

CHERN NUMBERS OF SMOOTH VARIETIES VIA HOMOTOPY CONTINUATION AND INTERSECTION THEORY

SANDRA DI ROCCO, DAVID EKLUND, CHRIS PETERSON, AND ANDREW J. SOMMESE

ABSTRACT. Homotopy continuation provides a numerical tool for computing the equivalence of a smooth variety in an intersection product. Intersection theory provides a theoretical tool for relating the equivalence of a smooth variety in an intersection product to the degrees of the Chern classes of the variety. A combination of these tools leads to a numerical method for computing the degrees of Chern classes of smooth projective varieties in \mathbb{P}^n . We illustrate the approach through several worked examples.

1. INTRODUCTION

In this paper we describe a numerical method for computing the degrees of the Chern classes of a smooth projective variety. The main tools that will be used are drawn from intersection theory and numerical algebraic geometry. In particular, we will be using a refined Bézout's theorem [8, 9, 31] together with ideas related to numerical polynomial algebra such as homotopy continuation, monodromy, and the numerical decomposition of zero sets [22, 25, 26, 28]. In addition, we will need to numerically determine the degree of a certain residual zero-scheme [3, 6, 18, 20, 21]. Implementations of many of the computational tools that will be used can be found in freely available software packages (e.g. [2, 5, 12, 13, 14, 19, 27, 30, 32, 34]). The algorithms developed in this paper primarily use the software package *Bertini* with help from the software package *Macaulay 2* [2, 12].

From the generators of the ideal of a smooth projective variety, Chern numbers can be computed through a purely symbolic computation. This naturally leads one to ask if a numerical approach is useful or needed. A partial answer is that in the numerical approach described in this paper, the algorithms work equally well whether the generators are sparse or dense and whether they have rational, algebraic or transcendental coefficients. This is not the case with purely symbolic methods. Furthermore, a surprising amount of information can be extracted from an ideal even in situations where the generators have inaccuracies in their coefficients [15, 17, 23]. Another important feature, that will likely play a dominant role in the future, is that homotopy continuation algorithms parallelize well. This will allow such algorithms to take advantage of the trend of more processors on a chip. These

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features lead us to believe that the introduction of a numerical approach is indeed useful and will allow experimentation in settings outside the domain of purely symbolic methods.

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2. INTERSECTION THEORY AND HOMOTOPY CONTINUATION

In this section, we describe the main tools that will be used from intersection theory and from numerical algebraic geometry. Let X_1, \dots, X_r be a set of hypersurfaces in \mathbb{P}^r . Let Z be a smooth connected component of the scheme defined by $X_1 \cap \dots \cap X_r$. We present a well known theorem from intersection theory that relates the equivalence of Z in the intersection product $X_1 \cdot \dots \cdot X_r$ to a formula involving the total Chern classes of certain bundles related to X_1, \dots, X_r, Z , and \mathbb{P}^r . With mild assumptions on the scheme determined by the hypersurfaces, this theorem can be translated into a numerical condition involving the degrees of the Chern classes of Z , the degrees of the X_i and the degree of a certain residual zero-scheme. Tools from numerical algebraic geometry can be used to compute the degree of the residual zero-scheme. By varying the degrees of the X_i , we obtain multiple numerical conditions that can be used to uniquely determine the degrees of the Chern classes of Z .

2.1. Intersection Theory. Intersection theory has been primarily developed from the two different viewpoints found in [8, 9, 31]. Though each viewpoint seems to have its own advantages and disadvantages, Leendert van Gastel showed that the two are closely related [29]. While either version would be adequate for our purposes, we follow the approach of Fulton-MacPherson as described in [9]. We first need to fix the notation and definitions that will be used in this section.

Definition 2.1. Let Z and X be projective varieties in $\mathbb{P}^r = \mathbb{P}_{\mathbb{C}}^r$ with $Z \subseteq X$.

- (1) Let $A_*(X) = \bigoplus_k A_k(X)$ where $A_k(X)$ denotes the Chow group of k -dimensional cycles on X modulo rational equivalence.
- (2) For $\alpha \in A_*(X)$, $\{\alpha\}_0$ denotes the component of α in $A_0(X)$. Let $\alpha \in A_k(X)$, $\alpha = \sum_{i=1}^m a_i V_i$ where $a_1, \dots, a_m \in \mathbb{Z}$ and $V_1, \dots, V_m \subseteq X$ are irreducible subvarieties. The degree of α is defined by $\deg \alpha = \sum_i a_i \deg V_i$.
- (3) $N_X \mathbb{P}^r$ denotes the normal bundle of X in \mathbb{P}^r , T_X denotes the tangent bundle of X and $T_X|_Z$ denotes its restriction to Z .
- (4) Given a vector bundle E of rank n on a smooth variety, $c_i(E)$ denotes the i^{th} Chern class of E and $c(E) = 1 + c_1(E) + \dots + c_n(E)$ denotes the total Chern class of E .
- (5) The i^{th} Chern class of a smooth variety X is defined as $c_i(T_X)$ and $\deg(c_i(T_X))$ is called a Chern number of X .

