WEIGHTED BLOW-UPS

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Abstract. We introduce and develop the theory of stack-theoretic weighted blow-ups which simultaneously generalize ordinary blow-ups, root stacks and Cartierification of $\mathbb{Q}$-Cartier divisors. A stack-theoretic weighted blow-up in a weighted smooth center is locally described as the toric stack corresponding to a star subdivision and is thus smooth. Such stack-theoretic weighted blow-ups were recently used in the weighted resolution of singularities in characteristic zero by Abramovich–Temkin–Włodarczyk. We also show that the Kummer log blow-ups, introduced by Abramovich–Temkin–Włodarczyk in their work on log resolution of singularities, are stack-theoretic weighted blow-ups. Finally, we show that GIT wall-crossings are given by stack-theoretic weighted blow-ups and blow-downs in weighted smooth centers. **DRAFT version with 2 missing sections.**

Contents

Introduction

Weighted blow-ups. Weighted blow-ups appear in the context of toric varieties and, more generally, in locally toric situations. Given a fan Σ, a cone σ and an interior lattice point $v \in \sigma$, we can form the *star subdivision* $\Sigma^*(v)$ which induces a map of toric varieties $X_{\Sigma^*(v)} \to X_\Sigma$ [CLS11, §11.1].

If σ is simplicial, then there is a unique way to write $v = \sum_i d_i b_i$ where the b_i are the primitive lattice points of the rays $\rho_1, \rho_2, \dots, \rho_n$ of σ and the d_i are positive integers. If $D_1, D_2, \dots, D_n$ are the corresponding toric

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divisors, then $X_{\Sigma^*(v)} \to X_\Sigma$ is the *weighted blow-up* in the weighted center $\frac{1}{d_1}D_1 \cap \cdots \cap \frac{1}{d_n}D_n$. One way to make this precise is to take the usual blow-up in the integral closure of the ideal $I_{\frac{N}{d_1}D_1} + \cdots + I_{\frac{N}{d_n}D_n}$ for a sufficiently divisible integer N. If every cone containing σ is smooth, then the D_i are Cartier divisors and if all the multiplicities d_i are equal, then $\Sigma^*(v) \to \Sigma$ is the blow-up in the smooth center $D_1 \cap D_2 \cap \cdots \cap D_n$.

Similarly, given a sequence $x_1, x_2, \dots, x_n$ of functions and positive integers $d_1, d_2, \dots, d_n$, the weighted blow-up in $\big(x_1^{1/d_1}, x_2^{1/d_2}, \dots, x_n^{1/d_n}\big)$ is the blow-up in the integral closure of the ideal $\big(x_1^{N/d_1}, x_2^{N/d_2}, \dots, x_n^{N/d_n}\big)$ for sufficiently divisible N.

Stack-theoretic weighted blow-ups. A simplicial toric variety is the coarse space of a smooth toric stack [BCS05, FMN10]. Unless $d_1 = d_2 = \cdots = d_n$, a weighted blow-up of a smooth toric variety is always singular. Thus, if we want to study weighted blow-ups of smooth objects, we are lead to consider stacks. The toric stack corresponding to the star subdivision $\Sigma^*(v)$ is the *stack-theoretic weighted blow-up* of $X_{\Sigma(v)}$ in the center given by the *Rees algebra*

$$I_\bullet = (I_{D_1}, d_1) + (I_{D_2}, d_2) + \cdots + (I_{D_n}, d_n).$$

This Rees algebra is the smallest filtration (I_n) of $\mathscr{O}_X$ containing I_{D_i} in degree d_i. The stack-theoretic weighted blow-up in $I_\bullet$ is the *stack-theoretic Proj*

$$\mathscr{P}\mathrm{roj}(I_\bullet) = \big[\mathrm{Spec}_X(I_\bullet) \smallsetminus V(I_+) \,/\, \mathbb{G}_m\big].$$

Similarly, the exceptional divisor is the stack-theoretic Proj of $\bigoplus_n I_{n+1}/I_n$ which is a weighted projective stack — a smooth stack whose coarse space is a weighted projective space.

Rees algebras give a framework for stack-theoretic weighted blow-ups for arbitrary algebraic stacks, without toric connections. This framework contains:

(i) the usual blow-up in the ideal I, which corresponds to the usual Rees algebra $I_\bullet = (I, 1) = \bigoplus_{n \geq 0} I^n$;
(ii) the dth root stack in the Cartier divisor D, which corresponds to the Rees algebra $I_\bullet = (I_D, d)$; and
(iii) the Cartierification of a $\mathbb{Q}$-Cartier divisor D, which corresponds to the Rees algebra $I_\bullet = (I_D, 1) + (I_{2D}, 2) + \cdots$.

Whereas usual blow-ups modifies the space but does not introduce any stackiness, root stacks and Cartierifications leave the coarse space unmodified and introduces stackiness in codimension 1 and codimension ≥ 2 respectively. Toric stacks, with trivial generic stabilizer, can be obtained from their coarse toric variety by taking Cartierifications and root stacks (Example 2.3.8). Every stack-theoretic weighted blow-up is a usual blow-up followed by a root stack up to normalization (Proposition 3.4.4).

The star subdivisions of toric stacks give rise to smooth weighted centers. These are the Rees algebras that locally can be written as $(x_1, d_1) + (x_2, d_2) + \cdots + (x_n, d_n)$ where $x_1, x_2, \dots, x_n$ is a regular sequence and $V(x_1, x_2, \dots, x_n)$

is smooth. Smooth weighted centers can also be characterized as the Rees algebras whose weighted normal cone $\operatorname{Spec}\bigl(\bigoplus_{n\geq 0} I_n/I_{n+1}\bigr)$ is a *twisted weighted vector bundle* and whose center $V(I_1)$ is smooth (§5).

Applications. Using weighted blow-ups instead of blow-ups (and root stacks) gives more flexibility and significantly simplifies many algorithms. The prominent example is weighted resolution of singularities by Abramovich, Temkin and Włodarczyk [ATW19] which uses stack-theoretic weighted blow-ups in weighted smooth centers and is far more effective than Hironaka's classical algorithm but at the expense of using smooth stacks.

Wall-crossing in GIT is described by a weighted blow-up followed by a weighted blow-down. Except in the case where all degrees are equal, these blow-ups are singular even if both sides of the wall are smooth. Using stack-theoretic weighted blow-ups gives a description of the wall-crossing in terms of smooth stacks and stack-theoretic weighted blow-ups in weighted smooth centers (§7). This in turn gives an algorithm for weak factorization of schemes and Deligne–Mumford stacks using stack-theoretic weighted blow-ups that is much more effective than the classical one [Ryd15b]. Similarly, we expect that the destackification algorithm of Bergh [Ber17] becomes simpler and requires fewer blow-ups if stack-theoretic weighted blow-ups are used.

Overview. In Section 1, we study the stack-theoretic Proj. In particular we describe its charts (§1.3) and its universal property (Proposition 1.5.1).

In Section 2, we give four examples of stack-theoretic Proj. The first example is weighted projective stacks, which includes root stacks of line bundles, and more generally twisted weighted affine and projective bundles. The second example is root stacks of generalized Cartier divisors. The third example is a construction making a $\mathbb{Q}$-invertible sheaf into an invertible sheaf (generalizing Cartierification). This was prominently used by Abramovich and Hassett [AH10] to treat families of $\mathbb{Q}$-Gorenstein varieties. The fourth example is a stack-theoretic amplification of GIT quotients.

In Section 3 we introduce Rees algebras and stack-theoretic weighted blow-ups (§3.1). In particular, we describe their universal property (Theorem 3.2.9) and a simplified description in the normal case (Theorem 3.4.3). We also describe how Rees algebras can be seen as certain valuative $\mathbb{Q}$-ideals using Zariski–Riemann spaces (§3.5).

In Section 4, we treat weighted normal cones. In particular, we describe how the *extended Rees algebra* $I_\bullet^{\mathrm{ext}}$ gives rise to the *deformation to the weighted normal cone* $\operatorname{Spec}(I_\bullet^{\mathrm{ext}})$ (§4.3).

In Section 5, we consider stack-theoretic blow-ups in regular centers. Firstly, we show that various notions of quasi-regularity coincide and are equivalent to the weighted normal cone being a twisted weighted projective stack (§5.1). Secondly, we show that when the weighted center is given by a regular sequence, then the extended Rees algebra has a very simple description (Proposition 5.2.2). This gives simple equations for charts of weighted blow-ups in regular sequences. Thirdly, we specialize to weighted blow-ups in regular weighted centers (5.3). This is the case where $I_\bullet$ is locally given by a regular sequence and the center $V(I_1)$ also is regular. Finally, we expand on some of the toric connections alluded to in the beginning of the introduction (5.4).

In Section 6, we show that the Kummer log blow-ups of [ATW20, §5] that were constructed by gluing partial coarsenings of stack quotients of ordinary blow-ups by Galois actions, has a much neater description as stack-theoretic weighted blow-ups. This was the initial motivation for writing this paper.

In Section 7, we show that a GIT wall-crossing between smooth Deligne–Mumford stacks is given by a stack-theoretic weighted blow-up followed by a stack-theoretic weighted blow-down, both in regular weighted centers.

Conventions. In general, we work over the base scheme $\mathrm{Spec}(\mathbb{Z})$ and occasionally we write $* = \mathrm{Spec}(\mathbb{Z})$ for the final object. The reader, if so inclined, may instead work over a base scheme $* = S$. The letter $\Bbbk$ denotes a field, and unless otherwise mentioned, X denotes a scheme, or more generally, an algebraic stack. By an ideal on X, we always mean a quasi-coherent, *finitely generated* ideal on X.

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1. Stack-theoretic Proj

1.1. **Graded algebras and $\mathbb{G}_m$-actions.** Let $R = \bigoplus_{n\geq 0} R_n$ be a quasi-coherent graded $\mathscr{O}_X$-algebra. The grading on R corresponds to a coaction of $\mathscr{O}_X[t^{\pm 1}]$ on R:

$$\beta\colon R \to R \otimes_{\mathscr{O}_X} \mathscr{O}_X[t^{\pm 1}] = R[t^{\pm 1}]$$

mapping a section r of R_n to rt^n, or equivalently, an action of $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t,t^{-1}])$ on $\mathrm{Spec}_X(R)$:

$$\alpha\colon \mathbb{G}_m \times \mathrm{Spec}_X(R) \to \mathrm{Spec}_X(R).$$

We will denote the ideal of R generated by $\bigoplus_{n\geq 1} R_n$ by R_+, and the d^{th} Veronese subalgebra $\bigoplus_{n\geq 0} R_{dn}$ by $R^{(d)}$.

1.1.1. *Stabilizers of the $\mathbb{G}_m$-action.* Let $x\colon \mathrm{Spec}(\Bbbk) \to \mathrm{Spec}_X(R)$ be a point. The stabilizer group scheme of x, denoted G_x, is a closed subgroup of $\mathbb{G}_m \times \mathrm{Spec}(\Bbbk)$, and sits in the cartesian diagram

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] & \mathrm{Spec}(\Bbbk) \arrow[d, "{(x,x)}"] \\ \mathbb{G}_m \times \mathrm{Spec}_X(R) \arrow[r, "{(\alpha,\pi_2)}"] & \mathrm{Spec}_X(R) \times_X \mathrm{Spec}_X(R). \end{tikzcd}$$

Thus, either $G_x = (\boldsymbol{\mu}_d)_k = \mathrm{Spec}\big(k[t]/(t^d - 1)\big)$ for some $d \geq 1$ or $G_x = (\mathbb{G}_m)_k$. Equivalently, the Cartier dual of G_x is either $\mathbb{Z}/d\mathbb{Z}$ or $\mathbb{Z}$.

Lemma 1.1.2. *The Cartier dual of G_x is $\mathbb{Z}/(d\colon x \notin V(R_d))$, that is,*

(i) *$G_x = \mathbb{G}_m$ if and only if $x \in V(R_+)$, and*
(ii) *$\boldsymbol{\mu}_d \subset G_x$ if and only if $x \in V(R_n)$ for all n such that $d \nmid n$.*

Proof. The question is local on X so we may assume that X is affine. Let $\varphi_x\colon R \to \Bbbk$ be the corresponding ring homomorphism. Then $\boldsymbol{\mu}_d \subset G_x$ if and only if x is $\boldsymbol{\mu}_d$-equivariant, or equivalently, if and only if φ_x is $\mathbb{Z}/d\mathbb{Z}$-graded. This happens precisely when the kernel of φ_x contains R_n for all n such that $d \nmid n$. $\square$

In particular, $V(R_+)$ precisely contains the points fixed by $\mathbb{G}_m$, whence the action of $\mathbb{G}_m$ on $\operatorname{Spec}_X(R)$ restricts to an action of $\mathbb{G}_m$ on $W := \operatorname{Spec}_X(R) \smallsetminus V(R_+)$. Moreover, if R is generated in degree 1, then this action of $\mathbb{G}_m$ on W is free, i.e., $G_x = \{1\}$ for all points $x \in W$. This is because in that case $(R_n \colon d \nmid n) = R_+$ whenever $d > 1$.

1.2. **Definition of the stack-theoretic Proj.** Let $W := \operatorname{Spec}_X(R) \smallsetminus V(R_+)$. The *stack-theoretic Proj* of R is the stack quotient

$$\mathscr{P}\mathrm{roj}_X(R) := [W \,/\, \mathbb{G}_m].$$

The $\mathbb{G}_m$-equivariant map $W \to X$, where X is equipped with the trivial action, gives a map $\mathscr{P}\mathrm{roj}_X(R) \to X \times \mathrm{B}\mathbb{G}_m$. We let $\pi\colon \mathscr{P}\mathrm{roj}_X(R) \to X$ and $q\colon \mathscr{P}\mathrm{roj}_X(R) \to \mathrm{B}\mathbb{G}_m$ be the induced maps. In particular, we have a cartesian square:

$$\begin{tikzcd} W \arrow[r] \arrow[d, "p"] & * \arrow[d] \\ \mathscr{P}\mathrm{roj}_X(R) \arrow[r, "q"] & \mathrm{B}\mathbb{G}_m \end{tikzcd}$$

By Lemma 1.1.2, $\mathscr{P}\mathrm{roj}_X(R)$ is a *tame* algebraic stack [AOV08]. If the orders of the stabilizer groups of the points of $\mathscr{P}\mathrm{roj}_X(R)$ are invertible on X, then $\mathscr{P}\mathrm{roj}_X(R)$ is a Deligne–Mumford stack. In particular, this holds in characteristic zero.

Since closed substacks of $\mathscr{P}\mathrm{roj}_X(R)$ correspond to $\mathbb{G}_m$-invariant closed subschemes of W, every closed substack of $\mathscr{P}\mathrm{roj}_X(R)$ can be written as $\mathscr{P}\mathrm{roj}_X(R/I)$ for a homogeneous ideal I of R.

The stack-theoretic Proj also makes sense when X is an algebraic stack. In this case $\operatorname{Spec}_X(R) \smallsetminus V(R_+)$ is an algebraic stack with an action of $\mathbb{G}_m$ [Rom05].

Finally, note that R is also an R_0-algebra, so π factors through $\operatorname{Spec}_X(R_0)$, and $\mathscr{P}\mathrm{roj}_X(R) \to \operatorname{Spec}_X(R_0)$ is the stack-theoretic Proj of R as an R_0-algebra. It is thus harmless to assume that $R_0 = \mathscr{O}_X$.

1.3. **Local charts.** We can give local charts of $\mathscr{P}\mathrm{roj}_X(R)$ as follows. Let $f_i \in R_+$ be homogeneous elements of degrees $d_i \geq 1$, indexed by some indexing set I, such that $R_+ \subset \sqrt{(f_i : i \in I)}$. Then $W = \operatorname{Spec}_X(R) \smallsetminus V(R_+) = \bigcup_{i\in I} \operatorname{Spec}_X(R_{f_i})$, so we have an open covering $\mathscr{P}\mathrm{roj}_X(R) = \bigcup_{i\in I} D_+(f_i)$, where the i^{th} chart is:

$$D_+(f_i) := \Big[\operatorname{Spec}_X(R_{f_i}) \,/\, \mathbb{G}_m\Big] = \big[\operatorname{Spec}_X\big(R_{f_i}/(f_i - 1)\big) \,/\, \boldsymbol{\mu}_{d_i}\big]\,. \tag{1.1}$$

The second equality follows from Lemma 1.3.1, with $A = \mathbb{Z}$, $a = d_i$, $R = R_{f_i}$ and $r = f_i$. The intersection of charts works as usual: $D_+(f_i) \cap D_+(f_j) = D_+(f_i f_j)$ and the open inclusion $D_+(f_i) \cap D_+(f_j) \subset D_+(f_i)$ is given by $f_j \neq 0$.

Lemma 1.3.1. *Let A be a finitely generated abelian group, with corresponding diagonalizable algebraic group $D(A)$. Let $R = \bigoplus_{\alpha\in A} R_\alpha$ be an A-graded algebra, and let $r \in R$ be a homogeneous element of degree $a \in A$. Then $R/(r-1)$ is an $A/\langle a\rangle$-graded algebra and the $A/\langle a\rangle$-graded homomorphism $R \to R/(r-1)$ induces a morphism of algebraic stacks*

$$\big[\big(\mathrm{Spec}(R/(r-1)\big) \,/\, D\big(A/\langle a\rangle\big)\big] \xrightarrow{\simeq} [\mathrm{Spec}(R) \,/\, D(A)].$$

This is an isomorphism if r is invertible and a has infinite order.

Note that as A-graded modules $R \simeq R/(r-1)[r, r^{-1}]$ but the algebra structures do not coincide. Similarly, $R/(r-1) \simeq \bigoplus_{[\alpha]\in A/\langle a\rangle} R_\alpha$ but only as $A/\langle a\rangle$-graded modules.

Proof. We need to prove that the natural $D(A)$-equivariant map

$$\mathrm{Spec}\big(R/(r-1)\big) \times^{D(A/\langle a\rangle)} D(A) \to \mathrm{Spec}(R)$$

is an isomorphism. Let us elaborate on the left hand side. We have two commuting actions on $\mathrm{Spec}\big(R/(r-1)\big) \times D(A) = \mathrm{Spec}(R/(r-1)[v^A]) := \mathrm{Spec}(R/(r-1)[v^\alpha\colon \alpha \in A])$:

(i) the diagonal $D(A/\langle a\rangle)$-action, given by $(y,t)\cdot s = (ys, s^{-1}t)$, where in the first factor the action corresponds to the induced $A/\langle a\rangle$-grading on $R/(r-1)$, and
(ii) the $D(A)$-action on the second factor given by $(y,t)\cdot s = (y, ts)$.

The $D(A/\langle a\rangle)$-action is free with quotient $\mathrm{Spec}\big(R/(r-1)\big)\times^{D(A/\langle a\rangle)} D(A) = \mathrm{Spec}(R^\circ)$ where R° is the degree 0 part of $R/(r-1)[v^A]$ with the $A/\langle a\rangle$-grading. The $D(A)$-action endows R° with the following A-grading

$$R^\circ = \bigoplus_{\alpha\in A} \big(R/(r-1)\big)_{[\alpha]} v^\alpha.$$

The natural A-graded algebra homomorphism $R \to R^\circ$ is thus an isomorphism. $\square$

Remark 1.3.2. Let $\mathscr{C}$ be the full sub-category of algebraic stacks whose objects are Zariski-locally of the form $[\mathrm{Spec}(B) \,/\, D(A)]$ for a finitely generated abelian group A with diagonalizable group scheme $D(A)$, and an A-graded ring B. Then we claim that $\mathscr{C}$ is closed under taking stacky Proj.

Indeed, let $X = [\mathrm{Spec}(B) \,/\, D(A)]$ as above, and let R be a quasi-coherent graded $\mathscr{O}_X$-algebra, i.e., a quasi-coherent $(A\times\mathbb{Z})$-graded B-algebra. For a collection of $(A\times\mathbb{Z})$-homogeneous elements $f_i \in R_+$ of degrees (a_i, d_i) such that $R_+ \subset \sqrt{(f_i\colon i\in I)}$, then $\mathscr{P}\mathrm{roj}_X(R)$ is covered by the charts $D_+(f_i) = [\mathrm{Spec}(R_{f_i}) \,/\, D(A\times\mathbb{Z})] = \big[\mathrm{Spec}(R_{f_i}/(f_i-1)) \,/\, D\big(A\times\mathbb{Z}/\langle(a_i,d_i)\rangle\big)\big]$.

1.4. **Tautological line bundles $\mathscr{O}(d)$.** Let as before $p\colon W \to \mathscr{P}\mathrm{roj}_X(R)$ denote the presentation. Pull-back of line bundles induces an isomorphism

$$\rho^*\colon\ \mathrm{Pic}\big(\mathscr{P}\mathrm{roj}_X(R)\big) \xrightarrow{\simeq} \mathrm{Pic}^{\mathbb{G}_m}(W)$$

where the right hand side denotes the $\mathbb{G}_m$-equivariant Picard group of W. Therefore, on the stack-theoretic Proj, there are tautological line bundles $\mathscr{O}(d)$ for each $d\in\mathbb{Z}$, arising from the shifts $R(d)$, as well as natural maps $\pi^* R_d \to \mathscr{O}(d)$ induced from the multiplication maps $R\otimes_{\mathscr{O}_X} R_d \to R(d)$. Note that $\mathscr{O}(1)$ is invertible and that $\mathscr{O}(d) = \mathscr{O}(1)^{\otimes d}$.

For each $d \in \mathbb{Z}$, we let $q_d\colon \mathscr{P}\mathrm{roj}_X(R) \to \mathrm{B}\mathbb{G}_m = [* \,/\, \mathbb{G}_m]$ be the morphism classifying the line bundle $\mathscr{O}(d)$. Then $q_1 = q$, that is, $\mathscr{O}(1)$ corresponds to the $\mathbb{G}_m$-torsor $W \to \mathscr{P}\mathrm{roj}_X(R)$. In particular, $q_d = (\cdot)^d \circ q$ where $(\cdot)^d\colon \mathrm{B}\mathbb{G}_m \to \mathrm{B}\mathbb{G}_m$ is induced by the d^{th} power morphism. Equivalently, $q_d\colon [W \,/\, \mathbb{G}_m] \to [* \,/\, \mathbb{G}_m]$ is induced by the structure morphism $W \to *$ and $(\cdot)^d\colon \mathbb{G}_m \to \mathbb{G}_m$. Therefore, the morphism also fits in the following cartesian square:

$$\begin{tikzcd}
{[W \,/\, \boldsymbol{\mu}_d]} \arrow[r] \arrow[d] & * \arrow[d] \\
\mathscr{P}\mathrm{roj}_X(R) \arrow[r, "q_d"] & \mathrm{B}\mathbb{G}_m.
\end{tikzcd}$$

1.5. **The universal property of the stack-theoretic Proj.** The stack-theoretic Proj satisfies the following universal property:

Proposition 1.5.1 (Universal property)**.** *Let R be a graded quasi-coherent $\mathscr{O}_X$-algebra. Given a scheme T with a morphism $f\colon T \to X$, a lift of f to $\mathscr{P}\mathrm{roj}_X(R)$ corresponds to the data of a line bundle $\mathscr{L}$ on T and a graded homomorphism $\varphi\colon f^*R \to \bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}$ of sheaves of algebras on T such that locally on T, $\varphi_n\colon f^*R_n \to \mathscr{L}^{\otimes n}$ is surjective for all sufficiently divisible n.*

Proof. To lift f to $\mathscr{P}\mathrm{roj}_X(R)$, one needs to supply a $\mathbb{G}_m$-torsor P over T mapping $\mathbb{G}_m$-equivariantly to W making the following diagram commute:

$$\begin{tikzcd}
P \arrow[r, dotted] \arrow[d, dotted] & W \arrow[r, hook] & \mathrm{Spec}_X(R) \arrow[d] \\
T \arrow[rr, "f"] & & X
\end{tikzcd}$$

Every line bundle $\mathscr{L}$ on T gives rise to a $\mathbb{G}_m$-torsor $P = \mathrm{Spec}_T\big(\bigoplus_{n\in\mathbb{Z}} \mathscr{L}^{\otimes n}\big)$ over T, and conversely any $\mathbb{G}_m$-torsor P over T arises from some line bundle $\mathscr{L}$ on T. Then a $\mathbb{G}_m$-equivariant morphism

$$P = \mathrm{Spec}_T\left(\bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}\right) \smallsetminus V\left(\bigoplus_{n\geq 1} \mathscr{L}^{\otimes n}\right) \longrightarrow \mathrm{Spec}_X(R) \smallsetminus V(R_+) = W$$

making the diagram above commute, is equivalent to a graded homomorphism $\varphi\colon f^*R \to \bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}$ such that $\bigoplus_{n\geq 1} \mathscr{L}^{\otimes n} \subset \sqrt{\varphi(f^*R_+)}$, i.e., $\mathscr{L} \subset \sqrt{\varphi(f^*R_+)}$. Given a trivialization t of $\mathscr{L}$ over an open subset $U \subset T$, there therefore exists a positive integer N such that the trivializing section $t^{\otimes N}$ of $\mathscr{L}^{\otimes N}$ over U lifts to f^*R_N. Thus, whenever N divides n, $\varphi_n\colon f^*R_n \to \mathscr{L}^{\otimes n}$ is surjective over U. □

1.6. **Properties of the stack-theoretic Proj and relations to the usual Proj.**

Proposition 1.6.1. *Let R be a graded $\mathscr{O}_X$-algebra.*

(i) *$\mathscr{P}\mathrm{roj}_X(R)$ has finite diagonal relative to X. In particular, $\mathscr{P}\mathrm{roj}_X(R)$ is separated over X.*
(ii) *If R is finitely generated, then $\mathscr{P}\mathrm{roj}_X(R)$ is proper over X.*
(iii) *The coarse space of $\mathscr{P}\mathrm{roj}_X(R)$, relative to X, is the usual $\mathrm{Proj}_X(R)$.*

(iv) *The morphism* $q_1\colon \mathscr{P}\mathrm{roj}_X(R) \to X \times \mathrm{B}\mathbb{G}_m$ *corresponding to* $\mathscr{O}(1)$ *is quasi-affine.*
(v) *The relative coarse space of* $q_d\colon \mathscr{P}\mathrm{roj}_X(R) \to X\times\mathrm{B}\mathbb{G}_m$ *is* $\mathscr{P}\mathrm{roj}_X(R^{(d)})$, *where* $R^{(d)} = \bigoplus_{n\geq 0} R_{dn}$.
(vi) *If* S *is another graded* $\mathscr{O}_X$*-algebra and* $\varphi\colon R \to S$ *is a graded homomorphism such that* $S_+ \subset \sqrt{\varphi(R_+)}$, *then there is an induced affine morphism* $f\colon \mathscr{P}\mathrm{roj}_X(S) \to \mathscr{P}\mathrm{roj}_X(R)$ *such that* $f^*\mathscr{O}(1) = \mathscr{O}(1)$. *If* R *and* S *are of finite type and* $R_0 \to S_0$ *is finite, then* f *is finite.*

Proof. The questions are local on X, so we may assume that $X = \mathrm{Spec}(A)$ is affine. For (i), we need to show that $D_+(f_if_j) \to D_+(f_i) \times_X D_+(f_j)$ is finite. Over the tautological $\mathbb{G}_m^2$-torsor this map is $\mathrm{Spec}(R_{f_if_j}[u,u^{-1}]) \to \mathrm{Spec}(R_{f_i}) \times_X \mathrm{Spec}(R_{f_j})$, where $\varphi\colon R_{f_i} \otimes_A R_{f_j} \to R_{f_if_j}[u,u^{-1}]$ is given by $\varphi(r\otimes s) = rsu^d$ if $s \in R_d$. It is finite since $u^{d_i} = \varphi(f_i^{-1}\otimes f_i)$ and $u^{-d_j} = \varphi(f_j \otimes f_j^{-1})$.

