descends to a unique ideal sheaf $\mathcal{I}$ on S which equals $\mathcal{O}_U$ over U.

Case (b$'$): Let $Z = X \setminus U$ and let $\mathrm{SEC} := \mathrm{SEC}^d_Z(S'/S) \subseteq (S'/S)^d$ be the open substack that parameterizes d sections of $S' \to S$ that are disjoint over Z and let $\mathrm{\acute{E}T} := \mathrm{\acute{E}T}^d_Z(S'/S) = [\mathrm{SEC}/\mathfrak{S}_d]$ [Ryd11, Def. 5.1]. Then the projections $\pi_i \colon \mathrm{SEC} \to X'$, $i = 1, 2, \dots, d$ are étale and surjective and $\mathrm{\acute{E}T} \to X$ is an étale neighborhood of Z [Ryd11, Lem. 5.3].

Let $\mathcal{I}' \subseteq \mathcal{O}_{S'}$ be the ideal sheaf defining the blow-up $X' \to S'$ so that $\mathcal{I}'$ equals $\mathcal{O}_{S'}$ over U'. Let $\mathcal{J} = \prod_{i=1}^d (\pi_i^{-1}\mathcal{I}')$. Then $\mathcal{J}$ is invariant under the action of $\mathfrak{S}_d$ on SEC and thus descends to an ideal sheaf $\mathcal{I}^{(d)}$ on ÉT. By the previous case, we have that $\mathcal{I}^{(d)}$ descends further to an ideal sheaf $\mathcal{I}$ on S.

By construction, the pull-back of $\mathcal{I}$ along $\mathrm{SEC} \to S' \to S$ equals $\mathcal{J}$ (where $\mathrm{SEC} \to S'$ is any projection). As $\pi_1^{-1}\mathcal{I}'$ divides $\mathcal{J}$ and π_1^{-1} is faithfully flat, it follows that the blow-up defined by $\mathcal{I}\mathcal{O}_{S'}$ dominates the blow-up defined by $\mathcal{I}'$.

Case (b$''$): Let $\emptyset = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_n = S$ be a filtration of open quasi-compact subsets such that $f|_{S_i \setminus S_{i-1}}$ is finite and étale of constant rank for every $i = 1, 2, \dots, n$ [Ryd11, Prop. 4.4]. Let $S'_i = f^{-1}(S_i)$, $U_i = U \cap S_i$, $U'_i = f^{-1}(U_i)$ and $X'_i = X' \times_{S'} S'_i$.

We will show the result on by induction on n. The case where $n = 0$ is trivial. Thus, assume that we have a blow-up $(X_{n-1}, U_{n-1}) \to (S_{n-1}, U_{n-1})$ that after pull-back to S'_{n-1} dominates $X'_{n-1} \to S'_{n-1}$. Let $V = U \cup S'_{n-1}$. Our purpose is now to reduce to the case where $U = V$.

By Lemma (4.8) we can extend the blow-up $(X_{n-1}, U_{n-1}) \to (S_{n-1}, U_{n-1})$ to a blow-up $(X, U) \to (S, U)$. We will now construct a blow-up $(Y, U) \to (X, U)$ such that $Y \times_S S' \to X \times_S S' \to S'$ dominates $X' \to S'$. For this purpose, we may replace S, S' and X' with their strict transforms by $X \to S$. We then have that $X' \to S'$ is an isomorphism over $V' = U' \cup S'_{n-1}$. After replacing (X', U') with a blow-up we can thus, by Corollary (4.11), assume that $U = V$. We conclude the induction step by case (b$'$).

Case (c): There is a factorization $S' \hookrightarrow S'_0 \to S$ of f where the first morphism is a closed immersion and the second morphism is unramified, representable and of finite presentation [Ryd09b, Thm. D]. The second morphism has a further factorization $S'_0 \hookrightarrow E \to S$ where the first morphism is a closed immersion and the second is étale, representable and of finite presentation [Ryd09a]. The result thus follows from cases (a) and (b). $\square$

Let $f \colon X \to S$ be a morphism locally of finite type. For a point $x \in |X|$ we let $\dim_{X/S}(x) := \dim_x(X_s)$ be its relative dimension. If $p \colon U \to X$ is a smooth presentation, then $\dim_{X/S} \circ |p| = \dim_{U/S} - \dim_{U/X}$ and it follows from Chevalley's theorem that $\dim_{X/S} \colon |X| \to \mathbb{Z}$ is upper semi-continuous. If f is of finite presentation, $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite type and $n \in \mathbb{Z}$, then we say that $\mathcal{F}$ is *S-flat in dimension* $\geq n$ if there is a quasi-compact open substack $V \subseteq X$ containing all points of relative dimension at least n such that $\mathcal{F}|_V$ is S-flat and of finite presentation [RG71, Déf. 5.2.1].

Proposition (4.14) immediately give us the following generalization of [RG71, Thm. 5.2.2] to Deligne–Mumford stacks.

Theorem (4.15). *Let $g\colon (Y,V) \to (S,U)$ be an unramified cartesian morphism of Deligne–Mumford stackpairs. Further, let $f\colon X \to Y$ be a morphism of finite presentation of algebraic stacks and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type and n an integer. If $\mathcal{F}|_{f^{-1}(V)}$ is V-flat in dimension $\geq n$, then there exists a blow-up $(\widetilde{S},U) \to (S,U)$ such that the strict transform $\widetilde{\mathcal{F}}$ is $\widetilde{Y}$-flat in dimension $\geq n$.*

Proof. The question is smooth-local on X (change n accordingly) and étale-local on Y, so we can assume that X and Y are (affine) schemes. Then, by Lemma (4.4) and the usual blow-up theorem [RG71, Thm. 5.2.2] there exists a blow-up $(\widetilde{Y},V) \to (Y,V)$ which flatifies f in dimension $\geq n$. This gives us, by Proposition (4.14), a blow-up $(\widetilde{S},U) \to (S,U)$ which flatifies f in dimension $\geq n$. $\square$

Corollary (4.16) (cf. [RG71, Cor. 5.7.10])**.** *Let (S,U) be a Deligne–Mumford stackpair and let $f\colon X \to S$ be a morphism of finite type.*

(i) *If $f|_V$ is flat and of finite presentation, then there exists a blow-up of (S,U) such that the strict transformation of f is flat and of finite presentation.*
(ii) *If X is Deligne–Mumford and $f|_V$ is quasi-finite, then there exists a blow-up of (S,U) such that the strict transformation of f is quasi-finite.*

Proof. By Proposition (4.14) the question is étale-local on S so we can assume that S is affine. We can also replace X by a presentation and assume that X is affine.

Then there exists a Y-scheme of *finite presentation* $X'_0 \to Y$ and a closed immersion $j\colon X' \to X'_0$ over Y. We then apply Theorem (4.15) to $j_*\mathcal{O}_{X'}$ and $X'_0 \to Y$ with $n=0$ and $n=1$ respectively. $\square$

Remark **(4.17).** The flatification theorem (4.15) is not valid relative to a finite flat morphism g. Indeed, let S be a smooth curve and let Y be two copies of S glued along a point so that $Y \to S$ is a ramified covering of degree 2. Let $X = S$ as one of the components of Y. Then $X \to Y$ cannot be flatified relative to S. This is related to the fact that the Weil-restriction of a proper morphism along a finite flat morphism need not be proper. Similarly, the theorem is not valid relative to a smooth g. This makes it difficult to directly deduce the flatification theorem for stacks from the flatification theorem for schemes. For stacks that are étale-locally a global quotient stack, a possible approach is to first prove an equivariant version of the flatification theorem and then proceed by étale dévissage.

We will now proceed to prove the flatification theorem for quasi-Deligne–Mumford stacks using Proposition (2.6).

Corollary (4.18). *Let (S, U) be a quasi-Deligne–Mumford stackpair and let $f\colon X \to S$ be a finite type morphism of quasi-Deligne–Mumford stacks. If $f|_U$ is quasi-finite, then there exists a blow-up of (S, U) such that the strict transform of f is quasi-finite.*

Proof. By [Ryd09b, Thm. D], there is a closed immersion $X \hookrightarrow X_0$ such that $X_0 \to S$ is of finite presentation and $X_0|_U \to U$ is quasi-finite. After replacing X with X_0, we can assume that $X \to S$ is of finite presentation. By [Ryd09b], S can be approximated by quasi-Deligne–Mumford stacks of finite type over $\operatorname{Spec}(\mathbb{Z})$. By a standard limit argument [Ryd09b, App. B], we can thus assume that S is of finite type over $\operatorname{Spec}(\mathbb{Z})$. By Proposition (2.6) there is a representable étale morphism $S' \to S$ such that S' has a coarse moduli space S'_0. We now apply [RG71, Cor. 5.7.10], cf. Corollary (4.16), to the finite type morphism $X \times_S S' \to S' \to S'_0$ which is quasi-finite over the image of U and obtain a blow-up of $(S', U \times_S S')$ such that the strict transform of $X \times_S S' \to S'$ is quasi-finite. By Proposition (4.14) we can find a blow-up of (S, U) that does the job. □

Proof of Theorem A. By Proposition (4.14) the question is local on S with respect to representable étale morphisms, so we can, by Proposition (2.6) assume that S has a finite flat presentation $g\colon S' \to S$ where S' is affine. Let $U' = g^{-1}(U)$. By Theorem (4.15) applied to $X \times_S S' \to S'$ with $n = -\infty$ there is a blow-up $(\widetilde{S'}, U') \to (S', U')$ that flatifies $f'\colon X \times_S S' \to S'$.

By Corollary (4.18), there is a blow-up of (S, U) such that the strict transform of $(\widetilde{S'}, U') \to (S, U)$ becomes quasi-finite and hence finite. As $(\widetilde{S'}, U') \to (S, U)$ restricts to the flat morphism $U' \to U$ there is, by Proposition (4.10), a blow-up of (S, U) that flatifies $(\widetilde{S'}, U') \to (S, U)$. After strict transformations we have then accomplished that $f\colon X \to S$ is flat and of finite presentation. □

5. Applications to flatification

The first important application of Raynaud–Gruson flatification is that U-admissible blow-ups are cofinal among U-admissible modifications:

Corollary (5.1). *Let (S, U) be a quasi-Deligne–Mumford stackpair and let $f\colon (X, U) \to (S, U)$ be a modification. Then there exist a blow-up $p\colon (X', U) \to (X, U)$ such that the composition $f \circ p$ is a blow-up.*

Proof. Let $(S', U) \to (S, U)$ be a blow-up such that the strict transform $X' \to S'$ of f is a flat modification so that $X' = S'$. As $X' \to X$ also is a blow-up, namely the blow-up centered at the inverse image of the center of $S' \to S$, the corollary follows. □

Corollary (5.1) for schemes also follows from Nagata's proof without using the flatification result, cf. [Con07, Thm. 2.11]. Indeed, this result is equivalent with the most important step in Nagata's theorem [Con07, Thm. 2.4] (with $U = V$). Therefore, using Raynaud–Gruson's flatification theorem significantly reduces the complexity of Nagata's proof. This explains why Lütkebohmert's proof [Lüt93] is much shorter.

The next result is a general form of "resolving the indeterminacy locus via blow-ups".

Corollary (5.2). *Let $(S', U') \to (S, U)$ be a cartesian étale morphism of quasi-Deligne–Mumford stackpairs. Let $X' \to S'$ be quasi-compact étale and let $Y' \to S'$ be proper and representable. Let $f_U \colon X'|_{U'} \to Y'|_{U'}$ be a U'-morphism. Then there exists a blow-up $(\widetilde{S}, U) \to (S, U)$ and a unique extension $f \colon X' \times_S \widetilde{S} \to Y' \times_S \widetilde{S}$ of f_U.*

Proof. After a blow-up as in Lemma (4.4) we can assume that (S, U) is strict. Let $\Gamma_{f_U} \hookrightarrow X' \times_{S'} Y' \times_S U$ be the graph of Γ_{f_U}. This is a closed immersion since $Y' \to S'$ is separated and representable. Let Γ be the schematic closure of Γ_{f_U} in $X' \times_{S'} Y'$. We note that $\Gamma \to X'$ is proper, representable and an isomorphism over $X'|_{U'}$. Thus there exists a blow-up $(\widetilde{S}, U) \to (S, U)$ such that the strict transform of $\Gamma \to X' \to S$ is flat. Then $\Gamma \to X'$ is flat, proper and birational, hence an isomorphism. The composition of the corresponding morphism $X' \times_S \widetilde{S} \to \Gamma$ and $\Gamma \to Y'$ is the requested extension. The uniqueness follows from the fact that $X' \times_S U \subseteq X'$ is schematically dense and $Y' \to S'$ is separated and representable. □

A stacky version of the previous result for $Y' \to S'$ proper but not necessarily representable, is given in Corollary (12.3).

As another application, we give a strong relative version of Chow's lemma, showing in particular that projective morphisms are cofinal among proper representable morphisms.

Theorem (5.3). *Let S be a quasi-compact quasi-Deligne–Mumford stack and let $f \colon X \to S$ be a representable morphism of finite type. Let $U \subseteq X$ be an open subset such that $f|_U$ is quasi-projective (a dense such U always exists). Then there exists a projective morphism $p \colon P \to S$, a U-admissible blow-up $\widetilde{X} \to X$ and a quasi-finite flat S-morphism $g \colon \widetilde{X} \to P$ which is an isomorphism over U. Moreover, if f is locally separated (resp. separated, resp. proper) then g can be chosen to be étale (resp. an open immersion, resp. an isomorphism).*

Proof. Proven exactly as [RG71, Cor. 5.7.13] using Theorem A. To see that there always is a dense open $U \subseteq X$ such that $f|_U$ is quasi-projective ... □

6. Compactification of algebraic spaces

In this section, we prove Theorem B — that any finite type, separated and *representable* morphism $f \colon X \to S$ of quasi-Deligne–Mumford stacks can be compactified. The main argument is due to Raoult who treated the case where S is the spectrum of an excellent ring [Rao74b, Rao74a] and the case where X is normal and S is a scheme [Rao71]. We give a simplified version of Raoult's argument and also generalize the result to allow the target to be a quasi-Deligne–Mumford stack.

Definition (6.1). A representable morphism $f \colon X \to S$ is *compactifiable* if there exists a factorization $f = \overline{f} \circ j$ where j is a quasi-compact open immersion and $\overline{f}$ is proper and representable.

Remark **(6.2).** Every compactifiable morphism is separated and of finite type. If there exists a factorization $f = \overline{f} \circ j$ where j is a quasi-compact open immersion and $\overline{f}\colon \overline{X} \to S$ is proper with *finite diagonal*, then f is compactifiable if S is noetherian or quasi-Deligne–Mumford. In fact, if $\widetilde{f}\colon \widetilde{X} \to S$ denotes the relative coarse moduli space of $\overline{f}$ and $\pi\colon \overline{X} \to \widetilde{X}$ is the moduli map, then $\pi \circ j$ is an open immersion and $\widetilde{f}$ is quasi-compact, universally closed, separated and representable. If S is not noetherian then $\widetilde{f}$ need not be of finite type but if S is quasi-Deligne–Mumford, then it is possible to approximate $\widetilde{f}$ with a proper representable morphism [Ryd09b].

We begin by proving that the theorem is étale-local on S.

Proposition (6.3). *Let S be a quasi-compact quasi-Deligne–Mumford stack and let $f\colon X \to S$ be a* representable *and separated morphism of finite type. Let $S' \to S$ be a representable, étale and surjective morphism. Then f is compactifiable if and only if $f'\colon X \times_S S' \to S'$ is compactifiable.*

Proof. The necessity is obvious and to prove the sufficiency we can assume that $S' \to S$ is quasi-compact. We will use étale dévissage in the form of Theorem (2.5). Let $\mathbf{D} \subseteq \mathbf{E} = \mathbf{Stack}_{\mathrm{qc,\acute{e}t}/S}$ be the full subcategory of quasi-compact étale morphisms $T \to S$ such that $X \times_S T \to T$ is compactifiable. Then $(S' \to S) \in \mathbf{D}$ by hypothesis. If $(T \to S) \in \mathbf{D}$ and $(T' \to T) \in \mathbf{E}$, then clearly $(T' \to S) \in \mathbf{D}$. Thus $\mathbf{D}$ satisfies axiom (D1) of Theorem (2.5). That $\mathbf{D}$ satisfies axioms (D2) and (D3) follow from the following two Lemmas (6.4) and (6.5). We deduce that $(S \to S) \in \mathbf{D}$ so that $X \to S$ has a compactification. □

Lemma (6.4). *Let $g\colon S' \to S$ be a finite étale morphism of algebraic stacks and let $f\colon X \to S$ be a representable and separated morphism of finite type. Then f is compactifiable if $f'\colon X \times_S S' \to S'$ is compactifiable.*

Proof. Let $f'\colon X \times_S S' \hookrightarrow Y' \to S'$ be a compactification. Consider the Weil restriction $\mathbf{R}_{S'/S}(Y') \to S$ (which is representable by [Ryd10] as can also be seen from the following description). By adjunction we have a morphism $X \to \mathbf{R}_{S'/S}(X \times_S S') \to \mathbf{R}_{S'/S}(Y')$. This morphism is an immersion and $\mathbf{R}_{S'/S}(Y')$ is proper and representable over S. Indeed, this can be checked fppf-locally on S so we can assume that $S' = S^{\amalg n}$ is n copies of S. Then $X \times_S S' = X^{\amalg n}$ and $Y' = Y_1 \amalg Y_2 \amalg \cdots \amalg Y_n$ so that the morphism described above becomes

$$X \xhookrightarrow{\Delta} X^n \hookrightarrow Y_1 \times_S \cdots \times_S Y_n.$$

The schematic closure of X in $\mathbf{R}_{S'/S}(Y')$ is thus a compactification of X. □

Lemma (6.5). *Let (S, U) be a quasi-Deligne–Mumford stackpair and let $S' \to S$ be an étale neighborhood of $S \setminus U$ so that*

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d] \arrow[dr, phantom, "\square"] & S' \arrow[d] \\ U \arrow[r, "j"'] & S \end{tikzcd}$$

is a bi-cartesian square. Let $f\colon X \to S$ be a representable and separated morphism of finite type. If the pull-backs $f'\colon X' \to S'$ and $f|_U\colon X|_U \to U$ are compactifiable, then so is f.

Proof. Let $f'\colon X' \hookrightarrow Y' \to S'$ and $f|_U\colon X|_U \hookrightarrow Y_U \to U$ be compactifications. To obtain a compactification of f we need to modify Y' and Y_U so that $Y'|_{U'} = Y_U \times_U U'$. We have a quasi-compact immersion $X'|_{U'} \to (Y_U \times_U U') \times_{U'} Y'|_{U'}$. Let Z be the schematic image of this immersion so that the two projections $Z \to Y_U \times_U U'$ and $Z \to Y'|_{U'}$ are proper and isomorphisms over the open subscheme $X'|_{U'}$.

By Corollary (5.1), we can assume that $(Z, X'|_{U'}) \to (Y_U \times_U U', X'|_{U'})$ and $(Z, X'|_{U'}) \to (Y'|_{U'}, X'|_{U'})$ are blow-ups after replacing $(Z, X'|_{U'})$ with a blow-up. Moreover, by Proposition (4.14) we can assume that $Z = Y_U \times_U U'$ after replacing $(Z, X'|_{U'})$ and $(Y_U, X|_U)$ with blow-ups. Finally, the blow-up $(Z, X'|_{U'}) \to (Y'|_{U'}, X'|_{U'})$ extends to a blow-up of (Y', X') by Lemma (4.8). After replacing Y' with this last blow-up we have obtained the cartesian diagram

$$\begin{tikzcd}
X|_U \arrow[d] & X'|_{U'} \arrow[l] \arrow[r] \arrow[d] & X' \arrow[d] \\
Y_U \arrow[d] & Z \arrow[l] \arrow[r] \arrow[d] & Y' \arrow[d] \\
U & U' \arrow[l] \arrow[r] & T'
\end{tikzcd}$$

where the first row of vertical arrows are open immersions and the second row are proper morphisms. The push-out $Y = Y_U \amalg_Z Y'$ of the second row exists by Theorem (2.4) and the natural cube diagrams are cartesian [Ryd11, Prop. 3.2]. We thus obtain a compactification $X \subseteq Y \to S$. $\square$

To prove the compactification theorem we need yet another lemma similar to the previous one. This time we use another kind of pushout that we call a pinching, cf. Appendix A. We will only use the following two lemmas when S is a scheme although they are stated more generally.