Let X_1, \dots, X_r be hypersurfaces in \mathbb{P}^r and let Z be a smooth connected component of $X_1 \cap \dots \cap X_r$. In [9] Fulton considers the intersection product $X_1 \cdot \dots \cdot X_r \in A_0(X_1 \cap \dots \cap X_r)$ and defines $(X_1 \cdot \dots \cdot X_r)^Z \in A_0(Z)$, the *equivalence* of Z for $X_1 \cdot \dots \cdot X_r$, as the part of the intersection product supported on Z . The following is a simplified version of Proposition 9.1.1 of [9] specialized to our setting.

Proposition 2.2. (Fulton) *Let X_1, \dots, X_r be hypersurfaces in \mathbb{P}^r and let Z be a smooth connected component of $X_1 \cap \dots \cap X_r$. Let N_i be the restriction of $N_{X_i} \mathbb{P}^r$ to Z . Then*

$$(1) \quad (X_1 \cdot \dots \cdot X_r)^Z = \{(\prod_{i=1}^r c(N_i))c(T_{\mathbb{P}^r}|_Z)^{-1}c(T_Z)\}_0.$$

We introduce notation for the purpose of stating a useful corollary of Proposition 2.2. Let $n_i = \deg X_i$, let $\dim Z = n$, and let c_0, \dots, c_n be the Chern classes of Z . The k^{th} elementary symmetric function in n_1, \dots, n_r will be denoted by σ_k . In other words,

$$\sigma_0 = 1, \quad \sigma_1 = \sum_i n_i, \quad \sigma_2 = \sum_{i < j} n_i n_j, \quad \sigma_3 = \sum_{i < j < k} n_i n_j n_k, \quad \dots$$

If we let

$$a_i = \sum_{j=0}^{n-i} (-1)^j \binom{r+j}{j} \sigma_{n-i-j}$$

then we obtain the following identity as a direct corollary of Proposition 2.2:

$$\text{Corollary 2.3.} \quad \deg(X_1 \cdot \dots \cdot X_r)^Z = \sum_{i=0}^n a_i \deg c_i.$$

Proof. Let H be the hyperplane class in \mathbb{P}^r and let $H|_Z$ be its restriction to Z . Because $c(T_{\mathbb{P}^r}|_Z) = (1 + H|_Z)^{r+1}$,

$$c(T_{\mathbb{P}^r}|_Z)^{-1} = \sum_{j=0}^n (-1)^j \binom{r+j}{j} H|_Z^j.$$

As cycle classes on \mathbb{P}^r , $N_{X_i} \mathbb{P}^r = X_i = n_i H$. Hence $c(N_i) = 1 + n_i H|_Z$ and we get that $\prod_{i=1}^r (1 + c(N_i)) = \sum_{k=0}^n \sigma_k H|_Z^k$. Thus the formula (1) can be written as

$$(X_1 \cdot \dots \cdot X_r)^Z = \left\{ \left(\sum_{k=0}^n \sigma_k H|_Z^k \right) \left(\sum_{j=0}^n (-1)^j \binom{r+j}{j} H|_Z^j \right) \left(\sum_{i=1}^n c_i \right) \right\}_0.$$

The statement now follows since $\deg c_i = \deg(c_i H|_Z^{n-i})$ and the component of $(X_1 \cdot \dots \cdot X_r)^Z$ in $A_0(Z)$ is given by the sum of terms where $k = n - i - j$. \square

The classical version of Bézout's theorem (Proposition 8.4 of [9]) states that

$$\deg(X_1 \cdot \dots \cdot X_r) = \prod_{i=1}^r n_i.$$

A refined version of Bézout's theorem (Proposition 9.1.2 of [9]) ties these results together with an extended Bézout formula.

Proposition 2.4. *Suppose $X_1 \cap \dots \cap X_r$ consists of a connected component Z and a finite set S . For $p \in S$ let $m_p = i(p, X_1 \cdot \dots \cdot X_r; \mathbb{P}^r)$ denote the intersection multiplicity of p in $X_1 \cdot \dots \cdot X_r$. Then $\deg(X_1 \cdot \dots \cdot X_r)^Z + \sum_{p \in S} m_p = \prod_{i=1}^r n_i$.*

If n_1, \dots, n_r and $\sum_{p \in S} m_p$ are known, then we can use Proposition 2.4 to solve for $\deg(X_1 \cdot \dots \cdot X_r)^Z$. If Z is smooth, Corollary 2.3 provides a linear relation among the degrees of the Chern classes of Z . We summarize these observations in the following theorem implicit in Fulton [9]:

Theorem 2.5. *Let F_1, \dots, F_r be homogeneous forms corresponding to hypersurfaces $X_1, \dots, X_r \subset \mathbb{P}^r$. Let $n_i = \deg X_i$, let σ_k be the k^{th} elementary symmetric polynomial in n_1, \dots, n_r and let $\vec{\mathbf{A}} = [a_0, \dots, a_n]$ with $a_k = \sum_{i=0}^{n-k} (-1)^i \binom{r+i}{i} \sigma_{n-k-i}$. Let Z be a smooth connected n -dimensional scheme with Chern classes c_0, \dots, c_n and let $\vec{\mathbf{C}} = [\deg c_0, \dots, \deg c_n]$. If the subscheme of \mathbb{P}^r defined by the ideal (F_1, \dots, F_r) is a disjoint union of Z and a (possibly empty) zero-scheme S , then $\vec{\mathbf{A}} \cdot \vec{\mathbf{C}} = \sigma_r - \deg S$.*

In order to compute the degrees of the Chern classes of an n -dimensional smooth variety, it is enough to determine $n+1$ independent linear relations that they satisfy. Theorem 2.5 provides a mechanism for producing the linear relations provided we are able to find a sufficient number of r -tuples of homogeneous forms that satisfy the conditions of Theorem 2.5 and provided we have a method for computing $\deg S$. In the next two sections, we present a collection of tools that allow us to complete the latter task.