For (iii), note that $\mathrm{Proj}_X(R) = \big(\mathrm{Spec}(R) \smallsetminus V(R_+)\big) \mathbin{/} \mathbb{G}_m$. Indeed, the coarse space of $D_+(f_i) = [\mathrm{Spec}(R_{f_i}) \mathbin{/} \mathbb{G}_m]$ is the spectrum of the invariant ring of R_{f_i}, that is, $R_{(f_i)}$. For (ii), if R is finitely generated, then $\mathrm{Proj}_X(R) \to X$ is proper. Coupled with the fact that the coarse space morphism $\mathscr{P}\mathrm{roj}_S(R) \to \mathrm{Proj}_X(R)$ is also proper, we conclude that $\mathscr{P}\mathrm{roj}_X(R)$ is proper over X. For (iv), it suffices to note that the total space of the $\mathbb{G}_m$-bundle corresponding to $\mathscr{O}(1)$ is the quasi-affine scheme $W = \mathrm{Spec}(R) \smallsetminus V(R_+)$, which we saw in §1.4.

For (v), the question can be checked flat-locally, so we may pass to the total space of the $\mathbb{G}_m$-bundle corresponding to $\mathscr{O}(d)$, which is $[W \mathbin{/} \boldsymbol{\mu}_d]$ (by §1.4 again), and which has coarse space $\mathrm{Spec}(R^{(d)})\smallsetminus V(R_+^{(d)})$. Consequently, the relative coarse space of q_d is $[\mathrm{Spec}(R^{(d)})\smallsetminus V(R_+^{(d)}) \mathbin{/} \mathbb{G}_m] = \mathscr{P}\mathrm{roj}_X(R^{(d)})$.

Finally, for (vi), we obtain an affine $\mathbb{G}_m$-equivariant morphism $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ such that the inverse image of $V(R_+)$ contains $V(S_+)$, whence we obtain an affine morphism $f\colon \mathscr{P}\mathrm{roj}_X(S) \to \mathscr{P}\mathrm{roj}_X(R)$ over $\mathrm{B}\mathbb{G}_m$. If R and S are of finite type and $R_0 \to S_0$ is finite, then both stacky Proj are proper over $\mathrm{Spec}(R_0)$, and hence f is finite (as f is proper and affine). ☐

Note that (vi) with $S = \bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}$ retrieves the universal property (Proposition 1.5.1). In the terminology of [AH10, §2.3], the stack-theoretic Proj is a *cyclotomic stack*, i.e., has stabilizers $\boldsymbol{\mu}_d$ (Lemma 1.1.2), which is *uniformized by* $\mathscr{O}(1)$, i.e., $q_1\colon \mathscr{P}\mathrm{roj}_X(R) \to \mathrm{B}\mathbb{G}_m$ is representable (Proposition 1.6.1(iv)).

Corollary 1.6.2. *Let* R *be a graded* $\mathscr{O}_X$*-algebra.* $\mathscr{P}\mathrm{roj}_X(R)$ *coincides with the usual* $\mathrm{Proj}_X(R)$ *if and only if the action of* $\mathbb{G}_m$ *on* W *is free, in which case* $\mathscr{O}(1)$ *is very ample relative to* X*. In particular, this happens when* R *is generated in degree* 1.

Proof. $\mathscr{P}\mathrm{roj}_X(R)$ coincides with the coarse space $\mathrm{Proj}_X(R)$ if and only if $\mathscr{P}\mathrm{roj}_X(R)$ is an algebraic space, if and only if the action of $\mathbb{G}_m$ on W is free. ☐

Recall that the shift $R(d)$ also gives rise to a coherent sheaf $\widetilde{R(d)}$ on $\mathrm{Proj}_X(R)$, but this sheaf is not always invertible if R is not generated in degree 1. There is a canonical morphism $\widetilde{R(d)} \otimes \widetilde{R(e)} \to \widetilde{R(d+e)}$ but this is also not an isomorphism in general. If $p\colon \mathscr{P}\mathrm{roj}_X(R) \to \mathrm{Proj}_X(R)$ denotes the coarsening morphism, then $\widetilde{R(d)} = p_*\mathscr{O}(d)$.

Proposition 1.6.3. *Let R be a graded $\mathscr{O}_X$-algebra. If X is quasi-compact and R is finitely generated, then $R^{(d)} = \bigoplus_{n\geq 0} R_{dn}$ is generated in degree 1 for all sufficiently divisible d. In particular, the usual $\mathrm{Proj}_X(R)$ agrees with the stack-theoretic $\mathscr{P}\mathrm{roj}_X(R^{(d)})$ and is the relative coarse space of q_d.*

Proof. This can be verified locally on X so we can assume that $X = \mathrm{Spec}(A)$ is affine. If R has generators $f_1, \dots, f_m$ with degrees $d_1, d_2, \dots, d_m$, then we claim that choosing $d = m\ell$ suffices, where ℓ is the least common multiple of the d_i. Indeed, for every $n \geq 0$, R_n is generated by $f_1^{a_1} \cdots f_m^{a_m}$ with $\sum_{i=1}^m a_i d_i = n$. If $n \geq m\ell$, then for each such generator $f_1^{a_1} \cdots f_m^{a_m}$, there exists some $1 \leq i \leq m$ such that $f_1^{a_1} \cdots f_m^{a_m}$ is divisible by $f_i^{\ell/d_i} \in R_\ell$. This shows that $R_n = R_{n-\ell}R_\ell$ whenever $n \geq m\ell$, which implies the claim. □

Remark 1.6.4. If R has generators of degrees $d_1, d_2, \dots, d_m$, then it is not sufficient to take d as the least common multiple ℓ of $d_1, d_2, \dots, d_m$ in Proposition 1.6.3.

1.7. Embeddings into the stack-theoretic Proj.

1.7.1. *Morphisms into Proj.* Let $f\colon X \to S$ be a qcqs morphism of algebraic stacks and $\mathscr{L}$ a line bundle on X. If for every $x \in X$, there exists a positive integer N such that $f^*f_*\mathscr{L}^{\otimes N} \to \mathscr{L}^{\otimes N}$ is surjective at x, then the homomorphism $\bigoplus_{n\geq 0} f^*f_*\mathscr{L}^{\otimes n} \to \bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}$ induces, via the universal property (Proposition 1.5.1), a morphism

$$\varphi_{\mathscr{L}}\colon X \to \mathscr{P}\mathrm{roj}_S\Big(\bigoplus_{n\geq 0} f_*\mathscr{L}^{\otimes n}\Big) \tag{1.2}$$

such that $\varphi_{\mathscr{L}}^*\mathscr{O}(1) = \mathscr{L}$. In particular, if $\mathscr{L}$ is *uniformizing* relative to S, that is, the induced morphism $X \to S \times B\mathbb{G}_m$ is representable, then so is the induced morphism $\varphi_{\mathscr{L}}\colon X \to \mathscr{P}\mathrm{roj}_S(\bigoplus_{n\geq 0} f_*\mathscr{L}^{\otimes n})$.

Setup 1.7.2. Let $f\colon X \to S$ be a morphism of quasi-compact algebraic stacks with finite diagonal. Then there is a relative coarse space $p\colon X \to X_{\mathrm{cs}/S}$ and $f_{\mathrm{cs}}\colon X_{\mathrm{cs}/S} \to S$ is separated. Let $\mathscr{L}$ be a line bundle on X. Then for sufficiently divisible k, the line bundle $\mathscr{L}^{\otimes k}$ *descends to* $X_{\mathrm{cs}/S}$ [Ryd15a]. To be precise, $p_*\mathscr{L}^{\otimes k}$ is a line bundle and $p^*p_*\mathscr{L}^{\otimes k} \to \mathscr{L}^{\otimes k}$ is an isomorphism.

Definition 1.7.3 (Ampleness)**.** In Setup 1.7.2, we say that $\mathscr{L}$ is *ample* relative to S if the line bundle $p_*\mathscr{L}^{\otimes k}$ is ample relative to S.

Lemma 1.7.4. *Keep the assumptions of Setup 1.7.2. If S is an affine scheme, then the following statements are equivalent:*

(i) *$\mathscr{L}$ is ample.*
(ii) *The open subsets X_f, for $f \in \Gamma(X, \mathscr{L}^{\otimes d})$ where d is a positive integer, form a basis for the topology on X.*

(iii) *There exists a positive integer d and finitely many sections $f_i \in \Gamma(X, \mathscr{L}^{\otimes d})$ such that $(X_{\mathrm{cs}})_{f_i}$ is affine for all i, and such that $X = \bigcup_i X_{f_i}$.*
(iv) *There exists a positive integer d and finitely many sections $f_i \in \Gamma(X, \mathscr{L}^{\otimes d})$ such that $(X_{\mathrm{cs}})_{f_i}$ is quasi-affine for all i, and such that $X = \bigcup_i X_{f_i}$.*

This can be verified via passage to the coarse space X_{cs}, and applying the analogous classical result for $p_*\mathscr{L}^{\otimes k}$. The following example shows that some caution is warranted though.

Remark 1.7.5. Retain the situation of Setup 1.7.2. If $\mathscr{L}$ is ample, and F is a quasi-coherent $\mathscr{O}_X$-module of finite type, there does *not* always exist a positive integer n_0 such that $F \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}$ is globally generated over S for all $n \geq n_0$. It is also *not* true that $F \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}$ is globally generated over S for sufficiently divisible n. For example, take $X = \mathrm{B}\boldsymbol{\mu}_d$, $\mathscr{L} = \mathscr{O}_X$, and F to be the universal torsion line bundle on X. Then $F \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n} = F$ for every integer n, and F has no global sections.

Proposition 1.7.6. *In Setup 1.7.2, the following holds.*

(i) *If $\mathscr{L}$ is ample, then $f^*f_*\mathscr{L}^{\otimes N} \to \mathscr{L}^{\otimes N}$ is surjective for all sufficiently divisible N and thus induces a morphism*

$$\varphi_{\mathscr{L}} \colon X \to \mathscr{P}\mathrm{roj}_S\big(\textstyle\bigoplus_{n\geq 0} f_*\mathscr{L}^{\otimes n}\big)$$

as in (1.2).
(ii) *If $\mathscr{L}$ is ample and uniformizing, the induced morphism $\varphi_{\mathscr{L}}$ is a quasi-compact, schematically dominant, open immersion (so in particular, quasi-affine). If in addition f is proper, this morphism is an isomorphism.*
(iii) *Assume there exists a positive integer N such that $f^*f_*\mathscr{L}^{\otimes N} \to \mathscr{L}^{\otimes N}$ is surjective. If the induced morphism $\varphi_{\mathscr{L}}$ is quasi-affine, then $\mathscr{L}$ is ample and uniformizing.*

Proof. For (i), denote the induced morphism $X_{\mathrm{cs}/S} \to S$ by f_{cs}. Fix a positive integer k such that $\mathscr{L}^{\otimes k}$ descends to $X_{\mathrm{cs}/S}$. Since $p_*\mathscr{L}^{\otimes k}$ is ample over S, there exists a positive integer N such that for all $n \geq N$, $p_*\mathscr{L}^{\otimes kn} = (p_*\mathscr{L}^{\otimes k})^{\otimes n}$ is globally generated over S, that is, $f_{\mathrm{cs}}^*(f_{\mathrm{cs}})_*p_*\mathscr{L}^{\otimes kn} \to p_*\mathscr{L}^{\otimes kn}$ is surjective. Applying p^* and noting $f = f_{\mathrm{cs}} \circ p$, we see that $f^*f_*\mathscr{L}^{\otimes kn} \to p^*p_*\mathscr{L}^{\otimes kn} \xrightarrow{\simeq} \mathscr{L}^{\otimes kn}$ is surjective, as desired.

For (ii), the question is local and so we may assume that $S = \mathrm{Spec}(A)$ is affine. Set $R = \bigoplus_{n\geq 0} \Gamma(X, \mathscr{L}^{\otimes n})$. Since $\mathscr{L}$ is ample, we may apply Lemma 1.7.4(iii) above and thus $X = \bigcup_i X_{f_i}$ for some homogeneous $f_i \in R$ with $(X_{f_i})_{\mathrm{cs}}$ affine. Since each $[\mathrm{Spec}(R_{f_i}) \,/\, \mathbb{G}_m]$ is an open substack of $\mathscr{P}\mathrm{roj}_X(R)$, it suffices to prove that the induced morphism $X_{f_i} \to [\mathrm{Spec}(R_{f_i}) \,/\, \mathbb{G}_m]$ is an isomorphism for every i. Therefore, we set up the following diagram (where the reader should first disregard the dotted arrows and fill them in as the argument progresses):

$$
\begin{tikzcd}
Z' \arrow[r, dotted, two heads, "\text{finite}"] \arrow[d, dotted, "\text{affine}"] & \mathrm{Spec}_{X_{f_i}}\left(\bigoplus_{n\in\mathbb{Z}}\mathscr{L}^{\otimes n}\right) \arrow[r] \arrow[d] & \mathrm{Spec}(R_{f_i}) \arrow[r] \arrow[d] & S \arrow[d] \\
Z \arrow[r, dotted, two heads, "\text{finite}"] \arrow[dr, dotted, "\text{integral}"'] & X_{f_i} \arrow[r, "\varphi_{\mathscr{L}}|_{X_{f_i}}"] \arrow[d] & \left[\mathrm{Spec}(R_{f_i})\ /\ \mathbb{G}_m\right] \arrow[r] \arrow[d] & \mathrm{B}\mathbb{G}_m \\
& (X_{\mathrm{cs}})_{f_i} \arrow[r] & \mathrm{Spec}\big((R_{f_i})_0\big) &
\end{tikzcd}
\tag{1.3}
$$

Since X_{f_i} has finite diagonal over S (by assumption of Setup 1.7.2), there exists a finite cover $Z \to X_{f_i}$ from a scheme Z. Then Z is affine, since the composition $Z \to X_{f_i} \to (X_{\mathrm{cs}})_{f_i}$ is integral, and $(X_{\mathrm{cs}})_{f_i}$ is affine. Set

$$Z' := Z \times_{X_{f_i}} \mathrm{Spec}_{X_{f_i}}\left(\bigoplus_{n\in\mathbb{Z}}\mathscr{L}^{\otimes n}\right),$$

which is also an affine scheme. Since $\mathscr{L}$ is uniformizing, X_{f_i} is representable over $\mathrm{B}\mathbb{G}_m$, so that $\mathrm{Spec}_{X_{f_i}}\left(\bigoplus_{n\in\mathbb{Z}}\mathscr{L}^{\otimes n}\right)$ is an algebraic space (over S). Moreover, it admits a finite surjection from the affine scheme Z', so Chevalley's Theorem implies that it is also an affine scheme. Hence, the morphism $\mathrm{Spec}_{X_{f_i}}\left(\bigoplus_{n\in\mathbb{Z}}\mathscr{L}^{\otimes n}\right) \to \mathrm{Spec}(R_{f_i})$, which induces an isomorphism on global sections

$$R_{f_i} = \left(\bigoplus_{n\geq 0}\Gamma(X,\mathscr{L}^{\otimes n})\right)_{f_i} \xrightarrow{\simeq} \bigoplus_{n\in\mathbb{Z}}\Gamma(X_{f_i},\mathscr{L}^{\otimes n}), \tag{1.4}$$

must be an isomorphism, and hence so is $X_{f_i} \to [\mathrm{Spec}(R_{f_i})\ /\ \mathbb{G}_m]$, as desired.

For (iii), the question is again local, so we assume that $S = \mathrm{Spec}(A)$ is affine. Evidently, $\varphi_{\mathscr{L}}$ is representable, whence so is the morphism $X \to \mathrm{B}\mathbb{G}_m$ induced by $\mathscr{L}$. Thus, $\mathscr{L}$ is uniformizing. To show that $\mathscr{L}$ is ample, let d be a positive integer, and let $f \in \Gamma(X, \mathscr{L}^{\otimes d})$. Then $\varphi_{\mathscr{L}}|_{X_f} \colon X_f \to D_+(f)$ is quasi-affine, whence $(X_{\mathrm{cs}})_f$ is quasi-affine over $\mathrm{Spec}\left((R_f)_0\right)$. In particular, (iii′) of Lemma 1.7.4 is satisfied, so $\mathscr{L}$ is ample. □

Remark 1.7.7. Let $f\colon X \to S$ and $\mathscr{L}$ be as §1.7.1. We warn the reader that $R = \bigoplus_{n\geq 0} f_*\mathscr{L}^{\otimes n}$ is in general *not* finitely generated, so the corresponding stack-theoretic Proj is in general *not* proper over S. However, if S *is qcqs*, then R is the union of its finitely generated, quasi-coherent graded $\mathscr{O}_S$-subalgebras R_λ. Since X is quasi-compact, there exists an index λ_0 such that for all $\lambda \geq \lambda_0$, the composition $f^*(R_\lambda)_n \to f^*f_*\mathscr{L}^{\otimes n} \to \mathscr{L}^{\otimes n}$ is surjective for all sufficiently divisible n. For $\lambda \geq \lambda_0$, we get a morphism $\varphi_\lambda\colon X \to \mathscr{P}\mathrm{roj}_S(R_\lambda)$, where the stack-theoretic Proj is now proper over S, and such that φ_λ factors (rationally) through $\varphi_{\mathscr{L}}\colon X \to \mathscr{P}\mathrm{roj}_S(R)$.

Now assume that Setup 1.7.2 holds, assume that $\mathscr{L}$ is ample and uniformizing, and that X is *of finite type* over S. By Proposition 1.7.6(ii), $\varphi_{\mathscr{L}}\colon X \to \mathscr{P}\mathrm{roj}_S(R)$ is an open immersion. In this case, we will now show that, after increasing λ_0, the induced morphism $\varphi_\lambda\colon X \to \mathscr{P}\mathrm{roj}_S(R_\lambda)$ is an immersion for every $\lambda \geq \lambda_0$.

This is a local question on S, so we may assume that $S = \mathrm{Spec}(A)$ is affine and that $R = \bigoplus_{n\geq 0} f_*\mathscr{L}^{\otimes n} = \bigoplus_{n\geq 0}\Gamma(X, \mathscr{L}^{\otimes n})$. We apply Lemma 1.7.4:

there exists a positive integer d, as well as finitely many $f_i \in \Gamma(X, \mathscr{L}^{\otimes d})$ such that $X = \bigcup_i X_{f_i}$ and each $(X_{\mathrm{cs}})_{f_i}$ is affine (and each $\varphi_{\mathscr{L}}|_{X_{f_i}} \colon X_{f_i} \to D_+(f_i)$ is an isomorphism).

Since X is of finite type over S, the $\mathscr{O}_S$-algebra $R_{f_i} \simeq \bigoplus_{n\in\mathbb{Z}} \Gamma(X_{f_i}, \mathscr{L}^{\otimes n})$ is generated by finitely many homogeneous elements b_{ij}. Using the isomorphism in (1.4), we may, for a positive integer m, lift each b_{ij} to some $\frac{s_{ij}}{f_i^m} \in (R_{f_i})_{\deg(b_{ij})}$ for some $s_{ij} \in \Gamma(X, \mathscr{L}^{\otimes md+\deg(b_{ij})})$. Thus, for sufficiently large $\lambda \geq \lambda_0$, there exist homogeneous elements s_{ij}^λ and f_i^λ of R_λ which respectively lift each s_{ij} and each f_i to R_λ. Therefore, for all such λ, φ_λ maps each X_{f_i} into $D_+(f_i^\lambda) \subset \mathscr{P}\mathrm{roj}_S(R_\lambda)$. To complete the proof, it suffices to show each $\varphi_\lambda|_{X_{f_i}}$ is a closed immersion. To this end, recall each $\varphi_\lambda|_{X_{f_i}}$ fits into the following cartesian square:

$$\begin{tikzcd} \mathrm{Spec}_{X_{f_i}}\left(\bigoplus_{n\in\mathbb{Z}} \mathscr{L}^{\otimes n}\right) \arrow[r] \arrow[d] & \mathrm{Spec}_S\left((R_\lambda)_{f_i^\lambda}\right) \arrow[d] \\ X_{f_i} \arrow[r, "\varphi_\lambda|_{X_{f_i}}"] & D_+(f_i^\lambda) \end{tikzcd} \tag{1.5}$$

and it suffices to show that the top row is a closed immersion. Since each X_{f_i} has finite diagonal and each $(X_{f_i})_{\mathrm{cs}}$ is affine, we may argue, as in (1.3), that the algebraic space $\mathrm{Spec}_{X_{f_i}}\left(\bigoplus_{n\in\mathbb{Z}} \mathscr{L}^{\otimes n}\right)$ is an affine scheme. Then the top row of (1.5) is a morphism of affine schemes, which induces a surjection on global sections

$$(R_\lambda)_{f_i^\lambda} \twoheadrightarrow R_{f_i} \xrightarrow{\simeq} \bigoplus_{n\in\mathbb{Z}} \Gamma(X_{f_i}, \mathscr{L}^{\otimes n}),$$

and hence is necessarily a closed immersion, as desired.

1.8. **Sequences of stack-theoretic Proj.** A sequence of stack-theoretic Proj is not a stack-theoretic Proj because it need not be cyclotomic: the stabilizers are subgroups of $\mathbb{G}_m^n$, not of $\mathbb{G}_m$. Instead of a single uniformizing line bundle $\mathscr{L}$, we have a uniformizing collection of line bundles $\mathscr{L}_1, \mathscr{L}_2, \dots, \mathscr{L}_n$ — the corresponding map to $\mathrm{B}\mathbb{G}_m^n$ is representable. In fact, this collection is even *generating* in the sense that, locally on X, every quasi-coherent sheaf on X' of finite type is a quotient of a direct sum of line bundles of the form $\mathscr{L}_1^{\otimes d_1} \otimes \cdots \otimes \mathscr{L}_n^{\otimes d_n}$, $d_i \in \mathbb{Z}$. Equivalently, the corresponding map to $\mathrm{B}\mathbb{G}_m^n$ is quasi-affine [Gro17, Cor. 6.7]. That is, X' is *divisorial*.