Lemma (6.6). *Let S be a noetherian quasi-Deligne–Mumford stack. Let*

$$\begin{tikzcd}
Z' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[d] \\
Z \arrow[r, hook] & X
\end{tikzcd}$$

be a cartesian diagram of stacks that are representable, separated and of finite type over S. Further assume that $Z \hookrightarrow X$ is a closed immersion, that $X' \to X$ is finite and that the diagram is a pinching. Then $X \to S$ is compactifiable if and only if $X' \to S$ and $Z \to S$ are compactifiable.

Proof. That the condition is necessary follows easily from Zariski's main theorem. It is thus enough to prove sufficiency. Let $Z \subseteq \overline{Z} \to S$ and $X' \subseteq \overline{X'} \to S$ be compactifications. Let $\overline{Z'}$ be the closure of Z' in $\overline{Z} \times_S \overline{X'}$. We will now use blow-ups to achieve the situation where $\overline{Z'} \to \overline{Z}$ is finite and $\overline{Z'} \to \overline{X'}$ is a closed immersion. This is the key idea in Raoult's proof, cf. [Rao74a, pf. Prop. 2].

Let $\widetilde{Z'} \hookrightarrow \overline{X'}$ be the image of $\overline{Z'}$ so that $(\overline{Z'}, Z') \to (\widetilde{Z'}, Z')$ is a modification. We can then by Corollary (5.1) replace $(\overline{Z'}, Z')$ by a blow-up and assume that $(\overline{Z'}, Z') \to (\widetilde{Z'}, Z')$ is a blow-up. The morphism $(\overline{Z'}, Z') \to (\overline{Z}, Z)$ is not a modification but it is finite over Z. We can thus by Corollary (4.18) assume that $(\overline{Z'}, Z') \to (\overline{Z}, Z)$ is finite after replacing the source and the target with blow-ups. Finally, the blow-up $(\overline{Z'}, Z') \to (\widetilde{Z'}, Z')$ extends to a blow-up of $(\overline{X'}, X')$ (use the same center) so after replacing $(\overline{X'}, X')$ with a blow-up we can assume that $\overline{Z'} \to \overline{X'}$ is a closed immersion. We now have the cartesian diagram

$$\begin{tikzcd} Z \arrow[d] & Z' \arrow[l] \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ \overline{Z} & \overline{Z'} \arrow[l] \arrow[r, hook] & \overline{X'} \end{tikzcd}$$

where the vertical morphisms are open immersions. Taking push-outs of the two rows gives us an open immersion $X \to \overline{X}$. The existence and the properness of $\overline{X}$ follow from Theorem (A.4) or, as the question is fppf-local on S, from either [Rao74a, Prop. 1] or [Art70, Thm. 6.1]. □

Lemma (6.7). *Let S be a noetherian quasi-Deligne–Mumford stack and let $f\colon X \to S$ be representable, separated and of finite type. Assume that*

(i) *There exists a finite morphism $p\colon X' \to X$ such that $f \circ p\colon X' \to S$ is compactifiable and $p|_U$ is an isomorphism over a non-empty open substack $U \subseteq X$.*
(ii) *For every closed substack $j\colon Z \hookrightarrow X$ such that $|Z| \neq |X|$ we have that $f \circ j\colon Z \to S$ is compactifiable.*

Then f is compactifiable.

Proof. Let $X'' \hookrightarrow X$ be the schematic image of p so that $X' \to X''$ is schematically dominant and $X'' \hookrightarrow X$ is a closed immersion that is an isomorphism over U. As the finite morphism $p\colon X' \to X''$ satisfies the conditions of the lemma, it is thus enough to prove the lemma when either p is schematically dominant or p is a closed immersion.

Case 1: *p is schematically dominant.* Let $\mathcal{J} \subseteq \mathcal{O}_X$ be the conductor of $p\colon X' \to X$ and let $Z = V(\mathcal{J})$. Then $\mathcal{J} \to p_*(\mathcal{J}\mathcal{O}_{X'})$ is an isomorphism and we have a pinching diagram

$$\begin{tikzcd} Z' \arrow[r, hook] \arrow[d] & X' \arrow[d, "p"] \\ Z \arrow[r, hook] & X. \end{tikzcd}$$

Case 2: *p is a closed immersion.* Let $\mathcal{I}$ be the kernel of $\mathcal{O}_X \twoheadrightarrow p_*\mathcal{O}_{X'}$ and let $\mathcal{J}$ be the annihilator of $\mathcal{I}$. Then by Artin–Rees lemma, there is an integer $k \geq 1$ such that for all $n > k$ we have that

$$\mathcal{J}^n \cap \mathcal{I} = \mathcal{J}^{n-k}(\mathcal{J}^k \cap \mathcal{I}) = 0.$$

Let $Z \hookrightarrow X$ be the closed subspace defined by $\mathcal{J}^n$ so that

$$\begin{tikzcd} Z' \arrow[r, hook] \arrow[d, hook] & X' \arrow[d, hook, "p"] \\ Z \arrow[r, hook] & X \end{tikzcd}$$

is a pinching diagram.

In both cases $Z \cap U = \emptyset$ so that $Z \to S$ is compactifiable by assumption (ii). That $X \to S$ is compactifiable then follows from Lemma (6.6). □

Proof of Theorem B. By [Ryd09b, Thm. D], the morphism f factors as a closed immersion $X \hookrightarrow X'$ followed by a representable and separated morphism $X' \to S$ of finite presentation. After replacing X with X' we can thus assume that f is of finite presentation.

First assume that S is Deligne–Mumford. By Proposition (6.3), we can then assume that S is an affine scheme. The reduction to the case where S is of finite type over $\mathrm{Spec}(\mathbb{Z})$ is then standard. We can thus assume that S is affine of finite type over $\mathrm{Spec}(\mathbb{Z})$ (although we will only use that S is noetherian) so that X is a separated algebraic space of finite type over S.

By noetherian induction we can assume that for every closed subspace $Z \hookrightarrow X$ such that $Z \neq X$, the morphism $Z \to S$ is compactifiable. By Lemma (6.7) we may then assume that X is irreducible.

It is well-known that there exists a *scheme* W and a finite morphism $W \to X$ which is étale over a (dense) open subspace $U \subseteq X$ [LMB00, Thm. 16.6]. By the classical Nagata theorem, there is a compactification $W \subseteq \overline{W} \to S$ where $\overline{W}$ is a scheme that is proper over S. Let d be the rank of $W|_U \to U$. After replacing W and $\overline{W}$ with $\mathrm{SEC}^d(W/X)$ and its closure in $(\overline{W}/S)^d$ we have that the symmetric group $\mathfrak{S}_d$ acts on W and $\overline{W}$ such that $U = [W|_U/\mathfrak{S}_d]$.

Let $X' = W/\mathfrak{S}_d$ and $\overline{X'} = \overline{W}/\mathfrak{S}_d$ be the geometric quotients in the category of separated algebraic spaces, cf. [Knu71, p. 183] and [Mat76, KM97, Ryd07]. Then $X' \to X$ is finite and an isomorphism over U and $\overline{X'}$ is a compactification of X'. That $X \to S$ is compactifiable now follows from Lemma (6.7).

Now assume that S is merely quasi-Deligne–Mumford. As before, but now using [Ryd09b, Thm. D], we can assume that S is a quasi-Deligne–Mumford stack of finite type over $\mathrm{Spec}(\mathbb{Z})$. By Propositions (2.6) and (6.3), we can also assume that X and S have coarse moduli spaces X_0 and S_0 and that $S \to S_0$ and $X_0 \to S_0$ are of finite type.

We have already shown that $X_0 \to S_0$ has a compactification $X_0 \subset \overline{X_0} \to S_0$. The morphism $X \to X_0 \times_{S_0} S$ is quasi-finite, proper and representable, hence finite. We can thus apply Zariski's main theorem to the representable, separated and quasi-finite morphism $X \to \overline{X_0} \times_{S_0} S$ and obtain a non-representable compactification $X \subseteq \overline{X} \to \overline{X_0} \times_{S_0} S \to S$. After replacing $\overline{X} \to S$ with its relative coarse moduli space we have a representable compactification of $X \to S$. □

Part 2. Stacky modifications, stacky blow-ups, tame étalification and compactification of tame Deligne–Mumford stacks

7. Stacky modifications

In this section we reinterpret the results on blow-ups given in Section 4 in a more categorical language that allow us to give a transparent generalization to stacky blow-ups and stacky modifications. A pesky detail is that while there is at most one morphism between two modifications, there can be several morphisms between stacky modifications. On the other hand, stacky modifications at least constitute a 1-category — it is even a directed category — and not a 2-category so we do not have to worry about 2-morphisms.

Definition (7.1). Let (S,U) be a stackpair. A *stacky modification* of (S,U) or a *U-admissible stacky modification* of S is a proper morphism $\pi\colon (X,U) \to (S,U)$ with *finite diagonal* such that (X,U) is strict and $\pi^{-1}(U) \to U$ is an isomorphism.

Lemma (7.2) (cf. [FMN10, Prop. 1.2])**.** *Let (X,U) be a* strict *stackpair and let Y be an algebraic stack with finite diagonal. Let $f,g\colon X \to Y$ be two morphisms and let $\tau_U\colon f|_U \Rightarrow g|_U$ be a 2-morphism.*

(i) *There is at most one 2-morphism $\tau\colon f \Rightarrow g$ extending τ_U.*
(ii) *There exists a universal finite U-admissible modification $\pi\colon X' \to X$ such that τ_U extends to a 2-morphism $\tau\colon f\circ\pi \Rightarrow g\circ\pi$.*
(iii) *If Y is an algebraic space, then $f=g$.*

Proof. Consider the 2-cartesian diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\Delta_Y"] \\ X \arrow[r, "{(f,g)}"] & Y\times Y \end{tikzcd}$$

where $Z \to X$ is finite since Δ_Y is finite. The 2-morphism τ_U corresponds to a section $s_U\colon U \to Z$ of h over U. Similarly, a 2-morphism $\tau\colon f \Rightarrow g$ corresponds to a section of h. A section s of h is determined by its graph $\Gamma_s \hookrightarrow Z$ and as $U \subseteq X$ is schematically dense, we have that Γ_s is the closure of $\Gamma_{s|_U}$. It follows that there is at most one section of h extending s_U and such a section exists if $X' := \overline{\Gamma_{s_U}} \to X$ is an isomorphism. Finally (iii) follows since $X'=X$ in this case. ☐

Definition (7.3). Let (S,U) be a stackpair. We let

$$\mathbf{Mod}(S,U) \subset \mathbf{Mod}_{\text{stacky}}(S,U) \subset \mathbf{Stack}_{/S}$$

denote the full subcategories of the 2-category $\mathbf{Stack}_{/S}$ with objects modifications and stacky modifications of (S,U).

Lemma (7.4). *Let (S,U) be a stackpair.*

(i) *The 2-category $\mathbf{Mod}_{\text{stacky}}(S,U)$ is equivalent to a 1-category.*
(ii) *If $\pi_i\colon (X_i,U) \to (S,U)$ are stacky modifications for $i=1,2$ and π_2 is representable, then the set $\operatorname{Hom}(\pi_1,\pi_2)$ has at most one element. In particular, the 1-category $\mathbf{Mod}(S,U)$ is a directed set.*

(iii) *The category* $\mathbf{Mod}_{\mathrm{stacky}}(S,U)$ *has all finite limits and equalizers are finite modifications.*

Proof. Let $\pi_i\colon (X_i,U)\to(S,U)$ be stacky modifications for $i=1,2$ and let $f,g\colon(X_1,U)\to(X_2,U)$ be S-morphisms. Lemma (7.2) (i) shows that there is at most one 2-morphism $\tau\colon f\Rightarrow g$ so that the groupoid $\mathbf{Hom}(\pi_1,\pi_2)$ is equivalent to a set. If π_2 is representable, then (iii) of the lemma shows that $f=g$. Finally, (ii) of the lemma shows that the equalizer of f and g is a finite modification $(X_1',U)\to(X_1,U)$. The product of π_1 and π_2 is the strictification of $(X_1\times_S X_2,U)$ or, equivalently, the strict transformation of π_1 along π_2. $\square$

Let $f\colon(S',U')\to(S,U)$ be a morphism of stackpairs. The strict transform induces a functor $f^*\colon\mathbf{Mod}_{\mathrm{stacky}}(S,U)\to\mathbf{Mod}_{\mathrm{stacky}}(S',U')$. This makes

$$\mathbf{Mod}_{\mathrm{stacky}}\colon\mathbf{StackP}\to\mathbf{Cat},\quad (S,U)\mapsto\mathbf{Mod}_{\mathrm{stacky}}(S,U)$$

into a 2-presheaf (or a pseudofunctor). It can be seen that this is a 2-sheaf (or a stack in categories) for the fppf topology on **StackP** but we will not make use of this.

Definition (7.5). Let P be a property of stacky modifications stable under compositions and strict transformations. We say that a stacky modification is a P-modification if it has property P. Let $\mathbf{Mod}_P\subset\mathbf{Mod}_{\mathrm{stacky}}$ be the subpresheaf of P-modifications, i.e., let $\mathbf{Mod}_P(S,U)\subset\mathbf{Mod}_{\mathrm{stacky}}(S,U)$ be the full subcategory with objects P-modifications $f\colon(S',U')\to(S,U)$ for every stackpair (S,U).

The salient example of a property P is the property "being a blow-up" so that $\mathbf{Mod}_P(S,U)\subset\mathbf{Mod}_{\mathrm{stacky}}(S,U)$ is the directed set of blow-ups.

Note that we do not impose that $\mathbf{Mod}_P$ has any sheaf property. In particular, we do not require that the property P can be checked étale-locally. In fact, we do not even require that a modification $p\colon(X,U)\to(S,U)$ has property P if $p\colon(X,V)\to(S,V)$ has property P for an open substack $V\subseteq U$ (this is false for blow-ups). Also note that $\mathbf{Mod}_P$ has fiber products since the fiber product of two stacky modifications f and g coincides with the composition of g and the strict transformation of f along g.

Definition (7.6). We say that a stacky modification $f\colon(X,U)\to(S,U)$ is P-dominated by $f\colon(Z,U)\to(S,U)$ if there is a P-modification $(X,U)\to(Z,U)$.

Lemma (7.7). *Let* $p_i\colon(X_i,U)\to(S,U)$ *be* P*-modifications for* $i=1,2$.

(i) *If* p_2 *is representable, then there is at most one* S*-morphism* $g\colon X_1\to X_2$ *and such a morphism is necessarily a* P*-modification.*
(ii) *If* p_2 *is arbitrary, then any* S*-morphism* $g\colon X_1\to X_2$ *is the composition of a finite modification* $(X_1,U)\to(Z,U)$ *followed by a* P*-modification* $(Z,U)\to(X_2,U)$.

Proof. Let $Z=X_1\times_S X_2$ so that the projections are P-modifications and let $s=(\mathrm{id}_{X_1},g)\colon X_1\to Z$ be the section corresponding to a morphism g so that $g = \pi_2 \circ s$. If p_2 is representable, then s is an isomorphism. If p_2 is arbitrary then $s\colon (X_1, U) \to (Z, U)$ is a finite modification. $\square$

So, while for example a blow-up that is dominated by another blow-up automatically is P-dominated (with P being "blow-up"), cf. Remark (4.13), this is not the case for other properties.

We will be interested in the following properties of P-modifications:

- (E_f) *Extension along* $f\colon (S', U') \to (S, U)$ — Every P-modification of (S', U') extends to a P-modification of (S, U).
- (QE_f) *Quasi-extension along* $f\colon (S', U') \to (S, U)$ — Every P-modification of (S', U') that is an isomorphism over $f^{-1}(U)$ is P-dominated by a P-modification of $(S', f^{-1}(U))$ that extends to a P-modification of (S, U).
- (QD_f) *Quasi-descent of domination along* $f\colon (S', U') \to (S, U)$ — If $\pi_i\colon (X_i, U) \to (S, U)$ are P-modifications for $i = 1, 2$ and $X_2 \times_S S'$ is P-dominated by $X_1 \times_S S'$, then X_2 is P-dominated by a P-modification of X_1.
- (CF) *Strong cofinality* — Every stacky modification is P-dominated by a P-modification.

We have seen that blow-ups have the extension property along open immersions (Lemma 4.8), the quasi-extension property for representable étale morphisms (Proposition 4.14) and that blow-ups are strongly cofinal among modifications (Corollary 5.1).

The following "quasi-descent" result is a stronger form of "quasi-descent of domination".

Lemma (7.8). *Let $\pi_i\colon (X_i, U) \to (S, U)$ be P-modifications and let $f\colon (S', U') \to (S, U)$ be surjective étale and cartesian. Assume that $X_2 \times_S S'$ is P-dominated by $X_1 \times_S S'$. If every blow-up has property P then X_2 is P-dominated by a blow-up of X_1.*

Proof. Let $g'\colon (X_1 \times_S S', U') \to (X_2 \times_S S', U')$ be a P-modification and let $g_1'', g_2''\colon (X_1 \times_S S'', U'') \to (X_2 \times_S S'', U'')$ be the pull-backs of g' along the two projections $S'' = S' \times_S S' \to S'$. Since the equalizer of g_1'' and g_2'' is a finite modification, there is a blow-up of $(X_1 \times_S S'', U'')$ that equalizes them. This blow-up can be quasi-extended to a blow-up of $p\colon (Y_1, U) \to (X_1, U)$ so that after replacing X_1 with Y_1 we have that $g_1'' = g_2''$. Then we obtain a modification $g\colon (X_1, U) \to (X_2, U)$ by étale descent. By Lemma (7.7) this modification factors as a finite modification followed by a P-modification. By strong cofinality of blow-ups among modifications, we have that g becomes a P-modification after replacing X_1 with yet another blow-up. $\square$

Proposition (7.9). *Let P be a property of stacky modifications of quasi-Deligne–Mumford stackpairs. Consider the following properties of P-modifications:*

- ($\mathrm{E}_{\mathrm{open}}$) *The extension property for cartesian open immersions.*
- ($\mathrm{QE}_{\mathrm{adm}}$) *The quasi-extension property for (non-cartesian) open immersions.*
- ($\mathrm{QE}_{\text{étnbhd}}$) *The quasi-extension property for étale neighborhoods, i.e., étale cartesian morphisms $f\colon (S', U') \to (S, U)$ such that $f|_{S \setminus U}$ is an isomorphism.*

(QE$_{\mathrm{Gal}}$) *The quasi-extension property for Galois covers, i.e., finite étale cartesian morphisms that are Galois.*

(QE$_{\text{ét}}$) *The quasi-extension property for representable étale morphisms.*

(QE$_{\mathrm{closed}}$) *The quasi-extension property for closed immersions.*

(QE$_{\mathrm{unram}}$) *The quasi-extension property for representable unramified morphisms.*

If every blow-up has property P then the first four properties imply the fifth property (QE$_{\text{ét}}$). *Properties* (QE$_{\text{ét}}$) *and* (QE$_{\mathrm{closed}}$) *imply* (QE$_{\mathrm{unram}}$).

Note that if P is "being a blow-up" then (E$_{\mathrm{open}}$) and (QE$_{\mathrm{adm}}$) are Lemma (4.8) and Corollary (4.11) and the rest of the properties are proved in Proposition (4.14). The following proof generalizes the proof of Proposition (4.14) to stacky modifications.