2.2. Homotopy Continuation. In homotopy continuation, a polynomial ideal, I , is cast as a member of a parameterized family of polynomial ideals one of which has known isolated solutions. Each of the known isolated solutions is tracked through a predictor/corrector method to a point which lies numerically close to the algebraic set $V(I)$ determined by I . These points can then be refined to lie within a prescribed tolerance of $V(I)$. An introduction to general continuation methods can be found in [1]. Through the basic algorithms of numerical algebraic geometry [26], from I it is then possible to produce a collection of subsets of points such that:

- The subsets are in one to one correspondence with the irreducible components of the algebraic set $V(I)$.
- The points in a subset all lie within a prescribed tolerance of the irreducible component to which it corresponds.
- The number of points in the subset is the same as the degree of the irreducible component.
- The subset is a numerical approximation of the intersection of the irreducible component with a known linear space of complementary dimension.

Note that the decomposition above allows one to determine both the dimension and degree of each algebraic variety appearing in the decomposition of an algebraic set.

2.3. Degree of a Zero Dimensional Scheme. To each irreducible component of an algebraic set, one can use the defining set of polynomials to attach a positive integer, called the *multiplicity*, that determines roughly how many times the component should be counted in a computation. In [6], Dayton and Zeng study the multiplicity inspired, to a large degree, by Macaulay's inverse systems approach. They provide an algorithm which yields as output the multiplicity of isolated solutions. This is essentially done by counting how many partial derivatives of the polynomials are forced to be zero. In [3], an alternate approach is presented, inspired, to a large degree, by certain Gröbner basis calculations coupled with a fundamental result of Bayer and Stillman on regularity [4]. Both of these algorithms have been implemented in *Bertini* [2]. Thus, there is an implemented algorithm in place that allows one to determine the degree of a zero-dimensional scheme.

3. ALGORITHMS

In this section, we present the pseudocode for three algorithms. The first algorithm computes the equivalence of a connected scheme Z in an intersection product. The second algorithm determines a linear relation satisfied by the degrees of the Chern classes of a smooth connected scheme. The third algorithm computes the degrees of the Chern classes of a smooth connected scheme. For each of the algorithms, the input is a set of non-zero homogeneous generators for an ideal I . It is an assumption of the algorithms that the scheme determined by I is a disjoint union of Z and a zero-scheme S . The second and third algorithms have the additional assumption that Z is smooth.

Algorithm 1. *Equivalence_of_Z* ($\{F_1, F_2, \dots, F_r\}; D$)

Input: A set of r homogeneous polynomials $\{F_1, F_2, \dots, F_r\} \subset \mathbb{C}[z_0, z_1, \dots, z_r]$. The polynomials should generate an ideal I whose corresponding scheme is the disjoint union of a connected scheme Z and a possibly empty zero-scheme S .

Output: $D = \deg(X_1 \cdots X_r)^Z$ where X_i is the hypersurface corresponding to F_i .

Algorithm:

- Determine the support of S .
- Determine the multiplicity of each point in the support of S .
- Add up the multiplicities of the points in the support of S and store in μ_S .
- For each i , determine the degree of F_i and store in n_i .
- Compute $T = \prod_i n_i$.
- Compute $T - \mu_S$ and store in D .

Algorithm 2. *Linear_Relation_On_Chern_Numbers* ($\{F_1, F_2, \dots, F_r\}; \vec{\mathbf{A}}, D$)

Input: A set of r homogeneous polynomials $\{F_1, F_2, \dots, F_r\} \subset \mathbb{C}[z_0, z_1, \dots, z_r]$. The polynomials should generate an ideal I whose corresponding scheme is the disjoint union of Z and S , where Z is a smooth connected scheme and S is a possibly empty zero-scheme.

Output: $\vec{\mathbf{A}} = [a_0, \dots, a_n]$ and $D \in \mathbb{Z}$, where n denotes the dimension of Z . If c_i denotes the i^{th} Chern class of Z and $\vec{\mathbf{C}} = [\deg c_0, \dots, \deg c_n]$ then the linear relation is $\vec{\mathbf{A}} \cdot \vec{\mathbf{C}} = D$.

Algorithm:

- Determine the dimension of Z and store in n .
- Compute *Equivalence_of_Z* ($\{F_1, F_2, \dots, F_r\}; D$).
- Compute the elementary symmetric functions $\sigma_0, \sigma_1, \dots, \sigma_n$ of n_1, \dots, n_r .
- Compute $\vec{\mathbf{A}} = [a_0, a_1, \dots, a_n]$ where $a_k = \sum_{i=0}^{n-k} (-1)^i \binom{r+i}{i} \sigma_{n-k-i}$.