Proposition 1.8.1. *Let $X' := X_n \to X_{n-1} \to \cdots \to X_1 \to X$ be a sequence of n stack-theoretic Proj. Let $\mathscr{O}_{X_i}(1) \in \mathrm{Pic}(X_i)$ denote the corresponding ample uniformizing line bundle and let $\mathscr{L}_i$ be the pull-back of $\mathscr{O}_{X_i}(1)$ to X'.*

(i) *$(\mathscr{L}_1, \mathscr{L}_2, \dots, \mathscr{L}_n)$ is generating, that is, the induced morphism $X' \to \mathrm{B}\mathbb{G}_m^n$ is quasi-affine.*

(ii) *If X is quasi-compact, then the line bundle $\mathscr{L}_1^{\otimes N_1} \otimes \mathscr{L}_2^{\otimes N_2} \otimes \cdots \otimes \mathscr{L}_N^{\otimes N_n}$ is ample relative to X for every $N_1 \gg N_2 \gg \cdots \gg N_n$.*

Proof. The first part is immediate and the second part follows from the following lemma and the classical result for compositions of projective morphisms [EGA$_{\mathrm{II}}$, Prop. 4.6.13 (ii)]. □

Lemma 1.8.2. *Let $f\colon X\to Y$ be a qcqs morphism of algebraic stacks with finite inertia and let $f_{\mathrm{cs}}\colon X_{\mathrm{cs}}\to Y_{\mathrm{cs}}$ be the induced morphism on coarse spaces. Let $\mathscr{L}$ be an invertible sheaf on X_{cs}. If $\mathscr{L}|_X$ is f-ample, then $\mathscr{L}$ is f_{cs}-ample.*

Proof. By definition of ample, we may replace X with $X_{\mathrm{cs}/Y}$ so that f becomes representable. The question is also local on Y_{cs} so we may assume that Y_{cs} is affine and that Y admits a finite flat presentation $Y'\to Y$ of constant rank d. Let $X'=X\times_Y Y'$ and note that $Y'\to Y_{\mathrm{cs}}$ and $X'\to X_{\mathrm{cs}}$ are affine. Let $x\in|X|$ be a point. We need to find an affine open neighborhood X_g of x for some $g\in\Gamma(X,\mathscr{L}^m)$ such that $(X_g)_{\mathrm{cs}}$ is affine, or equivalently, such that X'_g is affine.

Consider the preimage $Z\subset|X'|$ of x. Since Z is finite and $\mathscr{L}|_{X'}$ is ample, there exists a section $f\in\Gamma(X',\mathscr{L}^n)$ such that $Z\subset X'_f$ [EGA$_{\mathrm{II}}$, Cor. 4.5.4]. The norm of f along $X'\to X$ gives a section $g\in\Gamma(X,\mathscr{L}^{dn})$ such that $Z\subset X'_g\subset X'_f$ and X'_g is affine, cf. [EGA$_{\mathrm{II}}$, Cor. 6.5.7]. □

Remark 1.8.3. One can also describe the iterated stack-theoretic Proj $f\colon X'\to X$ as a single $\mathbb{N}^n$-graded stack-theoretic Proj. When X is noetherian, we can take $R=\bigoplus_{d_1,d_2,\dots,d_n\geq 0} f_*\left(\mathscr{L}_1^{\otimes d_1}\otimes\cdots\otimes\mathscr{L}_n^{\otimes d_n}\right)$. When X is merely quasi-compact and quasi-separated, we can replace R by a sufficiently large subalgebra of finite type.

2. Examples of stack-theoretic Proj

In this section we give four examples of stack-theoretic Proj: (1) twisted weighted projective stacks, which include root stacks of line bundles, (2) root stacks of generalized Cartier divisors, (3) stacks that make $\mathbb{Q}$-invertible sheaves invertible, and (4) stack-theoretic amplifications of GIT quotients.

2.1. **Weighted projective stacks, root stacks of line bundles and twisted weighted vector bundles.**

Example 2.1.1 (Weighted projective stacks [AH10, §2.1])**.** An important class of examples of stack-theoretic Proj is *weighted projective stacks.* Given weights $d_0,d_1,\dots,d_n\in\mathbb{Z}_{\geq 1}$ we have the smooth stack

$$\mathscr{P}(d_0,d_1,\dots,d_n)=\mathscr{P}\mathrm{roj}_X\big(\mathscr{O}_X[x_0,x_1,\dots,x_n]\big)$$

where the degree of x_i is d_i. The generic stabilizer is $\boldsymbol{\mu}_d$, where $d=\gcd(d_0,d_1,\dots,d_n)$, and the coarse space is the usual, singular, weighted projective space $\mathbb{P}(d_0,d_1,d_2,\dots,d_n)$. Slightly more general, given vector bundles $\mathscr{E}_1,\mathscr{E}_2,\dots,\mathscr{E}_r$ on X and weights $d_1,\dots,d_r\in\mathbb{Z}_{\geq 1}$, the *weighted* (or *graded*) vector bundle $\mathscr{E}=\mathscr{E}_1(-d_1)\oplus\cdots\oplus\mathscr{E}_r(-d_r)$ gives the smooth stack

$$\mathscr{P}(\mathscr{E})=\mathscr{P}\big(\mathscr{E}_1(-d_1)\oplus\cdots\oplus\mathscr{E}_r(-d_r)\big):=\mathscr{P}\mathrm{roj}_X\left(\bigotimes_{i=1}^{r}\mathrm{Sym}_{\mathscr{O}_X}\big(\mathscr{E}_i(-d_i)\big)\right).$$

The universal property of this stack is as follows: given a morphism $f\colon T\to X$, a lift to $\mathscr{P}(\mathscr{E})$ corresponds to the data of a line bundle $\mathscr{L}$ on T and homomorphisms $\varphi_i\colon f^*\mathscr{E}_i\to\mathscr{L}^{\otimes d_i}$ such that locally on T at least one of the φ_i is surjective. An isomorphism between two lifts $(\mathscr{L},\{\varphi_i\})$ and $(\mathscr{L}',\{\varphi'_i\})$ is an isomorphism $\mathscr{L}\simeq\mathscr{L}'$ compatible with the φ_i and φ'_i.

Example 2.1.2 (Root stacks of line bundles [Cad07, Def. 2.2.6])**.** A special case of the previous example is roots of line bundles. Given a line bundle $\mathscr{E}$ on X and a positive integer d, the stack $\mathscr{P}\big(\mathscr{E}(-d)\big)$ parameterizes, for a morphism of schemes $f\colon T \to X$, a line bundle $\mathscr{L}$ on T together with an isomorphism $f^*\mathscr{E} \xrightarrow{\simeq} \mathscr{L}^{\otimes d}$. The corresponding graded algebra is $R = \operatorname{Sym}_{\mathscr{O}_X}\big(\mathscr{E}(-d)\big)$. We call the corresponding stack-theoretic Proj the d^{th} *root stack of the line bundle* $\mathscr{E}$, and denote it by $X_{(\mathscr{E},d)}$ or $X\big(\sqrt[d]{\mathscr{E}}\big)$.

We will need the following generalization of weighted vector bundles:

Definition 2.1.3. A *twisted weighted vector bundle* on X is a smooth affine morphism $E \to X$ with a $\mathbb{G}_m$-action such that E is smooth-locally $\mathbb{G}_m$-equivariantly isomorphic to $X \times \mathbb{A}^n$ where $\mathbb{G}_m$ acts linearly with some weights $d_1, d_2, \dots, d_n \in \mathbb{Z}$.

The morphism $E \to X$ is called a $\mathbb{G}_m$-fibration in [BB73, §3]. Equivalently, $E = \operatorname{Spec}_X(R)$ where R is a quasi-coherent graded $\mathscr{O}_X$-algebra that smooth-locally looks like the symmetric algebra over $\mathscr{O}_X$ of a graded vector bundle.

We will only need the case where all weights are positive (called fully definite in [BB73, §2]). If X is a scheme and all weights are positive, then smooth-locally can be replaced with Zariski-locally, see Remark 2.1.8.

Example 2.1.4 (Białynicki-Birula decomposition [BB73, Thm. 4.1])**.** Let X be a smooth quasi-projective variety with an action of $\mathbb{G}_m$. Let $F \subset X^{\mathbb{G}_m}$ be a connected component of the fixed locus. Let $F^+ = \{x \in X \,:\, \lim_{t\to 0} t.x \in F\}$. Then F and F^+ are $\mathbb{G}_m$-equivariant, F is closed, F^+ is locally closed, F and F^+ are smooth, and the natural map $F^+ \to F$ is a twisted weighted vector bundle with strictly positive weights.

Definition 2.1.5. A *twisted weighted projective stack* over X is the stack-theoretic Proj of a graded algebra corresponding to a twisted weighted bundle on X with strictly positive weights.

In what follows, we always assume that $E = \operatorname{Spec}_X(R)$ is a twisted weighted bundle over a *connected* scheme X. Then there exist a Zariski open cover U_i of X, weights $\mathbf{d} = (d_1 < d_2 < \cdots < d_r)$, and dimensions $\mathbf{n} = (n_1, n_2, \dots, n_r) \in \mathbb{Z}_{\geq 1}$, such that for every i,

$$R|_{U_i} \simeq \operatorname{Sym}\left(\bigoplus_{i=1}^{r} \mathscr{O}_{U_i}^{\oplus n_i}(-d_i)\right) = \bigotimes_{i=1}^{r} \operatorname{Sym}\big(\mathscr{O}_{U_i}^{\oplus n_i}(-d_i)\big).$$

2.1.6. *An example.* For a non-trivial example of a twisted weighted bundle, let us consider the weights $\mathbf{d} = (d_1, d_2, d_3) = (1, 2, 4)$ and the dimensions $\mathbf{n} = (n_1, n_2, n_3) = (1, 1, 1)$. Over each U_i, we have a graded isomorphism:

$$\alpha_i\colon R|_{U_i} \xrightarrow{\simeq} \mathscr{O}_{U_i}[x_i, y_i, z_i],$$

where x_i has weight 1, y_i has weight 2, and z_i has weight 4. Over pairwise intersections $U_{ij} := U_i \cap U_j$, we then have the graded isomorphism on U_{ij}:

$$\alpha_{ij} = \alpha_j|_{U_{ij}} \circ \alpha_i|_{U_{ij}}^{-1}\colon \mathscr{O}_{U_{ij}}[x_i, y_i, z_i] \xrightarrow{\simeq} R|_{U_{ij}} \xrightarrow{\simeq} \mathscr{O}_{U_{ij}}[x_j, y_j, z_j].$$

By considering the linear relations among the weights 1, 2, and 4, we deduce that these α_{ij}'s have the following form in general:

$$\begin{aligned} x_i &\mapsto a_{ij} \cdot x_j \\ y_i &\mapsto b_{ij} \cdot \big(y_j + d_{ij} \cdot x_j^2\big) \\ z_i &\mapsto c_{ij} \cdot \big(z_j + e_{ij} \cdot x_j^4 + f_{ij} \cdot x_j^2 y_j + g_{ij} \cdot y_j^2\big) \end{aligned}$$

where $a_{ij}, b_{ij}, c_{ij} \in \Gamma(U_{ij}, \mathscr{O}_{U_{ij}}^{\times})$ and $d_{ij}, e_{ij}, f_{ij}, g_{ij} \in \Gamma(U_{ij}, \mathscr{O}_{U_{ij}})$. Note that the data $d_{ij}, e_{ij}, f_{ij}, g_{ij}$ are precisely the "twists" in the twisted weighted bundle R. Over triple intersections $U_{ijk} := U_i \cap U_j \cap U_k$, we have the following cocycle conditions:

$$\begin{aligned} a_{ik} &= a_{ij} a_{jk} \\ b_{ik} &= b_{ij} b_{jk} \\ c_{ik} &= c_{ij} c_{jk} \\ d_{ik} &= d_{jk} + \frac{a_{jk}^2}{b_{jk}} d_{ij} \end{aligned}$$

$$\begin{pmatrix} e_{ik} \\ f_{ik} \\ g_{ik} \end{pmatrix} = \begin{pmatrix} e_{jk} \\ f_{jk} \\ g_{jk} \end{pmatrix} + \frac{1}{c_{jk}} \begin{pmatrix} a_{jk}^4 & a_{jk}^2 b_{jk} \cdot d_{jk} & b_{jk}^2 \cdot d_{jk}^2 \\ 0 & a_{jk}^2 b_{jk} \cdot 1 & b_{jk}^2 \cdot 2d_{jk} \\ 0 & 0 & b_{jk}^2 \cdot 1 \end{pmatrix} \begin{pmatrix} e_{ij} \\ f_{ij} \\ g_{ij} \end{pmatrix}$$

Therefore, twisted weighted bundles R on X with weights $\mathbf{d} = (1, 2, 4)$ and dimensions $\mathbf{n} = (1, 1, 1)$ are globally characterized by the Čech cocycles in $\check{\mathrm{H}}^1\big(X, G\big)$, where

$$G = \big(\mathbb{G}_m \times ((\mathbb{G}_m \times \mathbb{G}_m) \ltimes \mathbb{G}_a)\big) \ltimes \mathbb{G}_a^3,$$

and the semidirect product are given by the actions:

(i) $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \to \mathrm{Aut}(\mathbb{G}_a)$ where $(a, b) \mapsto \frac{a^2}{b}$,
(ii) $\mathbb{G}_m \times ((\mathbb{G}_m \times \mathbb{G}_m) \ltimes \mathbb{G}_a) \to \mathrm{GL}_3 \to \mathrm{Aut}(\mathbb{G}_a^3)$ where

$$(c, a, b, d) \longmapsto \frac{1}{c} \begin{pmatrix} a^4 & a^2 bd & b^2 d^2 \\ 0 & a^2 b & 2b^2 d \\ 0 & 0 & b^2 \end{pmatrix}.$$

2.1.7. *General description of twisted weighted bundles.* In general, twisted weighted bundles R on X with weights $\mathbf{d}$ and dimensions $\mathbf{n}$ are globally characterized by their respective Čech cocycles in $\check{\mathrm{H}}^1(X, G_{\mathbf{d},\mathbf{n}})$, where the group $G_{\mathbf{d},\mathbf{n}}$ can be described as follows. If all weights are equal, that is, $r = 1$, then $G_{d_1,n_1} = \mathrm{GL}_{n_1}$ and twisted weighted vector bundles are just weighted vector bundles. If not, that is $r > 1$, set $\mathbf{d}' = (d_1, \ldots, d_{r-1})$, $\mathbf{n}' = (n_1, \ldots, n_{r-1})$, and $G_{\mathbf{d},\mathbf{n}}$ is a semidirect product of the form

$$G_{\mathbf{d},\mathbf{n}} = \big(\mathrm{GL}_{n_r} \times G_{\mathbf{d}',\mathbf{n}'}\big) \ltimes \mathbb{G}_a^{n_r N_r},$$

where N_r is the dimension of the ${d_r}^{\text{th}}$ degree piece of a graded polynomial algebra with free variables $\{x_{i,j} \colon 1 \leq i \leq r-1,\ 1 \leq j \leq n_i\}$, where $x_{i,j}$ is given weight d_i. That is,

$$N_r = \sum_{d_r = \sum_{m_i \geq 0} m_i d_i} \prod_{i=0}^{r-1} \binom{m_i + n_i - 1}{m_i}.$$

Remark 2.1.8. From the description of $G_{\mathbf{d},\mathbf{n}}$ we obtain an exact sequence

$$1 \longrightarrow U_{\mathbf{d},\mathbf{n}} \longrightarrow G_{\mathbf{d},\mathbf{n}} \longrightarrow \mathrm{GL}_{\mathbf{n}} \longrightarrow 1$$

where $U_{\mathbf{d},\mathbf{n}}$ is a smooth connected unipotent group scheme of dimension $N(\mathbf{d},\mathbf{n}) = \sum_{i=2}^{r} n_i N_i$ and $\mathrm{GL}_{\mathbf{n}} = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$. In particular, $G_{\mathbf{d},\mathbf{n}}$ is *special* in the sense of Serre, that is, the Čech cohomology $\check{\mathrm{H}}^1(X, G_{\mathbf{d},\mathbf{n}})$ can be calculated in the Zariski topology if X is a scheme.

In particular, if d_i is not in the $\mathbb{Z}_{\geq 0}$-linear span of $d_1, \ldots, d_{i-1}$, for every $1 \leq i \leq r$, then $N(\mathbf{d},\mathbf{n}) = 0$ and any twisted weighted bundle E with weights $(d_1, \ldots, d_r)$ *is a weighted bundle*, i.e., E "*splits*" as

$$E \simeq \mathrm{Spec}_X\big(\mathrm{Sym}_{\mathscr{O}_X}(\mathscr{E}_1(-d_1) \oplus \cdots \oplus \mathscr{E}_r(-d_r))\big)$$

for vector bundles $\mathscr{E}_1, \ldots, \mathscr{E}_r$ on X with respective dimensions $n_1, \ldots, n_r$.

2.1.9. *Associated weighted vector bundle.* To a twisted weighted vector bundle $E = \mathrm{Spec}_X(R)$ we can associate a weighted vector bundle $\mathscr{E} := R_+/R_+^2$ on X. This is nothing but the vector bundle corresponding to the image of E under $\check{\mathrm{H}}^1(X, G_{\mathbf{d},\mathbf{n}}) \to \check{\mathrm{H}}^1(X, \mathrm{GL}_{\mathbf{n}})$. Since $\mathscr{E}$ is locally free, the quotient morphism $R_+ \to R_+/R_+^2 = \mathscr{E}$ *locally* admits a graded section, which *locally* induces a graded isomorphism $\mathrm{Sym}_{\mathscr{O}_X}(\mathscr{E}) \xrightarrow{\simeq} R$. One can interpret the presence of the "twists" in the twisted weighted bundle R as the obstructions to patching these local isomorphisms to a global isomorphism. Note that the weights $\mathbf{d}$ and the dimension $\mathbf{n}$ can be read off from $\mathscr{E}$.

2.2. **Root stacks of (generalized) Cartier divisors.**

Definition 2.2.1. A *generalized (effective) Cartier divisor* on X is a pair $(\mathscr{L}, s)$ where $\mathscr{L}$ is a line bundle and $s \in \Gamma(X, \mathscr{L})$ is a global section. Equivalently, s gives a homomorphism $s^\vee \colon \mathscr{L}^\vee \to \mathscr{O}_X$. We say that $(\mathscr{L}, s)$ is *ordinary* if $s^\vee$ is injective, or equivalently, if $(\mathscr{L}, s) = (\mathscr{O}_X(D), s_D)$ for an effective Cartier divisor D. Then $s^\vee$ is the inclusion of the ideal $I_D = \mathscr{O}_X(-D)$.

Example 2.2.2 (Root stacks of generalized divisors [Cad07, Def. 2.2.1])**.** Given a generalized Cartier divisor $(\mathscr{L}, s)$ on X and a positive integer d, we consider the following graded $\mathscr{O}_X$-algebra

$$R = \bigoplus_{n \geq 0} \mathscr{L}^{-\lceil n/d \rceil}, \tag{2.1}$$

where the multiplication in this algebra makes sense by using the homomorphism $s^\vee \colon \mathscr{L}^\vee \to \mathscr{O}_X$ whenever applicable. For example, for $0 \leq k, \ell \leq d$, the multiplication $R_k \otimes R_\ell \to R_{k+\ell}$ is the canonical $\mathscr{L}^\vee \otimes \mathscr{L}^\vee \to (\mathscr{L}^\vee)^{\otimes 2}$ if $k + \ell > d$, and is given by $\mathscr{L}^\vee \otimes \mathscr{L}^\vee \to (\mathscr{L}^\vee)^{\otimes 2} \xrightarrow{\mathbb{1} \otimes s^\vee} \mathscr{L}^\vee$ if $k + \ell \leq d$. We denote the corresponding stack-theoretic Proj by $X_{(\mathscr{L}, s, d)}$, and call it the d^{th} *root stack of* $(\mathscr{L}, s)$.

The root stack $X_{(\mathscr{L}, s, d)}$ has the following universal property: if $f \colon T \to X$ is a morphism then a lift to the root stack is equivalent to giving a generalized Cartier divisor $(\mathscr{E}, t)$ on T together with an isomorphism $\varphi \colon f^*(\mathscr{L}, s) \xrightarrow{\simeq} (\mathscr{E}, t)^d$. The corresponding universal generalized Cartier divisor on the root stack is $(\mathscr{O}(-1), t)$ where $t^\vee$ is given by the natural map $R(1) \to R$. An isomorphism between two lifts $(\mathscr{E}, t, \varphi)$ and $(\mathscr{E}', t', \varphi')$ is an isomorphism $\mathscr{E} \to \mathscr{E}'$ compatible with t, t', φ and φ'.

Forgetting the section induces a morphism $X_{(\mathscr{L},s,d)} \to X_{(\mathscr{L},d)}$ to the root stack of Example 2.1.2. Note that $X_{(\mathscr{L},d)} \simeq X_{(\mathscr{L}^\vee,d)}$.

Since $R^{(d)} = \mathrm{Sym}_{\mathscr{O}_X}(\mathscr{L}^\vee)$, it follows that $\pi\colon X_{(\mathscr{L},s,d)} \to X$ is a coarse space. Since R is flat, π is also flat.

Remark 2.2.3. When $\mathscr{L}$ is trivial, then $\mathscr{P}\mathrm{roj}_X(R)$ is covered by a single chart as follows. Let $f \in \Gamma(X, \mathscr{L})$ be an everywhere non-vanishing section. Then $\mathscr{P}\mathrm{roj}_X(R) = D_+(f) = \big[\mathrm{Spec}_X\big(R/(f-1)\big) \,/\, \boldsymbol{\mu}_d\big]$. Note that $R/(f-1) \simeq \mathscr{O}_X[x]/(x^d - \frac{s}{f})$ where $\deg(x) = -1$.

More generally, if there exists a line bundle $\mathscr{E}$ such that $\mathscr{L} \simeq \mathscr{E}^d$, then we can write the root stack as a global quotient by $\boldsymbol{\mu}_d$ by first twisting R with $\mathscr{E}$ so that

$$\mathscr{P}\mathrm{roj}_X(R) = \left[\mathrm{Spec}_X\left(\bigoplus_{n=0}^{d-1} \mathscr{E}^{-n}\right) \,/\, \boldsymbol{\mu}_d\right]$$

where $\mathscr{E}^{-n}$ has degree $-n$ and the multiplication is induced by $s^\vee\colon \mathscr{E}^{-d} = \mathscr{L}^\vee \to \mathscr{O}_X$.

Example 2.2.4 (Root stacks of ordinary divisors)**.** If D is an effective Cartier divisor on X, the previous construction gives the following graded $\mathscr{O}_X$-graded algebra:

$$R = \bigoplus_{n \geq 0} I_D^{\lceil n/d \rceil}.$$

We sometimes denote $X_{(\mathscr{O}_X(D),s_D,d)}$ by $X_{(D,d)}$ or $X\big(\sqrt[d]{D}\big)$ instead, and call it the d^{th} root stack of D. For morphisms $f\colon T \to X$ such that $f^{-1}(D)$ is a Cartier divisor, the root stack has the following universal property: a lift to the root stack is equivalent to giving an effective Cartier divisor E on T such that $dE = f^{-1}(D)$. In particular, the groupoid $X_{(D,d)}(T \to X)$ is equivalent to a set in this case, that is, there are no non-trivial isomorphisms between lifts.

The morphism $\pi\colon X_{(D,d)} \to X$ is a flat coarse space which is an isomorphism outside D. The morphism $E \to D$ is a gerbe isomorphic to the d^{th} root stack of the line bundle $\mathscr{O}_D(D)$.

2.3. **Inverting $\mathbb{Q}$-invertible sheaves.**

Setup 2.3.1. Let X be a noetherian scheme satisfying Serre's condition S_2 (for example, a normal scheme). Let F be a coherent $\mathscr{O}_X$-module that is *generically locally free*, that is, there exists a dense open $j_V\colon V \hookrightarrow X$ on which F is locally free. If $F|_V$ is locally free of rank r, then we say that F has *rank* r. Let $\mathrm{tor}(F)$ be the torsion submodule of F, i.e.,

$$\mathrm{tor}(F) := \ker\big(F \to j_{V*}(F|_V)\big)$$

and set $F_{\mathrm{tf}} := F/\mathrm{tor}(F)$. Note that $\mathrm{tor}(F)$ is independent of V since X has no embedded points.

Suppose that $j\colon U \hookrightarrow X$ is an open subset whose complement *has codimension* ≥ 2 *in* X, and on which F_{tf} is locally free. If X is normal, then this can always be arranged for since F_{tf} is free at every point of codimension 1.

Lemma 2.3.2. *The canonical morphism $F \to F^{\vee\vee}$ can be identified with the canonical morphism $F \to F_{\mathrm{tf}} \to j_*(F_{\mathrm{tf}}|_U)$.*

We call $F^{\vee\vee} = j_*(F_{\mathrm{tf}}|_U)$ the *reflexive hull* of F. We say F is *reflexive*, if the canonical morphism $F \to F^{\vee\vee} = j_*(F_{\mathrm{tf}}|_U)$ is an isomorphism.

Proof. Firstly, since X is S_2, $F^{\vee\vee}$ is also S_2, i.e., $j_*(F^{\vee\vee}|_U) = F^{\vee\vee}$. Next, since $F^{\vee\vee}$ is torsion-free, $F \to F^{\vee\vee}$ factors through F_{tf}, and the resulting morphism $F_{\mathrm{tf}} \to F^{\vee\vee}$ induces a morphism $j_*(F_{\mathrm{tf}}|_U) \to j_*(F^{\vee\vee}|_U) = F^{\vee\vee}$ of S_2-sheaves on X that is an isomorphism on U, and hence is an isomorphism on X. □

For every integer $n \geq 0$, we define the *saturated n^{th} power* of F to be $F^{[n]} := (F^{\otimes n})^{\vee\vee}$. Note that $F^{[n]} = j_*(F_{\mathrm{tf}}^{\otimes n}|_U)$ since $F_{\mathrm{tf}}|_U$ is locally free.