Proof. Every morphism of stackpairs factors as an open immersion of the form $(S', V) \to (S', U)$ followed by a cartesian morphism. Using (QE$_{\mathrm{adm}}$) it is thus enough to prove property (QE$_{\text{ét}}$) and (QE$_{\mathrm{unram}}$) for cartesian morphisms.

Let $f\colon (S', U') \to (S, U)$ be a representable étale cartesian morphism. Let $\emptyset = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_n = S$ be a filtration of open quasi-compact subsets such that $f|_{S_i \setminus S_{i-1}}$ is finite and étale of constant rank for every $i = 1, 2, \ldots, n$ [Ryd11, Prop. 4.4]. Let $p'\colon (X', U') \to (S', U')$ be a P-modification.

We will show that p' is dominated by a P-modification that extends to (S, U) by induction on n. Thus assume that we have a cartesian diagram

$$\begin{tikzcd}
(X_{n-1} \times_{S_{n-1}} S'_{n-1}, U'_{n-1}) \arrow[r] \arrow[d] & (X_{n-1}, U_{n-1}) \arrow[dd, "p_{n-1}"] \\
(X'_{n-1}, U'_{n-1}) \arrow[d, "p'_{n-1}"] & \\
(S'_{n-1}, U'_{n-1}) \arrow[r] & (S_{n-1}, U_{n-1})
\end{tikzcd}$$

where the vertical morphisms are P-modifications and p'_{n-1} is the restriction of p'. Let $V = U \cup S_{n-1}$ and $V' = U' \cup S'_{n-1}$.

Step 1: *Reduce to the case where $U = V$ and $U' = V'$.* By (E$_{\mathrm{open}}$) we can extend p_{n-1} to a P-modification $p\colon (X, U) \to (S, U)$. After replacing (S, U), (S', U') and (X', U') with the strict transforms along p we can then assume that $p'\colon (X', U') \to (S', U')$ has a section over V'. Let $s\colon V' \to X'$ denote the section. As p' has finite diagonal, we have that s is representable and quasi-finite. We can thus use Zariski's main theorem [Ryd09b, Thm. 8.6]: the quasi-finite map $V' \to X'$ factors as a schematically dense open immersion $j\colon V' \hookrightarrow X'_1$ followed by a finite morphism $X'_1 \to X'$.

By strong cofinality for blow-ups, there is a blow-up $(X'_2, U') \to (X'_1, U')$ such that the composition $(X'_2, U') \to (X'_1, U') \to (X', U')$ is a blow-up. Now $(X'_2, U') \to (S', U')$ is no longer necessary an isomorphism over V' but $(X'_2|_{V'}, U') \to (V', U')$ is at least a blow-up. By (QE$_{\text{ét}}$) for blow-ups, there is a blow-up $(W, U) \to (S, U)$ such that after taking strict transforms along $W \to S$, the P-modification $(X'_2, U') \to (S', U')$ is an isomorphism over V'.

By ($\mathrm{QE}_{\mathrm{adm}}$) there is then a P-modification $(X_3', V') \to (X_2', V')$ such that the composition $(X_3', V') \to (S', V')$ is a P-modification and we can replace X', U' and U with X_3', V' and V.

Step 2: *Reduce to descent question.* Let $Z = X \setminus U$. By Step 1, we have that $f|_Z$ is finite of constant rank d. Let $\mathrm{SEC} := \mathrm{SEC}^d_Z(S'/S) \subseteq (S'/S)^d$ be the open substack that parameterizes d sections of $S' \to S$ that are disjoint over Z and let $\mathrm{ÉT} := \mathrm{ÉT}^d_Z(S'/S) = [\mathrm{SEC}/\mathfrak{S}_d]$ [Ryd11, Def. 5.1]. Then the first projection $\pi_1 \colon \mathrm{SEC} \to X'$ is étale and surjective and $\mathrm{ÉT} \to X$ is a (surjective) étale neighborhood of Z [Ryd11, Lem. 5.3]. Then by ($\mathrm{QE}_{\mathrm{Gal}}$) and ($\mathrm{QE}_{\text{étnbhd}}$) there is a P-modification $p \colon (X, U) \to (S, U)$ that fits in the following diagram:

$$\begin{tikzcd}
(X \times_S \mathrm{SEC}, U \times_S \mathrm{SEC}) \arrow[r] \arrow[d] & (X \times_S S', U') \arrow[r] \arrow[dd, bend left] & (X, U) \arrow[dd, "p"] \\
(X' \times_{S'} \mathrm{SEC}, U \times_S \mathrm{SEC}) \arrow[r] \arrow[d, "p'"] & (X', U') \arrow[d] & \\
(\mathrm{SEC}, U \times_S \mathrm{SEC}) \arrow[r] & (S', U') \arrow[r] & (S, U)
\end{tikzcd}$$

where all vertical morphisms are P-modifications. By Lemma (7.8) there is a blow-up $(Y', U') \to (X' \times_S S', U')$ and a P-modification $(Y', U') \to (X', U')$. By ($\mathrm{QE}_{\text{ét}}$) for blow-ups, we obtain a blow-up of $(Y, U) \to (X, U)$ such that (X', U') is P-dominated by $(Y \times_S S', U')$. □

8. Root stacks

In this section we recall some basic facts about root stacks, cf. [Cad07]. Although we will later exclusively work with ordinary Cartier divisors, we introduce generalized Cartier divisors. Their main virtue is that they admit pull-backs under arbitrary morphisms whereas Cartier divisors do not.

Definition (8.1). Let X be an algebraic stack. A *generalized effective Cartier divisor* is a pair $(\mathcal{L}, s)$ where $\mathcal{L}$ is an invertible sheaf and $s \colon \mathcal{O}_X \to \mathcal{L}$ is a, not necessarily regular, section. A morphism $(\mathcal{L}, s) \to (\mathcal{L}', s')$ is an isomorphism $\varphi \colon \mathcal{L} \to \mathcal{L}'$ such that $s' = \varphi \circ s$. We let $\mathfrak{Div}\, X$ denote the groupoid of generalized effective Cartier divisors on X. We also equip the category $\mathfrak{Div}\, X$ with the symmetric monoidal structure given by the tensor product $(L, s) \otimes (L', s') = (L \otimes L', s \otimes s')$. We let $\mathrm{Div}\, X$ denote the monoid of ordinary effective Cartier divisors.

Note that $\mathfrak{Div}\, X$ can be interpreted as the groupoid $\mathbf{Hom}(X, [\mathbb{A}^1/\mathbb{G}_m])$. Let $\hat{}r \colon \mathbb{G}_m \to \mathbb{G}_m$ be the r^{th} power morphism which sits in the *Kummer sequence*, i.e., the exact sequence of fppf-sheaves

$$1 \longrightarrow \mu_r \longrightarrow \mathbb{G}_m \xrightarrow{\hat{}r} \mathbb{G}_m \longrightarrow 1.$$

The r^{th} power morphism extends to a morphism $\hat{}r \colon \mathbb{A}^1 \to \mathbb{A}^1$ and we let $\hat{}r \colon [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ denote the morphism induced by these two morphisms.

Definition (8.2). The r^{th} root stack of X along $(\mathcal{L},s)\in\mathfrak{Div}\,X$, denoted $X_{(\mathcal{L},s),r}$ or $X(\sqrt{(\mathcal{L},s),r})$, is the stack $X\times_{[\mathbb{A}^1/\mathbb{G}_m],\hat{}r}[\mathbb{A}^1/\mathbb{G}_m]\to X$.

The r^{th} root stack has the following modular interpretation. If $f\colon S\to X$ is a morphism, then $\mathbf{Hom}_X(S,X_{(\mathcal{L},s),r})$ is the groupoid of pairs $\big((\mathcal{K},t),\varphi\big)$ where $(\mathcal{K},t)\in\mathfrak{Div}\,S$ and $\varphi\colon(\mathcal{K},t)^{\otimes r}\to f^*(\mathcal{L},s)$ is an isomorphism. In particular, there is a universal generalized effective Cartier divisor $(\mathcal{L}^{1/r},s^{1/r})\in\mathfrak{Div}\,X_{(\mathcal{L},s),r}$.

If $\mathcal{L}=\mathcal{O}_X$ is trivial, then

$$X_{(\mathcal{L},s),r}=[\operatorname{Spec}_X\big(\mathcal{O}_X[z]/(z^r-s)\big)/\mu_r]$$

where $\mu_r=\operatorname{Spec}(\mathbb{Z}[t]/t^r-1)$ acts by $z\mapsto z\otimes t$. It follows that $X_{(\mathcal{L},s),r}\to X$ is a proper and flat universal homeomorphism of finite presentation and that $X_{(\mathcal{L},s),r}\to X$ is an isomorphism over the non-vanishing locus of s.

From now on, we will restrict the discussion to ordinary Cartier divisors. Thus, let D be an effective Cartier divisor on X and let $s\colon\mathcal{O}_X\to\mathcal{O}_X(D)$ be a section corresponding to D. Then $(\mathcal{O}(D),s)$ is unique up to isomorphism and we let $X_{D,r}=X_{(\mathcal{O}(D),s),r}$ be the corresponding root stack. The generalized effective Cartier divisor $(\mathcal{O}(D)^{1/r},s^{1/r})$ is then an ordinary Cartier divisor, denoted $\frac{1}{r}D\in\operatorname{Div}X_{D,r}$, such that $r\left(\frac{1}{r}D\right)$ equals the pull-back of D. Root stacks of ordinary Cartier divisors have the following properties.

(i) $(X_{D,r},X_{D,r}\setminus\frac{1}{r}D)\to(X,X\setminus D)$ is a stacky modification.
(ii) If $f\colon S\to X$ is a morphism such that $f^{-1}(D)$ is a Cartier divisor, then the groupoid $\mathbf{Hom}_X(S,X_{D,r})$ is equivalent to the set of Cartier divisors $E\in\operatorname{Div}S$ such that $rE=f^{-1}(D)$.
(iii) $X_{D,r}\to X$ is tame and the relative coarse moduli space of $X_{D,r}\to X$ is X. If r is invertible on D, then $X_{D,r}\to X$ is Deligne–Mumford.
(iv) If $D=D_1\amalg D_2$ is a disjoint sum then $X_{D,r}=X_{D_1,r}\times_X X_{D_2,r}$.
(v) If $r=1$ then $X_{D,r}=X$.

Given effective Cartier divisors (D_i) and positive integers (r_i) for $i=1,2,\dots n$, we let

$$X_{(D_i,r_i)}=X_{D_1,r_1}\times_X X_{D_2,r_2}\times_X\cdots\times_X X_{D_n,r_n}.$$

By abuse of notation, we will also refer to $X_{(D_i,r_i)}$ as a root stack. If r_i is invertible on D_i for every i, then we will say that $X_{(D_i,r_i)}$ is a DM root stack. This happens exactly when $X_{(D_i,r_i)}\to X$ is Deligne–Mumford. Note that if $D=D_1+D_2$ and $r=r_1=r_2$, then the root stacks $X_{(D_i,r_i)}$ and $X_{D,r}$ are different. There is however a canonical morphism $X_{(D_i,r_i)}\to X_{D,r}$ induced by the Cartier divisor $\frac{1}{r}D_1+\frac{1}{r}D_2$.

Remark **(8.3).** Let X be a regular scheme and let $D=\sum_{i=1}^n D_i$ be a simple normal crossings divisor. For any tuple (r_i) of positive integers we then have the root stack $X_{(D_i,r_i)}$. It is easily seen that this root stack is regular. More generally, if we only require D to have normal crossings, then there is a similar construction using logarithmic geometry [MO05] which étale-locally is such a root stack. Also see [BV10] for related constructions.

Definition (8.4). If (X,U) is a stackpair then we let $\operatorname{Div}(X,U)\subseteq\operatorname{Div}X$ denote the set of Cartier divisors that are trivial over U. We say that a stacky modification $(X', U) \to (X, U)$ is a root stack (resp. a DM root stack) if $X' = X_{(D_i, r_i)}$ for some $D_i \in \mathrm{Div}(X, U)$ and $r_i \in \mathbb{Z}_+$ (resp. $r_i \in \mathbb{Z}_+$ invertible over D_i).

The following lemma is strictly speaking not necessary in the sequel but illustrates that flat, proper, birational morphism with tame abelian stabilizer groups are "twisted forms" of root stacks. This includes the stacks constructed by Olsson and Matsuki [MO05, Ols09] (but this is obvious).

Lemma (8.5). *Let (S, U) be a Deligne–Mumford strict stackpair and let $\pi\colon (X, U) \to (S, U)$ be a flat stacky modification. Further assume that the relative stabilizer groups of π are locally diagonalizable (resp. tame, abelian and locally constant). Then there exists an étale surjective cartesian morphism $(S', U') \to (S, U)$ such that $(X \times_S S', U') \to (S', U')$ is a root stack (resp. a DM root stack).*

Proof. Using [RG71, Cor. 2.6 and 3.4.2] it can be shown that π is finitely presented. By a standard limit argument, we may thus assume that $S = \mathrm{Spec}(A)$ where A is strictly henselian. Then there is a diagonalizable group scheme G/S and a representable G-torsor $V \to X$, so that $V \to X \to S$ is finite [AOV08]. Let $V = \mathrm{Spec}(B)$. Since S is henselian, V splits into a disjoint union $\coprod V_i$ of spectra of henselian local rings and we can replace (V, G) with (V_i, G_i) for some i, where $G_i \hookrightarrow G$ is the stabilizer of the closed point of V_i.

By assumption $V|_U \to X|_U = U$ is a G-torsor and hence $V \to S$ is flat of rank $|G|$ and G acts faithfully on B. Thus the free A-module B splits into 1-dimensional irreducible representations $B = \bigoplus_{\chi \in G^*} B_\chi$ where B_χ is a free A-module of rank 1 on which G acts by the character χ. Let $\mathfrak{m}_A$ and $\mathfrak{m}_B$ be the maximal ideals of A and B respectively and consider $B/\mathfrak{m}_A$ which is a $k = A/\mathfrak{m}_A$-module. Then $B/\mathfrak{m}_A$ has a filtration $B/\mathfrak{m}_A \supseteq \mathfrak{m}_B/\mathfrak{m}_A \supseteq \mathfrak{m}_B^2/(\mathfrak{m}_A \cap \mathfrak{m}_B^2) \supseteq \dots$ of k-modules stable under the action of G and this induces a grading on the character group G^*. Let $R \subseteq G^*$ be the characters of degree one (i.e., the set of characters χ such that $(\mathfrak{m}_B/(\mathfrak{m}_A + \mathfrak{m}_B^2))_\chi \neq 0$). Then R is a minimal generating set of G^*. Let $b_\chi \in B_\chi$ be a generator of B_χ as an A-module, let n_χ be the order of χ and let $a_\chi = b_\chi^{n_\chi} \in A$. Then

$$B = A[z_1, \dots, z_m]/(z_i^{n_{\chi_i}} - a_{\chi_i})$$

where $R = \{\chi_1, \dots, \chi_m\}$ and G acts on $\prod_i z_i^{\alpha_i}$ by the character $\prod_i \chi_i^{\alpha_i}$. We thus have that $X \to S$ is the root stack along (D_i, n_{χ_i}) where $D_i = V(a_{\chi_i}) \in \mathrm{Div}\, S$. □

9. Tame stacky blow-ups

Definition (9.1). Let X be an algebraic stack, let $Z \hookrightarrow X$ be a closed substack of finite presentation and let r be a positive integer. We let $\mathrm{Bl}_{Z,r}(X)$ denote the algebraic stack $\mathrm{Bl}_Z(X)_{E,r}$ where E is the exceptional divisor.

We note that $\mathrm{Bl}_{Z,1}(X) = \mathrm{Bl}_Z(X)$ is the usual blow-up and that there is a canonical factorization $\mathrm{Bl}_{Z,r}(X) \to \mathrm{Bl}_{Z,1}(X) \to X$. More generally, if $r = st$ then there is a canonical morphism $\mathrm{Bl}_{Z,r}(X) \to \mathrm{Bl}_{Z,s}(X)$ which identifies $\mathrm{Bl}_{Z,r}(X)$ with the root stack $\mathrm{Bl}_{Z,s}(X)_{\frac{1}{s}E,t}$.

Definition (9.2). We say that $\pi\colon X \to Y$ is a *Kummer blow-up* if there is a closed substack $Z \hookrightarrow Y$ and a positive integer r such that $X = \mathrm{Bl}_{Z,r}(Y)$. If r is invertible along Z, then we say that π is a DM Kummer blow-up. If $U \subseteq Y$ is an open substack and Z can be chosen such that $Z \cap U = \emptyset$ then we say that the Kummer blow-up is *U-admissible*. If U is schematically dense in $Y \setminus Z$ then we say that the Kummer blow-up is *strictly U-admissible*. We say that $(X, U) \to (Y, U)$ is a (DM) Kummer blow-up if $X \to Y$ is a strictly U-admissible (DM) Kummer blow-up.

Let $(X, U) \to (S, U)$ be a Kummer blow-up with center Z and weight r. If $g\colon (S', U') \to (S, U)$ is a morphism and (S', U') is strict, then $\mathbf{Hom}_S(S', X)$ is equivalent to the set of effective Cartier divisors $D' \in \operatorname{Div} S'$ such that $rD' = g^{-1}(Z)$.

Contrarily to blow-ups, a composition of two Kummer blow-ups is not necessarily a Kummer blow-up. We therefore make the following definition:

Definition (9.3). We say that $\pi\colon X \to Y$ is a *tame stacky blow-up* if $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_{n-1} \circ \pi_n$ where each π_i is a Kummer blow-up. We say that π is DM (resp. U-admissible, resp. strictly U-admissible) if all the π_i can be chosen to be so. We say that $(X, U) \to (Y, U)$ is a tame stacky blow-up if $X \to Y$ is a strictly U-admissible tame stacky blow-up.

Clearly, a tame stacky blow-up $(X, U) \to (Y, U)$ is a stacky modification. We will now show that tame stacky blow-ups have the properties of Proposition (7.9). We begin with properties ($\mathrm{E}_{\mathrm{open}}$) and ($\mathrm{QE}_{\mathrm{adm}}$), i.e., the analogues of Lemma (4.8) and Corollary (4.11) for tame stacky blow-ups.

Lemma (9.4). *Let (S, U) be a pseudo-noetherian stackpair and let $V \subseteq S$ be a quasi-compact open substack. Then any tame (DM) stacky blow-up of $(V, U \cap V)$ extends to a tame (DM) stacky blow-up of (S, U).*

Proof. First replace $(S, U \cup V)$ with a blow-up so that it becomes a strict stackpair. We can then extend any Kummer blow-up via Lemma (4.3) and the lemma follows. □

Lemma (9.5). *Let (S, U) be a pseudo-noetherian stackpair. Let $\pi\colon X \to S$ be a tame (DM) stacky blow-up such that $\pi|_U$ is an isomorphism. Then there is a tame (DM) stacky blow-up $(X', U) \to (X, U)$ such that the composition $(X', U) \to (X, U) \to (S, U)$ is a tame (DM) stacky blow-up.*

Proof. Let $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_{n-1} \circ \pi_n$ be a factorization into (DM) Kummer blow-ups $\pi_i\colon X_i \to X_{i-1}$ where $X = X_n$ and $S = X_0$. We we will prove the lemma by induction on n.

If we let $Y_i \to S$ denote the relative coarse space of $X_i \to S$, then $Y_i \to S$ is an isomorphism over U. In particular, if $Z_1 \hookrightarrow S$ is a center for the first stacky blow-up, then $Z_1 \cap U$ is a Cartier divisor. We can thus, by Corollary (4.11), replace (S, U) with a blow-up and assume that Z_1 is a Cartier divisor.