Algorithm 3. *Chern_Numbers* ($\{F_1, F_2, \dots, F_t\}; \vec{\mathbf{C}}$)

Input: A set of t homogeneous polynomials $\{F_1, F_2, \dots, F_t\} \subset \mathbb{C}[z_0, z_1, \dots, z_r]$. The polynomials should generate an ideal I whose corresponding scheme is the disjoint union of Z and S , where Z is a smooth connected scheme and S is a possibly empty zero-scheme.

Output: $\vec{\mathbf{C}} = [\deg c_0, \dots, \deg c_n]$ where n denotes the dimension of Z and c_i denotes the i^{th} Chern class of Z .

Algorithm:

Determine the dimension of Z and store in n .
Determine the degrees of F_1, \dots, F_t , and store the maximal degree as b .
Set $l_c = b$ for $c = 1, \dots, n + 1$.
For $i = 1$ to $n + 1$
 Choose r random elements from I of degrees l_1, l_2, \dots, l_r .
 Store the random elements as G_1, G_2, \dots, G_r .
 Compute *Linear_Relation_On_Chern_Numbers* ($\{G_1, G_2, \dots, G_r\}; \vec{\mathbf{A}}_i, D_i$).
 Let $l_i = l_i + 1$.
Next i
Build the $(n + 1) \times (n + 1)$ matrix M whose i^{th} row is $\vec{\mathbf{A}}_i$.
Build $\vec{\mathbf{D}} = [D_1, \dots, D_{n+1}]$.
Set up the linear system $M\vec{\mathbf{C}} = \vec{\mathbf{D}}$, with $\vec{\mathbf{C}}$ and $\vec{\mathbf{D}}$ as column vectors.
Solve the linear system for $\vec{\mathbf{C}}$.

There are two potential problems that should be addressed in the use of Algorithm 3. The first potential problem is whether the matrix M has full rank. The second potential problem is whether the random r -tuples G_1, G_2, \dots, G_r , produced in such a simple manner from I , satisfy the input requirements of Algorithm 2. These two questions are answered in Proposition 3.1 and Corollary 3.4, respectively.

Proposition 3.1. *The matrix M defined in Algorithm 3 satisfies $\det(M) = \pm 1$.*

Proof. View M as a function of the maximal degree b . Let σ_i be the i^{th} elementary symmetric polynomial in r variables b_1, \dots, b_r and let

$$\sigma_i^j = \sigma_i(b_1 + 1, b_2 + 1, \dots, b_j + 1, b_{j+1}, \dots, b_r).$$

Define $\tau_i^j \in \mathbb{Z}[b]$ by $\tau_i^j = \sigma_i^j(b, b, \dots, b)$, $\tau_i^0 = \sigma_i(b, b, \dots, b)$. Define a $(n + 1) \times (n + 1)$ matrix $N(b)$ by

$$N = \begin{pmatrix} \tau_n^0 & \tau_{n-1}^0 & \dots & \tau_1^0 & 1 \\ \tau_n^1 & \tau_{n-1}^1 & \dots & \tau_1^1 & 1 \\ & & \vdots & & \\ \tau_n^n & \tau_{n-1}^n & \dots & \tau_1^n & 1 \end{pmatrix}.$$

Observe first that M can be factored as $M = NR$, where

$$R = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -(r+1) & 1 & 0 & \dots & 0 & 0 \\ \binom{r+2}{2} & -(r+1) & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^n \binom{r+n}{n} & (-1)^{n-1} \binom{r+n-1}{n-1} & \dots & \binom{r+2}{2} & -(r+1) & 1 \end{pmatrix}.$$

Since $\det(R) = 1$, we have $\det(M) = \det(N)$.

One checks that $\det(N(0)) = \pm 1$. We shall show that the determinant of $N(b)$ does not depend on b by showing that

$$\frac{d(\det(N))}{db} = 0.$$

We shall use the following fact:

$$\frac{d\tau_i^j}{db} = (r-i+1)\tau_{i-1}^j.$$

Let C_i be the i^{th} column of N , $N = (C_0, C_1, \dots, C_n)$. Then we have that

$$\frac{d(\det(N))}{db} = \sum_{i=0}^n \det(C_0, C_1, \dots, C_{i-1}, \frac{dC_i}{db}, C_{i+1}, \dots, C_n),$$

but each term in the above sum is zero since $\frac{dC_i}{db} = (r-n+i+1)C_{i+1}$ for $0 \leq i \leq n-1$ and $\frac{dC_n}{db} = 0$. \square

Let I be an ideal defining a scheme $X \subset \mathbb{P}^r$ and let I_X be the homogeneous ideal of X . For $d \in \mathbb{N}$, we say that X is cut out scheme theoretically in degree d if the saturation of the ideal generated by $I(d)$ is equal to I_X .