We say that F is $\mathbb{Q}$*-invertible* if $F^{[N]}$ is invertible for some positive integer N, locally on X. Note that a $\mathbb{Q}$-invertible sheaf has rank 1. In what follows, we consider the graded $\mathscr{O}_X$-algebra

$$F^{[\bullet]} = \bigoplus_{n \geq 0} F^{[n]}.$$

Proposition 2.3.3 (cf. [AH10, Prop. 5.3.2])**.** *Let $X' = \mathscr{P}\mathrm{roj}_X(F^{[\bullet]})$ with structure morphism $\pi\colon X' \to X$. If F is $\mathbb{Q}$–invertible, then:*

(i) *$F^{[\bullet]}$ is finitely generated, and hence, X' is proper over X.*
(ii) *$\pi\colon X' \to X$ is a coarse space and an isomorphism over U. In particular, π is quasi-finite.*
(iii) *If $F^{[N]} \simeq \mathscr{O}_X$ for some $N \geq 1$, then $X' = \left[\mathrm{Spec}_X \left(\bigoplus_{n=0}^{N-1} F^{[n]}\right) / \boldsymbol{\mu}_N\right]$.*
(iv) *X' satisfies S_2. Moreover, if X is normal, so is X'.*
(v) *For every positive integer n, the canonical morphism $F^{[n]} \to \pi_*\mathscr{O}_{X'}(n)$ is an isomorphism, and the canonical morphism $\pi^*F^{[n]} \to \mathscr{O}_{X'}(n)$ is a reflexive hull.*
(vi) *$\pi\colon X' \to X$ satisfies the following universal property: if T is a scheme satisfying S_2 (resp. T is a normal scheme) which admits a morphism $f\colon T \to X$ such that $\mathrm{codim}_T\big(T \smallsetminus f^{-1}(U)\big) \geq 2$ (resp. $f^{-1}(U)$ is dense in T), then there is a lift of f to X', unique up to a unique 2-isomorphism, if and only if $(f^*F)^{\vee\vee}$ is invertible.*

Before proving the proposition, we note that in (vi), the hypothesis that $\mathrm{codim}_T(T \smallsetminus f^{-1}(U)) \geq 2$ is satisfied whenever f satisfies one of the following conditions:

(a) f is flat;
(b) f is dominant and integral and T is integral; or
(c) f is dominant and quasi-finite and T is integral.

Proof of Proposition 2.3.3. All statements, except (iii), are local on X so we may assume that $F^{[N]}$ is invertible for some integer N. For (i), note that the multiplication $F^{[kN]} \otimes F^{[n]} \to F^{[kN+n]}$ is an isomorphism for all integers $k, n \geq 0$. Thus $F^{[\bullet]}$ is generated in degrees $\leq N$. Since $F^{[n]}$ is coherent for every n, we deduce that $F^{[\bullet]}$ is finitely generated. Thus, X' is proper over X by Proposition 1.6.1(ii).

For (ii), we note that $F^{[N\bullet]}$ is generated in degree 1 and thus that the coarse space of X' is $\mathrm{Proj}_X(F^{[\bullet]}) = \mathrm{Proj}_X(F^{[N\bullet]}) = \mathbb{P}(F^{[N]}) = X$ (Proposition 1.6.1(iii)). Moreover, since $F_{\mathrm{tf}}|_U$ is invertible, π is an isomorphism over U. For (iii), this follows from $X' = D_+(f)$ where f is a nowhere vanishing section of $F^{[N]}$ (§1.3). For (iv), the question is local so we can assume that $F^{[N]} \simeq \mathscr{O}_X$ and hence that we have a faithfully flat presentation $\mathrm{Spec}_X\big(\bigoplus_{n=0}^{N-1} F^{[n]}\big) \to X'$. The result follows since $\bigoplus_{n=0}^{N-1} F^{[n]}$ is a coherent S_2-sheaf.

For (v), let $U' := \pi^{-1}(U)$, and consider the cartesian square:

$$\begin{tikzcd} U' \arrow[r, hook, "j'"] \arrow[d, "\simeq"'] & X' \arrow[d, "\pi"] \\ U \arrow[r, hook, "j"] & X. \end{tikzcd}$$

Since $F^{[n]}|_U = F_{\mathrm{tf}}^{\otimes n}|_U$ is invertible, the canonical morphism $\pi^*F^{[n]} \to \mathscr{O}_{X'}(n)$ is an isomorphism when restricted to U'. Moreover, since $\mathscr{O}_{X'}(n)$ is invertible, it is S_2 and thus $\mathscr{O}_{X'}(n) = j'_*(\pi^*F^{[n]}|_{U'})$, i.e., $\mathscr{O}_{X'}(n)$ is the reflexive hull of $\pi^*F^{[n]}$. We also have that:

$$F^{[n]} \simeq j_*j^*F^{[n]} \simeq \pi_*j'_*j'^*\pi^*F^{[n]} \simeq \pi_*\mathscr{O}_{X'}(n).$$

Finally, for (vi), by Proposition 1.5.1 a morphism $T \to X'$ corresponds to a line bundle $\mathscr{L}$ on T and a graded homomorphism $\varphi\colon f^*F^{[\bullet]} \to \bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}$ of sheaves on T such that $\varphi_n\colon f^*F^{[n]} \to \mathscr{L}^{\otimes n}$ is surjective for sufficiently divisible n, or equivalently, such that the induced $(\varphi_n)_{\mathrm{tf}}\colon (f^*F^{[n]})_{\mathrm{tf}} \to \mathscr{L}^{\otimes n}$ is surjective for sufficiently divisible n. Since $F^{[n]}|_U$ is invertible, $f^*F^{[n]}$ is invertible over $f^{-1}(U)$, so $(\varphi_n)_{\mathrm{tf}}|_{f^{-1}(U)}$ is an isomorphism for sufficiently divisible n, and hence an isomorphism for all n. This means the following:

(a) If T is S_2, then by hypothesis, $(\varphi_n)_{\mathrm{tf}}$ is an isomorphism away from codimension ≥ 2 for all n. Since $\mathscr{L}^{\otimes n}$ is invertible, $(\varphi_n)_{\mathrm{tf}}$ is the reflexive hull for all n, and thus so is φ_n.

(b) If T is normal, then by hypothesis, $(\varphi_n)_{\mathrm{tf}}$ is generically an isomorphism for all n. In addition, Serre's condition R_1 implies that $(f^*F^{[n]})_{\mathrm{tf}}$ is invertible in codimension 1, so $(\varphi_n)_{\mathrm{tf}}$ is an isomorphism in codimension 1 for sufficiently divisible n, and hence an isomorphism in codimension 1 for all n. In conclusion, $(\varphi_n)_{\mathrm{tf}}$ is an isomorphism away from codimension ≥ 2, and the same argument as in (a) shows that φ_n is the reflexive hull for all n.

In either case, we conclude that such an $\mathscr{L}$ and φ exist precisely when $(f^*F)^{\vee\vee}$ is invertible and then $\mathscr{L} = (f^*F)^{\vee\vee}$. Finally, note that if $F^{[N]}$ is invertible, then φ_N is an isomorphism and in particular surjective. $\square$

Remark 2.3.4. When $F^{[N]}$ is invertible, the construction $X' \to X$ that makes F invertible is closely related to taking the N^{th} root of the invertible sheaf $F^{[N]}$ (Example 2.1.2). Since $\mathscr{O}_{X'}(1)^{\otimes n} = \mathscr{O}_{X'}(n) = \pi^*F^{[N]}$, there is a canonical map $\varphi\colon X' \to X\big(\sqrt[N]{F^{[N]}}\big)$ over X. This map is representable, hence finite, since φ^* is compatible with tautological line bundles. That is, φ is also induced, via Proposition 1.6.1(vi), by the graded homomorphism

$$\mathrm{Sym}_{\mathscr{O}_X}\big(F^{[N]}(-N)\big) \to \bigoplus_{n\geq 0} F^{[N]}.$$

The finite morphism φ is *not* an isomorphism. In fact, the root stack is a gerbe, whereas $X' \to X$ is generically an isomorphism (a *stacky modification*, i.e., proper and generically an isomorphism). On the root stack, $F^{[N]}$ has an N^{th} root, but it does not coincide with the reflexive hull of the pull-back of F because it does not agree over U. This is explained by the presence of torsion in the Picard group in the root stack over U.

Example 2.3.5 ($\mathbb{Q}$-Gorenstein varieties)**.** We now apply Proposition 2.3.3 to the canonical sheaf. Let X be a $\mathbb{Q}$-*Gorenstein variety of index* N, that is, a normal variety of such that the N^{th} pluricanonical divisor NK_X is Cartier. Let $\omega_X^{[n]} = \mathscr{O}_X(nK_X)$ denote the n^{th} pluricanonical sheaf, or equivalently, $(\omega_X^{\otimes n})^{\vee\vee}$. Then $\omega_X^{[n]}$ is a reflexive sheaf of rank 1, which is invertible whenever N divides n. Let $X' = \mathscr{P}\mathrm{roj}_X\big(\omega_X^{[\bullet]}\big)$. Then $\pi\colon X' \to X$ is an isomorphism over the locus where ω_X is invertible, i.e., where X is quasi-Gorenstein, that is, $\mathbb{Q}$-Gorenstein of index 1. The coarse moduli space of X' is $\mathrm{Proj}_X\big(\omega_X^{[N\bullet]}\big)$ which equals X, and $\pi_*\mathscr{O}(n) = \omega_X^{[n]}$ for every positive integer n. The morphism $\pi\colon X' \to X$ only adds some stackiness in codimension ≥ 2. Finally, the canonical sheaf $\omega_{X'}$ is $(\pi^*\omega_X)^{\vee\vee}$, and hence is equal to $\mathscr{O}_{X'}(1)$.

Example 2.3.6 (Cartierification)**.** More generally, fix a normal noetherian scheme X, with an effective $\mathbb{Q}$-Cartier divisor $D \subset X$, say ND is Cartier. Let $I_D = \mathscr{O}_X(-D)$ be the ideal of $D \subset X$, which is a reflexive $\mathscr{O}_X$-submodule of $\mathscr{O}_X$ of rank one. Let U denote the largest open subset of X on which $D|_U$ is Cartier (i.e., $I_D|_U = I_{D|_U}$ is an invertible $\mathscr{O}_U$-submodule of $\mathscr{O}_U$), so that $I_D = j_*(I_{D|_U})$. Recall that $U \supset \mathrm{Reg}(X)$ (the latter has complement of codimension ≥ 2 in X), and moreover note that $U \supset Y \smallsetminus D$.

We apply Proposition 2.3.3 with $F = I_D$. Note that $F^{[n]} = I_D^{[n]}$ is precisely the n^{th} *symbolic power* I_{nD} of I_D, since all the associated points of I_D are non-embedded, and hence, are contained in U. The $\mathscr{O}_X$-algebra $F^{[\bullet]} = I_{\bullet D} = \bigoplus_{n\geq 0} I_{nD}$ is called the *symbolic Rees algebra* of D (or I_D). Let $X' = \mathscr{P}\mathrm{roj}_X(I_{\bullet D})$ with structure morphism $\pi\colon X' \to X$. The inverse image of D under π is a Cartier divisor (v), and π satisfies the following universal property (vi): if $f\colon T \to X$ is a morphism from a normal scheme T such that $f^{-1}(D)$ is nowhere dense in T, then f factors, uniquely, through π if and only the *inverse image* f^*D of D under f is an effective Cartier divisor on T. Here, f^*D is the Weil divisor on T whose underlying ideal sheaf is $(f^*I_D)^{\vee\vee}$.

As a final note, we will see later in Example 3.3.7 that $\bigoplus_{n\geq 0} I_{nD}$ is the integral closure of the $\mathscr{O}_X$-subalgebra $\bigoplus_{n\geq 0} I_{ND}^{\lceil n/N\rceil}$ of Example 2.2.4. In other words, there is a canonical finite morphism $X' = \mathscr{P}\mathrm{roj}_X(I_{\bullet D}) \to X\big(\sqrt[N]{ND}\big)$, which presents X' as the *normalization* of $X\big(\sqrt[N]{ND}\big)$.

Example 2.3.7 (Stacky modifications given by inverting $\mathbb{Q}$-invertible sheaves)**.** Suppose that $\pi\colon X' \to X$ is a proper quasi-finite morphism of noetherian stacks satisfying S_2, that $\mathscr{L} \in \operatorname{Pic}(X')$ is an ample and uniformizing line bundle relative to X, and that π is an isomorphism over an open substack $U \subset X$ and that U and $\pi^{-1}(U)$ have complements of codimension at least 2. Then $X' = \mathscr{P}\mathrm{roj}_X(F^{[\bullet]})$ where $F = \pi_*\mathscr{L}$. Indeed, first note that $X' \to X$ is a relative coarse space since $X'_{\mathrm{cs}/X} \to X$ is a finite morphism between S_2-stacks that is an isomorphism outside codimension 2, hence an isomorphism. It follows that $\pi_*\mathscr{L}^{\otimes N}$ is a line bundle for sufficiently divisible N (1.7.2). Moreover, $X' = \mathscr{P}\mathrm{roj}_X\big(\bigoplus_{n\geq 0} \pi_*\mathscr{L}^{\otimes n}\big)$ by Proposition 1.7.6(ii) so it suffices to note that $\pi_*\mathscr{L}^{\otimes n} = \pi_* j'_* j'^* \mathscr{L}^{\otimes n} = j_*(\pi|_U)_* j'^* \mathscr{L}^{\otimes n} = j_* j^* F^{\otimes n} = F^{[n]}$.

Example 2.3.8 (Toric varieties and toric stacks)**.** Let Σ be a simplicial fan. Then the associated toric variety X_Σ is normal and the toric divisors $D_1, D_2, \dots, D_n$ are $\mathbb{Q}$-Cartier. The corresponding toric stack $\mathscr{X}_\Sigma$ is smooth with smooth toric divisors. We thus get a map $\mathscr{X}_\Sigma \to X'$ where $X' \to X_\Sigma$ is the iterated stack-theoretic Proj that makes all the toric divisors Cartier (Example 2.3.6). Since $(\mathscr{O}(D_1), \mathscr{O}(D_2), \dots, \mathscr{O}(D_n))$ is uniformizing on the toric stack $\mathscr{X}_\Sigma$ (by the Cox construction) as well as on X' (see Proposition 1.8.1), it follows that $\mathscr{X}_\Sigma \to X'$ is a representable birational homeomorphism between normal stacks, hence an isomorphism.

The toric stack $\mathscr{X}_\Sigma$ is the *canonical stack* associated to the variety X_Σ with finite quotient singularities [FMN10, §4]. The Cartierification thus gives a different description of the canonical stack for a toric variety. If Σ is a *stacky fan*, then the associated toric stack can be described as the Cartierification of the toric divisors of the associated toric variety followed by taking root stacks of these divisors and then root stacks of line bundles [FMN10, Thm. 1].

2.4. **Stack-theoretic amplification of GIT quotients.** Let X be a projective variety with an action of a reductive group G and let $\mathscr{L}$ be an ample line bundle with a G-action. Then we can form the GIT quotient $X /\!\!/ G = \operatorname{Proj}\big(\bigoplus_{n\geq 0} \Gamma(X, \mathscr{L}^n)^G\big)$. If $X^{\mathrm{ss}} \subseteq X$ denotes the semi-stable locus of X, then $X^{\mathrm{ss}} \to X /\!\!/ G$ is a good quotient. This can also be phrased as saying that $[X^{\mathrm{ss}} / G] \to X /\!\!/ G$ is a good moduli space.

It is very natural to also consider the "stack-theoretic GIT quotient" $[X /\!\!/ G] = \mathscr{P}\mathrm{roj}\big(\bigoplus_{n\geq 0} \Gamma(X, \mathscr{L}^n)^G\big)$. Whereas $[X^{\mathrm{ss}} / G]$ is typically an Artin stack with infinite stabilizers, the stack $[X /\!\!/ G]$ is a tame Artin stack with finite cyclic stabilizers. To summarize, we have

$$[X^{\mathrm{ss}} / G] \xrightarrow{\text{rel. good mod. space}} [X^{\mathrm{ss}} /\!\!/ G] \xrightarrow{\text{tame coarse space}} X^{\mathrm{ss}} /\!\!/ G.$$

The stack-theoretic GIT quotient $[X /\!\!/ G]$ was studied by Hassett [Has05, §3.1] and Gulbrandsen [Gul11] when X is the projective space of hypersurfaces in $\mathbb{P}^n$ of degree d and $G = \mathrm{SL}(n+1)$ for small d and n.

3. Rees algebras and weighted blow-ups

3.1. Definition of Rees algebras.

Definition 3.1.1 (Rees algebras)**.** A *Rees algebra* on X is a quasi-coherent, finitely generated, graded $\mathscr{O}_X$-subalgebra $R = \bigoplus_{n\geq 0} I_n \cdot t^n$ of $\mathscr{O}_X[t]$ such that $I_0 = \mathscr{O}_X$ and $I_n \supset I_{n+1}$ for every $n \in \mathbb{N}$. Equivalently, a Rees algebra is a descending filtration

$$I_\bullet = (I_0 \supset I_1 \supset I_2 \supset \dots)$$

of ideals of $\mathscr{O}_X$, satisfying the following conditions:

(i) $I_0 = \mathscr{O}_X$;
(ii) $I_n I_m \subset I_{n+m}$ for every n, m;
(iii) locally on X, there exists a sufficiently large positive integer d such that for all integers $n \geq 1$,

$$I_n = \left(I_1^{\ell_1} I_2^{\ell_2} \cdots I_d^{\ell_d} \colon \ell_i \in \mathbb{N},\ \sum_{i=1}^{d} i\ell_i = n \right),$$

in which case, we say that $I_\bullet$ is *generated in degrees* $\leq d$. Equivalently, the graded module R_+/R_+^2 is concentrated in degrees $\leq d$.

Rees algebras are partially ordered by inclusion. The initial object is the *zero Rees algebra* which is $\mathscr{O}_X$ in degree 0 and zero in positive degrees. For any positive integer d, we write $I_{d\bullet}$ for the d^{th} Veronese subalgebra of $I_\bullet$.

Remark 3.1.2.

(i) That $I_\bullet$ is generated in degrees $\leq d$ does not imply that the Veronese subalgebra $I_{\ell\bullet}$ is generated in degree 1 for ℓ the least common multiple of $1, 2, \dots, d$, see Remark 1.6.4. But it does imply that the Veronese subalgebra $I_{d\bullet}$ is generated in degree 1 for sufficiently divisible d, see Proposition 3.1.7 below.
(ii) Rees algebras are called *idealistic filtrations* by Kawanoue [Kaw07]. Moreover, the element $gt^n \in I_n t^n$ is also written there as (g, n); however, we shall reserve that notation for the smallest Rees algebra containing gt^n.
(iii) Encinas–Villamayor [EV07] do not require their Rees algebras to satisfy $I_i \supset I_{i+1}$. This condition is, however, essential for the purpose of weighted blow-ups (Definition 3.2.1): without this condition, the exceptional divisor (Definition 3.2.2) of a weighted blow-up would not make sense.

It is also convenient to account for the condition that $I_\bullet$ is a descending filtration by extending Rees algebras trivially in negative degrees:

Definition 3.1.3 (Extended Rees algebras)**.** An *extended Rees algebra* on X is a quasi-coherent, finitely generated $\mathbb{Z}$-graded $\mathscr{O}_X[t^{-1}]$-subalgebra $I_\bullet^{\text{ext}} = \bigoplus_{n \in \mathbb{Z}} I_n^{\text{ext}} \cdot t^n$ of $\mathscr{O}_X[t, t^{-1}]$ such that $I_0^{\text{ext}} = \mathscr{O}_X$.

3.1.4. For an extended Rees algebra $I_\bullet^{\text{ext}}$ on X, $I_\bullet := \bigoplus_{n \geq 0} I_n^{\text{ext}} \cdot t^n$ is a Rees algebra on X in the sense of Definition 3.1.1. Conversely, every Rees algebra $I_\bullet$ on X can uniquely be extended to an extended Rees algebra $I_\bullet^{\text{ext}}$ on X by setting

$$I_n^{\text{ext}} := \begin{cases} \mathscr{O}_X, & \text{if } n < 0; \\ I_n, & \text{if } n \geq 0. \end{cases}$$

Definition 3.1.5. Given an ideal $J \subset \mathscr{O}_X$ and $d \geq 1$, we let (J, d) denote the smallest Rees algebra containing Jt^d. Given a finite collection of Rees algebras $I_{k,\bullet}$ we let $\sum_k I_{k,\bullet}$ denote the smallest Rees algebra containing all the $I_{k,\bullet}$.

The *marked ideal* (J, d), used in resolution of singularities, can be identified with the Rees algebra (J, d). Explicitly, we have that $(J, d)_n = J^{\lceil n/d \rceil}$, that is:

$$(J, d) = \mathscr{O}_X \oplus Jt \oplus Jt^2 \oplus \cdots \oplus Jt^d \oplus J^2 t^{d+1} \oplus J^2 t^{d+2} \oplus \cdots$$

and that

$$(I_{1,\bullet} + \cdots + I_{r,\bullet})_n = \sum_{n = n_1 + \cdots + n_r} I_{1,n_1} I_{2,n_2} \cdots I_{r,n_r}.$$

In particular, $I_\bullet$ is generated in degree $\leq d$ if and only if $I_\bullet = (I_1, 1) + (I_2, 2) + \cdots + (I_d, d)$.

At this point the reader should be familiar with some examples of Rees algebras:

Example 3.1.6 (Ordinary Rees algebras)**.** If a Rees algebra $I_\bullet$ is *generated in degree* 1, i.e., $I_n = I_1^n$ for all $n \geq 1$, then we say that $I_\bullet = I_1^\bullet = (I_1^n) = (I_1, 1)$ is the *Rees algebra of the ideal* $I_1 \subset \mathscr{O}_X$.

The next proposition is a direct translation of Proposition 1.6.3:

Proposition 3.1.7. *Let $I_\bullet$ be a Rees algebra on X and suppose that X is quasi-compact. Then for all sufficiently divisible d, we have $I_{rd} = (I_d)^r$ for all integers $r \geq 1$, i.e., $I_{d\bullet}$ is the Rees algebra of the ideal I_d.* □

Example 3.1.8. The graded $\mathscr{O}_X$-algebra of the d^{th} root stack of the divisor D (Example 2.2.4) is the Rees algebra (I_D, d). The symbolic Rees algebra $I_{\bullet D}$ of the Cartierification of D (Example 2.3.6) is also a Rees algebra.

We can also see (I_D, d) as a *dilation* of the ordinary Rees algebra $(I_D, 1)$:

Example 3.1.9 (Dilation)**.** Given a Rees algebra $I_\bullet$ on X, and a positive integer d, the d^{th} *dilation* of $I_\bullet$ is the Rees algebra $D_\bullet := I_{\lceil \bullet/d \rceil}$, i.e., $D_n := I_{\lceil n/d \rceil}$ for every integer $n \geq 0$. Note that the d^{th} Veronese subalgebra of $D_\bullet$ is $I_\bullet$, and $I_\bullet \subset D_\bullet$.

Taking the d^{th} dilation of the d^{th} Veronese subalgebra of a Rees algebra $I_\bullet$ on X gives:

Example 3.1.10 (Truncation)**.** Given a Rees algebra $I_\bullet$ on X, and a positive integer d, the d^{th} *truncation* of $I_\bullet$ is the Rees algebra $T_\bullet := I_{d\lceil \bullet/d \rceil} = \sum_{d|n} (I_n, n)$, i.e., $T_n = I_{d\lceil n/d \rceil}$ for every integer $n \geq 0$. Note that $T_{d\bullet} = I_{d\bullet}$, and $T_\bullet \subset I_\bullet$.

Remark 3.1.11. The Rees algebra $\bigoplus_{n \geq 0} I_{ND}^{\lceil n/N \rceil}$ mentioned at the end of Example 2.3.6 is precisely (I_{ND}, N), i.e., the N^{th} truncation of $I_{\bullet D}$, since ND is a Cartier divisor.

3.2. Weighted blow-ups.

Definition 3.2.1 (Weighted blow-ups)**.** If $I_\bullet$ is a Rees algebra on X, the *(stack-theoretic) weighted blow-up* of X along $I_\bullet$ is defined as the morphism

$$\mathrm{Bl}_{I_\bullet} X = \mathscr{P}\mathrm{roj}_X(I_\bullet) \longrightarrow X$$

which is proper (Proposition 1.6.1(ii)). Note that $\sqrt{I_n} = \sqrt{I_1}$ for any positive integer n. We call $V(I_1)$ the *center* of the weighted blow-up (or the Rees algebra).

Definition 3.2.2 (Exceptional divisor)**.** Let $X' := \mathrm{Bl}_{I_\bullet} X \xrightarrow{\pi} X$. The natural inclusion $I_{\bullet+1} \hookrightarrow I_\bullet$ corresponds to the inclusion $\mathscr{O}_{X'}(1) \hookrightarrow \mathscr{O}_{X'}(0) = \mathscr{O}_{X'}$ of invertible sheaves, and defines an effective Cartier divisor E on X' such that $\mathscr{O}_{X'}(1) = \mathscr{O}_{X'}(-E)$. We call E the *exceptional divisor* of $\mathrm{Bl}_{I_\bullet} X$.