Then $X_1 = S_{Z_1,r}$ is a root stack for some positive integer r. Since $X \to S$ is an isomorphism over U, it follows that there is a Cartier divisor $E_U \hookrightarrow U$ such that $rE_U = Z_1 \cap U$. We can extend E_U to a finitely presented closed subscheme $E \hookrightarrow S$ by Lemma (4.3). By Corollary (4.11), we can then assume that $E \hookrightarrow S$ is a Cartier divisor after replacing (S, U) with a blow-up.

It follows that $Z_1 = W_1 + rE$ where W_1 is a Cartier divisor disjoint from U. After replacing (S, U) with the root stack along W_1 we can thus assume that $X_1 \to S$ has a section s. The section s is a finite modification $(S, U) \to (X_1, U)$ so by Proposition (4.10) there is a blow-up $(S', U) \to (S, U)$ such that $(S', U) \to (S, U) \to (X_1, U)$ is a blow-up. We can then assume that $(X_1, U) = (S', U) = (S, U)$ so that π is a composition of $n - 1$ Kummer blow-ups. By induction we can then find a tame (DM) stacky blow-up $(X', U) \to (X, U)$ such that $(X', U) \to (S, U)$ is a tame (DM) stacky blow-up. □

The following lemma will only be used when f is representable and étale.

Lemma (9.6). *Let (S, U) be a stackpair and let $f\colon (S', U') \to (S, U)$ be an unramified cartesian morphism of finite presentation. Let $j\colon D' \hookrightarrow S'$ be an effective Cartier divisor such that $D' \cap U' = \emptyset$ and such that $D' \hookrightarrow S' \to S$ is finite. Then there exists a blow-up $(\widetilde{S}, U) \to (S, U)$ and Cartier divisors $D_1, D_2, \dots, D_n \in \operatorname{Div}(\widetilde{S}, U)$ such that $D' \times_S \widetilde{S}$ is a sum of connected components of the Cartier divisors $D_i \times_S S'$.*

Proof. To illustrate the method that we are going to apply, first assume that $S' = \coprod_{i=1}^n S$ so that $D' = D'_1 \amalg D'_2 \amalg \dots \amalg D'_n$. Then we will first blow-up the intersection of all the D'_i's and subtract one copy of the exceptional divisor from the total transforms of the D'_i's. We then continue by blowing up the union of the intersections of $n - 1$ of these smaller divisors and so on.

In the general case, note that $f \circ j\colon D' \to S$ is finite and of finite presentation. Let $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \dots \subseteq \mathcal{I}_m \subsetneq \mathcal{I}_{m+1} = \mathcal{O}_S$ be the Fitting ideals of $f_* j_* \mathcal{O}_{D'}$ (these are of finite type) and let $\emptyset = Z_{m+1} \hookrightarrow Z_m \hookrightarrow \dots \hookrightarrow Z_1 \hookrightarrow S$ be the corresponding closed substacks so that $f \circ j|_{Z_k}$ has rank at least k. We will prove the lemma by induction on m.

Note that $(f \circ j)|_{Z_m}$ has constant rank m (i.e., is flat, finite and of finite presentation with constant rank m) and in particular $(f \circ j)|_{Z_m}$ is finite and étale. Now let $(\widetilde{S}, U)$ be the blow-up of $\mathcal{I}_m$ and let $D_m = V(\mathcal{I}_m \mathcal{O}_{\widetilde{S}})$ be the exceptional divisor. Let $\widetilde{D'} = D' \times_S \widetilde{S}$ be the *total transform* of D'. Consider the pull-back of $f \circ j$ to D_m:

$$\widetilde{D'}|_{D_m} \hookrightarrow S' \times_S D_m \to D_m.$$

As the composition is étale and the second morphism is unramified, it follows that the first morphism is an open and closed immersion. In particular, we have that $\widetilde{D'}|_{D_m} \hookrightarrow S'$ is a Cartier divisor. We can thus write $\widetilde{D'} = \widetilde{D'}|_{D_m} + R'$. Then $R' \hookrightarrow \widetilde{S}' \to \widetilde{S}$ has rank strictly smaller than m. Indeed, to show this we can assume that S is the spectrum of a strictly henselian local ring. Then $\widetilde{D'} = \widetilde{D'_1} \amalg \widetilde{D'_2} \amalg \dots \amalg \widetilde{D'_m}$ where $\widetilde{D'_i} \hookrightarrow \widetilde{S}$ are closed subschemes. Moreover $D_m = \bigcap_{i=1}^m \widetilde{D'_i}$ and $\widetilde{D'_i} = D_m + R'_i$. But then $\bigcap_{i=1}^m R'_i = \emptyset$ and hence $R' = \coprod_{i=1}^m R'_i \to \widetilde{S}$ has rank strictly smaller than m.

By induction we have, after replacing $(\widetilde{S}, U)$ with a blow-up and S' with the total transform, Cartier divisors $D_1, D_2, \dots, D_{m-1}$ such that R'

can be written as a sum of connected components of the $D_1 \times_S S', D_2 \times_S S', \dots, D_{m-1} \times_S S'$ and the result follows. $\square$

We can now prove the analogue of Proposition (4.14).

Proposition (9.7). *Let $f\colon (S',U') \to (S,U)$ be a cartesian morphism of pseudo-noetherian stacks. Let $(X',U') \to (S',U')$ be a tame (DM) stacky blow-up. Assume that one of the following holds:*

- (a) *f is a closed immersion.*
- (b) *f is representable and étale and S is quasi-Deligne–Mumford.*
- (b′) *f is representable, étale and $f|_{X\setminus U}$ is finite of constant rank d.*
- (b″) *f is representable, étale and $f|_{X\setminus U}$ is an isomorphism.*
- (c) *f is representable and unramified and S is quasi-Deligne–Mumford.*

Then there exists a tame (DM) stacky blow-up $(X,U) \to (S,U)$ and a tame (DM) stacky blow-up $(X \times_S S', U') \to (X',U')$ over (S',U'). If (S,U) is strict, and (a) *or* (b″) *holds, then it can further be arranged so that $X \times_S S' = X'$.*

Proof. It is enough to show the proposition when $(X',U') \to (S',U')$ is a (DM) Kummer blow-up so that $X' = \mathrm{Bl}_{Z',r} S'$ for some closed substack $Z' \hookrightarrow S'$ and positive integer r. We can further assume that (S,U) is strict using Lemma (4.4). After replacing (S,U) with a blow-up, we can further assume that Z' is a Cartier divisor by Proposition (4.14) so that $X' = S'_{Z',r}$ is a root stack.

In case (a) the proof of Lemma (4.5) gives an extension of $Z' \hookrightarrow S'$ to a finitely presented closed substack $Z \hookrightarrow S$, so after a blow-up we can assume that $Z' \hookrightarrow S'$ extends to a Cartier divisor $Z \hookrightarrow S$. If we let $X = S_{Z,r}$ we are then done.

In case (b″) we let $Z \hookrightarrow S$ be the image of $Z' \hookrightarrow S' \to S$. Then $Z' = Z \times_S S'$ and $X = S_{Z,r}$ is the required root stack.

In case (b′) we use Lemma (9.6) and conclude that after replacing S with a blow-up there are Cartier divisors $D_1, D_2, \dots, D_m \hookrightarrow Z$ such that Z' is a sum of connected components of the $D_i \times_S S'$. We thus have a morphism $(S_{(D_i,r)} \times_S S', U') \to (X',U')$. This morphism is the composition of a finite modification followed by a tame (DM) stacky blow-up, cf. Lemma (7.7). We can thus, by Proposition (4.14) find a blow-up of $(X,U) \to (S_{(D_i,r)}, U)$ such that $(X \times_S S', U') \to (X',U')$ is a tame (DM) stacky blow-up and we are done in this case.

Cases (b) and (c) now follow from Lemmas (9.5) and (9.4), the cases above and Proposition (7.9). $\square$

The following preliminary result will not be used in the sequel. Using the tame étalification theorem, we will instead prove a stronger result for stacks with tame but not necessarily abelian stabilizers, cf. Section 12.

Theorem (9.8). *Let (S,U) be a Deligne–Mumford stackpair and let $f\colon (X,U) \to (S,U)$ be a stacky modification such that the relative stabilizers of f are locally diagonalizable (resp. tame, abelian and locally constant). Then there exists a tame stacky blow-up (resp. a tame DM stacky blow-up) $(X',U) \to$*

(X,U) such that the composition $(X',U) \to (X,U) \to (S,U)$ is a tame stacky blow-up (resp. a tame DM stacky blow-up).

Proof. After replacing (S,U) with a blow-up, we may assume that f is flat. Then by Lemma (8.5) there is a representable étale surjective cartesian morphism $(S',U') \to (S,U)$ and a collection (D'_i, r_i) where $D'_1, D'_2, \dots, D'_n \in \mathrm{Div}(S',U')$ and $r_i \in \mathbb{Z}_+$ such that $X \times_S S' = S'_{(D'_i,r_i)}$. In particular, we have that $(X \times_S S', U') \to (S',U')$ is a tame (DM) stacky blow-up and the theorem follows by Proposition (9.7). $\square$

For completeness we also mention the following more restrictive class of tame stacky blow-ups that will not be used in the sequel.

Definition (9.9). A stacky modification $(X,U) \to (S,U)$ is a *normalized tame (DM) stacky blow-up* if it factors as $(X,U) \to (S_2,U) \to (S_1,U) \to (S,U)$ where

- $(S_1,U) \to (S,U)$ is a blow-up,
- $(S_2,U) \to (S_1,U)$ is a (DM) root stack given by a collection (D_i, r_i) where $D_1, D_2, \dots, D_n \in \mathrm{Div}(S_1,U)$.
- $(X,U) \to (S_2,U)$ is a blow-up that is finite.

A composition of two normalized tame stacky blow-ups is not necessarily a normalized tame stacky blow-up. However, using Lemmas (8.5) and (9.6) together with an étale dévissage similar as in the proof of Proposition (7.9) it can be shown that if $(X,U) \to (S,U)$ is a tame (DM) stacky blow-up then there exists a normalized tame (DM) stacky blow-up $(X',U) \to (X,U)$ such that the composition $(X',U) \to (S,U)$ is a normalized tame (DM) stacky blow-up.

We will need the following limit result for stacky blow-ups.

Lemma (9.10). *Let (S,U) be a quasi-Deligne–Mumford stackpair and let $(S',U') = \varprojlim_\lambda (S_\lambda, U_\lambda)$ be an inverse limit of affine cartesian morphisms $(S_\lambda, U_\lambda) \to (S,U)$. Let $\pi' \colon (X',U') \to (S',U')$ be a tame (DM) stacky blow-up. Then there exists an index α and a tame (DM) stacky blow-up $\pi_\alpha \colon (X_\alpha, U_\alpha) \to (S_\alpha, U_\alpha)$ such that the strict transform of π_α along $(S',U') \to (S_\alpha, U_\alpha)$ is π'. Moreover, if we let $\pi_\lambda \colon (X_\lambda, U_\lambda) \to (S_\lambda, U_\lambda)$ denote the strict transform of π_α then $(X',U') = \varprojlim_\lambda (X_\lambda, U_\lambda)$.*

Proof. It is enough to prove the lemma for the case where π' is a (DM) Kummer blow-up, i.e., $X' = \mathrm{Bl}_{Z',r} S'$ for some finitely presented closed substack $Z' \hookrightarrow S'$ disjoint from U'. We can then find a index α and a finitely presented closed substack $Z_\alpha \hookrightarrow S_\alpha$ disjoint from U_α. Letting π_λ be the (DM) Kummer blow-up in (Z_α, r) answers the first part.

To prove the second claim let $X'_\lambda = X_\lambda \times_{S_\lambda} S'$. We have to show that the inverse system $X' \hookrightarrow X'_\alpha$ of closed immersions have limit $X' \hookrightarrow X'_\alpha$. But X'_α is pseudo-noetherian so $X' \hookrightarrow X'_\alpha$ is the inverse limit of finitely presented closed immersions $X'_\gamma \hookrightarrow X'_\alpha$ that are isomorphisms over U'. For every γ and sufficiently large λ, we then have a closed immersion $X'_{\gamma\lambda} \hookrightarrow X_\alpha \times_{S_\alpha} S_\lambda$ that pull-backs to $X'_\gamma \hookrightarrow X'_\alpha$ and is an isomorphism over U_λ. As X_λ is the closure of U_λ in $X_\alpha \times_S S_\lambda$, it follows that the X'_λ's are cofinal among the X'_γ's. $\square$

10. Tame ramification

We start this section with some basic facts from valuation theory, in particular on extensions of valuations. Then we define tamely ramified extensions and morphisms and give a criterion via tame DM stacky blow-ups, cf. Proposition (11.7).

Let V be a valuation ring. By $K(V)$, $\kappa(V)$, $\mathfrak{m}_V$ and Γ_V we denote the fraction field, the residue field, the maximal ideal and the value group of V. For any ring A, we denote the group of invertible elements by A^*. Recall that the value group Γ_V is the abelian group $K(V)^*/V^*$ with the total order given by $\overline{f} \geq \overline{g}$ if $f, g \in K(V)^*$ and $f/g \in V$. Note that Γ_V is torsion-free and that Γ_V is the group of Cartier divisors on $\operatorname{Spec}(V)$. A small warning: V is often not noetherian so Krull's Hauptidealsatz do not apply. This means that Cartier divisors on $\operatorname{Spec}(V)$ often have codimension > 1!

The *valuation* of $K(V)$ corresponding to V is the homomorphism $v\colon K(V)^* \to \Gamma_V$ and the *value* of an element is the image of this map. Note that $v(f) = 0$ if and only if $f \in V^*$ and $v(f) \geq 0$ if and only if $f \in V$.

Valuation rings are Bézout domains, i.e., every finitely generated ideal is principal. It follows that a module over a valuation ring is flat if and only if it is torsion-free. We will also use that a finitely generated and flat algebra over an integral domain is finitely presented. This is a non-trivial result due to Raynaud–Gruson [RG71, Cor. 3.4.7].

A homomorphism of valuation rings $V \to W$ is an *extension* if $V \to W$ is injective and local, or equivalently, if $V \to W$ is faithfully flat. For such an extension we have field extensions $\kappa(V) \hookrightarrow \kappa(W)$ and $K(V) \hookrightarrow K(W)$ and a group homomorphism $\Gamma_V \to \Gamma_W$. Moreover, by faithful flatness $V = W \cap K(V)$ and $V^* = W^* \cap K(V)$ so that $\Gamma_V \to \Gamma_W$ is injective.

Let V be a valuation ring, let $K'/K(V)$ be an algebraic field extension and let $V' = \operatorname{norm}_{K'} V$ be the integral closure of V in K'. Then every local ring of V' is a valuation ring and there is a one-to-one correspondence between maximal ideals of V' and valuations of K' extending V [Bou64, Ch. VI, §8, Prop. 6]. If $K'/K(V)$ is finite, then there is a finite number of maximal ideals of V' [Bou64, Ch. VI, §8, Thm. 1].

A valuation ring V is henselian if and only if the integral closure of V in any finite field extension $K'/K(V)$ is a local ring, i.e., if and only if there exists a unique valuation of K' extending V. If V is a valuation ring, then the henselization ${}^{\mathrm{h}}V$ and the strict henselization ${}^{\mathrm{sh}}V$ are valuation rings. The induced group homomorphisms $\Gamma(V) \hookrightarrow \Gamma({}^{\mathrm{h}}V) \hookrightarrow \Gamma({}^{\mathrm{sh}}V)$ are isomorphisms.

Let $V \hookrightarrow W$ be an extension of valuation rings. We use the standard notation

$$e = e(W/V) = |\Gamma_W/\Gamma_V|$$
$$f = f(W/V) = [\kappa(W) : \kappa(V)]$$

(these cardinals may be infinite). It is clear that $e(W/V) = e({}^{\mathrm{h}}W/{}^{\mathrm{h}}V) = e({}^{\mathrm{sh}}W/{}^{\mathrm{sh}}V)$ and $f(W/V) = f({}^{\mathrm{h}}W/{}^{\mathrm{h}}V)$. If the generic degree $n = [K(W) : K(V)]$ is finite, then $ef \leq n$ [Bou64, Ch. VI, §8, Thm. 1]. If n is finite and W is the only valuation of $K(W)$ extending V, then $ef | n$ and $d = n/ef$ is the *defect* of W/V [Bou64, Ch. VI, §8, Exer. 9a)]. For W/V such that

$[K(^{\mathrm{sh}}W) : K(^{\mathrm{sh}}V)]$ is finite we define the defect as

$$d = d(W/V) = [K(^{\mathrm{sh}}W) : K(^{\mathrm{sh}}V)]/ef.$$

The defect is always a power of p, the exponential characteristic of the residue field $\kappa(V)$. We say that W/V is *defectless* if $d = 1$. We note that $d(W/V) = d(^{\mathrm{h}}W/^{\mathrm{h}}V) = d(^{\mathrm{sh}}W/^{\mathrm{sh}}V)$. If $K'/K(V)$ is a finite field extension and $\{W_1, W_2, \dots, W_m\}$ are the valuations of K' extending V then

$$\sum_i d(W_i/V)e(W_i/V)f(W_i/V) = n.$$

Definition (10.1). An extension of valuation rings $V \hookrightarrow W$ is *unramified* (resp. *tamely ramified*) if

(i) $K(^{\mathrm{sh}}W)/K(^{\mathrm{sh}}V)$ is a finite field extension.
(ii) $K(W)/K(V)$ and $\kappa(W)/\kappa(V)$ are separable.
(iii) $e = 1$ (resp. e is prime to the exponential characteristic of $\kappa(V)$).
(iv) The extension is defectless.

Let P be either the property "unramified" or the property "tamely ramified". From the above considerations it follows that W/V is P if and only if $^{\mathrm{h}}W/^{\mathrm{h}}V$ is P and if and only if $^{\mathrm{sh}}W/^{\mathrm{sh}}V$ is P. In particular, W/V is unramified if and only if $^{\mathrm{sh}}W = {}^{\mathrm{sh}}V$, or equivalently, if and only if $V \hookrightarrow W$ is ind-étale (i.e., a limit of étale extensions). It follows that if W/V is unramified and $K(W)/K(V)$ is finite, then W/V is essentially étale (i.e., a localization of an étale extension).

It is also not difficult to see that if V is henselian, then W/V is tamely ramified if and only if W/V is a composition of an unramified extension W_1/V and an extension W/W_1 such that $[K(W) : K(W_1)]$ is finite and prime to the residue field characteristic.

For a field K, let $\mu(K) \subset K^*$ denote the torsion subgroup.

Proposition (10.2). *Let W/V be a tamely ramified extension of valuation rings. Choose a decomposition $\Gamma_W/\Gamma_V \cong \mathbb{Z}/q_1\mathbb{Z} \oplus \mathbb{Z}/q_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/q_k\mathbb{Z}$ and positive elements $(\gamma_i) \in (\Gamma_W)^k$ such that the image of γ_i is a generator of the i^{th} factor of Γ_W/Γ_V.*

(i) *Assume that V is strictly henselian so that $d = f = 1$ and $n = e$ is prime to p. Given elements $a_1, a_2, \dots, a_k \in V$ such that $v(a_i) = q_i\gamma_i$ in Γ_W, we then have that*

$$K(W) \cong K(V)[a_1^{1/q_1}, \dots, a_k^{1/q_k}].$$

Moreover, $K(W)/K(V)$ is Galois and there is a natural isomorphism

$$\mathrm{Gal}\big(K(W)/K(V)\big) \to \mathrm{Hom}_{\mathbb{Z}}\Big(\Gamma_W/\Gamma_V, \mu\big(\kappa(V)\big)\Big).$$

(ii) *Let E_i be the effective Cartier divisor on $\mathrm{Spec}(W)$ corresponding to γ_i so that q_iE_i is the inverse image of the effective Cartier divisor D_i on $\mathrm{Spec}(V)$ corresponding to $q_i\gamma_i \in \Gamma_V$. The induced morphism of stacks*

$$\mathrm{Spec}(W) \to \mathrm{norm}\left(\mathrm{Spec}(V)_{(D_i,q_i)}\right)$$

is essentially étale.