Proposition 3.2. *Let I be an ideal defining a scheme in \mathbb{P}^r that is the disjoint union of a smooth connected scheme Z and a possibly empty zero-scheme S . Let $G_1, \dots, G_k \in I$, $1 \leq k \leq r$, generate an ideal that defines a scheme X , such that the singular locus X_{sing} satisfies $\text{codim}(X_{\text{sing}} \setminus S) > k$, and $\text{codim}(X \setminus Z) \geq k$. Let $d \in \mathbb{N}$ be such that $Z \cup S$ is cut out scheme theoretically in degree d and fix integers n_{k+1}, \dots, n_r with $n_i \geq d$ for all i . If G_{k+1}, \dots, G_r are general forms in I with $\deg G_i = n_i$, then the ideal $J = (G_1, G_2, \dots, G_r)$ defines the disjoint union of Z and a possibly empty zero-scheme S' . If S is non-singular or empty, then S' is non-singular.*

Proof. Suppose $k \neq r$. A generic form $G_{k+1} \in I$ of degree n_{k+1} cuts down the dimension of every component of $X \setminus (Z \cup S)$ as well as every component of $X_{\text{sing}} \setminus S$. By induction, the scheme Y defined by J is a disjoint union of Z and a zero-scheme S' , and Y is nonsingular away from S . In particular Y is non-singular on Z . \square

Corollary 3.3. *Suppose I and d are as in Proposition 3.2. If $G_1, \dots, G_k \in I$ is a regular sequence in I such that the scheme X defined by the ideal (G_1, \dots, G_k) is generically smooth, then for general forms G_{k+1}, \dots, G_r with $\deg G_i = n_i \geq d$, the ideal $J = (G_1, G_2, \dots, G_r)$ defines the disjoint union of Z and a possibly empty zero-scheme S' .*

Proof. Every component of X has codimension k and is of multiplicity 1. We may therefore apply Proposition 3.2. \square

The following corollary implies that for generic choices of the forms G_1, \dots, G_r in Algorithm 3, the assumptions on the input of Algorithm 2 are satisfied.

Corollary 3.4. *Let $I \neq 0$ be as in Proposition 3.2 and let $\{F_1, \dots, F_t\}$ be a set of non-zero generators of I . Put $d = \max\{\deg F_1, \dots, \deg F_t\}$ and fix integers n_1, \dots, n_r with $n_i \geq d$ for all i . If G_1, \dots, G_r are general forms in I with $\deg G_i = n_i$, then the ideal $J = (G_1, G_2, \dots, G_r)$ defines the disjoint union of Z and a possibly empty zero-scheme S' .*

Proof. Observe that the ideal generated by $I(d)$ is equal to $\bigoplus_{e \geq d} I(e)$. Since the ideal $\bigoplus_{e \geq d} I(e)$ and I have the same saturation, namely the homogeneous ideal of $Z \cup S$, the scheme $Z \cup S$ is cut out scheme theoretically in degree d . \square

For reasons of efficiency it is desirable to keep the degrees of the forms G_1, \dots, G_r in Algorithm 3 as low as possible. We give two examples that have bearing on the question of how much Corollary 3.4 can be strengthened.

Example 3.5. Consider a conic $Z \subset \mathbb{P}^3$. The conic is cut out by a plane P and a quadric Q . The corresponding forms of degree 1 and 2 generate the homogeneous ideal I of Z . In this case we cannot choose the generic forms $G_1, G_2, G_3 \in I$ of degrees $(3, 1, 1)$. A cubic and two planes, all three containing the conic, will intersect in the conic union a line, since the planes are necessarily both equal to P . The residue in this case is thus a line and not a finite scheme as required.

Example 3.6. This example is along the same lines as Example 3.5. Let A be a 2×3 matrix of general linear forms in $\mathbb{C}[x_0, \dots, x_4]$. Let F_2, F_3, F_4 be the 2×2 minors of A and let F_1 be a general form of degree 3. The ideal $I = (F_1, F_2, F_3, F_4)$ is the homogeneous ideal of a curve Z in \mathbb{P}^4 and the minors F_2, F_3, F_4 define a surface. We thus have a minimal generating set of I of degrees $(3, 2, 2, 2)$ but 4 general forms of degrees $(4, 2, 2, 2)$ will define a union of the curve Z and another curve, violating the input requirements of Algorithm 2.

4. EXAMPLES

In this section we present several examples of computations of Chern numbers which illustrate the algorithms of the previous section.

4.1. Curves. We consider the case of a smooth connected curve in \mathbb{P}^r ([9] Example 9.1.1). If Z is a smooth curve of genus g in \mathbb{P}^r then the first Chern class of Z is $-K_Z$ and $\deg(-K_Z) = 2 - 2g$. If $(F_1, F_2, \dots, F_r) \subset \mathbb{C}[z_0, z_1, \dots, z_r]$ is a homogeneous ideal I whose corresponding scheme is the disjoint union of the curve Z and a zero-scheme S then Corollary 2.3 leads to

$$\deg(X_1 \cdot \dots \cdot X_r)^Z = (n_1 + \dots + n_r - (r + 1)) \deg Z + 2 - 2g$$

where X_i denotes the hypersurface corresponding to F_i and $n_i = \deg X_i$. From Theorem 2.5, we can also write the equation as

$$(2) \quad \prod_i n_i - \deg S = (n_1 + \dots + n_r - (r + 1)) \deg Z + 2 - 2g.$$

Example 4.1. The homogeneous ideal of the twisted cubic curve in \mathbb{P}^3 is $I = (x^2 - wy, y^2 - xz, wz - xy) \subset \mathbb{C}[w, x, y, z]$. It is well known that this curve has degree 3 and genus 0. If we choose $F_1, F_2, F_3 \in I$ of degrees $(2, 2, 2)$ then the numerical irreducible decomposition implemented in [2] determines that the corresponding

scheme consists of a degree 3 curve and no additional points (S is empty). Equation (2) gives the relation

$$2 \cdot 2 \cdot 2 - 0 = (2 + 2 + 2 - (3 + 1)) \cdot 3 + 2 - 2g$$

which we can solve to get $g = 0$.