Remark 3.2.3. Explicitly, if we write $I_\bullet$ locally as $(f_1, d_1) + \cdots + (f_m, d_m)$, then the ideal sheaf I_E of E can be described locally on $\mathrm{Bl}_{I_\bullet} X$ as follows. On the chart

$$D_+(f_i \cdot t^{d_i}) = \left[\mathrm{Spec}_X\big(I_\bullet[(f_i \cdot t^{d_i})^{-1}]\big) \;/\; \mathbb{G}_m\right]$$

of $\mathrm{Bl}_{I_\bullet} X$, the ideal sheaf I_E is generated by $t^{-1} = \frac{f_i \cdot t^{d_i - 1}}{f_i \cdot t^{d_i}} \in I_\bullet[(f_i \cdot t^{d_i})^{-1}]$. In particular, the Cartier divisor $d_i E$ is principal and given by the vanishing of $\pi^{-1}(f_i)$ on this chart. Thus, for all N divisible by $d_1, d_2, \ldots, d_m$, the Cartier divisor NE has ideal sheaf

$$I_{NE} = \big(\pi^{-1}(f_i)^{N/d_i} \;:\; i = 1, 2, \ldots, m\big).$$

Remark 3.2.4 (Weighted blow-ups in terms of extended Rees algebras)**.** Note that if $I_\bullet^{\mathrm{ext}}$ denotes the extended Rees algebra of $I_\bullet$ (3.1.4), then:

$$\mathrm{Bl}_{I_\bullet} X = \mathscr{P}\mathrm{roj}_X(I_\bullet^{\mathrm{ext}}) := \left[\mathrm{Spec}_X(I_\bullet^{\mathrm{ext}}) \smallsetminus V(I_+^{\mathrm{ext}}) \;/\; \mathbb{G}_m\right].$$

Indeed, if we write $I_\bullet$ locally as $(f_1, d_1) + \cdots + (f_m, d_m)$, then one has, for each $1 \le i \le m$, that $I_\bullet^{\mathrm{ext}}[(f_i \cdot t^{d_i})^{-1}] = I_\bullet[(f_i \cdot t^{d_i})^{-1}]$, and thus

$$D_+(f_i \cdot t^{d_i}) = \left[\mathrm{Spec}_X\big(I_\bullet^{\mathrm{ext}}[(f_i \cdot t^{d_i})^{-1}]\big) \;/\; \mathbb{G}_m\right].$$

Evidently these identifications are compatible with each other.

Note too that the exceptional divisor E of $\mathrm{Bl}_{I_\bullet} X$ is induced by the principal divisor given by $t^{-1} = 0$ on $\mathrm{Spec}_X(I_\bullet^{\mathrm{ext}})$ whereas the ideal sheaf $I_{\bullet+1}$ on $\mathrm{Spec}_X(I_\bullet)$, which is not even invertible, only becomes principal over the localizations $I_\bullet[(f_i \cdot t^{d_i})^{-1}]$ (Remark 3.2.3).

The next proposition, like Proposition 3.1.7, is a direct translation of Proposition 1.6.3:

Proposition 3.2.5. *Let $I_\bullet$ be a Rees algebra on X. The coarse space of $\mathrm{Bl}_{I_\bullet} X$, relative to X, is the ordinary blow-up $\mathrm{Bl}_{I_d} X$ for any positive integer d such that $I_{d\bullet}$ is generated in degree 1. Such a d always exists if X is quasi-compact (Proposition 3.1.7).* □

Example 3.2.6 (Ordinary blow-ups)**.** Let $I \subset \mathscr{O}_X$ be an ideal. The weighted blow-up $\mathrm{Bl}_{I^\bullet} X$ of X along the Rees algebra $I^\bullet = (I, 1)$ of I is the usual blow-up $\mathrm{Bl}_I X$ of X along I.

Example 3.2.7 (Root stack of a divisor)**.** Given an effective Cartier divisor D on X, and a positive integer d, the root stack $X\big(\sqrt[d]{D}\big)$ is the weighted blow-up $\mathrm{Bl}_{(I_D,d)} X$ (see Examples 2.2.4 and 3.1.8).

Example 3.2.8 (Cartierification)**.** If X is normal and noetherian and D is an effective $\mathbb{Q}$-Cartier divisor, then the *Cartierification* of D in X is $\mathrm{Bl}_{I_{\bullet D}} X$ (see Examples 2.3.6 and 3.1.8).

Theorem 3.2.9 (Universal property of weighted blow-ups)**.** *Let $I_\bullet$ be a Rees algebra, and let $\pi\colon X' = \mathrm{Bl}_{I_\bullet} X \to X$ be the corresponding weighted blow-up.*

(i) *For every $n \in \mathbb{N}$ we have an inclusion of ideals $\pi^{-1}(I_n)\cdot\mathscr{O}_{X'} \subset I_E^n$, which is an equality for all sufficiently divisible n (locally on X).*
(ii) *Let $f\colon T \to X$ be a morphism such that $U := T \smallsetminus f^{-1}\big(V(I_1)\big)$ is schematically dense. The groupoid of factorizations through π is equivalent to the set of effective Cartier divisors D on T such that $f^{-1}(I_n)\cdot\mathscr{O}_T \subset I_D^n$ for all n with equality for all sufficiently divisible n (locally on T). If $f = \pi \circ g$, then $D = g^{-1}(E)$.*

Proof. By Proposition 1.5.1, a factorization of f through π corresponds to a line bundle $\mathscr{L}$ on T together with a graded algebra homomorphism $\varphi\colon \bigoplus_{n\geq 0} f^*I_n \to \bigoplus_{n\geq 0}\mathscr{L}^{\otimes n}$ which is surjective for all sufficiently divisible n. The case $T = \mathrm{Bl}_{I_\bullet} X$ corresponds to $\mathscr{L} = \mathscr{O}_{X'}(1) = \mathscr{O}_{X'}(-E)$ with the canonical map φ.

Let N be a sufficiently divisible integer. We begin by noting that $\varphi_N|_U$ is an isomorphism and hence that $\varphi|_U$ is an isomorphism. Since $j\colon U \to T$ is schematically dominant, we have that $\mathscr{L}^n \to j_*j^*\mathscr{L}^n = j_*j^*\mathscr{O}_T$ is injective whereas the image of $f^*I_n \to j_*j^*f^*I_n = j_*j^*\mathscr{O}_T$ is $f^{-1}(I_n)\cdot\mathscr{O}_T$. It follows that φ factors through an injective graded homomorphism

$$\psi\colon \bigoplus_{n\geq 0} f^{-1}(I_n)\cdot\mathscr{O}_T \to \bigoplus_{n\geq 0}\mathscr{L}^{\otimes n}.$$

In particular, ψ_N is an isomorphism. The composition of ψ_N^{-1}, the inclusion $f^{-1}(I_N)\cdot\mathscr{O}_T \subset f^{-1}(I_{N-1})\cdot\mathscr{O}_T$ and ψ_{N-1} gives an injective homomorphism $\mathscr{L}^N \hookrightarrow \mathscr{L}^{N-1}$, or equivalently, an injective homomorphism $s\colon \mathscr{L} \hookrightarrow \mathscr{O}_T$. This defines the Cartier divisor D. Note that $s|_U = (\psi_1|_U)^{-1}$ so ψ_n together with s^n gives the inclusion of ideals $f^{-1}(I_n)\cdot\mathscr{O}_T \hookrightarrow \mathscr{L}^n = \mathscr{O}_T(-nD) \hookrightarrow \mathscr{O}_T$. $\square$

Remark 3.2.10. For all n, we have a commutative diagram

$$\begin{tikzcd} nE \arrow[r] \arrow[d, hook] & V(I_n) \arrow[d, hook] \\ \mathrm{Bl}_{I_\bullet} X \arrow[r, "\pi"] & X \end{tikzcd}$$

which is cartesian for sufficiently divisible n. Unlike the usual blow-up, the diagram is *not always cartesian* for $n = 1$. Nevertheless:

(i) π is an isomorphism away from $V(I_1)$.
(ii) $E_{\mathrm{red}} = \pi^{-1}(V(I_1))_{\mathrm{red}}$.

(iii) $\pi^{-1}(V(I_1))$ is of codimension 1 in $\mathrm{Bl}_{I_\bullet} X$ and its complement is schematically dense in $\mathrm{Bl}_{I_\bullet} X$.

Remark 3.2.11. If one locally writes $I_\bullet$ as $(f_1, d_1) + \cdots + (f_m, d_m)$, then the condition in Theorem 3.2.9(ii) that $f^{-1}(I_n) \subset I_D^n$ for all n with equality for sufficiently divisible n (locally on T) can be explicated as the following equivalent condition: $f^{-1}(f_i) \in I_D^{d_i}$ for all $1 \le i \le m$ and locally on T there exists an i such that $I_D^{d_i} = \big(f^{-1}(f_i)\big)$. The latter occurs on the preimage of the chart $D_+(f_i \cdot t^{d_i})$ of $\mathrm{Bl}_{I_\bullet} X$ (Remark 3.2.3). Thus, $f^{-1}(I_n) \cdot \mathscr{O}_T = I_D^n$ for every n divisible by the $d_1, d_2, \ldots, d_m$.

The next corollary generalizes Example 3.2.7.

Corollary 3.2.12. *Let $I \subset \mathscr{O}_X$ be an ideal, and fix a positive integer d. Then $\mathrm{Bl}_{(I,d)} X$ coincides with the d^{th} root stack (Example 2.2.4) of the exceptional divisor of the usual blow-up $\mathrm{Bl}_I X$ of X along I.*

Proof. Let X' denote the d^{th} root stack of the exceptional divisor of $\mathrm{Bl}_I X$. For a morphism $T \to X$:

(a) The groupoid of factorizations through $X' \to X$ is equivalent to the set of effective Cartier divisors D on T such that $f^{-1}(I) \cdot \mathscr{O}_T = I_D^d$ (Example 2.2.4).

(b) The groupoid of factorizations through $\mathrm{Bl}_{(I,d)} X \to X$ is equivalent to the set of effective Cartier divisors D on T such that $f^{-1}(I^{\lceil n/d \rceil}) \subset I_D^n$ for every n with equality whenever $d \mid n$ (Remark 3.2.11).

The groupoids in (a) and (b) are equivalent, and the corollary follows. ☐

Remark 3.2.13. The corollary shows that our definition of $\mathrm{Bl}_{(I,d)} X$ agrees with the definition of $\mathrm{Bl}_{(I,d)} X$ as the dth root stack of the usual blow-up in [Ryd09]. The weighted blow-up $\mathrm{Bl}_{(I,d)} X$ is called the *d^{th} Kummer blow-up of X along I* in [Ryd09].

Corollary 3.2.14 (Functoriality for weighted blow-ups)**.** *Let $f\colon Y \to X$ be a morphism of schemes, and let $I_\bullet$ be a Rees algebra on X. There is a unique morphism $g\colon \mathrm{Bl}_{f^{-1}(I_\bullet)\cdot\mathscr{O}_Y} Y \to \mathrm{Bl}_{I_\bullet} X$ making the following diagram commute:*

$$\begin{tikzcd} \mathrm{Bl}_{f^{-1}(I_\bullet)\cdot\mathscr{O}_Y} Y \arrow[r, hook, dotted, "\iota"] \arrow[rr, bend left, dotted, "g"] \arrow[dr, "\pi_Y"'] & (\mathrm{Bl}_{I_\bullet} X) \times_X Y \arrow[r] \arrow[d] & \mathrm{Bl}_{I_\bullet} X \arrow[d, "\pi"] \\ & Y \arrow[r, "f"] & X \end{tikzcd} \tag{3.1}$$

and the morphism ι is a closed immersion. Hence:

(i) *$\mathrm{Bl}_{f^{-1}(I_\bullet)\cdot\mathscr{O}_Y} Y$ is the schematic closure of the complement of $E \times_X Y$ in $(\mathrm{Bl}_{I_\bullet} X) \times_X Y$.*

(ii) *If f is a closed immersion, then so is $g\colon \mathrm{Bl}_{f^{-1}(I_\bullet)\cdot\mathscr{O}_Y} Y \to \mathrm{Bl}_{I_\bullet} X$.*

(iii) *If f is flat (e.g., f is an open immersion), then ι is an isomorphism.*

Proof. The existence and uniqueness of g follow from Theorem 3.2.9. To see that ι is a closed immersion, note that ι is induced by the natural surjective morphism $f^*I_\bullet \to f^{-1}(I_\bullet)\cdot\mathscr{O}_Y$ of graded $\mathscr{O}_Y$-algebras, i.e.,

$$\iota\colon \mathrm{Bl}_{f^{-1}(I_\bullet)\cdot\mathscr{O}_Y} Y = \mathscr{P}\mathrm{roj}_Y(f^{-1}(I_\bullet)\cdot\mathscr{O}_Y) \hookrightarrow \mathscr{P}\mathrm{roj}_Y(f^*I_\bullet) = (\mathrm{Bl}_{I_\bullet} X)\times_X Y.$$

Then parts (i) and (ii) are immediate. For part (iii), note that if f is flat, then $f^*I_\bullet \to f^{-1}(I_\bullet)\cdot\mathscr{O}_Y$ is an isomorphism. □

Definition 3.2.15 (Transforms under weighted blow-ups)**.** In the above corollary, $\mathrm{Bl}_{f^{-1}(I_\bullet)\cdot\mathscr{O}_Y} Y$ is known as the *proper* (or *strict*) *transform* of $Y \to X$ under the weighted blow-up $\mathrm{Bl}_{I_\bullet} X \xrightarrow{\pi} X$, while $\pi^{-1}(Y) = (\mathrm{Bl}_{I_\bullet} X)\times_X Y$ is the *total transform* of $Y \to X$ under the weighted blow-up $\mathrm{Bl}_{I_\bullet} X \xrightarrow{\pi} X$.

3.3. **Integral closure of Rees algebras.**

Definition 3.3.1 (Integral closure)**.** For a Rees algebra $I_\bullet$ (or more generally, a quasi-coherent $\mathscr{O}_X$-subalgebra of $\mathscr{O}_X[t]$) on X, we denote by $\mathrm{IC}(I_\bullet)$ the *integral closure* of $I_\bullet$ in $\mathscr{O}_X[t]$. We say $I_\bullet$ is *integrally closed* if $\mathrm{IC}(I_\bullet) = I_\bullet$.

Note that if X is normal, then $I_\bullet$ is integrally closed if and only if $\mathrm{Spec}_X(I_\bullet)$ is normal.

Remark 3.3.2. For a Rees algebra $I_\bullet$ on X, note that (by definition) the integral closure of $I_{d\bullet}$ is the d^{th} Veronese subalgebra of $\mathrm{IC}(I_\bullet)$.

Remark 3.3.3. Given a Rees algebra $I_\bullet$, the integral closure $\mathrm{IC}(I_\bullet)$ is always a quasi-coherent graded $\mathscr{O}_X$-subalgebra of $\mathscr{O}_X[t]$ but not necessarily of finite type. However, if X is *integral and Nagata*, then we claim that $\mathrm{IC}(I_\bullet)$ is of finite type over $\mathscr{O}_X$, and hence a Rees algebra on X.

Indeed, the question is local, so we may assume that $X = \mathrm{Spec}(A)$ is affine. Let K be the fraction field of A. Let $\overline{\mathrm{IC}}(I_\bullet)$ be the integral closure of $I_\bullet$ in $K(t)$. Since A is Nagata, so is $I_\bullet$ (being a finitely generated A-subalgebra of $A[t]$). Since $K(t)$ is also the field of fractions of $I_\bullet$[1], we conclude that $\overline{\mathrm{IC}}(I_\bullet)$ is finite over $I_\bullet$. In particular, it is a noetherian $I_\bullet$-module, so its $I_\bullet$-submodules (e.g., $\mathrm{IC}(I_\bullet)$) are finitely generated $I_\bullet$-modules, and hence, finitely generated A-algebras.

3.3.4. *Integral closure of ideals.* For an ideal I on X, the t^1-graded piece of $\mathrm{IC}(I^\bullet)$ is known in the literature (e.g., [Laz04, 9.6.A]) as the *integral closure* $\mathrm{IC}(I)$ of the ideal $I \subset \mathscr{O}_X$. Note that $I \subset \mathrm{IC}(I) \subset \sqrt{I}$.

Example 3.3.5. Let X be a normal scheme. If E is an effective Cartier divisor on X, then the ordinary Rees algebra $(I_E, 1)$ of $I_E = \mathscr{O}_X(-E)$ on X is integrally closed. Indeed, locally on X, we have that $I_E^\bullet \simeq \mathscr{O}_X[t]$ which is integrally closed.

Example 3.3.6 (Integral closure of truncations)**.** Let $I_\bullet$ be a Rees algebra on X, and for any positive integer d, let $T_\bullet$ be the d^{th} truncation of $I_\bullet$ (Example 3.1.10). Then $\mathrm{IC}(I_\bullet) = \mathrm{IC}(T_\bullet)$. For this, it suffices to observe that the d^{th} Veronese subalgebra of $\mathrm{IC}(I_\bullet)$ coincides with that of $\mathrm{IC}(T_\bullet)$.

Example 3.3.7 (Cartierification, II)**.** Adopt the set-up in Example 2.3.6. We shall now show that $I_{\bullet D} = \bigoplus_{n\geq 0} I_{nD}$ is the integral closure in $\mathscr{O}_X[t]$ of $(I_{ND}, N) = \bigoplus_{n\geq 0} I_{ND}^{\lceil n/N\rceil}$. Since (I_{ND}, N) is the N^{th} truncation of $I_{\bullet D}$

[1]If $I_n = 0$ for $n > 0$, then $\mathrm{IC}(I_\bullet) = I_\bullet$ and there is nothing to prove.

(Remark 3.1.11(i)), it remains to show (because of Example 3.3.6) that $I_{\bullet D}$ is integrally closed.

For this, let U be as in Example 2.3.6. On U, we have $I_{\bullet D}|_U = (I_D, 1)|_U$, and hence, by Example 3.3.5, $I_{\bullet D}|_U$ is integrally closed. Thus, so is $I_{\bullet D} = j_*(I_{\bullet D}|_U)$.

3.4. **Normalized weighted blow-ups.** In this subsection we assume that X is normal.

Definition 3.4.1 (Normalized weighted blow-ups)**.** The *normalized weighted blow-up* of X along a Rees algebra $I_\bullet$ on X is the normalization of $\mathrm{Bl}_{I_\bullet} X$ (denoted $\mathrm{Bl}^{\mathrm{norm}}_{I_\bullet} X$), or equivalently, $\mathscr{P}\mathrm{roj}_X\,(\mathrm{IC}(I_\bullet))$.

Note that $\mathrm{IC}(I_\bullet)$ is not always a Rees algebra (Remark 3.3.3), and thus the normalized weighted blow-up $\mathrm{Bl}^{\mathrm{norm}}_{I_\bullet} X$ of X is not always proper (not always of finite type but always separated, quasi-compact and universally closed) over X.

Example 3.4.2. Adopt the set-up in Example 2.3.6. Then the Cartierification (Example 3.2.8) of D in X is the normalized weighted blow-up of X along (I_{ND}, N) by Example 3.3.7. In other words,

$$\mathrm{Bl}_{I_{\bullet D}} X = \mathrm{Bl}^{\mathrm{norm}}_{(I_{ND},N)} X = \Big(X\big(\sqrt[N]{ND}\big)\Big)^{\mathrm{norm}}.$$

The integral closure $I_{\bullet D} = \mathrm{IC}\big((I_{ND}, N)\big)$ is always finitely generated and hence a Rees algebra by Proposition 2.3.3(i).

The universal property of weighted blow-ups in Theorem 3.2.9 has a neater re-interpretation after passage to normalizations:

Theorem 3.4.3 (Universal property of normalized weighted blow-ups)**.** *For a Rees algebra $I_\bullet$ on X, the normalized weighted blow-up $\pi\colon \mathrm{Bl}^{\mathrm{norm}}_{I_\bullet} X \to X$ satisfies the following universal property. Let $f\colon T \to X$ be a morphism, where T is normal and such that $f^{-1}(V(I_1))$ is nowhere dense. Then there exists at most one lift $g\colon T \to \mathrm{Bl}^{\mathrm{norm}}_{I_\bullet} X$ of f, and such a lift exists if and only if $\mathrm{IC}(f^{-1}(I_\bullet)\cdot \mathscr{O}_T) = (I_D, 1)$ for some effective Cartier divisor D on T. If this is the case, then $D = g^{-1}(E)$.*

Proof. By Theorem 3.2.9, the lifts $T \to \mathrm{Bl}^{\mathrm{norm}}_{I_\bullet} X$ are equivalent to the set of Cartier divisors D such that $f^{-1}(\mathrm{IC}(I_n)) \cdot \mathscr{O}_T \subset I_D^n$ for every $n \geq 1$ with equality for sufficiently divisible n (locally on T). Since T is normal, the Rees algebra $(I_D, 1)$ is integrally closed and the condition is equivalent to $\mathrm{IC}\big(f^{-1}(I_\bullet)\cdot\mathscr{O}_T\big) \subset (I_D, 1)$ with equality for sufficiently divisible n. This means that we have an equality of Rees algebras (Examples 3.3.5 and 3.3.6). In particular, D is unique. □

A partial converse to Corollary 3.2.12 is:

Proposition 3.4.4. *Let X be a normal, quasi-compact scheme. Every normalized weighted blow-up of X is a normalized Kummer blow-up $\mathrm{Bl}^{\mathrm{norm}}_{(I,d)} X$ of X.*

Proof. Let $I_\bullet$ be a Rees algebra on X, and let $X' = \mathrm{Bl}_{\mathrm{IC}(I_\bullet)} X$. By Proposition 3.1.7, there exists a positive integer d such that $I_{d\bullet}$ is generated in degree

1. Let $T_\bullet$ be the d^{th} truncation of $I_\bullet$. By Example 3.3.6, $\mathrm{IC}(I_\bullet) = \mathrm{IC}(T_\bullet)$, so $X' = \mathrm{Bl}_{\mathrm{IC}(T_\bullet)} X$. By definition, $T_\bullet$ is the d^{th} dilation of $I_{d\bullet}$. Thus, $X' = \mathrm{Bl}_{\mathrm{IC}(T_\bullet)} X = \mathrm{Bl}_{\mathrm{IC}(I_d,d)} X = \mathrm{Bl}^{\mathrm{norm}}_{(I_d,d)} X$. $\square$

Under an additional hypothesis on stackiness, we can also describe a Kummer blow-up as an ordinary blow-up followed by a Cartierification.

Proposition 3.4.5. *Let X be a normal scheme, $I \subset \mathscr{O}_X$ be an ideal, and d be a positive integer. If the normalized Kummer blow-up $\mathrm{Bl}^{\mathrm{norm}}_{(I,d)} X \to X$ is representable over an open subset $U \subset \mathrm{Bl}^{\mathrm{norm}}_{(I,d)} X$ with complement of codimension ≥ 2, then:*

- (i) *$\mathrm{Bl}^{\mathrm{norm}}_I X$ has an effective $\mathbb{Q}$-Cartier divisor D such that dD is the exceptional divisor on $\mathrm{Bl}^{\mathrm{norm}}_I X$.*
- (ii) *$\mathrm{Bl}^{\mathrm{norm}}_{(I,d)} X \to X$ can be factored as follows:*

$$\mathrm{Bl}^{\mathrm{norm}}_{(I,d)} X \xrightarrow{\ p\ } \mathrm{Bl}^{\mathrm{norm}}_I X \xrightarrow{\ q\ } X$$

where q is the normalized blow-up of X along I, and p is the Cartierification of D in $\mathrm{Bl}^{\mathrm{norm}}_I X$.

Proof. Let E (resp. E') denote the exceptional divisor on $\mathrm{Bl}^{\mathrm{norm}}_I X$ (resp. $\mathrm{Bl}^{\mathrm{norm}}_{(I,d)} X$). By the universal property of $\mathrm{Bl}^{\mathrm{norm}}_I X$, the map $\mathrm{Bl}^{\mathrm{norm}}_{(I,d)} X \to X$ factors uniquely through $\mathrm{Bl}^{\mathrm{norm}}_I X$ as follows:

$$\mathrm{Bl}^{\mathrm{norm}}_{(I,d)} X \xrightarrow{\ p\ } \mathrm{Bl}^{\mathrm{norm}}_I X \xrightarrow{\ q\ } X.$$

Since $\mathrm{Bl}^{\mathrm{norm}}_{(I,d)} X$ has no relative stackiness over X in codimension 1, and since p is a coarse moduli space (Proposition 1.6.1(iii)), p is an isomorphism in codimension 1, and thus p induces an identification of Weil divisors of both sides. Since $p^{-1}(E) = dE'$ with E' a Cartier divisor on $\mathrm{Bl}^{\mathrm{norm}}_{(I,d)} X$, there exists a Weil divisor D on $\mathrm{Bl}^{\mathrm{norm}}_I X$ such that $dD = E$.

Next, set $Y := \mathrm{Bl}^{\mathrm{norm}}_I X$, and we show that p can be identified with the Cartierification of D, i.e., $\pi\colon \mathrm{Bl}_{I_{\bullet D}} Y \to Y$. We do so by comparing the universal properties of π and $q \circ p$. Let $f\colon T \to Y$ be a morphism from a normal scheme T, where $f^{-1}(D)$ is nowhere dense in T, i.e., $(q \circ f)^{-1}(V(I))$ is nowhere dense in T (for example, $f = \pi$ or $f = p$). Then:

- (a) f factors uniquely through π if and only if f^*D is an effective Cartier divisor on T (Example 2.3.6).
- (b) f factors uniquely through p if and only if

$$\begin{aligned}\mathrm{IC}\big((q \circ f)^{-1}(I,d) \cdot \mathscr{O}_T\big) &= \mathrm{IC}(f^{-1}(I_E,d) \cdot \mathscr{O}_T)\\ &= \mathrm{IC}(f^{-1}(I_{\bullet D}) \cdot \mathscr{O}_T) = I_{\bullet f^*D}\end{aligned}$$

  is generated in degree 1 by the underlying ideal of an effective Cartier divisor on T (Theorem 3.4.3). Note that the last equality above holds since both sides are integrally closed Rees algebras whose d^{th} Veronese subalgebras coincide.