Proof. (i) Choose elements $b_1, b_2, \dots, b_k \in W$ with values $\gamma_1, \gamma_2, \dots, \gamma_k \in \Gamma_W$. Since $n = e$, we have that $K(W)$ is generated by $b_1, b_2, \dots, b_k$ over $K(V)$. Moreover, we have that $b_i^{q_i} = u_i a_i$ for some invertible elements $u_1, u_2, \dots, u_k \in W^*$. Since $p \nmid q_i$ and W is strictly henselian, we have that $u_i = v_i^{q_i}$ for some $v_1, v_2, \dots, v_k \in W$. Thus, we have that $(b_i v_i^{-1})^{q_i} = a_i$ and hence an induced isomorphism

$$K(V)[a_1^{1/q_1}, \dots, a_k^{1/q_k}] \to K(W).$$

From this description it is clear that $K(W)/K(V)$ is Galois with Galois group isomorphic to $\mathbb{Z}/q_1\mathbb{Z} \oplus \mathbb{Z}/q_2\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/q_k\mathbb{Z}$ and the second statement follows from [GR03, Cor. 6.2.14] (this is also easily to verify directly).

(ii) We have a finite étale presentation

$$\operatorname{norm}\Big(\operatorname{Spec}(V[a_1^{1/q_1}, \dots, a_k^{1/q_k}])\Big) \to \operatorname{norm}\big(\operatorname{Spec}(V)_{(D_i,q_i)}\big)$$

since normalization commutes with étale morphisms. As taking root stacks and normalizing commute with the essentially étale base change $V \to {}^{\mathrm{sh}}V$, we can assume that V is strictly henselian. Then W is the normalization of $V[a_1^{1/q_1}, \dots, a_k^{1/q_k}]$ by (i) so that W is a finite étale presentation of the normalization of the root stack. $\square$

Definition (10.3). We say that a valuation ring V is *Kummer* if it is strictly henselian and $\Gamma_V \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ is divisible.

Every valuation ring V has a unique (up to non-unique isomorphism) extension $V \hookrightarrow {}^{\mathrm{t}}V$, where ${}^{\mathrm{t}}V$ is the maximal tamely ramified extension, a Kummer valuation ring [GR03, Def. 6.2.16]. We have a factorization $V \hookrightarrow {}^{\mathrm{sh}}V \hookrightarrow {}^{\mathrm{t}}V$.

The valuative criteria for separated and universally closed morphisms are about the uniqueness and existence of liftings of valuation rings. We will now define tamely ramified extensions by a similar valuative criterion.

Lemma (10.4). *Let $S = \operatorname{Spec}(V)$ be the spectrum of a valuation ring with generic point η and closed points s. Let $f\colon X \to S$ be a morphism of schemes such that $f^{-1}(\eta) \to \operatorname{Spec}(k(\eta))$ is étale. Let $x \in |X_s|$ be a point and let $g\colon X' \to X$ be the normalization of X in $f^{-1}(\eta)$. The following are equivalent*

(i) *For every $\xi \in f^{-1}(\eta)$ and every valuation ring $W \subseteq k(\xi)$ centered on x, the extension W/V is tamely ramified.*
(ii) *For every $x' \in g^{-1}(x)$ there exists a tamely ramified extension W/V of valuation rings and an S-morphism $\operatorname{Spec}(W) \to X'$ taking the closed point to x'.*
(iii) *For every $x' \in g^{-1}(x)$ there is an S-morphism $\operatorname{Spec}({}^{\mathrm{t}}V) \to X'$ taking the closed point to x'.*

Proof. Clearly (i) $\implies$ (ii) $\implies$ (iii). That (iii) $\implies$ (i) follows immediately from the following two facts. If $W'/W/V$ are extensions of valuation rings and W'/V is tamely ramified, then so is W/V. The local rings $\mathcal{O}_{X',x'}$ are the valuation rings of $k(\xi)$ centered on x. $\square$

When the equivalent conditions of Lemma (10.4) hold, then we say that f is *tamely ramified* at x. Let $p\colon Z \to X$ be an étale morphism and let

$z \in |Z|$ be a point above x. Then f is tamely ramified at x if and only if $f \circ p$ is tamely ramified at z. We can thus extend the definition of tamely ramified points to the case where X is a Deligne–Mumford stack. Also note that criteria (ii) and (iii) applies to X a Deligne–Mumford stack and is equivalent to the étale-local definition. We now define tamely ramified over general bases:

Definition (10.5). Let $f\colon X \to S$ be a morphism of algebraic stacks that is relatively Deligne–Mumford (i.e., the diagonal is unramified). Let $U \subseteq S$ be an open substack such that $f|_U$ is étale. Let $x \in |X|$ be a point.

(i) We say that f is *tamely ramified outside U at x* if for every valuation ring V and morphism $g\colon \operatorname{Spec}(V) \to S$ mapping the closed point to $f(x)$ and such that the image of g meets U, we have that $X \times_S \operatorname{Spec}(V) \to \operatorname{Spec}(V)$ is tamely ramified at all points above x.
(ii) We say that f is *tamely ramified outside U* if f is tamely ramified at every point.

Let $f\colon (X, V) \to (S, U)$ be a Deligne–Mumford morphism of stackpairs. We say that f is tamely ramified if f is cartesian, if $f|_V\colon V \to U$ is étale and f is tamely ramified outside U.

Remark **(10.6).** We have that $f\colon X \to S$ is tamely ramified outside U if and only if for every Kummer valuation ring V and morphism $g\colon \operatorname{Spec}(V) \to S$ such that the image of g meets U, there are "sufficiently many liftings" to X.

Remark **(10.7).** Let (S, U) be a stackpair and let $f\colon (X, X|_U) \to (Y, Y|_U)$ and $g\colon (Y, Y|_U) \to (S, U)$ be cartesian morphisms that are relatively Deligne–Mumford.

(i) If f and g are tamely ramified then so is $g \circ f$.
(ii) If f is surjective and $g \circ f$ is tamely ramified then so is f.
(iii) Every modification $(S', U) \to (S, U)$ is tamely ramified.
(iv) Let $h\colon (S', U') \to (S, U)$ be a cartesian morphism. If g is tamely ramified, then so is the base change $g'\colon (Y', Y|_{U'}) \to (S', U')$. If h is surjective and tamely ramified (e.g., étale or a modification), then the converse holds.

Lemma (10.8). *Let $f\colon (X, U) \to (S, U)$ be a tame modification of Deligne–Mumford stackpairs. Then f is tamely ramified. In particular, any tame DM stacky blow-up is tamely ramified.*

Proof. We can assume that S is the spectrum of a Kummer valuation ring V. In particular, we have that f is flat and of finite presentation and that S is strictly henselian. Then S is the coarse moduli space of X. As S is strictly henselian we have a lifting $x\colon \operatorname{Spec}(k(s)) \to X$. Let G be the stabilizer group of x. As S is strictly henselian we can find a scheme Z with an action of G such that $X = [Z/G]$. Then $Z \to S$ is finite and of degree $|G|$ over U. As f is tame we have that $|G|$ is prime to the characteristic of $k(s)$ so that $Z \to S$ is tamely ramified. It follows that f is tamely ramified. $\square$

11. Tame étalification by stacky blow-ups

Let X be a quasi-compact Deligne–Mumford stack, let $U \subseteq X$ be an open quasi-compact subset and let $E_U \to U$ be a finite étale covering. We would like to know when we can extend E_U to a finite étale cover of X. We will show that if $E_U \to U$ is tamely ramified, e.g., if X has characteristic zero, then this is possible after a U-admissible stacky blow-up on X (Corollary D).

Let us first make some initial remarks:

(i) If X is *normal* and $E \to X$ is an étale covering, then E is normal. Thus, there is a single candidate for the extension — the normalization of X in E_U. If X is not normal, then in general the extension is not unique but any two extensions coincide after a U-admissible blow-up, cf. Corollary (5.2).

(ii) For arbitrary X and $E_U \to U$ finite and étale, if $\operatorname{Spec}(D) \to X$ is a DVR whose generic point maps to U, then the étale cover $E_U \to U$ comes with a ramification index over D (1 if D is centered on U). This is the obstruction to extending the étale cover over the closed point of $\operatorname{Spec}(D)$: if $\mathscr{D} \to \operatorname{Spec}(D)$ is an r^{th} root of the closed point, then there is extension of E_U over $\mathscr{D}$ if and only if r is a multiple of the ramification index.

The étalification theorem is motivated by the following stacky version of Abhyankar's lemma based upon the previous remarks, cf. [SGA$_1$, XIII], [GM71], [Bor09, Prop. 3.2.2], [LO10, App. A] and [KL10, §3].

Theorem (11.1) (Generalized Abhyankar Lemma). *Let X be a regular scheme and let $U \subseteq X$ be an open subscheme such that the complement $D = (X \setminus U)_{\text{red}} = \sum_{i=1}^{n} D_i$ is a simple normal crossings divisor. Let Z be a normal scheme and $Z \to X$ a finite morphism such that $Z|_U \to U$ is étale. For every codimension one point $z \in Z$ we let $e(z)$ be the ramification index. Assume that $Z \to X$ is tamely ramified, i.e., that for every $i = 1, 2, \dots, n$ and point z above D_i*

(i) *The ramification index $e(z)$ is invertible over D_i.*

(ii) *The field extension $k(z)/K(D_i)$ is separable.*

Choose positive integers r_i such that $e(z) | r_i$ for every z above D_i. Then $\operatorname{norm}\big(Z \times_X X_{(D_i,r_i)}\big)$ *is étale over the regular scheme $X_{(D_i,r_i)}$.*

Proof. Follows immediately from the above remarks and the Zariski–Nagata purity of branch locus. □

Remark **(11.2).** *Tame G-torsors* — Theorem (11.1) can be partly generalized to non-étale coverings as follows. Let $G \to X$ be a tame group scheme (e.g., μ_p) and let $E_U \to U$ be a $G|_U$-torsor. Then there exists integers (r_i) such that $E_U \to U$ extends to a G-torsor over the root stack $X_{(D_i,r_i)}$ [Ols09]. When D has non-simple normal crossings, there is a similar canonical construction which étale-locally is a root stack [MO05, Ols09].

Remark **(11.3).** *Alterations* — Note that it is trivial that E_U extends after an alteration of X. Indeed, we have the following stronger result (similar to the flatification of proper morphisms using Hilbert schemes). Let $E \to X$ be a finite morphism compactifying $E_U \to U$ as exists by Zariski's main theorem. Consider the d^{th} stacky symmetric product $[\mathrm{Sym}^d(E/X)]$ which parameterizes finite étale families $Z \to T$ of rank d together with a map $Z \to E$, cf. [Ryd10, §5]. The family $E_U \to U$ induces a quasi-finite morphism $U \to [\mathrm{Sym}^d(E/X)]$ and Zariski's main theorem gives a stacky U-admissible modification $\widetilde{X} := \overline{U} \to [\mathrm{Sym}^d(E/X)] \to X$. By the universal property we have a finite étale cover $\widetilde{E} \to \widetilde{X}$ which restricts to $E_U \to U$. To obtain a generically étale alteration instead of a stacky modification, we replace $\widetilde{X}$ with the pull-back of the étale presentation $(E/X)^d \to [\mathrm{Sym}^d(E/X)]$.

Exactly as in the flatification theorem, the merit of the étalification theorem is not mainly that we can handle the non-proper case but that the solution is given by a *stacky blow-up* $\widetilde{X} \to X$ and not merely a proper birational morphism.

Remark **(11.4).** If we have strong resolution of singularities, e.g., if X is of characteristic zero, and U is regular, then we can first resolve (X, U) by a blow-up $(\widetilde{X}, U) \to (X, U)$ so that $\widetilde{X}$ is regular and $\widetilde{X} \setminus U$ is a simple normal crossings divisor. We can then apply the generalized Abhyankar lemma as above. When U is not regular it seems difficult to reduce to the regular case. Using valuation theory, we do not need resolution of singularities and can both treat the singular case and the tame case in positive characteristic.

From now on we will leave the proper case and prove Theorem C. We begin with showing that the theorem is étale-local, that it behaves well under limits and that it can be solved when the base is the spectrum of a valuation ring. The theorem is then proved using Riemann–Zariski spaces.

Lemma (11.5). *Let $f\colon (X, W) \to (S, U)$ be a cartesian morphism of finite type between quasi-Deligne–Mumford stackpairs such that $f|_U$ is étale. Let*

$$\begin{tikzcd}
(X', W') \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, "\circ"] & (X, W) \arrow[d, "f"] \\
(S', U') \arrow[r] & (S, U)
\end{tikzcd}$$

be a commutative diagram where the horizontal morphisms are representable, étale, surjective and cartesian. Assume that there exists a tame (DM) stacky blow-up $(\widetilde{S'}, U') \to (S', U')$ and a blow-up $(\widetilde{X'}, W') \to (X' \times_{S'} \widetilde{S'}, W')$ such that $\widetilde{X'} \to \widetilde{S'}$ is étale. Then there exists a tame (DM) stacky blow-up $(\widetilde{S}, U) \to (S, U)$ and a blow-up $(\widetilde{X}, W) \to (X \times_S \widetilde{S}, W)$ such that $\widetilde{X} \to \widetilde{S}$ is étale.

Proof. By Proposition (9.7), there is a tame (DM) stacky blow-up $(\widetilde{S}, U) \to (S, U)$ and stacky blow-up $(\widetilde{S} \times_S S', U') \to (\widetilde{S'}, U')$. After replacing (S, U) with $(\widetilde{S}, U)$ we can thus assume that $\widetilde{S'} = S'$.

By Proposition (4.14) there is then a blow-up $(\widetilde{X}, W) \to (X, W)$ and a blow-up $(\widetilde{X} \times_S S', W') \to (\widetilde{X'}, W)$. Since $\widetilde{X'} \to S' \to S$ is étale, we can by the same proposition find a blow-up $(\widetilde{S}, U) \to (S, U)$ such that after passing to strict transforms, we have that the blow-up $(\widetilde{X} \times_S S', W') \to (\widetilde{X'}, W)$ is an isomorphism so that $\widetilde{X} \to S$ is étale. $\square$

Lemma (11.6). *Let $f\colon (X,W)\to(S,U)$ be a cartesian morphism of finite type between quasi-Deligne–Mumford stackpairs. Let $S'=\varprojlim_\lambda S_\lambda$ be an inverse limit of affine morphisms $S_\lambda\to S$ and let $U_\lambda=U\times_S S_\lambda$, $U'=U\times_S S'$, $(X_\lambda,W_\lambda)=(X,W)\times_{(S,U)}(S_\lambda,U_\lambda)$ and $(X',W')=(X,W)\times_{(S,U)}(S',U')$. Assume that there exists a tame (DM) stacky blow-up $(\widetilde{S'},U')\to(S',U')$ and a blow-up $(\widetilde{X'},W')\to(X'\times_{S'}\widetilde{S'},W')$ such that $\widetilde{X'}\to\widetilde{S'}$ is étale. Then for every sufficiently large λ, there exists a tame (DM) stacky blow-up $(\widetilde{S_\lambda},U)\to(S_\lambda,U)$ and a blow-up $(\widetilde{X_\lambda},W)\to(X_\lambda\times_{S_\lambda}\widetilde{S_\lambda},W)$ such that $\widetilde{X_\lambda}\to\widetilde{S_\lambda}$ is étale.*

Proof. First approximate the stacky blow-up $(\widetilde{S'},U')\to(S',U')$ to a stacky blow-up $(\widetilde{S_\alpha},U_\alpha)\to(S_\alpha,U_\alpha)$ for some index α and then for every $\lambda\geq\alpha$ let $(\widetilde{S_\lambda},U_\lambda)\to(S_\lambda,U_\lambda)$ denote the strict transform so that $\widetilde{S'}=\varprojlim_\lambda\widetilde{S_\lambda}$, cf. Lemma (9.10). We can then replace S, S_λ and S' with $\widetilde{S_\alpha}$, $\widetilde{S_\lambda}$ and $\widetilde{S'}$ respectively and assume that $\widetilde{S'}=S'$.

Next, approximate the étale morphism $\widetilde{X'}\to S'$ with an étale morphism $\widetilde{X_\lambda}\to S_\lambda$. After increasing λ we can also assume that $(\widetilde{X_\lambda},W_\lambda)\to(X_\lambda,W_\lambda)$ is a (finite) modification. There is a blow-up $(Y_\lambda,W_\lambda)\to(\widetilde{X_\lambda},W_\lambda)$ such that the composition $(Y_\lambda,W_\lambda)\to(X_\lambda,W_\lambda)$ is a blow-up. By Proposition (4.14), we can replace (S_λ,U_λ) with a blow-up so that $Y_\lambda\to\widetilde{X_\lambda}$ becomes an isomorphism after strict transforms and we are done. $\square$

Proposition (11.7). *Let (S,U) be a stackpair where $S=\operatorname{Spec}(V)$ is the spectrum of a valuation ring V with closed point s. Let $f\colon(X,W)\to(S,U)$ be a finite cartesian morphism such that $f|_U$ is étale. Then f is tamely ramified if and only if there exists a commutative diagram*

$$\begin{tikzcd}
(\widetilde{X},W) \arrow[r, "h"] \arrow[dr, "\widetilde{f}"'] & (X\times_V\widetilde{S},W) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (X,W) \arrow[d, "f"] \\
& (\widetilde{S},U) \arrow[r, "g"] & (S,U)
\end{tikzcd}$$

where g is a tame DM stacky blow-up, h is a blow-up and $\widetilde{f}$ is finite and étale.

Proof. The sufficiency follows from Lemma (10.8). To see the necessity, let $X'\to X$ be the normalization of X in W and let $x'_1,x'_2,\dots,x'_k\in X'$ be the closed points of X'. By assumption, the valuation ring $\mathcal{O}_{X',x'_i}$ is a tamely ramified extension of V for every $i=1,\dots,k$. For each i we can thus, by Proposition (10.2) choose divisors $D_{i1},D_{i2},\dots,D_{in_i}\in\operatorname{Div}(S,U)$ and positive integers $r_{i1},r_{i2},\dots,r_{in_i}$ such that there is an essentially étale morphism $\operatorname{Spec}(\mathcal{O}_{X',x'_i})\to\operatorname{norm}S_{(D_{ij},r_{ij})_j}$ for every i. Let $\widetilde{S}=\operatorname{norm}S_{(D_{ij},r_{ij})_{ij}}$. As $\operatorname{norm}\Bigl(\operatorname{Spec}(\mathcal{O}_{X',x'_i})_{(D_{ij},r_{ij})_j}\Bigr)\to\operatorname{Spec}(\mathcal{O}_{X',x'_i})$ is an isomorphism, we then have that $\widetilde{X}:=\operatorname{norm}(X'\times_S\widetilde{S})\to\widetilde{S}$ is finite étale.

We have that $(\widetilde{S},U)$ is the inverse limit of all finite modifications $(\widetilde{S}_\lambda,U)\to(S_{(D_{ij},r_{ij})_{ij}},U)$. Every modification of $(S_{(D_{ij},r_{ij})_{ij}},U)$ is finite since every modification is quasi-finite over S. As blow-ups are cofinal among modifications, we can write $(\widetilde{S}, U)$ as the inverse limit of blow-ups $(\widetilde{S}_\lambda, U) \to (S, U)$. The proposition thus follows by Lemma (11.6). $\square$

Proof of Theorem C. By Lemma (11.5), the theorem is étale-local on S and X so we can assume that S is an (affine) scheme. After flatification, we can also assume that $f\colon X \to S$ is flat, quasi-finite and of finite presentation. After further étale-localization, using [$\mathrm{EGA}_{\mathrm{IV}}$, Thm. 18.12.1], we can assume that f is also finite.