If we choose $F_1, F_2, F_3 \in I$ of degrees $(2, 2, 3)$ then we obtain a degree 3 curve and one additional point of multiplicity 1. Equation (2) gives the relation

$$2 \cdot 2 \cdot 3 - 1 = (2 + 2 + 3 - (3 + 1)) \cdot 3 + 2 - 2g$$

again leading to $g = 0$.

If we did not know the degree of Z , the two computations in this example would yield

$$2 \cdot 2 \cdot 2 - 0 = (2 + 2 + 2 - (3 + 1)) \deg Z + 2 - 2g,$$

$$2 \cdot 2 \cdot 3 - 1 = (2 + 2 + 3 - (3 + 1)) \deg Z + 2 - 2g.$$

The unique solution to these two equations is $\deg Z = 3$ and $g = 0$.

4.2. Surfaces. Now consider the case where Z is a smooth connected surface in \mathbb{P}^r ([9] Example 9.1.5). Let $I = (F_1, F_2, \dots, F_r) \subset \mathbb{C}[z_0, z_1, \dots, z_r]$ be a homogeneous ideal. If the scheme corresponding to I is the disjoint union of the surface Z and a zero-scheme S then Corollary 2.3 and Theorem 2.5 lead to the equation

$$(3) \quad \prod_i n_i - \deg S = \deg (X_1 \cdot \dots \cdot X_r)^Z = a_2 \deg c_2 + a_1 \deg c_1 + a_0 \deg Z,$$

where

$$a_2 = 1, \quad a_1 = \sum_{i=1}^r n_i - (r + 1) \quad \text{and} \quad a_0 = \sum_{i < j} n_i n_j - (r + 1) \sum_{i=1}^r n_i + \binom{r + 2}{2}.$$

Example 4.2. In characteristic zero and up to standard modifications, the Horrocks-Mumford bundle E is the only known indecomposable rank 2 vector bundle on \mathbb{P}^4 [16]. For a general section s of E , the zero set $V(s)$ is a smooth surface in \mathbb{P}^4 called a Horrocks-Mumford surface. We shall compute the degree of a smooth Horrocks-Mumford surface Z as well as the degrees of its Chern classes c_1 and c_2 . It is known that Z is an Abelian surface of degree 10 [16]. Hence $\deg c_0 = 10$ and $c_1 = c_2 = 0$. The homogeneous ideal I of Z is generated by three quintics and fifteen sextics and Z is cut out scheme theoretically by three quintics and one sextic. Generators for I can be found by a variety of methods, e.g. by a Beilinson monad [7]. In this example we do not follow Algorithm 3 in detail but the example is covered by Proposition 3.2.

The following table shows the number of solutions resulting from zero-dimensional, 128-bit precision runs in [2] on 4 random elements of I for various choices of degrees. We let (n_1, n_2, n_3, n_4) denote the degrees of these elements.

(n_1, n_2, n_3, n_4)	non-singular points	singular points
$(5, 5, 5, 6)$	0	750
$(5, 5, 6, 6)$	40	860
$(5, 6, 6, 6)$	100	980

In each of these computations the zero set consists of the Abelian surface together with a finite set S of points of multiplicity one. In each row of the table the entry

in the middle column is the cardinality of S and the entry in the rightmost column is the Bézout number $\prod_{i=1}^4 n_i$ minus the entry in the middle column, i.e.

$$n_1 n_2 n_3 n_4 - \sum_{p \in S} m_p.$$

By Proposition 2.4, this is also the degree of the equivalence of Z . Evaluating (3) in each case gives:

$$\begin{aligned} \deg c_2 + 16 \deg c_1 + 75 \deg Z &= 750 \\ \deg c_2 + 17 \deg c_1 + 86 \deg Z &= 860 \\ \deg c_2 + 18 \deg c_1 + 98 \deg Z &= 980. \end{aligned}$$

This system has the unique solution $\deg Z = 10$, $\deg c_1 = \deg c_2 = 0$.

4.3. Higher Dimensional Varieties. If Z is a smooth n -dimensional variety in \mathbb{P}^r then the algorithm proceeds in a similar manner to the curve and surface cases. We use homotopy continuation to determine the equivalence of Z in various intersection products. This combines with the formulas appearing in Corollary 2.3 and Theorem 2.5 to produce a linear system involving the degrees of the Chern classes of Z . Finally, we solve the linear system to determine these Chern numbers.

Example 4.3. Let I be the ideal defined by the 4×4 minors of a 4×5 matrix of general linear forms in $\mathbb{C}[x_0, x_1, \dots, x_5]$ and let Z be the corresponding threefold in \mathbb{P}^5 . The following table shows the number of solutions resulting from zero-dimensional, 128-bit precision runs in [2] on 5 random elements of I for various choices of degrees. We let $(n_1, n_2, n_3, n_4, n_5)$ denote the degrees of these elements.