Evidently the universal properties in (a) and (b) coincide, as desired. $\square$

3.5. **Valuative $\mathbb{Q}$-ideals.** Let X be (unless otherwise specified) an *integral* and separated scheme, with field of fractions $K(X)$. As outlined in [ATW19] and explicated in [Que20], integrally closed Rees algebras on X have an equivalent formulation as *valuative* $\mathbb{Q}$*-ideals* over X. We recall this equivalence in this subsection, before relating it back to the previous subsection.

3.5.1. *Zariski–Riemann spaces.* We denote by $\mathrm{ZR}(X)$ the *Zariski–Riemann space* of X, which is the inverse limit of the system of modifications $X' \to X$ in the category of locally ringed spaces over X. As a set, its elements are valuations ν of $K(X)$ that possess a (necessarily *unique*) center x_ν on X, i.e., R_ν dominates some local ring $\mathscr{O}_{X,x_\nu}$ on X.

As a topological space, it has a basis of open sets of the form $U(x_1, x_2, \ldots, x_n)$ for an integer $n \geq 0$ and $x_1, x_2, \ldots, x_n \in K(X)^\times$, where

$$U(x_1, x_2, \ldots, x_n) := \big\{\nu \in \mathrm{ZR}(X)\colon\ x_i \in R_\nu \text{ for every } 1 \leq i \leq n\big\}.$$

If X is quasi-compact, then so is $\mathrm{ZR}(X)$. Next, its structure sheaf $\mathscr{O}_{\mathrm{ZR}(X)}$, evaluated over an open subset U of $\mathrm{ZR}(X)$, is given by the intersection $\bigcap_{\nu \in U} R_\nu$ in $K(X)$. There is also a canonical morphism of locally ringed spaces

$$\begin{aligned} \pi_X\colon\ \mathrm{ZR}(X) &\longrightarrow X \\ \nu &\longmapsto x_\nu \end{aligned}$$

sending a valuation ν to its unique center x_ν on X, and whose morphism of structure sheaves $\pi_X^\sharp\colon \mathscr{O}_X \to (\pi_X)_*\mathscr{O}_{\mathrm{ZR}(X)}$ is given on stalks by the canonical inclusion $\mathscr{O}_{X,x_\nu} \hookrightarrow R_\nu$. If X is *normal*, a standard result in commutative ring theory implies that $\pi_X^\sharp$ is an isomorphism. These facts and more can be found in [Que20, Appendix A].

Finally, the Zariski–Riemann space $\mathrm{ZR}(X)$ carries a sheaf of partially ordered groups $\Gamma_X = K(X)^\times/\mathscr{O}_{\mathrm{ZR}(X)}^\times$, whose stalk at $\nu \in \mathrm{ZR}(X)$ is given by the (totally ordered) value group $\Gamma_\nu := K(X)^\times/R_\nu^\times$ of ν. Here $K(X)^\times$ denotes the constant sheaf on $\mathrm{ZR}(X)$ with value $K(X)^\times$. There is a canonical morphism of sheaves $\mathrm{val}\colon K(X)^\times \twoheadrightarrow \Gamma_X$, and the image $\mathrm{val}(\mathscr{O}_{\mathrm{ZR}(X)} \smallsetminus \{0\})$ is the sheaf of monoids consisting of non-negative sections of Γ_X, denoted $\Gamma_{X,+}$. Note that Γ_X is the sheaf of Cartier divisors on $\mathrm{ZR}(X)$ and $\Gamma_{X,+}$ is the sheaf of effective Cartier divisors. In what follows, we will also need the tensor product $\Gamma_{X,\mathbb{Q}} := \Gamma_X \otimes \mathbb{Q}$, and similarly its subsheaf $\Gamma_{X,\mathbb{Q}+}$ consisting of non-negative sections of $\Gamma_{X,\mathbb{Q}}$.

Definition 3.5.2 (Valuative $\mathbb{Q}$-ideals [ATW19, Section 2.2])**.** A *valuative* $\mathbb{Q}$*-ideal* over X is a section $\gamma \in \mathrm{H}^0(\mathrm{ZR}(X), \Gamma_{X,\mathbb{Q}+})$. For $\nu \in \mathrm{ZR}(X)$, we write γ_ν for the stalk of γ at ν.

Remark 3.5.3. The notion of valuative $\mathbb{Q}$-ideals over X still makes sense when we drop the separatedness assumption on X, i.e., X is just an integral scheme. In this case, we just work affine-locally. Namely, a *valuative* $\mathbb{Q}$*-ideal over* X is a collection of sections $(\gamma_{\mathrm{Spec}(A)})_{\mathrm{Spec}(A) \subset X}$, where the indexing set consists of the nonempty open affine subsets $\mathrm{Spec}(A) \subset X$, and each $\gamma_{\mathrm{Spec}(A)}$ is a valuative $\mathbb{Q}$-ideal over $\mathrm{Spec}(A)$, such that $(\gamma_U)|_{\mathrm{ZR}(V)} = \gamma_V$ for every inclusion $V \hookrightarrow U$ of open affine subsets of X. This definition is easily seen to be equivalent to Definition 3.5.2 if X is separated. In this way, our results below can be reduced to the case when X is also *separated*.

3.5.4. *Idealistic exponents.* Given a non-zero Rees algebra $I_\bullet$ on X, we associate a valuative $\mathbb{Q}$-ideal $\gamma_{I_\bullet}$ over X stalk-wise:

$$\gamma_{I_\bullet} := (\gamma_{I_\bullet,\nu})_{\nu\in \mathrm{ZR}(X)} \in \prod_{\nu\in\mathrm{ZR}(X)} (\mathbb{Q}\otimes\Gamma_\nu)_+,$$

where for each $\nu \in \mathrm{ZR}(X)$,

$$\gamma_{I_\bullet,\nu} := \min\left\{\frac{\nu(I_n)}{n} : n \geq 1\right\}, \text{ where } \nu(I_n) := \min\bigl\{\nu(g) : 0 \neq g \in (I_n)_{x_\nu}\bigr\}.$$

One can show that this minimum exists in $(\mathbb{Q}\otimes\Gamma_\nu)_+$, and that $(\gamma_{I_\bullet,\nu})_{\nu\in\mathrm{ZR}(X)}$ is a compatible collection of germs of $\Gamma_{X,\mathbb{Q}+}$ [Que20, Section 2.2]. This defines a map:

$$\left\{\begin{matrix}\text{non-zero Rees algebras}\\ I_\bullet \text{ on } X\end{matrix}\right\} \to \bigl\{\text{valuative } \mathbb{Q}\text{-ideals } \gamma \text{ over } X\bigr\}. \tag{3.2}$$

The valuative $\mathbb{Q}$-ideals in the image of this map (3.2) are called *idealistic exponents* over X.

Remark 3.5.5. A valuative $\mathbb{Q}$-ideal γ is an effective Cartier divisor with $\mathbb{Q}$-coefficients on $\mathrm{ZR}(X)$. If X is quasi-compact, we can also see γ as a Cartier divisor with $\mathbb{Q}$-coefficients on a modification of X. Indeed, there exists a positive integer N such that $N\gamma$ locally is the image under val of some element in $\mathscr{O}_{\mathrm{ZR}(X)}$. Since $\mathrm{ZR}(X)$ is quasi-compact, finitely many representatives suffice, and therefore, we may take some modification $X' \to X$ such that these representatives are regular on X'. In other words, γ is the idealistic exponent associated to the Rees algebra (I_D, N) for some effective Cartier divisor D on X'.

3.5.6. Ideally, we want a map going backwards in (3.2) as well. Given a valuative $\mathbb{Q}$-ideal γ over X, an obvious candidate for a filtration $I_{\gamma,\bullet}$ of ideals of $\mathscr{O}_X$ associated to γ is:

$$I_{\gamma,n}(U) := \bigl\{g \in \mathscr{O}_X(U) : \nu(g) \geq n\cdot\gamma_\nu \text{ for every } \nu \in \pi_X^{-1}(U)\bigr\}. \tag{3.3}$$

Note that $I_{\gamma,\bullet}$ is quasi-coherent as an $\mathscr{O}_X$-algebra. Indeed, fix a modification $\pi\colon X' \to X$ such that $N\gamma$ is the idealistic exponent associated to an effective Cartier divisor on X' (Remark 3.5.5). Then the $\mathscr{O}_{X'}$-algebra $I_{\gamma,\bullet}^{(X')}$ associated to γ is quasi-coherent (for example, by Theorem 3.5.7(ii)), and hence, the $\mathscr{O}_X$-algebra $I_{\gamma,\bullet} = \pi_* I_{\gamma,\bullet}^{(X')} \cap \mathscr{O}_X$ is also quasi-coherent.

However, we caution that $I_{\gamma,\bullet}$ is *not* in general a Rees algebra on X, because it may not be finitely generated. Nevertheless, the constructions in 3.5.4 and (3.3) provide a numerical approach to computing integral closures of Rees algebras. Moreover, under the additional assumption that X is *Nagata*, both maps (3.2) and (3.3) restrict to an order-reversing one-to-one correspondence between

$$\left\{\begin{matrix}\text{integrally closed, non-zero}\\ \text{Rees algebras } I_\bullet \text{ on } X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{idealistic exponents } \gamma\\ \text{over } X\end{matrix}\right\}. \tag{3.4}$$

Indeed, both statements are consequences of:

Theorem 3.5.7. *Let X be an integral scheme.*

(i) *If γ is a valuative $\mathbb{Q}$-ideal over X, then the $\mathscr{O}_X$-subalgebra $I_{\gamma,\bullet}$ of $\mathscr{O}_X[t]$ is integrally closed in $\mathscr{O}_X[t]$.*
(ii) *If $\gamma = \gamma_{I_\bullet}$ is an idealistic exponent over X, then $I_{\gamma,\bullet} = \mathrm{IC}(I_\bullet)$. In particular, if X is Nagata, $I_{\gamma,\bullet}$ is an integrally closed Rees algebra on X (Remark 3.3.3).*

Proof. Both parts can be proven affine-locally (Remark 3.5.3), so we may assume X is separated, in which case we refer the reader to [Que20, Lemma 2.2.1 and Theorem 2.2.2]. □

Definition 3.5.8 (cf. [ATW19, Section 3.3])**.** Let X be normal and let γ be an idealistic exponent over X. The weighted blow-up of X along the idealistic exponent γ is

$$\mathrm{Bl}_\gamma X := \mathrm{Bl}_{I_{\gamma,\bullet}} X = \mathscr{P}\mathrm{roj}_X(I_{\gamma,\bullet}),$$

where $I_{\gamma,\bullet}$ is the quasi-coherent graded $\mathscr{O}_X$-subalgebra of $\mathscr{O}_X[t]$ associated to γ (see 3.5.6).

If $I_\bullet$ is a Rees algebra on X and $\gamma = \gamma_{I_\bullet}$ is the corresponding idealistic exponent, then Theorem 3.5.7(ii) says that $\mathrm{Bl}^{\mathrm{norm}}_{I_\bullet} X = \mathrm{Bl}_\gamma X$. The universal property of the normalized blow-up (Theorem 3.4.3) can be reformulated as follows:

Theorem 3.5.9 (cf. [ATW19, Section 3.3])**.** *Let γ be an idealistic exponent on an integral scheme X. The weighted blow-up $\pi\colon \mathrm{Bl}_\gamma X \to X$ satisfies the following universal property. Let $f\colon T \to X$ be a dominant morphism, with T normal and integral. Then there exists at most one lift $g\colon T \to \mathrm{Bl}_\gamma X$ of f and such a lift exists if and only if $\gamma\mathscr{O}_T$ is equal to the idealistic exponent over T associated to an effective Cartier divisor D on T (3.4). If this is the case, then $D = g^{-1}(E)$.* □

4. Weighted normal cones

4.1. **Weighted embeddings.** There is a one-to-one correspondence between closed embeddings (of finite presentation) and quasi-coherent ideal sheaves (of finite type). Similarly, we introduce the geometric objects corresponding to Rees algebras:

Definition 4.1.1. A *weighted closed embedding* $Z_\bullet \hookrightarrow X$ is a sequence of closed embeddings $\{Z_n \hookrightarrow X\}_{n\geq 0}$, such that if I_n denotes the ideal sheaf of $Z_n \hookrightarrow X$, then $I_\bullet$ is a Rees algebra.

A weighted closed embedding thus consists of a closed embedding $Z_1 \hookrightarrow X$, together with a choice of infinitesimal thickenings $Z_n \hookrightarrow X$ for $n \geq 1$ satisfying the compatibility conditions of a Rees algebra. Note that $Z_0 = \emptyset$.

We say that $Z_\bullet$ is the *weighted center* of the weighted blow-up $\mathrm{Bl}_{I_\bullet} X$ (or the Rees algebra $I_\bullet$). We also write $\mathrm{Bl}_{Z_\bullet} X$ for $\mathrm{Bl}_{I_\bullet} X$.

4.2. **Weighted normal cones.** Given a weighted closed embedding $Z_\bullet \hookrightarrow X$, we can forget the weighting and form

(i) the conormal algebra $\mathscr{C}_{Z_1/X} := \bigoplus_{n\geq 0} (I_1)^n/(I_1)^{n+1}$, and
(ii) the conormal sheaf $\mathscr{N}^\vee_{Z_1/X} := I_1/I_1^2$,

(iii) the normal cone $C_{Z_1}X := \operatorname{Spec}_X(\mathscr{C}_{Z_1/X})$.

For every n, the weighting defines a decreasing filtration F on $(I_1)^n/(I_1)^{n+1}$ where for $d \in \mathbb{N}$, the dth filtered piece is $F^d = \big((I_1)^n \cap I_d + (I_1)^{n+1}\big)/(I_1)^{n+1}$.

We next introduce the weighted analogues of the above constructions:

Definition 4.2.1. Associated to a weighted closed embedding $Z_\bullet \hookrightarrow X$ are

(i) its *weighted conormal algebra* $\mathscr{C}_{Z_\bullet/X} := \bigoplus_{n\geq 0} I_n/I_{n+1} = I_\bullet^{\mathrm{ext}}/(t^{-1})$,

(ii) its *weighted conormal sheaf* $\mathscr{N}^\vee_{Z_\bullet/X} := (\mathscr{C}_{Z_\bullet/X})_+/(\mathscr{C}_{Z_\bullet/X})_+^2$, and

(iii) its *weighted normal cone* $C_{Z_\bullet}X := \operatorname{Spec}_X(\mathscr{C}_{Z_\bullet/X})$.

Note that the sheaf $\mathscr{N}^\vee_{Z_\bullet/X}$ has a natural $\mathbb{N}$-grading induced from that of the weighted conormal algebra. Recall too from Definition 3.2.2 that:

(iv) the *projectivized* weighted normal cone $\mathscr{P}\mathrm{roj}_X(\mathscr{C}_{Z_\bullet/X})$ of $Z_\bullet \hookrightarrow X$ is the *exceptional divisor* E of the weighted blow-up $\mathrm{Bl}_{Z_\bullet} X \to X$.

In the event that Z_1 is a point, (iv) is also known as the *projectivized weighted tangent cone* of $Z_\bullet \hookrightarrow X$.

4.3. **Deformation to the weighted normal cone.** We will now generalize the classical deformation to the normal cone [Ful84, §5.2] to the weighted case[2]. Recall that in the classical case, given a closed embedding $i\colon Z \hookrightarrow X$, there is a flat morphism $\pi\colon D_Z X \to \mathbb{A}^1$ such that $\pi^{-1}(0) = C_Z X$ is the normal cone and $\pi^{-1}(p) = X$ for all $p \neq 0$. More precisely, there is a closed embedding $k\colon Z \times \mathbb{A}^1 \hookrightarrow D_Z X$ such that outside $0 \in \mathbb{A}^1$, the embedding k is identified with $k|_{\mathbb{A}^1\smallsetminus 0} = i \times \mathbb{1}\colon Z \times \mathbb{G}_m \hookrightarrow X \times \mathbb{G}_m$ and over 0, the embedding k is the zero section $k|_0 = s\colon Z \hookrightarrow C_Z X$ of the normal cone. One can also replace $\mathbb{A}^1$ with $\mathbb{P}^1$.

Let us first begin with some generalities.

Definition 4.3.1 (Weighted cones)**.** For a quasi-coherent, finitely generated graded $\mathscr{O}_X$-algebra R with $R_0 = \mathscr{O}_X$, we call $C = \operatorname{Spec}_X(R)$ the *weighted cone* of R.

If R is generated in degree 1, we simply call C the *cone* of R. Note that every twisted weighted bundle (Definition 2.1.3) is a weighted cone.

4.3.2. *Zero section.* Every weighted cone $C = \operatorname{Spec}_X(R)$ over X admits a *zero section* $s\colon X \hookrightarrow C$, induced by the surjection $R \twoheadrightarrow R_0 = \mathscr{O}_X$. This closed embedding $s\colon X \hookrightarrow C$ fits into a weighted closed embedding $s_\bullet\colon X_\bullet \hookrightarrow C$ with $s_1 = s$. This weighted embedding $s_\bullet\colon X_\bullet \hookrightarrow C$ is defined by the Rees algebra $I_\bullet$ on C given by

$$I_d := \bigoplus_{n\geq d} R_n \qquad \text{for } d \in \mathbb{N}.$$

Note that the weighted normal cone $C_{X_\bullet}C$ recovers C. Finally, $C \to X$ is a twisted weighted bundle, if and only if $s_\bullet$ is a regular weighted embedding.

Definition 4.3.3. Let $i_\bullet\colon Z_\bullet \hookrightarrow X$ be a weighted embedding. The *deformation to the weighted normal cone* is $D_{Z_\bullet}X = \operatorname{Spec}_X(I_\bullet^{\mathrm{ext}})$.

We make the following observations:

[2] We use the notation $D_Z X \subset \overline{D_Z X}$ instead of $M^\circ_Z X \subset M_Z X$ and the special point 0 instead of ∞.

(i) The grading equips $D_{Z_\bullet}X$ with a $\mathbb{G}_m$-action.
(ii) The inclusion $\mathscr{O}_X[t^{-1}] \subset I_\bullet^{\mathrm{ext}}$ induces a $\mathbb{G}_m$-equivariant map $p\colon D_{Z_\bullet}X \to X \times \mathbb{A}^1$.
(iii) The induced morphism $\pi\colon D_{Z_\bullet}X \to \mathbb{A}^1$ is flat since t^{-1} is regular on $I_\bullet^{\mathrm{ext}}$.
(iv) The filtration $(I_{\geq d}^{\mathrm{ext}})$ induces a weighted closed $\mathbb{G}_m$-equivariant embedding $k_\bullet\colon (Z \times \mathbb{A}^1)_\bullet \hookrightarrow D_{Z_\bullet}X$.

Note that k_1 is induced by the surjection $I_\bullet^{\mathrm{ext}} \to I_\bullet^{\mathrm{ext}}/(I_+^{\mathrm{ext}}) = (\mathscr{O}_X/I_1)[t^{-1}]$ so $(Z \times \mathbb{A}^1)_1 = Z_1 \times \mathbb{A}^1$. In general, however, $(Z \times \mathbb{A}^1)_n \neq Z_n \times \mathbb{A}^1$, e.g., $I_\bullet^{\mathrm{ext}}/(I_{\geq 2}^{\mathrm{ext}})$ is $\mathscr{O}_X/I_2$ in degrees ≤ 0 and I_1/I_2 in degree 1.

Proposition 4.3.4 (Deformation to the weighted normal cone)**.** *Consider the sequence* $(Z \times \mathbb{A}^1)_\bullet \xrightarrow{k_\bullet} D_{Z_\bullet}X \xrightarrow{p} X \times \mathbb{A}^1 \to \mathbb{A}^1$.

(i) *Outside* $0 \in \mathbb{A}^1$, *this restricts to*
$$Z_\bullet \times \mathbb{G}_m \xhookrightarrow{i_\bullet \times \mathbb{1}} X \times \mathbb{G}_m \xrightarrow{\mathbb{1}} X \times \mathbb{G}_m \to \mathbb{G}_m$$
(ii) *Over* $0 \in \mathbb{A}^1$, *this restricts to*
$$Z_\bullet \xhookrightarrow{s_\bullet \times \mathbb{1}} C_{Z_\bullet}X \to X \to \{0\}$$
where $s_\bullet$ *is the weighted zero section.*

Proof. Outside 0, we are inverting t^{-1}. This gives
$$I_\bullet^{\mathrm{ext}}[t] = \mathscr{O}_X[t, t^{-1}] \text{ and } (I_{\geq d}^{\mathrm{ext}}) = I_d[t, t^{-1}].$$
Over 0, we are taking the quotient with t^{-1} which gives
$$I_\bullet^{\mathrm{ext}}/(t^{-1}) = \bigoplus_{n \geq 0} I_n/I_{n+1} = \mathscr{C}_{Z_\bullet/X} \text{ and } (I_{\geq d}^{\mathrm{ext}}) = \bigoplus_{n \geq d} I_n/I_{n+1}. \qquad \square$$

4.3.5. *Compactified version.* There is also a compactified version of the deformation to the normal cone which is projective over $X \times \mathbb{A}^1$ (or $X \times \mathbb{P}^1$). This is constructed as $\overline{D_Z X} = \mathrm{Bl}_{Z \times \{0\}}(X \times \mathbb{A}^1)$. Similarly, given a weighted embedding $i_\bullet\colon Z_\bullet \to X$, we let
$$\overline{D_{Z_\bullet}X} = \mathrm{Bl}_{Z_\bullet \times \{0\}}(X \times \mathbb{A}^1)$$
where $Z_n \times \{0\} \hookrightarrow X \times \mathbb{A}^1$ is the natural map. To describe $\overline{D_{Z_\bullet}X}$, we first need to introduce projective completions of weighted cones.

Definition 4.3.6 (Projective completion)**.** Fix a quasi-coherent, finitely generated graded $\mathscr{O}_X$-algebra R with $R_0 = \mathscr{O}_X$, with corresponding weighted cone C. Let y be an indeterminate (with degree 1), and let $R[y]$ denote the quasi-coherent graded $\mathscr{O}_X$-algebra whose degree d piece is:
$$R[y]_d := R_d \oplus R_{d-1} \cdot y \oplus \cdots \oplus R_1 \cdot y^{d-1} \oplus R_0 \cdot y^d.$$
Then the *projective completion* of C, denoted $\mathscr{P}(C \oplus \mathbb{1})$, is defined as
$$\mathscr{P}(C \oplus \mathbb{1}) := \mathscr{P}\mathrm{roj}_X(R[y]).$$
The *projectivized weighted cone* $\mathscr{P}(C) := \mathscr{P}\mathrm{roj}_X(R)$ sits inside $\mathscr{P}\mathrm{roj}_X(R[y])$ as the "hyperplane at infinity" cut out by $y = 0$, whose complement is the y-chart $D_+(y) := [\mathrm{Spec}_X(R[y]_y) \,/\, \mathbb{G}_m] = \mathrm{Spec}_X(R)$, which coincides with the cone C, cf. §1.3.

Proposition 4.3.7. *Let $Z_\bullet \hookrightarrow X$ be a weighted embedding, and let π denote the composition*

$$\overline{D_{Z_\bullet}X} := \mathrm{Bl}_{Z_\bullet \times \{0\}}(X \times \mathbb{A}^1) \xrightarrow{p} X \times \mathbb{A}^1 \xrightarrow{\mathrm{pr}_2} \mathbb{A}^1.$$

Then:

(i) *The exceptional divisor of $\mathrm{Bl}_{Z_\bullet \times \{0\}}(X \times \mathbb{A}^1)$ is canonically identified with the projective completion $\mathscr{P}(C_{Z_\bullet}X \oplus \mathbb{1})$ of the weighted normal cone $C_{Z_\bullet}X$.*

(ii) *$\pi^{-1}(0) = p^{-1}(X \times \{0\})$ is canonically identified with*

$$\mathrm{Bl}_{Z_\bullet} X \cup_E \mathscr{P}(C_{Z_\bullet}X \oplus \mathbb{1})$$

where $E = \mathscr{P}(C_{Z_\bullet}X)$ is the exceptional divisor of $\mathrm{Bl}_{Z_\bullet} X$.

(iii) *The deformation to the weighted normal cone $D_{Z_\bullet}X$ is naturally identified as the open substack $\overline{D_{Z_\bullet}X} \smallsetminus \mathrm{Bl}_{Z_\bullet} X$.*

Proof. Let $\mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[u]$. Let $J_\bullet$ be the Rees algebra of the embedding $Z_\bullet \times \{0\} \hookrightarrow X \times \mathbb{A}^1$. Then $J_\bullet^{\mathrm{ext}} = I_\bullet^{\mathrm{ext}}[u, U]/(t^{-1}U - u)$. In particular, $\mathscr{C}_{Z_\bullet \times \{0\}/X \times \mathbb{A}^1} = J_\bullet^{\mathrm{ext}}/(t^{-1}) = \mathscr{C}_{Z_\bullet/X}[U]$ and the part (i) follows.