Now consider the Riemann–Zariski space RZ of (S, U) and let (V, f) be a point of RZ. Recall that the local ring A of RZ at (V, f) is given by the bi-cartesian diagram

$$\begin{tikzcd} k(u) & \mathcal{O}_{U,u} \arrow[l, two heads] \\ V \arrow[u, hook] & A. \arrow[l, two heads] \arrow[u, hook] \end{tikzcd}$$

By assumption, the morphism $X \times_S \operatorname{Spec}(V) \to \operatorname{Spec}(V)$ is tamely ramified. By Proposition (11.7), we can thus find a stacky blow-up $\widetilde{Y_V} \to \operatorname{Spec}(V)$ and a blow-up $\widetilde{X_V} \to X \times_S \widetilde{Y_V}$ such that $\widetilde{X_V} \to \widetilde{Y_V}$ is étale.

By Corollary (C.15), we can extend the stacky blow-up to a stacky blow-up $\widetilde{Y_A} \to \operatorname{Spec}(A)$ and the blow-up to a blow-up $\widetilde{X_A} \to X \times_S \widetilde{Y_A}$. Then $\widetilde{X_V} \to \widetilde{Y_V}$ is étale by Proposition (C.9).

The local rings of the Riemann–Zariski space are inverse limits of local rings of modifications (or equivalently, of blow-ups) so using the quasi-compactness of the Riemann–Zariski space and Lemma (11.6) we obtain a blow-up $(S', U) \to (S, U)$, an open covering $S'' \to S'$, a stacky blow-up $(\widetilde{S''}, U'') \to (S'', U'')$ and a blow-up $(\widetilde{X''}, W'') \to (X \times_S \widetilde{S''})$. Finally we use Lemma (11.5) to eliminate the open covering $S'' \to S'$. $\square$

12. Applications to tame étalification

In this section we give some applications to the tame étalification theorem. We begin with proving Corollaries D (extension of finite étale covers) and E (cofinality of stacky blow-ups).

Proof of Corollary D. Recall that we are given a Deligne–Mumford stack-pair (X, U) and a finite étale cover $E_U \to U$ tamely ramified over $X \setminus U$. Choose a finite compactification $E \to X$ so that $(E, E_U) \to (X, U)$ is tamely ramified. Then by the tame étalification theorem, there exists a stacky blow-up $(\widetilde{X}, U) \to (X, U)$ and a blow-up $(\widetilde{E}, E_U) \to (E \times_X \widetilde{X}, E_U)$ such that $\widetilde{E} \to X$ is étale. $\square$

Proof of Corollary E. Recall that we are given a stacky tame (DM) modification $f\colon (X, U) \to (S, U)$. By the tame étalification theorem there is a stacky blow-up $(\widetilde{S}, U) \to (S, U)$ and a blow-up $(\widetilde{X}, U) \to (X \times_S \widetilde{S}, U)$ such that $\widetilde{f}\colon \widetilde{X} \to \widetilde{S}$ is étale. Then $\widetilde{f}$ is an isomorphism so that $(\widetilde{X}, U) \to (X, U) \to (S, U)$ is a stacky blow-up. $\square$

Corollary (12.1) (Extension of tame étale group schemes)**.** *Let (X,U) be a Deligne–Mumford stackpair. Let $G_U \to U$ be a finite étale and tame group scheme. Then there exists a stacky blow-up $(\widetilde{X},U) \to (X,U)$ such that $G_U \to U$ extends to a finite étale and tame group scheme $\widetilde{G} \to \widetilde{X}$. Similarly, if $R_U \rightrightarrows U$ is a finite étale and tame groupoid, there is a stacky blow-up and a finite étale tame groupoid $\widetilde{R} \rightrightarrows \widetilde{X}$ extending $R_U \rightrightarrows U$.*

Proof. This follows immediately from Corollary D and Corollary (5.2). □

Let $\mathbf{F\acute{E}T}(X)$ denote the category of finite étale morphisms $E \to X$ and let $\mathbf{F\acute{E}T}_{\mathrm{tame}}(X,U) \subseteq \mathbf{F\acute{E}T}(U)$ denote the category of finite étale morphisms $E_U \to U$ that are tamely ramified over $X \setminus U$.

Corollary (12.2). *Let (X,U) be a Deligne–Mumford stackpair. Then*

$$\varinjlim_{(\widetilde{X},U)} \mathbf{F\acute{E}T}(\widetilde{X}) \to \mathbf{F\acute{E}T}_{\mathrm{tame}}(X,U)$$

is an equivalence of categories. Here the limit is over all tame DM stacky blow-ups $(\widetilde{X},U) \to (X,U)$. In particular, if $u\colon \mathrm{Spec}(k) \to U$ is a point, then we have an isomorphism of pro-finite groups

$$\pi_1^{\mathrm{tame}}(U;u) \to \varprojlim_{(\widetilde{X},U)} \pi_1(\widetilde{X};u).$$

We have the following non-representable version of Corollary (5.2).

Corollary (12.3). *Let $(S',U') \to (S,U)$ be a cartesian étale morphism of Deligne–Mumford stackpairs. Let $X' \to S'$ be quasi-compact étale and let $Y' \to S'$ be proper and tame (and Deligne–Mumford). Let $f_U\colon X' \times_S U \to Y' \times_S U$ be a morphism. Then there exists a tame (DM) stacky blow-up $(\widetilde{S},U) \to (S,U)$ and an extension*[1] *$f\colon X' \times_S \widetilde{S} \to Y' \times_S \widetilde{S}$ of f_U.*

Proof. By Proposition (9.7) we can assume that $S = S'$. Consider $X|_U \to X \times_S Y \to X$. This is a quasi-finite morphism which is an isomorphism over U. Thus $X|_U \to X \times_S Y$ is quasi-finite, and finite over the inverse image of U. Zariski's Main theorem then gives a factorization $X|_U \subseteq \Gamma \to X \times_S Y$ where the first morphism is an open immersion which is an isomorphism over U and the second is a finite morphism. In particular, $(\Gamma,U) \to (X,U)$ is a tame (DM) stacky modification. By cofinality of stacky blow-ups, Corollary E, and Proposition (9.7), there is a stacky blow-up of (S,U) such that after passing to the strict transform, $\Gamma \to X$ has a section s. The composition of s with $\Gamma \to X \times_S Y \to Y$ is an extension of f_U. □

Remark **(12.4).** There is also another similar result. Let $f,g\colon X' \to Y'$ be two morphisms and $\tau_{U'}\colon f|_{U'} \to g|_{U'}$ a 2-isomorphism. Then there is a stacky blow-up such that $\tau_{U'}$ extends to a 2-isomorphism between f and g.

13. Compactification of tame Deligne–Mumford stacks

Definition (13.1). A morphism $f\colon X \to S$ of stacks is *tamely DM-compactifiable* if there exists a factorization $f = \overline{f} \circ j$ where j is a quasi-compact open immersion and $\overline{f}$ is proper, tame and Deligne–Mumford.

[1] Add something about (non-)uniqueness.

The proof of the following proposition is almost identical to Proposition (6.3) using stacky blow-ups instead of blow-ups.

Proposition (13.2). *Let $f\colon X \to S$ be a separated morphism of finite type between quasi-compact Deligne–Mumford stacks. Let $S' \to S$ be a representable, étale and surjective morphism. If $f'\colon X \times_S S' \to S'$ is tamely DM-compactifiable then so is f.*

Proof. We will use étale dévissage [Ryd11, Thm. D]. Let $\mathbf{D} \subseteq \mathbf{E} = \mathbf{Stack}_{\mathrm{qc,ét}/S}$ be the full subcategory of quasi-compact étale morphisms $T \to S$ such that $X \times_S T \to T$ has a tame Deligne–Mumford compactification. Then $(S' \to S) \in \mathbf{D}$ by hypothesis. That $\mathbf{D}$ satisfies axioms (D1) and (D2) follows exactly as in Proposition (6.3) and Lemma (6.4) with one modification. The morphism $X \times_S T \to \mathbf{R}_{T'/T}(X \times_S T')$ is not a closed immersion but only finite (and unramified but we do not need that). We thus obtain a quasi-finite representable morphism $X \times_S T \to \mathbf{R}_{T'/T}(Y)$ where $\mathbf{R}_{T'/T}(Y) \to T$ is proper, tame and Deligne–Mumford. Using Zariski's main theorem, we obtain a compactification of $X \times_S T \to T$. Axiom (D3) follows from the following lemma and we deduce that $(S \to S) \in \mathbf{D}$ and the proposition follows. □

Lemma (13.3). *Let (S, U) be a Deligne–Mumford stackpair and let $S' \to S$ be an étale neighborhood of $S \setminus U$ so that*

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d] \arrow[dr, phantom, "\square"] & S' \arrow[d] \\ U \arrow[r, "j"'] & S \end{tikzcd}$$

is a bi-cartesian square. Let $f\colon X \to S$ be a separated morphism of finite type. If the pull-backs $f'\colon X' \to S'$ and $f|_U\colon X|_U \to U$ are tamely DM-compactifiable, then so is f.

Proof. Let $f'\colon X' \hookrightarrow Y' \to S'$ and $f|_U\colon X|_U \hookrightarrow Y_U \to U$ be compactifications. We have a quasi-finite representable morphism $X'|_{U'} \to (Y_U \times_U U') \times_{U'} Y'|_{U'}$. By Zariski's main theorem, there is a factorization $X'|_{U'} \subseteq Z \to (Y_U \times_U U') \times_{U'} Y'|_{U'}$ where the first morphism is an open immersion and the second is finite. The two projections $Z \to Y_U \times_U U'$ and $Z \to Y'|_{U'}$ are thus proper and isomorphisms over the open subscheme $X'|_{U'}$.

$X \times_S U'$-admissible

By Corollary E we can replace $(Z, X'|_{U'})$ with a tame DM stacky blow-up and assume that the induced morphisms $(Z, X'|_{U'}) \to (Y_U \times_U U', X'|_{U'})$ and $(Z, X'|_{U'}) \to (Y'|_{U'}, X'|_{U'})$ tame DM stacky blow-ups. By Proposition (9.7), we can further assume that $Z = Y_U \times_U U'$ after replacing $(Z, X'|_{U'})$ and $(Y_U, X|_U)$ with tame DM stacky blow-ups. We can then extend $(Z, X'|_{U'}) \to (Y'|_{U'}, X'|_{U'})$ to a tame DM stacky blow-up of (Y', X') by Lemma (9.4) and assume that $Z = Y'|_{U'}$. We then obtain a compactification of $f\colon X \to S$ by taking the pushout $Y = Y_U \amalg_Z Y'$.

As Y_U and Y' are tame and Deligne–Mumford and $Y' \to Y$ is an isomorphism over the closed substack $Y \setminus Y_U$ it follows that $Y \to T$ is tame and Deligne–Mumford so that f is tamely DM compactifiable. □

Before proving Theorem F, let us first study various compactifications of uniformizable stacks, i.e., stacks which admit a finite étale presentation.

Lemma (13.4). *Let (S,U) be a Deligne–Mumford stackpair. Let $\pi\colon X \to U$ be a proper morphism of Deligne–Mumford stacks and let $Z \to X$ be a finite étale surjective morphism such that $Z \to U$ is finite. Let $\overline{Z} \to S$ be a compactification (e.g., as given by Zariski's main theorem). Then:*

(i) *If $X = [Z/G]$ where $G \to S$ is a constant group scheme (or constant on the connected components of S), then $X = [Z/G]$ can be compactified by a stack $[\widetilde{Z}/G]$ where $\widetilde{Z} \to \overline{Z}$ is a Z-admissible blow-up.*

(ii) *If $X = [Z/G]$ where $G \to U$ is a finite étale and tame group scheme, then there is a tame DM stacky blow-up $(\widetilde{S},U) \to (S,U)$, a finite étale and tame group scheme $\widetilde{G} \to \widetilde{S}$, a blow-up $(\widetilde{Z},Z) \to (\overline{Z} \times_S \widetilde{S}, Z)$ and a compactification $[\widetilde{Z}/\widetilde{G}]$.*

(iii) *If $Z \to X$ is tame (resp. is tamely ramified over S), then there is a tame DM stacky blow-up $(\widetilde{Z},Z) \to (\overline{Z},Z)$ and a compactification $\widetilde{X}$ of X with a finite étale and tame (resp. finite étale) presentation $\widetilde{Z} \to \widetilde{X}$.*

(iv) *In general, there is a compactification $[\overline{W}/\mathfrak{S}_d]$ where d is the rank of $Z \to X$.*

The situation in mind is when U is the coarse moduli space of a uniformizable stack X with presentation $Z \to X$. Note the decreasing control of the compactification. In particular, if $Z \to X$ is tame then the compactification is tame in (i)–(iii) but in (iv) we have almost no control over the compactification except that we know that it is uniformizable.

Proof of Lemma (13.4). For (i), we note that the action $G \times_S Z \to Z$ extends to an action $G \times_S \overline{Z} \to \overline{Z}$ after replacing $(\overline{Z},Z)$ with a blow-up by Corollary (5.2).

For (ii), the extension of G/U to $\widetilde{G}/\widetilde{S}$ follows from Corollary (12.1) and the compactification is obtained as in (i).

In (iii), we use Corollary (12.1) to extend the groupoid $Z \times_X Z \rightrightarrows Z$ to a finite étale groupoid $\widetilde{R} \rightrightarrows \widetilde{Z}$ and let $\widetilde{X}$ be the quotient.

Finally in (iv), we can take $W = \mathrm{SEC}^d(Z/X)$ so that $X = \mathrm{\acute{E}T}^d(Z/X) = [W/\mathfrak{S}_d]$ and the result follows from (i). □

Lemma (13.5). *Let k be a field and let $\mathscr{G}$ be a tame Deligne–Mumford gerbe over k. Then there exists a tame separable field extension k'/k, i.e., $p \nmid [k':k]$, such that $\mathscr{G}(k') \neq \emptyset$.*

Proof. Let K/k be a normal separable field extension such that $\mathscr{G}$ has a K-point $x\colon \mathrm{Spec}(K) \to \mathscr{G}$ and such that $\mathrm{Aut}(x)$ is constant, i.e., $\mathscr{G} = [\mathrm{Spec}(K)/G]$ for a constant étale group G. Let $N \lhd G$ be the inertia subgroup and let $H = G/N = \mathrm{Gal}(K/k)$. Since $\mathscr{G}$ is tame, we have that $p \nmid |N|$. Let $G_p < G$ be a Sylow p-group. Then $N \cap G_p = 0$ and thus $G_p \to G \twoheadrightarrow H$ is injective and such that $[H : G_p]$ is prime to p. Since $N \cap G_p = 0$, we have that G_p acts freely on K so that $[\mathrm{Spec}(K)/G_p] = \mathrm{Spec}(k')$ where $k' = K^{G_p}$. The natural morphism $[\mathrm{Spec}(K)/G_p] \to [\mathrm{Spec}(K)/G] = \mathscr{G}$ gives a k'-point of $\mathscr{G}$ and $[k':k] = [H : G_p]$ is prime to p. □

Note that the lemma only states that we can neutralize $\mathscr{G}$ as an abstract gerbe after a tame field extension, not trivialize it as a G-gerbe. That is, we do not claim that that the automorphism group scheme of the point $x\colon \mathrm{Spec}(k') \to \mathscr{G}$ is constant.

Lemma (13.6). *Let X be a separated Deligne–Mumford stack with coarse moduli space $\underline{X}$ and moduli map $\pi\colon X \to \underline{X}$.*

(i) *Let $x\colon \mathrm{Spec}(k') \to X$ be a point. Then there exists an étale morphism $U \to \underline{X}$, a lifting $y\colon \mathrm{Spec}(k') \to U$ of $\pi \circ x$, a finite étale tame group scheme $G \to U$ and a G-torsor $T \to X \times_{\underline{X}} U$ such that T is a scheme.*

(ii) *Let $\underline{x}\colon \mathrm{Spec}(k) \to \underline{X}$ be a point. Then there exists an étale morphism $U \to \underline{X}$, a lifting $y\colon \mathrm{Spec}(k) \to U$ of $\underline{x}$ and a finite étale presentation $T \to X \times_{\underline{X}} U$ such that T is a scheme.*

Proof. (i) follows from rigidity of reductive groups: First consider the reductive group scheme $\mathrm{Stab}(x) \to \mathrm{Spec}(k')$. This extends to a reductive group scheme $G \to U$ for some étale $U \to X$ as in (i). By rigidity, the $\mathrm{Stab}(x)$-torsor $x\colon \mathrm{Spec}(k') \to (X \times_{\underline{X}} \mathrm{Spec}(k'))_{\mathrm{red}}$ extends to a G-torsor $V \to X \times_{\underline{X}} U$ and V is a scheme in a neighborhood of x.

(ii) Either reason as in the Keel–Mori lemma (using Hilbert schemes to produce finite étale coverings) or first take any $U \to \underline{X}$ as in (i) and then replace U with $\mathrm{\acute{E}T}^{d}(U/\underline{X})$ where d is the rank of $U \to \underline{X}$ at the point $\underline{x}$. $\square$

Proof of Theorem F. Let $X \to S$ be a separated morphism of Deligne–Mumford stacks. If $X \to S$ has a tame Deligne–Mumford compactification then it is obvious that $X \to S$ is strictly tame. Conversely, assume that $X \to S$ is strictly tame.

By Proposition (13.2) the question whether $X \to S$ has a tame compactification is étale-local so we can assume that S is affine. Using approximation [Ryd09b, Thm. D], we can reduce to the case where $X \to S$ is of finite presentation and the reduction to the case where S is of finite type over $\mathrm{Spec}(\mathbb{Z})$ is then standard[2]. Finally, let $X \to X_{\mathrm{cms}} \to S$ be the coarse moduli space of X. Recall that $X_{\mathrm{cms}} \to S$ is separated, representable and of finite type and that $X \to X_{\mathrm{cms}}$ is proper. We now apply the compactification theorem for algebraic spaces, Theorem B, and obtain a compactification $X_{\mathrm{cms}} \subseteq Y \to S$. We may now replace S with Y and hence assume that $X \to S$ is quasi-finite and that there exists an open $U \subseteq S$ such that $X \to U$ is a coarse moduli space. Applying Proposition (13.2) once again, we can continue to assume that S is an (affine) scheme.

We will now use the Riemann–Zariski space RZ of (S, U). Let (V, f) be a point of RZ so that the local ring A at (V, f) is given by a bi-cartesian

[2]The details about strictly tameness and approximation should be written out. For tameness, which coincides with strict tameness in the equicharacteristic case, this is done in [Ryd09b].

diagram

$$\begin{tikzcd} k(u) & \mathcal{O}_{U,u} \arrow[l, two heads] \\ V \arrow[u, hook] & A. \arrow[l, two heads] \arrow[u, hook] \end{tikzcd}$$

Note that the stack $X \times_U \operatorname{Spec}(\mathcal{O}_{U,u})$ has coarse moduli space $\operatorname{Spec}(\mathcal{O}_{U,u})$. By Lemma (13.5) there is a tame separable field extension $k/k(u)$ such that X_u has a k-point. By Lemma (13.6), there is an affine étale morphism $E_U \to \operatorname{Spec}(\mathcal{O}_{U,u})$ with $E_u = \operatorname{Spec}(k)$, a tame étale group scheme $G \to E_U$ and a G-torsor $T \to X \times_U E_U$ with T an (affine) scheme. In particular we have a tame étale cover $E_u = \operatorname{Spec}(k) \to \operatorname{Spec}(k(u))$. By the étalification theorem, there is a tame DM stacky blow-up $(\widetilde{Y}, u) \to (Y, u)$, where $Y = \operatorname{Spec}(V)$, and an extension of $E_u \to \operatorname{Spec}(k(u))$ to a finite étale cover $E_{\widetilde{Y}} \to \widetilde{Y}$.