$(n_1, n_2, n_3, n_4, n_5)$	non-singular points	singular points
(4, 4, 4, 4, 4)	0	1024
(4, 4, 4, 4, 5)	1	1279
(4, 4, 4, 5, 5)	6	1594
(4, 4, 5, 5, 5)	21	1979

In each of these computations the zero set consists of the threefold together with a finite set S of points of multiplicity one. The formulas from Corollary 2.3 and Theorem 2.5 lead to:

$$\begin{aligned} \deg c_3 + 14 \deg c_2 + 61 \deg c_1 + 44 \deg Z &= 1024 \\ \deg c_3 + 15 \deg c_2 + 71 \deg c_1 + 65 \deg Z &= 1279 \\ \deg c_3 + 16 \deg c_2 + 82 \deg c_1 + 92 \deg Z &= 1594 \\ \deg c_3 + 17 \deg c_2 + 94 \deg c_1 + 126 \deg Z &= 1979. \end{aligned}$$

The unique solution is $\deg Z = 10$, $\deg c_1 = 0$, $\deg c_2 = 45$, $\deg c_3 = -46$.

Example 4.4. Let V be the image of the Segre embedding $i : \mathbb{P}^1 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^7$. Let Z be the intersection of V with a general hyperplane and let I be the homogeneous ideal of Z . The following table shows the number of solutions resulting from zero-dimensional, 128-bit precision runs in [2] on 6 random elements of I for various choices of the degrees (n_1, \dots, n_6) of these elements.

$(n_1, n_2, n_3, n_4, n_5, n_6)$	non-singular points	singular points
(2, 2, 2, 2, 2, 2)	0	64
(2, 2, 2, 2, 2, 3)	0	96
(2, 2, 2, 2, 3, 3)	2	142
(2, 2, 2, 3, 3, 3)	10	206

In each of these computations the zero set of the 6 random elements consists of the threefold together with a finite set of points of multiplicity one. The formulas from Corollary 2.3 and Theorem 2.5 lead to:

$$\begin{aligned} \deg c_3 + 5 \deg c_2 + 4 \deg c_1 - 8 \deg Z &= 64 \\ \deg c_3 + 6 \deg c_2 + 7 \deg c_1 - 10 \deg Z &= 96 \\ \deg c_3 + 7 \deg c_2 + 11 \deg c_1 - 11 \deg Z &= 142 \\ \deg c_3 + 8 \deg c_2 + 16 \deg c_1 - 10 \deg Z &= 206. \end{aligned}$$

The unique solution is $\deg Z = 4$, $\deg c_1 = 10$, $\deg c_2 = 10$, $\deg c_3 = 6$.

5. CONCLUSION

The results of this paper demonstrate the viability of computing Chern numbers of smooth varieties through numerical homotopy continuation. Homotopy continuation via square systems is a natural venue through which to compute the equivalence of a scheme in the context of an important generalized Bézout's theorem from intersection theory. It should be noted that the algorithms of this paper could be implemented in a purely symbolic setting as well. The advantages of the numerical approach is that the algorithms work equally well whether the generators are sparse or dense and whether they have rational, algebraic or transcendental coefficients. In addition, meaning can often be attached to the computations in situations where the generators have inaccuracies in their coefficients. Finally, homotopy continuation algorithms parallelize well allowing such algorithms to take full advantage of multi-processor machines. These features suggest the complementary nature of the approach to purely symbolic methods.

In future work, the authors intend to extend the approach to take further advantage of the ideas of intersection theory. In particular, intersection theory for non-square systems leads to a method for computing intersection numbers of Chern classes.

REFERENCES

- [1] E. Allgower, K. Georg, *Introduction to numerical continuation methods*, Classics in Applied Mathematics 45, SIAM Press, Philadelphia (2003).
- [2] D.J. Bates, J.D. Hauenstein, A.J. Sommese, C.W. Wampler, *Bertini: Software for Numerical Algebraic Geometry*, Available at <http://www.nd.edu/~sommese/bertini>.
- [3] D. Bates, C. Peterson, A. Sommese, *A numerical-symbolic algorithm for computing the multiplicity of a component of an algebraic set*, Journal of Complexity 22 (4) (2006), 475-489.
- [4] D. Bayer, M. Stillman, *A criterion for detecting m-regularity*, Invent. Math. 87 (1987), 1-11.
- [5] *CoCoA: a system for doing Computations in Commutative Algebra*, Available at <http://cocoa.dima.unige.it>.
- [6] B. Dayton, Z. Zeng, *Computing the multiplicity structure in solving polynomial systems*, Proceedings of ISSAC 2005, 116-123, ACM, New York, 2005.
- [7] W. Decker, L. Ein, F.-O. Schreyer, *Construction of Surfaces in \mathbb{P}^4* , Journal of Algebraic Geometry 2 (1993), 185-237.
- [8] H. Flenner, L. O'Carroll, W. Vogel, *Joins and intersections*, Springer Monographs in Mathematics. Springer-Verlag, Berlin (1999).
- [9] W. Fulton, *Intersection Theory*, Springer-Verlag (1998).
- [10] A. Galligo, D. Rupprecht, *Irreducible decomposition of curves*, Journal of Symbolic Computation Volume 33 , Issue 5, May 2002, 661-677.