For part (ii), the fiber $\pi^{-1}(0)$ corresponds to $t^{-1}U = u = 0$ and thus splits up in two components. The first, $t^{-1} = 0$, is the exceptional divisor $\mathscr{P}(C_{Z_\bullet}X \oplus \mathbb{1})$, the second, $U = 0$, is $\mathrm{Bl}_{Z_\bullet} X = \mathscr{P}\mathrm{roj}_X(I_\bullet^{\mathrm{ext}})$ and their intersection $t^{-1} = U = 0$ is the exceptional divisor E.

For part (iii), the open set in question is

$$D_+(U) = \left[\operatorname{Spec}_X(I_\bullet^{\mathrm{ext}}[U, U^{-1}]) \,/\, \mathbb{G}_m\right] = \operatorname{Spec}_X(I_\bullet^{\mathrm{ext}}) = D_{Z_\bullet}X. \qquad \square$$

5. Weighted blow-ups along regular embeddings

Recall that if $Z \hookrightarrow X$ is a regular embedding, then it is a quasi-regular embedding [Stacks, 00LN], i.e., the conormal sheaf I/I^2 is locally free and the canonical surjection $\mathrm{Sym}_{\mathscr{O}_Z}(I/I^2) \twoheadrightarrow \mathscr{C}_{Z/X}$ is an isomorphism. In this section we will discuss the notions of *quasi-regular weighted embeddings* and *regular weighted embeddings*. However, we forewarn the reader that even if $Z_\bullet \hookrightarrow X$ is a quasi-regular weighted embedding, there are only local isomorphisms between $\mathrm{Sym}_{\mathscr{O}_{Z_1}}(\mathscr{N}^\vee_{Z_\bullet/X})$ and $\mathscr{C}_{Z_\bullet/X}$, which in general are not compatible with each other. The former is a weighted (or graded) vector bundle (2.1.9) while the latter in general is a *twisted* weighted vector bundle (Proposition 5.1.4).

In the noetherian case, a weighted embedding is regular, if and only if it is quasi-regular (Corollary 5.2.4). In this case there is a simple description of the extended Rees algebra $I_\bullet^{\mathrm{ext}}$ (Proposition 5.2.2) and the charts of the weighted blow-up (Corollary 5.2.5).

5.1. **Quasi-regular weighted embeddings.**

Suppose that we have global sections $x_1, x_2, \ldots, x_m \in \mathscr{O}_X$ and positive integers $d_1, d_2, \ldots, d_m$ such that $I_\bullet = (x_1, d_1) + \cdots + (x_m, d_m)$. We consider the graded polynomial ring $(\mathscr{O}_X/I_1)[X_1, X_2, \ldots, X_m]$ where X_i has degree d_i. We have a natural graded homomorphism

$$\alpha\colon (\mathscr{O}_X/I_1)[X_1, X_2, \ldots, X_m] \to \mathscr{C}_{Z_\bullet/X} = \bigoplus_{n \geq 0} I_n/I_{n+1}$$

taking X_i to $x_i \in I_{d_i}/I_{d_i+1}$. This map is evidently surjective.

Definition 5.1.1. We say that $(x_1, d_1), \ldots, (x_m, d_m)$ is a *quasi-regular* sequence if α is bijective.

When $d_1 = d_2 = \cdots = d_m = 1$, then $I_n = I_1^n$ and this is the usual notion of a quasi-regular sequence $x_1, x_2, \ldots, x_m$.

Proposition 5.1.2. *The sequence $(x_1, d_1), \ldots, (x_m, d_m)$ is quasi-regular if and only if $x_1, x_2, \ldots, x_m$ is quasi-regular.*

Proof. It suffices to prove that if $(x_1, d_1), \ldots, (x_m, d_m)$ is quasi-regular, then so is $(x_1, e_1), \ldots, (x_m, e_m)$ for every sequence $e_1, e_2, \ldots, e_m \in \mathbb{Z}_{>0}$. Let $J_\bullet = (x_1, e_1) + \cdots + (x_m, e_m)$ and note that $I_1 = J_1$. For a multi-index $\alpha \in \mathbb{N}^m$, let $|\alpha|_d = d_1\alpha_1 + \cdots + d_m\alpha_m$ and $|\alpha|_e = e_1\alpha_1 + \cdots + e_m\alpha_m$.

By quasi-regularity of $(x_1, d_1), \ldots, (x_m, d_m)$, we have that I_a/I_{a+1} is a free $\mathscr{O}_X/I_1$-module with generators x^α such that $|\alpha|_d = a$. We have a filtration

$$I_a/I_{a+1} = I_a \cap J_1/I_{a+1} \supseteq (I_a \cap J_2 + I_{a+1})/I_{a+1} \supseteq \cdots$$

By definition, we have the inclusion

$$\bigoplus_{\substack{|\alpha|_d = a \\ |\alpha|_e \geq b}} (\mathscr{O}_X/I_1)x^\alpha \subset (I_a \cap J_b + I_{a+1})/I_{a+1}. \tag{5.1}$$

Claim. The inclusion (5.1) is an equality.

Proof of Claim. To see this, let $z \in I_a \cap J_b$. Then by definition of $J_\bullet$, we have that $z = \sum_{|\alpha|_e \geq b} z_\alpha x^\alpha$ for some (non-unique) $z_\alpha \in \mathscr{O}_X$. Let $c = \min\{|\alpha|_d : z_\alpha \neq 0\}$. If $c \geq a$, then z mod I_{a+1} is in the left hand side and we are done.

If $c < a$ and there exists α with $|\alpha|_d = c$ and $z_\alpha \notin I_1$, then it follows by quasi-regularity of $(x_1, d_1), \ldots, (x_m, d_m)$ that $z \in I_c \smallsetminus I_{c+1}$. This contradicts that $z \in I_a$.

Finally, suppose that $c < a$ and $z_\alpha \in I_1 = (x_1, x_2, \ldots, x_m)$ for all α such that $|\alpha|_d = c$. Then z can be written as a polynomial $\sum_\alpha \tilde{z}_\alpha x^\alpha$ where the non-zero terms have $|\alpha|_d > c$ and $|\alpha|_e \geq b$. Repeating the argument with this expression of z increases c and eventually $c \geq a$ and we get the desired equality. △

Next, we show that the equality (5.1) implies that $(x_1, e_1), \ldots, (x_m, e_m)$ is quasi-regular. Indeed, let $f(X) = \sum_{|\alpha|_e = b} f_\alpha X^\alpha \in \mathscr{O}_X[X_1, X_2, \ldots, X_m]$ be a polynomial that is homogeneous of degree b with respect to the e_i-grading, and such that the image of f in $(\mathscr{O}_X/J_1)[X_1, X_2, \ldots, X_m]$ is non-zero. We want to show that $f(x) \notin J_{b+1}$. Set $a = \min\{|\alpha|_d \colon f_\alpha \notin I_1\}$, so that by the quasi-regularity of $(x_1, d_1), \ldots, (x_m, d_m)$, $f(x) \in I_a \smallsetminus I_{a+1}$. Then the image of $f(x)$ in $(I_a \cap J_b + I_{a+1})/I_{a+1}$ is non-zero, and is not contained in $(I_a \cap J_{b+1} + I_{a+1})/I_{a+1}$, by the claim and the hypothesis that $f(X)$ is homogeneous of degree b. In other words, $f(x) \notin J_{b+1}$, as desired. □

Definition 5.1.3. We say that a weighted closed embedding $Z_\bullet \hookrightarrow X$ is *quasi-regular* if at every point $p \in |Z_1|$, there exists a smooth neighborhood $U \to X$ of p, a quasi-regular sequence $x_1, x_2, \ldots, x_m$ on U, and positive integers $d_1, d_2, \ldots, d_m$, such that $I_\bullet|_U = (x_1, d_1) + \cdots + (x_m, d_m)$.

Proposition 5.1.4. *A weighted closed embedding $Z_\bullet \hookrightarrow X$ is quasi-regular if and only if the weighted normal cone $C_{Z_\bullet/X} \to Z_1$ is a twisted weighted vector bundle.*

Proof. If $I_\bullet = (x_1, d_1) + \cdots + (x_m, d_m)$ for a quasi-regular sequence, then we have seen that $\mathscr{C}_{Z_\bullet/X}$ is a graded polynomial ring. Conversely, if $C_{Z_\bullet/X} \to Z_1$ is a twisted weighted vector bundle, then locally on X we have that $\mathscr{C}_{Z_\bullet/X}$ is a graded polynomial ring $\mathscr{O}_{Z_1}[X_1, X_2, \ldots, X_m]$. If we take any preimages $x_i \in I_{d_i}$ of $X_i \in I_{d_i}/I_{d_i+1}$, then $(x_1, d_1), \ldots, (x_m, d_m)$ is quasi-regular. $\square$

Remark 5.1.5. Recall that the weighted conormal sheaf $\mathscr{N}^\vee_{Z_\bullet/X}$ is a graded vector bundle whereas the *unweighted* conormal sheaf $\mathscr{N}^\vee_{Z_1/X}$ is equipped with a filtration (§4.2). There is a canonical surjection $\mathscr{N}^\vee_{Z_\bullet/X} \to \mathrm{Gr}_F(\mathscr{N}^\vee_{Z_1/X})$ but it is not an isomorphism of $\mathscr{O}_{Z_1}$-modules in general, see Example 5.1.7. For $d \geq 1$, the dth graded pieces are as follows:

$$(\mathscr{N}^\vee_{Z_\bullet/X})_d \simeq \frac{I_d/I_{d+1}}{(I_d/I_{d+1}) \cap (\mathscr{C}_{Z_\bullet/X})^2_+}$$

$$\mathrm{Gr}^d_F(\mathscr{N}^\vee_{Z_1/X}) \simeq \frac{I_d}{(I_1^2 + I_{d+1}) \cap I_d} \simeq \frac{I_d/I_{d+1}}{\big((I_1^2 \cap I_d) + I_{d+1}\big)/I_{d+1}}.$$

Remark 5.1.6. If $Z_\bullet \hookrightarrow X$ is quasi-regular, then the canonical surjection $\mathscr{N}^\vee_{Z_\bullet/X} \twoheadrightarrow \mathrm{Gr}_F(\mathscr{N}^\vee_{Z_1/X})$ in Remark 5.1.5 is an isomorphism. Indeed, we may assume we are in the local situation where $I_\bullet = (x_1, d_1) + \cdots + (x_m, d_m)$ for a quasi-regular sequence $x_1, x_2, \ldots, x_m \in \mathscr{O}_X$, and positive integers $d_1, d_2, \ldots, d_m$. By Definition 5.1.1, $(\mathscr{N}^\vee_{Z_\bullet/X})_d$ is only non-zero (in which case it is locally free as an $\mathscr{O}_{Z_1}$-module) in degrees $d \in \{d_1, d_2, \ldots, d_m\}$, and the same holds for $\mathrm{Gr}^d_F(\mathscr{N}^\vee_{Z_1/X})$. It remains to note that the canonical surjection carries the free generators x_i in each non-zero degree of $\mathscr{N}^\vee_{Z_\bullet/X}$ to free generators in the corresponding degree of $\mathrm{Gr}^d_F(\mathscr{N}^\vee_{Z_1/X})$.

Example 5.1.7. For a counterexample, consider on $X = \mathbb{A}^1_{\Bbbk} = \mathrm{Spec}(\Bbbk[x])$ the Rees algebra $I_\bullet = (x, 1) + (x^2, 3)$. Then $(\mathscr{N}^\vee_{Z_\bullet/X})_d \neq 0$ if and only if $d \in \{1, 3\}$, where $(\mathscr{N}^\vee_{Z_\bullet/X})_1 = (x)/(x^2)$ and $(\mathscr{N}^\vee_{Z_\bullet/X})_3 = (x^2)/(x^3)$. On the other hand, $\mathrm{Gr}^1_F(\mathscr{N}^\vee_{Z_1/X}) = (x)/(x^2)$, and $\mathrm{Gr}^d_F(\mathscr{N}^\vee_{Z_1/X}) = 0$ for $d \neq 1$.

5.2. **Regular weighted embeddings.** As before, suppose we have global sections $x_1, x_2, \ldots, x_m \in \mathscr{O}_X$ and positive integers $d_1, d_2, \ldots, d_m$ such that $I_\bullet = (x_1, d_1) + \cdots + (x_m, d_m)$. We have the graded polynomial ring

$$\mathscr{O}_X[t^{-1}, X_1, X_2, \ldots, X_m]$$

where we let $\deg(t^{-1}) = -1$ and $\deg(X_i) = d_i$. There is a natural map

$$\mathscr{O}_X[t^{-1}, X_1, X_2, \ldots, X_m] \to I^{\mathrm{ext}}_\bullet \subset \mathscr{O}_X[t, t^{-1}]$$

of $\mathscr{O}_X$-algebras that takes X_i to $x_i t^{d_i}$ and t^{-1} to t^{-1}. We note that $t^{-d_i} X_i - x_i$ is in the kernel so that we obtain a map

$$\beta\colon B := \mathscr{O}_X[t^{-1}, X_1, X_2, \ldots, X_m]/(t^{-d_i} X_i - x_i \colon 1 \leq i \leq m) \to I^{\mathrm{ext}}_\bullet.$$

Note that β is surjective.

The following lemma generalizes [Stacks, 0G8S].

Lemma 5.2.1. *The kernel of β equals the kernel of $B \to B_{t^{-1}}$, that is, $\ker\beta = \bigcup_{n\geq 1}\operatorname{Ann}_B(t^{-n})$. In particular, β is bijective if and only if $t^{-1}\in B$ is a non-zero divisor.*

Proof. The composition $B \xrightarrow{\beta} I_\bullet^{\mathrm{ext}} \hookrightarrow \mathscr{O}_X[t,t^{-1}]$ has the same kernel as β, and factors through the localization $B \to B_{t^{-1}}$. The result follows since $B_{t^{-1}} \to \mathscr{O}_X[t,t^{-1}]$ is an isomorphism. □

The following proposition generalizes [Stacks, 0BIQ]. For the definition of H_1-regular sequences, see [Stacks, 062D]. For noetherian stacks, H_1–regular is equivalent to regular.

Proposition 5.2.2. *If $x_1, x_2, \dots, x_m$ is an H_1-regular sequence, then β is bijective, so that*

$$\mathrm{Bl}_{I_\bullet} X = \mathscr{P}\mathrm{roj}_X(I_\bullet^{\mathrm{ext}}) \xrightarrow{\simeq} \mathscr{P}\mathrm{roj}_X\left(\frac{\mathscr{O}_X[t^{-1}, X_1, X_2, \dots, X_m]}{(t^{-d_i}X_i - x_i\colon 1\leq i\leq m)}\right).$$

Proof. If $x_1, x_2, \dots, x_m \in \mathscr{O}_X$ is an H_1-regular sequence, then the sequence $x_1, x_2, \dots, x_m, t^{-1}$ in $\mathscr{O}_X[t^{-1}, X_1, X_2, \dots, X_m]$ is H_1-regular as well [Stacks, 0668]. The sequence $t^{-d_1}X_1 - x_1, t^{-d_2}X_2 - x_2, \dots, t^{-d_m}X_m - x_m, t^{-1}$ generates the same ideal and is thus also H_1-regular. It follows that $t^{-1}\in B$ is a non-zero divisor [Stacks, 068L]. □

Question 5.2.3. *Is $x_1, x_2, \dots, x_m$ always an H_1-regular sequence if β is bijective?*

The following corollary shows that the answer is 'yes' in the noetherian case.

Corollary 5.2.4. *Let $I_\bullet = (x_1, d_1) + \dots + (x_m, d_m)$ as before. Consider the conditions*

(i) *$x_1, x_2, \dots, x_m$ is an H_1-regular sequence.*
(ii) *β is bijective.*
(iii) *$x_1, x_2, \dots, x_m$ is a quasi-regular sequence.*

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). If X is locally noetherian, then the three conditions are equivalent.

Proof. We have seen that (i) $\Longrightarrow$ (ii) and if X is locally noetherian then (iii) $\Longrightarrow$ (i). It thus remains to prove (ii) $\Longrightarrow$ (iii). But if β is bijective, then the weighted conormal algebra becomes

$$\begin{aligned}\mathscr{C}_{Z_\bullet/X} = I_\bullet^{\mathrm{ext}}/t^{-1}I_\bullet^{\mathrm{ext}} &= \mathscr{O}_X[t^{-1}, X_1, X_2, \dots, X_m]/(t^{-1}, x_1, x_2, \dots, x_m)\\ &= \mathscr{O}_{Z_1}[X_1, X_2, \dots, X_m]\end{aligned}$$

so the sequence $x_1, x_2, \dots, x_m$ is quasi-regular. □

As a consequence of Proposition 5.2.2 and Lemma 1.3.1 (with $A = \mathbb{Z}$, $a = d_i \in \mathbb{Z}$, and $r = x_i \cdot t^{d_i}$), we obtain

Corollary 5.2.5 (Charts for blow-ups along regular weighted embeddings)**.** *If $x_1, x_2, \dots, x_m$ is an H_1-regular sequence, then for each $1 \leq i \leq m$, the* *chart* $D_+(x_i \cdot t^{d_i})$ *of* $\mathrm{Bl}_{I_\bullet} X$ *is:*

$$\left[\mathrm{Spec}_X \left(\frac{\mathscr{O}_X[t^{-1}, X_1, X_2, \ldots, X_m][X_i^{-1}]}{(t^{-d_j} X_j - x_j \colon 1 \leq j \leq m)}\right) \ / \ \mathbb{G}_m\right]$$
$$= \left[\mathrm{Spec}_X \left(\frac{\mathscr{O}_X[t^{-1}, X_1, X_2, \ldots \widehat{X_i}, \ldots, X_m]}{(t^{-d_i} - x_i) + (t^{-d_j} X_j - x_j \colon 1 \leq j \leq m, \ j \neq i)}\right) \ / \ \boldsymbol{\mu}_{d_i}\right]$$

where $\widehat{X_i}$ *means* X_i *omitted.* □

Slightly more generally, let A be a finitely generated abelian group, $D(A)$ be the corresponding diagonalizable algebraic group, and let $D(A)$ act on X. Assume that $x_1, x_2, \ldots, x_m \in \mathscr{O}_X$ is an A-homogeneous H_1-regular sequence, with weights $\mathrm{wt}_A(x_i) = a_i$ for $1 \leq i \leq m$, so that $I_\bullet = (x_1, d_1) + \cdots + (x_m, d_m)$ is an A-graded Rees algebra on X defining a regular weighted embedding $Z_\bullet \hookrightarrow X$. Thus, $I_\bullet$ descends to a Rees algebra $\mathscr{I}_\bullet$ on $[X \ / \ D(A)]$, which defines a regular weighted embedding $[Z_\bullet \ / \ D(A)] \hookrightarrow [X \ / \ D(A)]$. Consider the diagram:

$$\begin{tikzcd}
D_+(x_i \cdot t^{d_i}) \arrow[r, hook] \arrow[d, two heads] & \mathrm{Bl}_{I_\bullet} X \arrow[r] \arrow[d, two heads] & X \arrow[d, two heads] \\
{[D_+(x_i \cdot t^{d_i}) \ / \ D(A)]} \arrow[r, hook] & {[\mathrm{Bl}_{I_\bullet} X \ / \ D(A)]} \arrow[r] & {[X \ / \ D(A)]}
\end{tikzcd}$$

where the right morphism in the bottom row is also the weighted blow-up $\mathrm{Bl}_{\mathscr{I}_\bullet}[X \ / \ D(A)] \to [X \ / \ D(A)]$. The next corollary obtains a description for $[D_+(x_i \cdot t^{d_i}) \ / \ D(A)]$ which is analogous to that for $D_+(x_i \cdot t^{d_i})$ in the previous corollary:

Corollary 5.2.6 (Charts for blow-ups along regular weighted embeddings, with respect to a $D(A)$-action)**.** *For* $1 \leq i \leq m$*, the* i^{th} *chart* $[D_+(x_i \cdot t^{d_i}) \ / \ D(A)]$ *of* $\mathrm{Bl}_{\mathscr{I}_\bullet}[X \ / \ D(A)]$ *is:*

$$\left[\mathrm{Spec}_X \left(\frac{\mathscr{O}_X[t^{-1}, X_1, X_2, \ldots \widehat{X_i}, \ldots, X_m]}{(t^{-d_i} - x_i) + (t^{-d_j} X_j - x_j \colon 1 \leq j \leq m, \ j \neq i)}\right) \ / \ D(A')\right]$$

where $A' = A\langle -\frac{a_i}{d_i}\rangle := (A \oplus \mathbb{Z})/\langle (a_i, d_i)\rangle$ *and the action of* $D(A')$ *corresponds to the weights* $\mathrm{wt}_{A'}(X_j) = a_j - d_j \frac{a_i}{d_i}$ *for* $j \neq i$*, and* $\mathrm{wt}_{A'}(t^{-1}) = \frac{a_i}{d_i}$*.*

Proof. Note that the i^{th} chart is, by Proposition 5.2.2:

$$\left[\mathrm{Spec}_X \left(\frac{\mathscr{O}_X[t^{-1}, X_1, X_2, \ldots, X_m][X_i^{-1}]}{(t^{-d_j} X_j - x_j \colon 1 \leq j \leq m)}\right) \ / \ D(A) \times \mathbb{G}_m\right]$$

where the action of $D(A) \times \mathbb{G}_m$ is expressed via the weights $\mathrm{wt}_{A\oplus\mathbb{Z}}(X_j) = (a_j, d_j)$ for $1 \leq j \leq m$, and $\mathrm{wt}_{A\oplus\mathbb{Z}}(t^{-1}) = (0, -1)$. The corollary thus follows from Lemma 1.3.1, with A there replaced by $A \oplus \mathbb{Z}$ here, a there replaced by (a_i, d_i) here, and $r = X_i$. □

Motivated by the results above, we conclude the subsection with the following definition:

Definition 5.2.7. A weighted closed embedding $Z_\bullet \hookrightarrow X$ is called *regular* (resp. H_1*-regular*) if at every point $p \in |Z_1|$, there exists a smooth neighborhood $U \to X$ of p, a regular (resp. H_1-regular) sequence $x_1, x_2, \ldots, x_m$ on U, and positive integers $d_1, d_2, \ldots, d_m$, such that $I_\bullet|_U = (x_1, d_1) + \cdots + (x_m, d_m)$.

5.3. **Regular weighted centers.** In this subsection we assume that X is noetherian and regular, and we let $Z_\bullet \hookrightarrow X$ be a weighted closed embedding.

Definition 5.3.1. We say that the weighted center $Z_\bullet$ is *regular*, if Z_1 is regular and $Z_\bullet \hookrightarrow X$ is a regular weighted embedding.

Recall that the latter is equivalent to the weighted normal cone $C_{Z_\bullet/X} \to Z_1$ being a twisted weighted vector bundle (Proposition 5.1.4). Note too that while Z_1 being regular ensures that $Z_1 \hookrightarrow X$ is a regular embedding, it does not imply that $Z_\bullet \hookrightarrow X$ is a regular weighted embedding (Example 5.1.7).

This means that smooth locally around each point $p \in |Z_1|$, $I_\bullet = (x_1, d_1) + \cdots + (x_m, d_m)$, where $x_1, x_2, \ldots, x_m$ is a regular sequence that can be extended to a regular system of parameters at p, and $d_1, d_2, \ldots, d_m$ are positive integers. Note that our notion of regular weighted centers here is precisely the notion of "centers" in [ATW19, Section 2.4].

Corollary 5.3.2. *If $Z_\bullet$ is a regular weighted center, then the deformation to the weighted normal cone $D_{Z_\bullet} X = \mathrm{Spec}_X(I_\bullet^{\mathrm{ext}})$ is regular. In particular, the stack-theoretic weighted blow-up $\mathrm{Bl}_{Z_\bullet} X \subset \left[D_{Z_\bullet} X \,/\, \mathbb{G}_m\right]$ is regular.*

Proof. By Proposition 5.1.4, the weighted normal cone $N_{Z_\bullet} X$ is a twisted weighted vector bundle over Z_1, hence regular if Z_1 is regular. Since $N_{Z_\bullet} X$ is the Cartier divisor $t^{-1} = 0$ in $D_{Z_\bullet} X$ and its complement is $X \times \mathbb{G}_m$, it follows that $D_{Z_\bullet} X$ is regular. □

For the remainder of this subsection, we focus on the local case where X is a scheme, and $I_\bullet = (x_1, d_1) + \cdots + (x_m, d_m)$ for a regular sequence $x_1, x_2, \ldots, x_m \in \mathscr{O}_X$ that admits an extension $x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_r$ on X that is a regular system of parameters at some point $p \in |Z_1|$. Adopting the notation in Corollary 5.2.5, Corollary 5.3.2 can be made more precise as follows:

Proposition 5.3.3. *Let $1 \le i \le m$. On $D_{Z_\bullet} X$, the sequence*

$$t^{-1}, X_1, X_2, \ldots, X_m, x_{m+1}, \ldots, x_r$$

is a $\mathbb{Z}$-homogeneous regular system of parameters at any preimage of p in $t^{-1} = 0$. On the i^{th} chart $D_+(x_i \cdot t^{d_i})$ of $\mathrm{Bl}_{I_\bullet} X$, there is a $\boldsymbol{\mu}_{d_i}$-torsor and the induced $\mathbb{G}_m$-torsor as in Corollary 5.2.5.