Now, since V is a valuation ring, any blow-up of $Y = \operatorname{Spec}(V)$ is trivial. By the flatification theorem, it follows that the stacky blow-up $(\widetilde{Y}, u) \to (Y, u)$ is flat, quasi-finite and of finite presentation. By Corollary (C.15), this stacky blow-up descends to a tame DM stacky blow-up $\widetilde{W} \to W = \operatorname{Spec}(A)$ that is flat, quasi-finite and of finite presentation. Moreover, the cartesian diagram

$$\begin{tikzcd} \operatorname{Spec}(k(u)) \arrow[r, hook] \arrow[d] & \operatorname{Spec}(\mathcal{O}_{U,u}) \arrow[d] \\ \widetilde{Y} \arrow[r, hook] & \widetilde{W} \end{tikzcd}$$

is bi-cartesian. By Proposition (C.9), the affine étale morphisms $E_{\widetilde{Y}} \to \widetilde{Y}$ and $E_U \to U$ glue to an affine étale morphism $E \to \widetilde{W}$.

By a limit argument (W is the inverse limit of local rings of blow-ups of (S, U)) using Lemma (9.10), there is a blow-up $(S_{(V,f)}, U) \to (S, U)$, a cartesian open immersion $(S'_{(V,f)}, U') \subseteq (S_{(V,f)}, U)$, a stacky blow-up $(\widetilde{S'_{(V,f)}}, U') \to S'_{(V,f)}$ an étale surjective morphism $E_{(V,f)} \to \widetilde{S'_{(V,f)}}$, a finite étale tame group scheme $G_{(V,f)} \to E_{(V,f)} \times_S U$ and a representable G-torsor $T_{(V,f)} \to X \times_S E_{(V,f)}$.

By the quasi-compactness of the RZ space and Lemma (9.4), we thus obtain a tame DM stacky blow-up $(S', U) \to (S, U)$, an étale surjective morphism $E' \to S'$, a finite tame étale group scheme $G \to E' \times_S U$ and a representable G-torsor $T \to X \times_S E'$. By Lemma (13.4) (ii), there is then a tame compactification of $X \times_S E'$ over E' by a quotient stack $[\overline{T}/\overline{G}]$ (here $\overline{G}$ is a tame étale group scheme over a stacky blow-up of E'). Finally, by Proposition (13.2) we obtain a tame compactification $\overline{X'}$ of $X \times_S S'$. Since $(S', U) \to (S, U)$ is a tame DM stacky blow-up, $X \subseteq \overline{X'} \to S' \to S$ is a tame compactification of X. $\square$

Remark **(13.7).** In characteristic zero, we can take a field extension $k/k(u)$ such that X_u has a point x with $\operatorname{Stab}(x)$ a constant group scheme. We can then assume that G is a constant group scheme and we can compactify $X \times_S E' = [T/G]$ by a stack of the form $[\overline{T}/G]$. With this approach we can do without the stacky blow-up needed to extend the group scheme $G \to E' \times_S U$ to a group scheme $\overline{G} \to \widetilde{E'}$.

Remark **(13.8).** *Non-tame DM-stacks* — First note that, for any DM-stack, we can get a compactification "étale-locally on the Riemann–Zariski space". Indeed, by Lemma (13.6) (ii) there is an étale *neighborhood* $E \to \operatorname{Spec}(\mathcal{O}_{U,u})$ (i.e., $E_u = \operatorname{Spec}(k(u))$) such that $X \times_U E$ has a finite étale presentation by a scheme T. This gives us a blow-up $(S', U) \to (S, U)$ and an étale covering $E' \to S'$ such that $X \times_S E'$ has a finite étale presentation. We can then compactify $X \times_S E' \to E'$ by Lemma (13.4) (iv).

In the tame case, this is not sufficient since we have no control over this compactification and cannot assert that it is tame so that Proposition (13.2) applies. However, granted a non-tame analogue of Proposition (13.2) (where we could also assume that the compactification of $X \times_S S' \to S'$ is uniformizable and in particular a global quotient stack) the compactification of arbitrary DM-stacks follows.

Corollary (13.9). *Let (X, U) be a Deligne–Mumford stackpair. Let $E_U \to U$ be a tame étale gerbe. Then there is a tame DM stacky blow-up $(\widetilde{X}, U) \to (X, U)$ and a tame étale gerbe $\widetilde{E} \to \widetilde{X}$ extending $E_U \to U$.*

Proof. Choose a tame DM compactification $\overline{E_U} \to X$ and étalify it to an étale morphism $\widetilde{E} \to \widetilde{X}$. The étale morphism $\widetilde{E} \to \widetilde{X}$ is automatically an étale gerbe since the diagonal $\Delta_{\widetilde{E}/\widetilde{X}}$ is finite and étale, and the restriction to the open dense subset U is surjective so that $\Delta_{\widetilde{E}/\widetilde{X}}$ is surjective. □

Remark **(13.10).** Note that unless E_U is uniformizable, it is a priori not at all obvious that a compactification of E_U exists and the fact that $E_U \to U$ is a gerbe does not seem to help very much.

Appendix A. Pinchings of algebraic stacks

In this appendix we show that *pinchings* — the push-out of a closed immersion and a finite morphism — exist in the category of algebraic stacks. As an exception, we do not assume that all stacks are quasi-separated.

The following is a standard lemma for which we have not found a suitable reference.

Lemma (A.1). *Let $f\colon Z' \to Z$ be a finite morphism of algebraic stacks. Let $V' \to Z'$ be smooth and surjective. Then there exists a smooth presentation $W \to Z$ and a smooth morphism $W \times_Z Z' \to V'$ over Z'.*

Proof. First let $W \to Z$ be a smooth presentation and let $W' = W \times_Z Z'$. We may then replace Z', Z and V' with W', W and $W' \times_{Z'} V'$ respectively and assume that Z is a scheme. Then Z' is also a scheme.

Let $z \in Z$ be any point so that $f^{-1}(z) = \{z'_1, z'_2, \dots, z'_n\}$ is a finite set of points in Z'. It is enough to construct a smooth neighborhood $W \to Z$ of z such that $W \times_Z Z' \to Z'$ factors via a smooth map $W \times_Z Z' \to V'$.

For every point z'_i choose a preimage v'_i in V', an open neighborhood $v'_i \in V'_i \subseteq V'$ and an étale morphism $V'_i \to \mathbb{A}^{r_i}_{Z'}$ over Z'. We may replace V'

with the disjoint union $\coprod V_i'$ after replacing Z with an open neighborhood of z such that $V' \to Z'$ remains surjective.

Now let r be an integer greater than all the r_i's and choose smooth maps $\mathbb{A}^r_{Z'} \to \mathbb{A}^{r_i}_{Z'}$. We may then replace V_i' with $V_i' \times_{\mathbb{A}^{r_i}_{Z'}} \mathbb{A}^r_{Z'}$ and assume that $V' \to Z'$ factors through $\mathbb{A}^r_{Z'}$. After replacing Z' and Z with $\mathbb{A}^r_{Z'}$ and $\mathbb{A}^r_Z$ respectively, we can then assume that $V' \to Z'$ is étale.

Let $Z_{\overline{z}}$ denote the strict henselization of Z at z so that $Z' \times_Z Z_{\overline{z}} = \coprod_{i=1}^n Z'_{\overline{z}_i'}$ is a disjoint union of spectra of strictly local rings. It follows that the étale morphism $V' \times_Z Z_{\overline{z}} \to Z' \times_Z Z_{\overline{z}}$ has a section. By a limit argument, there is thus an étale neighborhood $W \to Z$ of z such that $V' \times_Z W \to Z' \times_Z W$ has a section. The composition of this section (which is open) and the projection $V' \times_Z W \to V'$ gives the required étale morphism $W \times_Z Z' \to V'$. $\square$

Lemma (A.2) ([Rao74a, Lemme])**.** *Let $Z' \hookrightarrow X'$ be a closed immersion and let $f\colon Z' \to Z$ be a finite morphism of algebraic stacks. Then there exists disjoint unions of affine schemes W, W', U' and smooth presentations $W \to Z$, $W' \to Z'$ and $U' \to X'$ such that*

$$\begin{tikzcd} W \arrow[d] \arrow[dr, phantom, "\square"] & W' \arrow[l] \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U' \arrow[d] \\ Z & Z' \arrow[l] \arrow[r, hook] & X' \end{tikzcd}$$

is cartesian.

Proof. It is enough to construct, for every point $z \in Z$, smooth morphisms $W \to Z$, $W' \to Z'$ and $U' \to X'$ with W, W' and U' affine such that the image of $W \to Z$ contains z. Indeed, we can then take the disjoint union of all these neighborhoods and add a presentation of $X' \setminus Z'$ to U'.

Let $U' \to X'$ be a smooth presentation. Then by the previous lemma, there exists a smooth presentation $W \to Z$ and a smooth morphism $W' = W \times_Z Z' \to U' \times_{X'} Z'$. Let $w \in W$ so that $f^{-1}(w) = \{w_i'\}$ is a finite set of points in W'. After replacing W with an étale neighborhood of w, we can assume that the $W' = \coprod_i W_i'$ such that $w_i' \in W_i'$.

For every i, there is an open neighborhood $V_i' \subseteq W_i'$ of w_i' such that there exists a smooth morphism $U_i' \to U'$ which restricts to $V_i' \subseteq W' \to U' \times_{X'} Z'$ over Z' [EGA$_{\text{IV}}$, Prop. 18.1.1]. Replace U_i' and V_i' with affine neighborhoods of w_i'. After replacing W with an open affine neighborhood of w, we can finally assume that $V_i' = W_i'$ and we are done. $\square$

Lemma (A.3). *Let A be a ring and let $B \hookrightarrow C$ be an extension of A-algebras such that C is integral over B. If C is noetherian and an A-algebra of finite type, then B is noetherian and an A-algebra of finite type.*

Proof. Let $c_1, c_2, \dots, c_n$ be generators of C and let $B_0 \subseteq B$ be a finitely generated A-algebra containing the coefficients of the integral equations of the c_i's so that $B_0 \hookrightarrow C$ is finite. Since $B_0 \hookrightarrow C$ is finite and injective it follows that B_0 is noetherian by the Eakin–Nagata theorem [EGA$_{\text{I}}$, Prop. 0.6.4.9]. Thus $B_0 \hookrightarrow B$ is finite and the lemma follows. $\square$

Theorem (A.4) (Existence of pinchings)**.** *Let S be an algebraic stack and let X', Z', Z be algebraic stacks over S. Let $j'\colon Z' \hookrightarrow X'$ be a closed immersion and let $f'\colon Z' \to Z$ be a finite morphism. Then the push-out X of j' and f' exists in the category of algebraic stacks and fits into the bi-cartesian diagram*

$$\begin{tikzcd} Z' \arrow[r, hook, "j'"] \arrow[d, "f'"] & X' \arrow[d, "f"] \\ Z \arrow[r, hook, "j"] & X. \end{tikzcd}$$

Furthermore,

(i) *f is integral, j is a closed immersion and $f \amalg j\colon X' \amalg Z \to X$ is integral, schematically dominant and surjective.*
(ii) *f is an isomorphism over $X \setminus Z$.*
(iii) *If $\mathcal{I}$ and $\mathcal{I}'$ are the ideals defining $Z \hookrightarrow X$ and $Z' \hookrightarrow X'$ respectively, then $\mathcal{I} \to f_*\mathcal{I}' = f_*(\mathcal{I}\mathcal{O}_{X'})$ is the identity.*
(iv) *The square remains co-cartesian after flat base change $T \to S$.*
(v) *The square of associated topological spaces is co-cartesian and this holds after arbitrary base change $T \to S$.*
(vi) *The following square of quasi-coherent sheaves on X*

$$\begin{tikzcd} \mathcal{O}_{Z'} & \mathcal{O}_{X'} \arrow[l, two heads] \\ \mathcal{O}_Z \arrow[u] & \mathcal{O}_X \arrow[l, two heads] \arrow[u] \end{tikzcd}$$

is cartesian.
(vii) *Let P be one of the properties: affine,* AF*-scheme, algebraic space, Deligne–Mumford stack, has quasi-finite diagonal, quasi-separated, separated, quasi-compact. Then X has P if and only if X' and Z have P.*
(viii) *If S is locally noetherian and X' and Z are locally of finite type (resp. of finite type, resp. proper) over S, then so is X.*

Proof. (v) and (vi): The questions are smooth-local, so we can assume that X is affine. The statement is then [Fer03, Thm. 5.1].

(vii) The condition is clearly necessary. If Z, Z' and X' are affine, then by construction X is affine (as also follows from Chevalley's theorem). If Z, Z' and X' are AF-schemes, then so is X by [Fer03, Thm. 5.4]. The last six properties follow from the fact that $X' \amalg Z \to X$ is finite and surjective.

(viii) If $X' \amalg Z \to S$ is of finite type, then so is $X \to S$ by Lemma (A.3). The proper case follows from the observation that if $X' \amalg Z \to S$ is universally closed then so is $X \to S$. □

Appendix B. Riemann–Zariski spaces

Let X be a quasi-compact and quasi-separated *scheme* and let $U \subseteq X$ be an open quasi-compact subset. A U-modification is a proper morphism $\pi\colon X' \to X$ of schemes such that $\pi|_U$ is an isomorphism. The Riemann–Zariski space $RZ = RZ(U \subseteq X)$ is the inverse limit of all U-modifications in the category of *locally ringed spaces*. This limit is filtered as the category of U-modifications has fiber products. Note that the subcategory of U-admissible blow-ups is cofinal by [RG71, Cor. 5.7.12], cf. Corollary (5.1), and hence RZ is also the inverse limit of all U-admissible blow-ups and again this is a filtered limit.

Note that RZ is not necessarily a scheme since the transition morphisms are not affine (as in [EGA$_{\mathrm{IV}}$, §8]). If $X = \varprojlim_\lambda X_\lambda$ is an inverse limit of schemes with affine transition morphisms, then X is also the inverse limit in the category of locally ringed spaces [EGA$_{\mathrm{IV}}$, Rmk. 8.2.14].

We begin by studying the topological properties of RZ. If $X = \varprojlim_\lambda X_\lambda$ is any inverse limit of locally ringed spaces, then we have that $|X| = \varprojlim_\lambda |X_\lambda|$ where $|X|$ denotes the underlying topological space. If the X_λ's are quasi-compact and quasi-separated schemes then we also have the finer constructible topology $|X_\lambda|^{\mathrm{cons}}$ which is compact and Hausdorff. It is well-known that $|X|^{\mathrm{cons}} := \varprojlim_\lambda |X_\lambda|^{\mathrm{cons}}$ is compact and Hausdorff and it follows that the coarser topology $|X|$ is quasi-compact.

Recall that a topological space is *quasi-separated* if the intersection of any two quasi-compact open subsets is quasi-compact. If $U \subseteq X$ is quasi-compact, then there exists a λ and $U_\lambda \subseteq X_\lambda$ such that U is the inverse image of U_λ. Since $|X_\lambda|$ is generated by quasi-compact open subsets we can also choose U_λ to be quasi-compact. It then follows that U is closed and hence compact in $|X|^{\mathrm{cons}}$. Thus, if U and V are quasi-compact open subsets then $U \cap V$ is quasi-compact and we have shown that:

Lemma (B.1). *Let X be quasi-compact and quasi-separated. Then the Riemann–Zariski space $RZ(U \subseteq X)$ is quasi-compact and quasi-separated.*

Next, we will describe the points of $RZ(U \subseteq X)$. Let V be a valuation ring and let $f\colon \mathrm{Spec}(V) \to X$ be a morphism such that the generic point $\xi \in \mathrm{Spec}(V)$ maps into U. Then f lifts uniquely to any U-modification and hence lifts to a unique map $\mathrm{Spec}(V) \to RZ$ (of locally ringed spaces). Conversely, we have the following result:

Proposition (B.2) ([Tem08, Cor. 3.4.7])**.** *There is a one-to-one correspondence between points of RZ and pairs (V, f) where V is a valuation ring and $f\colon \mathrm{Spec}(V) \to X$ is a morphism such that $f^{-1}(U) = \{\xi\}$ is the generic point and the residue field extension $k(f(\xi)) \hookrightarrow k(\xi)$ is trivial.*

Proof. The open immersion $U \subseteq X$ is decomposable (after a blow-up of the complement of U, the inclusion $U \to X$ is affine) and hence [Tem08, Cor. 3.4.7] applies. □

Finally, let us describe the local rings of RZ. If $X = \varprojlim_\lambda X_\lambda$ is an inverse limit of locally ringed spaces then the local ring at $x \in X$ is $\mathcal{O}_{X,x} = \varinjlim_\lambda \mathcal{O}_{X_\lambda, x_\lambda}$ where x_λ is the image of x by $X \to X_\lambda$.

We have the following descriptions of the local rings of RZ:

Proposition (B.3) ([Tem08, Prop. 2.2.1])**.** *Let $z = (V, f)$ be a point of RZ. Then the local ring $\mathcal{O}_{RZ,z}$ at z fits into the following bi-cartesian diagram*

$$\begin{tikzcd} k(\xi) = k(u) & \mathcal{O}_{U,u} \arrow[l, two heads] \\ V \arrow[u, hook] & \mathcal{O}_{RZ,z} \arrow[l, two heads] \arrow[u, hook] \end{tikzcd}$$

where $u = f(\xi)$.

Note that cartesian means that $\mathcal{O}_{RZ,z}$ is the inverse image of V in $\mathcal{O}_{U,u}$ and co-cartesian means that $k(u)$ is the tensor product. Also note that even if V is a DVR, then $\mathcal{O}_{RZ,z}$ is non-noetherian.

Note that the generic point $\{\xi\} = f^{-1}(U)$ is open in $\mathrm{Spec}(V)$.

Appendix C. A descent result for push-outs

In this section we study various descent properties for push-outs of the following type: Let $f'\colon Z \hookrightarrow X$ be a closed immersion and let $g'\colon Z \to Y$ be a flat, affine, schematically dominant, monomorphism. Equivalently, g' is affine and $\mathcal{O}_Y \to g'_*\mathcal{O}_Y$ is a flat injective epimorphism of rings. The main examples are affine schematically dominant open immersions and the inclusion of the generic point of an integral scheme.

We will show that for such pairs (f', g') of morphisms of *affine schemes*, the push-out $X \amalg_Z Y$ in the category of affine schemes is also the push-out in the category of algebraic stacks. By base change we then deduce the existence of similar push-outs of morphisms of stacks.

It is easily seen that, except in trivial cases, these push-outs never are noetherian. Nevertheless, we will show that the push-outs enjoy several nice properties. Push-outs of this kind was introduced by M. Temkin and the results in this section generalizes [Tem10, Prop. 2.4.3] and [Tem08, Lem. 2.3.1]. The main application is the push-out in Proposition (B.3) and flat base changes of this. In these applications $g'\colon Z \to Y$ is an affine schematically dominant open immersion.

(I recently realized that most of these results can be found in [Fer03].)

Recall that the pull-back in the category of rings coincides with the pull-back in the category of modules (and sets) and that $M = N_1 \times_P N_2$ if and only if

$$M \longrightarrow N_1 \times N_2 \rightrightarrows P$$

is exact, or equivalently if and only if

$$0 \longrightarrow M \longrightarrow N_1 \oplus N_2 \longrightarrow P$$

is exact (the last morphism is the difference of the two projections). (The first sequence makes sense in the category of rings but not the latter.) From the exact sequence, it follows that pull-backs of modules (and rings) commute with flat base change.

Lemma (C.1). *Let*

$$\begin{tikzcd} D & C \arrow[l, two heads, "f'"'] \\ B \arrow[u, hook, "g'"] & A \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

be a cartesian diagram of rings such that f' is surjective and g' is injective. Then

(i) $A = f'^{-1}(B)$, *f is surjective and g is injective.*
(ii) $\ker(f) \to \ker(f')$ *and* $\operatorname{coker}(g) \to \operatorname{coker}(g')$ *are bijective.*
(iii) *The diagram is co-cartesian.*

If in addition g' is an epimorphism (resp. a flat epimorphism) then so is g.