- [11] S. Gao, E. Kaltofen, J.P. May, Z. Yang, L. Zhi, *Approximate factorization of multivariate polynomials via differential equations*, In ISSAC 2004 Proc. 2004 Internat. Symp. Symbolic Algebraic Comput., pages 167-174.
- [12] D. Grayson, M. Stillman, *Macaulay 2: a software system for research in algebraic geometry*, Available at <http://www.math.uiuc.edu/Macaulay2>.
- [13] G.M. Greuel, G. Pfister, H. Schönemann, *Singular 3.0: A Computer Algebra System for Polynomial Computations*, Available at <http://www.singular.uni-kl.de>.
- [14] T. Gunji, S. Kim, M. Kojima, A. Takeda, K. Fujisawa; T. Mizutani, *PHoM: a polyhedral homotopy continuation method for polynomial systems*. Computing 73 (2004), no. 1, 57-77.
- [15] J. Hauenstein, J. Migliore, C. Peterson, A. SommeSE, *Numerical Computation of the Dimension of the Cohomology of Twists of Ideal Sheaves*, Contemporary Mathematics 496, "Interactions of Classical and Numerical Algebraic Geometry (Notre Dame, IN 2008)", pg 235-242, (2009).
- [16] G. Horrocks, D. Mumford, *A Rank 2 Vector Bundle on \mathbb{P}^4 with 15,000 Symmetries*, Topology Vol. 12 (1973), 63-81.
- [17] B. Huber, F. Sottile, B. Sturmfels, *Numerical Schubert Calculus*, Journal of Symbolic Computation Volume 26, Issue 6, December 1998, 767-788.
- [18] G. Lecerf, *Quadratic Newton Iteration for Systems with Multiplicity*, FoCM Volume 2, Number 3, August 2002, 247-293.
- [19] T.L. Lee, T.Y. Li, C.H. Tsai. *HOM4PS-2.0: a software package for solving polynomial systems by the polyhedral homotopy continuation method*. Computing 83 (2008), no. 2-3, 109-133.
- [20] A. Leykin, *Numerical primary decomposition*, In Proceedings of ISSAC 2008, pages 165-172, 2008.
- [21] A. Leykin, J. Verschelde, A. Zhao, *Newton's method with deflation for isolated singularities of polynomial systems*, Theoretical Computer Science, 359(1-3):111-122, 2006.
- [22] T.Y. Li, *Numerical solution of multivariate polynomial systems by homotopy continuation methods*, Acta Numerica, 6:399-436, 1997.
- [23] P. Niyogi, S. Smale, S. Weinberger, *Finding the Homology of Submanifolds with High Confidence from Random Samples*, Discrete and Computational Geometry, Vol. 39, No. 1-3. (March 2008), pp. 419-441.
- [24] G. Reid, L. Zhi, *Solving polynomial systems via a symbolic-numeric elimination method*, To appear in the Journal of Symbolic Computation.
- [25] A.J. SommeSE, J. Verschelde, C.W. Wampler, *Numerical Decomposition of the Solution Sets of Polynomial Systems into Irreducible Components*, SIAM Journal on Numerical Analysis, v.38 n.6, 2022-2046, 2001.
- [26] A.J. SommeSE, C.W. Wampler, *The Numerical solution of systems of polynomials arising in engineering and science*, World Scientific, 2005.
- [27] W. Stein, D. Joyner, *SAGE: System for Algebra and Geometry Experimentation*, Available at <http://www.sagemath.org>.
- [28] H.J. Stetter. Numerical Polynomial Algebra, SIAM, 2004.
- [29] L.J. van Gastel, *Excess intersections and a correspondence principle*, Invent. Math. 103 (1991), no. 1, 197-222.
- [30] J. Verschelde, *PHCpack: a general-purpose solver for polynomial systems by homotopy continuation*, Available at <http://www.math.uic.edu/~jan>.
- [31] W. Vogel, *Lectures on results on Bézout's theorem*, Notes by D. P. Patil. Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 74. Published for the Tata Institute of Fundamental Research, Bombay; by Springer-Verlag, Berlin, 1984.
- [32] L.T. Watson, M. Sosonkina, R.C. Melville, A.P. Morgan, H.F. Walker, *Algorithm 777: HOMPACK90: a suite of Fortran 90 codes for globally convergent homotopy algorithms*. ACM Trans. Math. Software 23 (1997), no. 4, 514-549.
- [33] W. Wu, G. Reid, *Application of Numerical Algebraic Geometry and Numerical Linear Algebra to PDE*, Proc. International Symposium on Symbolic and Algebraic Computation (ISSAC 2006), Edited by Jean-Guillaume Dumas, 345-352, ACM Press. 2006.
- [34] Z. Zeng, *ApaTools: a software toolbox for approximate polynomial algebra*, Available at <http://www.neiu.edu/~zzeng>.

DEPARTMENT OF MATHEMATICS, KTH, 100 44 STOCKHOLM, SWEDEN
E-mail address: dirocco@math.kth.se
URL: <http://www.math.kth.se/~sandra>

DEPARTMENT OF MATHEMATICS, KTH, 100 44 STOCKHOLM, SWEDEN
E-mail address: daek@math.kth.se
URL: <http://www.math.kth.se/~daek>

DEPARTMENT OF MATHEMATICS, COLORADO STATE UNIVERSITY, FORT COLLINS, CO 80523
E-mail address: peterson@math.colostate.edu
URL: <http://www.math.colostate.edu/~peterson>

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, IN 46556
E-mail address: sommese@nd.edu
URL: <http://www.nd.edu/~sommese>