(i) *On the total space of the $\mathbb{G}_m$-torsor,*

$$t^{-1}, X_1, X_2, \ldots, \widehat{X}_i, \ldots, X_m, x_{m+1}, \ldots, x_r$$

is a $\mathbb{Z}$-homogeneous regular sequence at any preimage of p that cuts out the $\mathbb{G}_m$-orbit $\mathrm{Spec}\big(\kappa(p)[X_i, X_i^{-1}]\big)$.

(ii) *On the total space of the $\boldsymbol{\mu}_{d_i}$-torsor,*

$$t^{-1}, X_1, X_2, \ldots, \widehat{X}_i, \ldots, X_m, x_{m+1}, \ldots, x_r$$

is a $\mathbb{Z}/d_i\mathbb{Z}$-homogeneous regular system of parameters at any preimage of p.

In both cases, $\mathrm{wt}(t^{-1}) = -1$ *and* $\mathrm{wt}(X_j) = d_j$.

Proof. Let q be a preimage of p in the $\mathbb{G}_m$-torsor (resp. the $\boldsymbol{\mu}_{d_i}$-torsor). Then the dimension at q is $r+1$ (resp. r). By Corollary 5.2.5, the sequences cut out the same scheme as $x_1, x_2, \dots, x_r$ cuts out on X, and the proposition follows. $\square$

Slightly more generally, adopt the hypotheses and notations of Corollary 5.2.6, and moreover assume that $x_1, x_2 \dots, x_m, x_{m+1} \dots, x_r$ is an A-homogeneous regular system of parameters on X at p. Then it is immediate that:

Corollary 5.3.4. *On the* i^{th} *chart* $D_+(x_i \cdot t^{d_i})$ *of* $\mathrm{Bl}_{\mathscr{I}_\bullet}[X \,/\, D(A)]$*, the sequence* $t^{-1}, X_1, X_2, \dots \widehat{X}_i, \dots, X_m, x_{m+1}, \dots, x_r$ *is an* A'*-homogeneous regular system of parameters (on the total space of the* A'*-torsor) with* $\mathrm{wt}(t^{-1}) = \frac{a_i}{d_i}$ *and* $\mathrm{wt}(X_j) = a_j - d_j \frac{a_i}{d_i}$. $\square$

The sequences above can also be interpreted as sequences of sections of line bundles on the stack $D_+(x_i \cdot t^{d_i})$ itself. Similarly, in the A-graded case $x_i \cdot t^{d_i}$ is also only a section of a line bundle.

5.4. **Toric interpretation.** In this subsection, we consider toric varieties and stacks over some base scheme (e.g., $\operatorname{Spec}\mathbb{C}$ or $\operatorname{Spec}\mathbb{Z}$). Let N be a lattice and let Σ be a fan in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. This defines a toric variety X_Σ. If Σ is simplicial, then the corresponding toric stack $\mathscr{X}_\Sigma$ is smooth with finite stabilizers and coarse space X_Σ.

More generally, let $\boldsymbol{\Sigma} = (N, \Sigma, \beta)$ be a *stacky fan* [BCS05, FMN10]. Recall that this consists of a finitely generated abelian group N, a *simplicial* fan Σ in $N_{\mathbb{R}}$ and a homomorphism $\beta\colon \mathbb{Z}^{\Sigma(1)} \to N$ with finite cokernel. Here $\Sigma(1)$ denotes the rays of Σ and for every $\rho \in \Sigma(1)$, the image of $\beta(\mathbf{e}_\rho)$ in $N_{\mathrm{tf}} = N/\mathrm{tor}(N) \subset N_{\mathbb{R}}$ is required to be a lattice point on the ray ρ. Thus, if N is free, then β is the choice of a lattice point on every ray of Σ. A stacky fan gives rise to a *smooth* toric stack $X_{\boldsymbol{\Sigma}}$ with finite abelian stabilizers and coarse space X_Σ (a possibly singular toric variety). The toric stack $\mathscr{X}_\Sigma$ alluded to above is the toric stack corresponding to the stacky fan where we for every ray ρ have chosen the generator of $N \cap \rho$.

Example 5.4.1 (Weighted blow-up of smooth toric variety)**.** Let N be a lattice, and let Σ be a smooth fan in $N_{\mathbb{R}}$. Let σ be a cone in Σ with rays $\rho_1, \rho_2, \dots, \rho_n$ generated by primitive lattice points $b_1, b_2, \dots, b_n$. Let v be a lattice point contained in the cone σ. Then it can be uniquely expressed as $v = \sum_{i=1}^n d_i b_i$ for some $d_i \in \mathbb{N}$.

Consider the *subdivision* $\Sigma^*(v)$ of Σ at v, i.e., this is the set of all cones in $\Sigma \smallsetminus \{\sigma\}$, as well as the cones generated by subsets of $\{b_1, b_2, \dots, b_n, v\}$ not containing $\{b_1, b_2, \dots, b_n\}$. Letting $X_{\Sigma^*(v)}$ be the corresponding toric variety, the identity map on N is compatible with the fans $\Sigma^*(v)$ and Σ, and thus induces a toric morphism

$$X_{\Sigma^*(v)} \to X_\Sigma$$

which is simply the *coarse space* of the weighted blow-up of X_Σ along the regular weighted center $I_\bullet = (I_{D_1}, d_1) + \cdots + (I_{D_n}, d_n)$, where each I_{D_i} is the ideal of the toric divisor D_i corresponding to the ray ρ_i. The new ray ρ_E in $\Sigma^*(v)$, generated by v, corresponds to the exceptional divisor E.

To recover the Deligne–Mumford stack and not the coarse space, let $\Sigma^*(v)(1) = \Sigma(1) \cup \{\rho_E\}$[3] denote the rays of $\Sigma^*(v)$, and consider the surjective homomorphism $\beta^*(v)\colon \mathbb{Z}^{\Sigma^*(v)(1)} \to N$, which sends $\mathbf{e}_\rho$ for each $\rho \in \Sigma(1)$ to the first lattice point of ρ and sends $\mathbf{e}_{\rho_E}$ to v. The associated toric stack $X_{\mathbf{\Sigma}^*(v)}$ coincides with the stack-theoretic weighted blow-up.

Example 5.4.2 (Weighted blow-up of toric stack)**.** More generally, let $\mathbf{\Sigma} = (N, \Sigma, \beta)$ be a stacky fan and let $\sigma \in \Sigma$ be a cone generated by rays $\rho_1, \rho_2, \dots, \rho_n$ with lattice points $b_i = \beta(\mathbf{e}_{\rho_i})$ and pick a lattice point v contained in σ such that it can be written as $v = \sum_{i=1}^n d_i b_i$ for some, necessarily unique, $d_i \in \mathbb{N}$. Then there is a subdivided stacky fan $\mathbf{\Sigma}^*(v) = \big(N, \Sigma^*(v), \beta^*(v)\big)$ where $\Sigma^*(v)$ is as above and $\beta^*(v)$ extends β by assigning $\beta^*(v)(\mathbf{e}_{\rho_E}) = v$. The identity map on N induces a morphism of toric stacks $X_{\mathbf{\Sigma}^*(v)} \to X_{\mathbf{\Sigma}}$ and this is the weighted blow-up along the regular weighted center $I_\bullet = (I_{D_1}, d_1) + \cdots + (I_{D_n}, d_n)$.

Remark 5.4.3 (Partial Cox construction)*.* We can also describe the weighted blow-up using a partial Cox construction. Let Σ be a smooth fan[4] and let $\sigma \in \Sigma$ be a cone with rays ρ_i and primitive generators b_i as before. Let $v = \sum_i d_i b_i$ be a lattice point in σ. Consider the lattice $N' = N \oplus \mathbb{Z}$ and the homomorphism $\gamma\colon N' \to N$ defined by $\gamma(x, 0) = x$ and $\gamma(0,1) = \sum_i d_i b_i$. This gives the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ \alpha\ } N' \xrightarrow{\ \gamma\ } N \longrightarrow 0$$

where $\alpha(1) = (\sum_i d_i b_i, -1)$. We lift the fan $\Sigma^*(v)$ in $N_\mathbb{R}$ to a fan $\Sigma'(v)$ in $N'_\mathbb{R}$ by lifting every ray $\rho \in \Sigma(1)$ with generator b to the ray through $(b, 0)$ and the ray ρ_E to the ray through $(0, 1)$. Note that $\Sigma'(v) \to \Sigma^*(v)$ is a bijection on cones. We can also partially compactify $\Sigma'(v)$ to $\overline{\Sigma'(v)}$ by adding the cones $\sigma = \{\rho_1, \dots, \rho_n\}$ and $\{\rho_1, \dots, \rho_n, \rho_E\}$. Note that $\overline{\Sigma'(v)} \to \Sigma$ induces a bijection on maximal cones.

Then $X_{\overline{\Sigma'(v)}} = \operatorname{Spec}(I_\bullet^{\mathrm{ext}})$ and $X_{\Sigma'(v)} = \operatorname{Spec}(I_\bullet^{\mathrm{ext}}) \smallsetminus V(I_+)$. The homomorphism α corresponds to the $\mathbb{G}_m$-action on $\operatorname{Spec}(I_\bullet^{\mathrm{ext}})$ and the homomorphism γ induces $\operatorname{Spec}(I_\bullet^{\mathrm{ext}}) \to X_\Sigma$. We can thus describe the weighted blow-up as the stack-theoretic quotient $\big[X_{\Sigma'(v)} \,/\, \mathbb{G}_m\big]$.

To see this, let $\tau \in \Sigma$ be a maximal cone and let $\tau' \in \overline{\Sigma'(v)}$ be the corresponding cone. This induces a morphism of affine toric varieties $U_{\tau'} \to U_\tau$ given by the morphism of monoids $M_\tau = (\tau^\vee \cap M) \to M_{\tau'} = (\tau'^\vee \cap M')$ induced by $\gamma^\vee\colon M \to M' = M \oplus \mathbb{Z}$ where $\gamma^\vee(m) = \big(m, m(v)\big)$. Let $\iota\colon M \to M'$ be given by $\iota(m) = (m, 0)$.

(i) If $\tau \neq \sigma$, then $M_{\tau'} = \iota(M_\tau) \oplus \mathbb{Z} = \gamma(M_\tau) \oplus \mathbb{Z}$ so $U_{\tau'} = U_\tau \times \mathbb{G}_m$ where the $\mathbb{G}_m$-action is on the second factor.

(ii) If $\tau = \sigma$, then $M_{\tau'} = \iota(M_\tau) \oplus \mathbb{N}$. If $m_1, m_2, \dots, m_n$ is a dual basis to $b_1, \dots, b_n$, then the $\mathbb{G}_m$-action on $U_{\tau'}$ corresponds to the weights

[3] If σ is a ray, then $\Sigma^*(v)(1) = \big(\Sigma(1) \setminus \{\sigma\}\big) \cup \{\rho_E\}$.

[4] More generally, it is enough that every cone containing σ is smooth: this ensures that the D_i are Cartier divisors in a neighborhood of $D_1 \cap \cdots \cap D_n$. Alternatively, we could start with a stacky fan $\mathbf{\Sigma}$ and the corresponding smooth toric stack.

$\mathrm{wt}(m_i, 0) = d_i$ and $\mathrm{wt}(0,1) = -1$ and $\gamma^\vee(m_i) = (m_i, d_i)$. Thus, $U_\tau = \operatorname{Spec} k[m_1, m_2, \dots, m_n]$ and $U_{\tau'} = \operatorname{Spec} k[M_1, M_2, \dots, M_n, u]$ where $m_i = u^{d_i} M_i$ in agreement with Proposition 5.2.2.

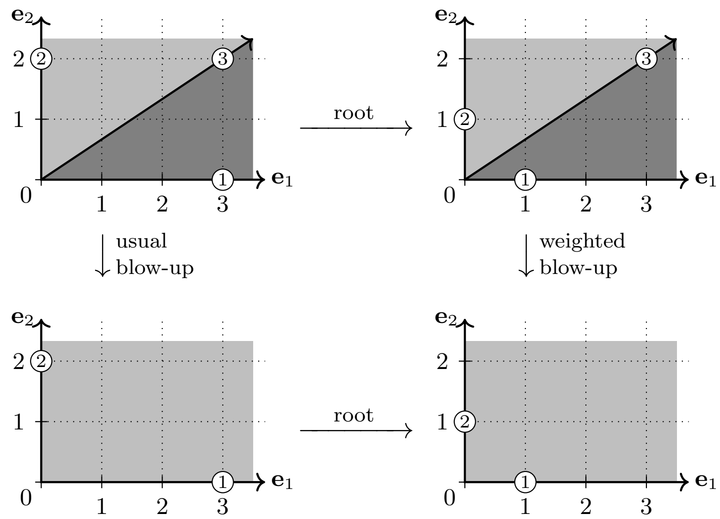
Moreover, if σ_i is the unique maximal cone in $\Sigma'(v)$ above σ not containing ρ_i, then U_{σ_i} is exactly the ith chart $D_+(x_i \cdot t^{d_i})$ of the weighted blow-up where $I_{D_i} = (x_i)$.

Example 5.4.4 (A local description via root stacks and ordinary blow-ups)**.** The above toric interpretation also motivates a *local* description of weighted blow-ups along regular weighted centers via a sequence of *root stacks* (Example 2.2.4), followed by a *usual blow-up*, and finally a sequence of *"de-rootings"*. For convenience, let us illustrate this via an example, which is no less informative than the general case.

Let $X = \mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec}(\mathbb{C}[x_1, x_2])$, i.e., the toric variety associated to the standard fan Σ in $\mathbb{R}^2$. Consider the weighted blow-up of X along the regular weighted center $I_\bullet = (x_1, 3) + (x_2, 2)$. Let $D_i = V(x_i)$ for $i = 1, 2$. Then there exists a canonical identification as shown in the dotted arrow below:

$$\begin{tikzcd}
\mathrm{Bl}_{\left(x_1^{\frac{1}{3}}, x_2^{\frac{1}{2}}\right)} X\left(\sqrt[3]{D_1}, \sqrt[2]{D_2}\right) \arrow[r, leftrightarrow, dotted, "\simeq"] \arrow[d, "\text{usual blow-up}"] & (\mathrm{Bl}_{I_\bullet} X)\left(\sqrt[3]{D_1'}, \sqrt[2]{D_2'}\right) \arrow[r, "\text{sequence of}", "\text{root stacks}"'] & \mathrm{Bl}_{I_\bullet} X \arrow[d, "\text{weighted blow-up}"'] \\
X\left(\sqrt[3]{D_1}, \sqrt[2]{D_2}\right) \arrow[rr, "\text{sequence of root stacks}"'] & & X
\end{tikzcd}$$

where D_i' is the proper transform of D_i in $\mathrm{Bl}_{I_\bullet} X$. The corresponding stacky fans are shown below:



In the diagram above, each corner illustrates a fan Σ in the usual lattice $N = \mathbb{Z}^2$, as well as some markings which define a homomorphism $\beta\colon \mathbb{Z}^k \to N$, where $\beta(\mathbf{e}_i)$ is marked with the circle that is labeled i. The data (Σ, β) at each corner then defines a stacky fan (N, Σ, β).

This also means that there are compatible canonical identifications between the $(x_1' := x_1 \cdot t^3)$-chart (resp. $(x_2' := x_2 \cdot t^2)$-chart) of $\mathrm{Bl}_{I_\bullet} X$ and the $(x_1^{1/3})'$-chart (resp. $(x_2^{1/2})'$-chart) of $\mathrm{Bl}_{(x^{1/3}, y^{1/2})} X(\sqrt[3]{D_1}, \sqrt[2]{D_2})$. Indeed, this can be sketched as follows:

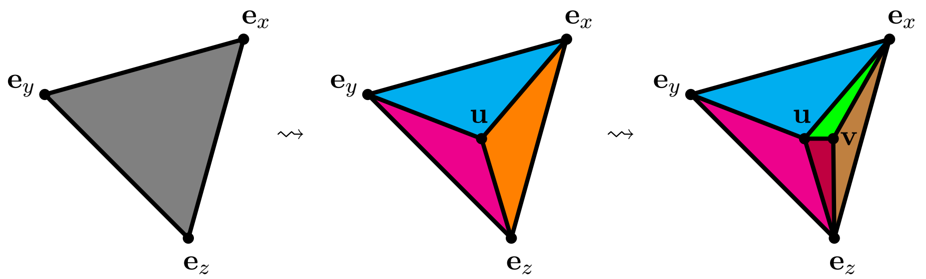
$$(x_1, x_2) \overset{\text{root}}{\rightsquigarrow} \left(x_1^{1/3}, x_2^{1/2}\right) \overset{\text{blow-up}}{\rightsquigarrow} \begin{cases} \left(x_1^{1/3}, \frac{x_2^{1/2}}{x_1^{1/3}}\right) & \text{on the } (x_1^{1/3})'\text{-chart} \\ \left(\frac{x_1^{1/3}}{x_2^{1/2}}, x_2^{1/2}\right) & \text{on the } (x_2^{1/2})'\text{-chart} \end{cases}$$

$$\overset{\text{de-root}}{\rightsquigarrow} \begin{cases} \left(x_1^{1/3}, \frac{x_2}{\left(x_1^{1/3}\right)^2}\right) & \text{on the } x_1'\text{-chart} \\ \left(\frac{x_1}{\left(x_2^{1/2}\right)^3}, x_2^{1/2}\right) & \text{on the } x_2'\text{-chart} \end{cases}$$

Example 5.4.5. We conclude this subsection with an example. Let $X = \mathbb{A}^3_{\mathbb{C}} = \operatorname{Spec}(\mathbb{C}[x,y,z])$, i.e., the toric variety associated to the standard fan Σ in $\mathbb{R}^3$. Let us compute the logarithmic embedded resolution algorithm of [Que20] for the closed subscheme $Z = V(x^2 + yz^2 + y^5) \subset X$. The algorithm first blows up X in the weighted center $\boldsymbol{I_\bullet} = (x,3) + (y,2) + (z,2)$:

$$\operatorname{Bl}_{I_\bullet} X = \mathscr{P}\mathrm{roj}_X \left(\frac{\Bbbk[x,y,z][x',y',z',u]}{(u^3x' - x, u^2y' - y, u^2z' - z)} \right) \longrightarrow X$$

where $V(u)$ denotes the exceptional divisor of $\operatorname{Bl}_{I_\bullet} X$. The proper transform of Z is $V(x'^2 + y'z'^2 + y'^5u^4) \subset \operatorname{Bl}_{I_\bullet} X$, which is smooth on the x'-chart and the z'-chart, but not on the y'-chart. The algorithm then resolves the isolated singularity in the y'-chart by blowing up in the weighted center $I'_\bullet = (x',2) + (z',2) + (u,1)$. In terms of fans, the algorithm is depicted by the following subdivisions of the standard fan Σ:



Here, $\mathbf{u} = 3\mathbf{e}_x + 2\mathbf{e}_y + 2\mathbf{e}_z = (3,2,2)$ and $\mathbf{v} = 2\mathbf{e}_x + 2\mathbf{e}_z + \mathbf{u} = (5,2,4)$. After the first subdivision at $\mathbf{u}$, the unique maximal cone that does not contain $\mathbf{e}_y$ gives rise (via Remark 5.4.3(iv)) to the y'-chart of $\operatorname{Bl}_{I_\bullet} X$. Thus, the second weighted blow-up of the y'-chart along $I'_\bullet$ corresponds to the subdivision of that maximal cone at $\mathbf{v}$.

Example 5.4.6. In the example above, let us also demonstrate the utility of Corollary 5.2.6 by using it to write down a presentation for one of the charts of the second weighted blow-up. First recall from Corollary 5.2.5 that the y'-chart of the first weighted blow-up $\operatorname{Bl}_{I_\bullet} X$ is:

$$\left[\operatorname{Spec}_X \left(\frac{\Bbbk[x,y,z][u,x',z']}{(u^3x' - x, u^2 - y, u^2z' - z)} \right) \Big/ \boldsymbol{\mu}_2 \right]$$

where $\mathrm{wt}_{\mathbb{Z}/2}(u) = \mathrm{wt}_{\mathbb{Z}/2}(x') = 1$ and $\mathrm{wt}_{\mathbb{Z}/2}(z') = 0$. Next, we directly apply Corollary 5.2.6 to see that the x''-chart of the second weighted blow-up is:

$$\left[\mathrm{Spec}_X \left(\frac{\Bbbk[x,y,z][v,u',z'']}{(v^5u'^3 - x, v^2u'^2 - y, v^4u'^2z'' - z)} \right) \, / \, \boldsymbol{\mu}_4 \right]$$

where $\mathrm{wt}_{\mathbb{Z}/4}(u) = \mathrm{wt}_{\mathbb{Z}/4}(x') = 2$, $\mathrm{wt}_{\mathbb{Z}/4}(z') = 0$, $\mathrm{wt}_{\mathbb{Z}/4}(v) = 1$, $\mathrm{wt}_{\mathbb{Z}/4}(u') = 1$, and $\mathrm{wt}_{\mathbb{Z}/4}(z'') = 2$. Here, we have used the isomorphism $(\mathbb{Z}/2)\langle -\frac{1}{2}\rangle = \big((\mathbb{Z}/2) \oplus \mathbb{Z}\big)/\langle (1,2)\rangle \xrightarrow{\simeq} \mathbb{Z}/4$ that maps $[(a,b)] \mapsto 2a - b$.

6. Kummer log blow-ups (not complete)

NOT COMPLETE. REMOVED IN THIS VERSION.

7. Geometric invariant theory (yet missing)

TO BE WRITTEN.

References

[AH10] Dan Abramovich and Brendan Hassett, *Stable varieties with a twist*, Classification of Algebraic Varieties, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2010, pp. 1–38.

[AOV08] Dan Abramovich, Martin Olsson, and Angelo Vistoli, *Tame stacks in positive characteristic*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 4, 1057–1091.

[ATW19] Dan Abramovich, Michael Temkin, and Jarosław Włodarczyk, *Functorial embedded resolution via weighted blowings up*, Preprint, Jun 2019, arXiv:1906.07106.

[ATW20] Dan Abramovich, Michael Temkin, and Jarosław Włodarczyk, *Toroidal orbifolds, destackification, and Kummer blowings up*, Algebra Number Theory **14** (2020), no. 8, 2001–2035, With an appendix by David Rydh.

[BB73] A. Białynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2) **98** (1973), 480–497.

[BCS05] Lev A. Borisov, Linda Chen, and Gregory G. Smith, *The orbifold Chow ring of toric Deligne-Mumford stacks*, J. Amer. Math. Soc. **18** (2005), no. 1, 193–215.

[Ber17] Daniel Bergh, *Functorial destackification of tame stacks with abelian stabilisers*, Compos. Math. **153** (2017), no. 6, 1257–1315.

[Cad07] Charles Cadman, *Using stacks to impose tangency conditions on curves*, Amer. J. Math. **129** (2007), no. 2, 405–427.

[CLS11] David Cox, John Little, and Hal Schenck, *Toric varieties*, American Mathematical Society, 2011.

[$\mathrm{EGA_{II}}$] A. Grothendieck, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 8, 222.

[EV07] Santiago Encinas and Orlando Villamayor, *Rees algebras and resolution of singularities*, Preprint, Feb 2007, arXiv:math.AG/0702836v1.

[FMN10] Barbara Fantechi, Etienne Mann, and Fabio Nironi, *Smooth toric Deligne-Mumford stacks*, J. Reine Angew. Math. **648** (2010), 201–244.

[Ful84] William Fulton, *Intersection theory*, Springer-Verlag Berlin Heidelberg, 1984.

[Gro17] Philipp Gross, *Tensor generators on schemes and stacks*, Algebr. Geom. **4** (2017), no. 4, 501–522.

[Gul11] Martin G. Gulbrandsen, *Stack structures on GIT quotients parametrizing hypersurfaces*, Math. Nachr. **284** (2011), no. 7, 885–898.

[Has05] Brendan Hassett, *Classical and minimal models of the moduli space of curves of genus two*, Geometric methods in algebra and number theory, Progr. Math., vol. 235, Birkhäuser Boston, Boston, MA, 2005, pp. 169–192.

[Kaw07] Hiraku Kawanoue, *Toward resolution of singularities over a field of positive characteristic. I. Foundation; the language of the idealistic filtration*, Publ. Res. Inst. Math. Sci. **43** (2007), no. 3, 819–909.

[Laz04] Robert Lazarsfeld, *Positivity in algebraic geometry* II*: positivity for vector bundles, and multiplier ideals*, Springer-Verlag Berlin Heidelberg, 2004.

[Que20] Ming Hao Quek, *Logarithmic resolution via weighted toroidal blowings up*, Preprint, May 2020, arXiv:2005.05939.

[Rom05] Matthieu Romagny, *Group actions on stacks and applications*, Michigan Math. J. **53** (2005), no. 1, 209–236.

[Ryd09] David Rydh, *Compactification of tame Deligne–Mumford stacks*, Draft, available on https://people.kth.se/~dary/papers.html, May 2009.

[Ryd15a] David Rydh, *Do line bundles descend to coarse moduli spaces of Artin stacks with finite inertia?*, MathOverflow answer at https://mathoverflow.net/a/206117/40, 2015.

[Ryd15b] David Rydh, *Weak factorization of Deligne–Mumford stacks*, In preparation, Oct 2015.

[Stacks] The Stacks Project Authors, *Stacks Project*, http://stacks.math.columbia.edu.

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