Proof. The first three statements are straight-forward to check. For the last assertion, let $I = \ker(f)$. Then $IC = I$ by (ii) and it follows that the natural homomorphism $I \to I(C \otimes_A C)$ is bijective. We have the following homomorphism of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & I(C \otimes_A C) \arrow[r] \arrow[d, equal] & C \otimes_A C \arrow[r] \arrow[d, two heads] & D \otimes_B D \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & I \arrow[r] & C \arrow[r] & D \arrow[r] & 0. \end{tikzcd}$$

Thus $g\colon A \to C$ is an epimorphism if and only if $g'\colon B \to D$ is an epimorphism. Now assume that g' is a flat epimorphism. It is enough to check that the epimorphism g is flat on local rings. Let $\mathfrak{p} \subseteq C$ be a prime ideal and let $\mathfrak{q} = g^{-1}(\mathfrak{p})$. Since g is an epimorphism, we have that $C_\mathfrak{p} = C \otimes_A A_\mathfrak{q}$. If $I \nsubseteq \mathfrak{p}$ then $D_\mathfrak{p} = D_\mathfrak{q} = 0$ and it follows that $B_\mathfrak{q} = 0$ and $A_\mathfrak{q} = C_\mathfrak{q}$. If $I \subseteq \mathfrak{p}$ then $B_\mathfrak{q} \hookrightarrow D_\mathfrak{q}$ is faithfully flat and hence an isomorphism and it follows that $A_\mathfrak{q} = C_\mathfrak{q}$. $\square$

Remark **(C.2).** If g' is flat but not an epimorphism, then easy examples show that g need not be flat.

(C.3) From now on, we will only consider pull-backs such that f' is surjective and g' is a flat injective epimorphism, i.e., bi-cartesian squares

$$\begin{tikzcd} D & C \arrow[l, two heads, "f'"'] \\ B \arrow[u, hook, "g'"] & A \arrow[l, two heads, "f"] \arrow[u, hook, "g"'] \end{tikzcd}$$

of rings where f and f' are surjective and g and g' are flat injective epimorphisms. Dually, we will consider 2-cartesian squares of algebraic stacks

$$\begin{tikzcd} Z \arrow[r, hook, "f'"] \arrow[d, "g'"] & X \arrow[d, "g"] \\ Y \arrow[r, hook, "f"] & W \end{tikzcd}$$

where f and f' are closed immersions and g and g' are flat, affine, schematically dominant, monomorphisms such that

$$\begin{tikzcd} f_*g'_*\mathcal{O}_Z & g_*\mathcal{O}_X \arrow[l, two heads] \\ f_*\mathcal{O}_Y \arrow[u, hook] & \mathcal{O}_W \arrow[l, two heads] \arrow[u, hook] \end{tikzcd}$$

is bi-cartesian.

We note that $X \to W$ is an isomorphism over $W \setminus Y$ so $X \amalg Y \to W$ is surjective. It is not difficult to see that $X \amalg Y \to W$ is submersive and we will show that it is in fact universally submersive.

Definition (C.4). Let $j\colon U \to X$ be a flat affine schematically dominant monomorphism. We say that a quasi-coherent sheaf $\mathcal{F}$ on X is *U-admissible* (or simply admissible) if $\mathcal{F} \to j_*j^*\mathcal{F}$ is injective. If $X' \to X$ is a morphism (of stacks), then we say that X' is U-admissible if $X' \times_X U \to X'$ is schematically dominant (i.e., if $\mathcal{O}_{X'} \to j'_*j'^*\mathcal{O}_{X'}$ is injective).

Lemma (C.5). *Let A, B, C, D form a bi-cartesian square of rings as in* (C.3)*.*

(i) *Let M be a C-admissible A-module. Then $M_B = M \otimes_A B$ is D-admissible and $M = M_B \times_{M_D} M_C$.*
(ii) *Let M_B be a B-module, let M_C be a C-module and let $\theta\colon M_B \otimes_B D \to M_C \otimes_C D$ be an isomorphism of D-modules. Let M_D be a module isomorphic to this D-module so that we have canonical induced homomorphisms $M_B \to M_D$ and $M_C \to M_D$. If M_B is D-admissible, i.e., if $M_B \to M_D$ is injective, then $M := M_B \times_{M_D} M_C$ is a C-admissible A-module such that $M_B = M \otimes_A B$, $M_C = M \otimes_A C$ and $M_D = M \otimes_A D$.*

Proof. We have an exact sequence

$$0 \longrightarrow A \longrightarrow B \oplus C \longrightarrow D \longrightarrow 0.$$

If M is a C-admissible A-module, i.e., if $M \to M \otimes_A C$ is injective, then

$$0 \longrightarrow M \longrightarrow M_B \oplus M_C \longrightarrow M_D \longrightarrow 0 \tag{C.5.1}$$

is exact so that $M = M_B \times_{M_D} M_C$. The injectivity of $M_B \to M_D$ then follows from the injectivity of $M \to M_C$. This settles (i).

In (ii), the equation (C.5.1) is exact by assumption. After tensoring with the flat monomorphism $A \hookrightarrow C$, we obtain the exact sequences

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & M_B \oplus M_C \arrow[r] \arrow[d, hook] & M_D \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \otimes_A C \arrow[r] & M_D \oplus M_C \arrow[r] & M_D \arrow[r] & 0 \end{tikzcd}$$

which shows that $M \otimes_A C \to M_C$ is bijective and that $M \to M_C$ is injective. Now applying (i) to M, we obtain the exact sequence

$$0 \longrightarrow M \longrightarrow M \otimes_A B \oplus M_C \longrightarrow M_D \longrightarrow 0$$

which shows that $M \otimes_A B = M_B$. $\square$

Lemma (C.5) says that there is an equivalence between C-admissible A-modules and triples (M_B, M_C, θ) where M_B is a D-admissible B-module, M_C is a C-module and $\theta\colon M_B \otimes_B D \to M_C \otimes_C D$ is an isomorphism. This is most vividly expressed as follows: $\mathrm{Spec}(B \times C) \to \mathrm{Spec}(A)$ is a morphism of effective descent for admissible quasi-coherent sheaves. To abbreviate, we say that a triple (M_B, M_C, θ) as above is *admissible*.

Also note that if A' is an admissible A-algebra, then the pull-back of the bi-cartesian square along $A \to A'$ remain bi-cartesian.

The following proposition follows from [Fer03, Thm. 5.1, Thm. 7.1]:

Proposition (C.6). *Let A, B, C, D form a bi-cartesian square of rings as in* (C.3)*. Then* $\mathrm{Spec}(B \times C) \to \mathrm{Spec}(A)$ *is universally submersive.*

Proof. By [Pic86, Thm. 37] it is enough to check that $\mathrm{Spec}((B \times C) \otimes_A V) \to \mathrm{Spec}(V)$ is submersive for every valuation ring V and ring homomorphism $A \to V$. If $C \otimes_A V = 0$ then $\mathrm{Spec}(V) \to \mathrm{Spec}(A)$ factors through $\mathrm{Spec}(B) \hookrightarrow \mathrm{Spec}(A)$ and it is obvious. If $C \otimes_A V \neq 0$, then V is admissible and we obtain a bi-cartesian square

$$\begin{tikzcd} V \otimes_A D & V \otimes_A C \arrow[l, two heads] \\ V \otimes_A B \arrow[u, hook] & V \arrow[l, two heads] \arrow[u, hook] \end{tikzcd}$$

where the vertical homomorphism are flat injective epimorphisms. It follows that the vertical homomorphisms are localizations and that all four rings are valuation rings. Since $V \otimes_A D \neq 0$, it follows that $\mathrm{Spec}((B \times C) \otimes_A V) \to \mathrm{Spec}(V)$ is submersive. Indeed, if $Z \subseteq \mathrm{Spec}(V)$ is a subset such that $Z \cap \mathrm{Spec}(V \otimes_A C)$ and $Z \cap \mathrm{Spec}(V \otimes_A B)$ are closed, then Z is closed. $\square$

Most parts of the following proposition follows from [Fer03, Thm. 2.2].

Proposition (C.7). *Let A, B, C, D form a bi-cartesian square of rings as in* (C.3) *and let M, N and P be C-admissible A-modules and let R be a C-admissible A-algebra. Then*

(i) *A homomorphism $M \to N$ is surjective if and only if $M_B \to N_B$ and $M_C \to N_C$ are surjective.*
(ii) *If $0 \to M \to N \to P \to 0$ is an exact sequence and P_C is a flat A-module, then $0 \to M_B \to N_B \to P_B \to 0$ and $0 \to M_C \to N_C \to P_C \to 0$ are exact.*
(iii) *M is of finite type if and only if M_B and M_C are of finite type.*
(iv) *Assume that M_C is flat, then M is of finite presentation if and only if M_B and M_C are of finite presentation.*
(v) *R is of finite type if and only if R_B and R_C are of finite type.*
(vi) *Assume that R_C is flat, then R is of finite presentation if and only if R_B and R_C are of finite presentation.*
(vii) *M is flat if and only if M_B is a flat B-module and M_C is a flat C-module.*

Proof. (i) Let $I = \ker(A \to B) = \ker(C \to D)$ and assume that $M_B \to N_B$ and $M_C \to N_C$ are surjective. Since $IM = \ker(M \to M_B) = \ker(M_C \to M_D) = IM_C$ and $IN = \ker(N \to N_B) = \ker(N_C \to N_D) = IN_C$ we have that $IM \to IN$ is surjective. It thus follows from the exact sequences

$$\begin{tikzcd} 0 \arrow[r] & IM \arrow[r] \arrow[d, two heads] & M \arrow[r] \arrow[d] & M_B \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & IN \arrow[r] & N \arrow[r] & N_B \arrow[r] & 0 \end{tikzcd}$$

that $M \to N$ is surjective.

(ii) Since C is flat over A, we have that $0 \to M_C \to N_C \to P_C \to 0$ is exact. Since P_C is a flat A-module, we have that $0 \to M_D \to N_D \to P_D \to 0$ is exact as well. Since M is admissible, we have that $M_B \hookrightarrow M_D$ is injective and it follows that $M_B \hookrightarrow N_B$.

(iii) If M_B and M_C are of finite type, then by an easy limit argument there exists a free A-module F of finite rank and a homomorphism $F \to M$ such that $F_B \to M_B$ and $F_C \to M_C$ are surjective. It follows that M is of finite type by (i).

(iv) By (iii) we know that M is of finite type. Let F be finite free A-module with a surjection onto M and let K be the kernel so that $0 \to K \to F \to M \to 0$ is exact. It then follows from (i) and (ii) that K is of finite type and hence that M is of finite presentation.

(v) Reason as (iii).

(vi) Reason as in (iv).

(vii) Assume that M_B and M_C are flat and let N be a, not necessarily admissible, A-module. The exact sequence $0 \to A \to B \oplus C \to D \to 0$ shows that $\mathrm{Tor}_i^A(B, N) \to \mathrm{Tor}_i^A(D, N)$ is injective for all $i \geq 1$ and bijective for $i \geq 2$. We note that

$$\mathrm{Tor}_i^A(B, N) \otimes_A M = \mathrm{Tor}_i^A(B, N) \otimes_B M_B = \mathrm{Tor}_i^A(M_B, N)$$
$$\mathrm{Tor}_i^A(D, N) \otimes_A M = \mathrm{Tor}_i^A(D, N) \otimes_D M_D = \mathrm{Tor}_i^A(M_D, N)$$

since M_B and M_D are flat. Also since M_B is a flat B-module it follows that $\mathrm{Tor}_i^A(M_B, N) \to \mathrm{Tor}_i^A(M_D, N)$ is injective for all $i \geq 1$ and bijective for $i \geq 2$. As M_C is a flat C-module, and a fortiori a flat A-module, the exact sequence $0 \to M \to M_B \oplus M_C \to M_D \to 0$ gives us the long exact sequence

$$\longrightarrow \mathrm{Tor}_i^A(M, N) \longrightarrow \mathrm{Tor}_i^A(M_B, N) \longrightarrow \mathrm{Tor}_i^A(M_D, N) \longrightarrow$$

and it follows that $\mathrm{Tor}_i^A(M, N) = 0$ for all $i \geq 1$, hence that M is a flat A-module. □

We now reformulate the above results for schemes and stacks:

Definition (C.8). Consider a bi-cartesian square as in (C.3). An *admissible triple* is a triple $(\mathcal{G}, \mathcal{H}, \theta)$ where $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_X$-module, $\mathcal{H}$ is a quasi-coherent Z-admissible $\mathcal{O}_Y$-module and θ is an isomorphism $f'^*\mathcal{G} \to g'^*\mathcal{H}$.

Proposition (C.9). *Consider a bi-cartesian square as in* (C.3).

(i) *The morphism* $X \amalg Y \to W$ *is universally submersive.*
(ii) *The pull-back functor* $\mathcal{F} \mapsto (f^*\mathcal{F}, g^*\mathcal{F}, \theta)$ *(where* θ *is the gluing isomorphism on* $X \times_W Y = Z$*) induces an equivalence between the*

category of X-admissible quasi-coherent sheaves on W and admissible triples. Moreover, $\mathcal{F}$ is flat (resp. of finite type, resp. flat and of finite presentation) if and only if $f^*\mathcal{F}$ and $g^*\mathcal{F}$ are so.

(iii) *The pull-back functor $T \mapsto (T \times_W X, T \times_W Y, \theta)$ induces an equivalence between the category of affine admissible morphisms $T \to W$ and affine admissible triples $(T_X \to X, T_Y \to Y, \theta)$. Moreover, $T \to W$ is flat (resp. of finite type, resp. flat and of finite presentation, resp. finite, resp. étale, resp. smooth, resp. unramified) if and only if $T_X \to X$ and $T_Y \to Y$ are so.*

Proof. Follows easily from the affine case, noting that being étale (resp. smooth, resp. unramified) can be checked on fibers (since we have the descent result for flatness and finite type/presentation) and that $X \amalg Y \to W$ is surjective. □

Remark **(C.10).** M. Temkin claims in [Tem08, Lem. 2.3.1] that $X \amalg Y \to W$ satisfies effective descent for the category of quasi-compact strongly representable morphisms. This question reduces to quasi-affine morphisms but there appears to be a crucial error in the proof for quasi-affine morphisms. D. Ferrand has a necessary and sufficient condition in [Fer03, Thm. 7.1].

Definition (C.11). Let $f \colon Z \hookrightarrow X$ be a closed immersion. We say that a quasi-coherent sheaf of ideals $\mathcal{J} \subseteq X$ is Z-admissible if $f^*\mathcal{J} \twoheadrightarrow \mathcal{J}\mathcal{O}_Z$ is an isomorphism (i.e., if $f^*\mathcal{J}$ is an ideal of $\mathcal{O}_Z$). We say that a closed subscheme $X_0 \hookrightarrow X$ is admissible if it is defined by an admissible ideal sheaf.

If Z is defined by the ideal sheaf $\mathcal{I}$ then $\mathcal{J}$ is Z-admissible if and only if $\mathcal{I}\mathcal{J} = \mathcal{I} \cap \mathcal{J}$.

Proposition (C.12). *Consider a bi-cartesian square as in* (C.3)*. There is a one-to-one correspondence between Y-admissible ideals of $\mathcal{O}_W$ and pairs $(\mathcal{J}_X, \mathcal{J}_Y)$ such that $\mathcal{J}_X \subseteq \mathcal{O}_X$ is a Z-admissible ideal and $\mathcal{J}_Y \subseteq \mathcal{O}_Y$ is an ideal such that $\mathcal{J}_Y|_Z = \mathcal{J}_X|_Z$ (as ideals of $\mathcal{O}_Z$). Under this correspondence an ideal $\mathcal{J} \subseteq \mathcal{O}_W$ is of finite type if and only if $\mathcal{J}|_X$ and $\mathcal{J}|_Y$ are of finite type. If $\mathcal{J}|_X$ is flat then $\mathcal{J}$ is of finite presentation if and only if $\mathcal{J}|_X$ and $\mathcal{J}|_Y$ are of finite presentation.*

Proof. Since $g \colon X \to W$ is flat and schematically dominant, every ideal $\mathcal{J} \subseteq \mathcal{O}_W$ is X-admissible and every ideal $\mathcal{J}|_Y \subseteq \mathcal{O}_Y$ is Z-admissible. Thus if $(\mathcal{J}_X, \mathcal{J}_Y)$ is a pair as in the proposition, then they glue to an ideal $\mathcal{J} \subseteq \mathcal{O}_W$ such that $\mathcal{J}|_Y = \mathcal{J}_Y$ and hence $\mathcal{J}$ is Y-admissible. Conversely, if $\mathcal{J}$ is Y-admissible, then $\mathcal{J}|_Z$ is an ideal and hence $\mathcal{J}|_X$ is X-admissible. The last assertions follow from Proposition (C.7). □

Corollary (C.13). *Consider a bi-cartesian square as in* (C.3)*. There is a one-to-one correspondence between Y-admissible ideals $\mathcal{J} \subseteq \mathcal{O}_W$ such that $\mathcal{J}|_X = \mathcal{O}_X$ and ideals $\mathcal{J}_Y \subseteq \mathcal{O}_Y$ such that $(\mathcal{J}_Y)|_Z = \mathcal{O}_Z$. Under this correspondence $\mathcal{J}$ is of finite type (resp. of finite presentation, resp. invertible) if and only if $\mathcal{J}_Y$ is so.*

Remark **(C.14).** Note that under the correspondence of the previous Corollary, a closed subscheme $Y_0 \hookrightarrow Y$ corresponds to a closed subscheme $W_0 \hookrightarrow$

W such that $Y_0 = W_0 \cap Y$ and $Y_0 \to W_0$ is a nil-immersion (a bijective closed immersion). In general, $Y_0 \hookrightarrow Y \hookrightarrow W$ is not of finite presentation and $Y_0 \neq W_0$.

Corollary (C.15). *Consider a bi-cartesian square as in* (C.3)*. There is a one-to-one correspondence between X-admissible blow-ups (resp. stacky blow-ups) $\widetilde{W} \to W$ and Z-admissible blow-ups (resp. stacky blow-ups) $\widetilde{Y} \to Y$. This correspondence sends $\widetilde{W}$ to $\widetilde{W} \times_W Y$ (there is no need to take the strict transform). Under this correspondence $\widetilde{W} \to W$ is of finite presentation if and only if $\widetilde{Y} \to Y$ is of finite presentation.*

Corollary (C.16). *Consider a square as in* (C.3)*. Then W is a push-out in the category of algebraic stacks (with quasi-affine diagonals).*

Proof. This is a standard argument: Let T be a stack with morphisms $X \to T$ and $Y \to T$ such that $Z \hookrightarrow X \to T$ coincides with $Z \to Y \to T$ (up to a 2-isomorphism?). We want to show that there is a unique morphism $W \to T$. This question is fppf-local on W so we can assume that W is affine and that T is quasi-compact. Let $T' \to T$ be a smooth (or flat) presentation with T' affine so that $T' \to T$ is quasi-affine. Let T'^{aff} be the affine hull of T' in T so that $T' \subseteq T'^{\mathrm{aff}}$ is an open immersion. By pull-back we obtain $Z' \subseteq Z'^{\mathrm{aff}} \to Z$, $X' \subseteq X'^{\mathrm{aff}} \to X$ and $Y' \subseteq Y'^{\mathrm{aff}} \to Y$. We note that Y' is admissible but not necessarily Y'^{aff}. After replacing Y'^{aff} with the schematic closure of Z'^{aff} we obtain two admissible triples (X', Y', Z') and $(X'^{\mathrm{aff}}, Y'^{\mathrm{aff}}, Z'^{\mathrm{aff}})$. The second is affine and hence glues to an affine morphism $W'^{\mathrm{aff}} \to W$. The first triple is an open substack of the second and hence glues to an open substack W' of W'^{aff}. Since $W' \to W$ is smooth, we can replace W with W' and T with T' and the result follows from the affine case and fppf descent. □

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