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PROJECTIVE SURFACES WITH k-VERY AMPLE LINE BUNDLES OF GENUS $\leq 3k+1$

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Let $\mathcal{G}$ be the set of surfaces, S, polarized by a k-very ample line bundle, L, with genus$\leq 3k+1$. All the elements (S, L) of $\mathcal{G}$ are listed. The classification of surfaces polarized by a k-very ample line bundle of degree $\leq 4k+4$ is completed by proving that this class of surfaces is a subset of $\mathcal{G}$.

INTRODUCTION

Let L be a line bundle on a smooth connected surface S. According to [8] L is k-very ample, for an integer $k \geq 0$, if for any zero-scheme $(Z, \mathcal{O}_Z)$ of length $h^0(\mathcal{O}_Z) = k+1$ the induced map $H^0(L) \to H^0(L_Z)$ is surjective. This definition generalizes the notions of very-ampleness and spannedness for line bundles. A 0-very ample line bundle is in fact spanned and a 1-very ample line bundle is very ample. Moreover the k-very ampleness of a line bundle is closely related to the existence of multi-secants and gives an embedding of the Hilbert scheme of zero-dimensional subschemes of length k into the Grassmannian of k-th quotient subspaces of $H^0(L)$ (cf. 1).

A natural question that arises from the classical adjunction theory is the classification of surfaces polarized by k-very ample line bundles, (S, L), based on the degree or on the genus of L.

In [5] k-very ample line bundles up to genus 5 are classified. If we restrict the problem to $k = 2$ a classification up to genus 8 can be found in [17, 1]. The sufficiency of such lists has been recently completed by Franciosi in [14]. We point out that in the lists mentioned for $k = 2$ the case of $K3$-surfaces has been overlooked.

In This paper we classify surfaces, S, polarized by a k-very ample line bundle L with sectional genus $g \leq 3k + 1$ and we find that if S is not a minimal $K3$ surface k cannot be greater than 9; more specifically: Let $\mathcal{G} = \{(S, L) | g(L) \leq 3k + 1\}$, then $(S, L) \in \mathcal{G}$ is as in TABLE 1.

Notice that, since a $(k + 1)$-very ample line bundle is clearly k-very ample, for each k the table shows the line bundles which are strictly k-very ample, not to include repetitions. the reader should then consider for each k the cases above it. Except for the $K3$ case all the line bundles in the table are proved to be k-very ample.

In section 2 we study the general hyperplane section $C \in |L|$ and use induced bounds on the Clifford index and gonality to rule out the case $h^1(L_C) \geq 2$.

In section 3 we classify special classes of surfaces that occur in the case when $kK_S + L$ is not nef and big.

TABLE 1 collects the classification results worked out in section section 4 and section 5.

The 'Reider type Theorem' ([7]) gives a method to determine the k-very ampleness of the adjoint bundle if the degree is $\geq 4k+5$. It is interesting then to classify k-very ample line bundles of degree up to $4k+4$. This classification has been partially done in [3], where interesting asymptotic properties are proved. For example Theorem 6.1 gives $k \leq 8$.

In section 6 we use the classification obtained to classify surfaces polarized by k-very ample line bundles of degree $\leq 5k + 1$.

As a corollary we obtain the complete classification for degree $\leq 4k+4$ (see TABLE 2).

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TABLE 1

k	S	L
$9,8$	$\mathbb{P}^2$	$\mathcal{O}_{\mathbb{P}^2}(k)$
$7,6$	$\mathbb{P}^2$	$\mathcal{O}_{\mathbb{P}^2}(k)$, $\mathcal{O}_{\mathbb{P}^2}(k+1)$
5	$\mathbb{P}^2$	$\mathcal{O}_{\mathbb{P}^2}(5)$
	$\mathbb{F}_0$	$5E_0+5f$
4	$\mathbb{P}^2$	$\mathcal{O}_{\mathbb{P}^2}(4)$
	$\mathbb{F}_0$	$4E_0+bf$, $b=4,...,7$
	$Bl_7(\mathbb{P}^2)$	$-4K_S$
3	$\mathbb{P}^2$	$\mathcal{O}_{\mathbb{P}^2}(3)$
	$\mathbb{F}_0$	$3E_0+bf$, $b=3,...,6$
	elliptic $\mathbb{P}^1$-bundle, $e=-1$	$3E_0+2f$
	$Bl_7(\mathbb{F}_e)$, $e=0,1,$	$6E_0+(3e+7)f-\sum_1^7 3E_i$
	$Bl_8(\mathbb{P}^2)$	$-4K_S+E_i$
	$Bl_7(\mathbb{P}^2)$	$-3K_S$
	$Bl_6(\mathbb{P}^2)$	$-3K_S$
2	$\mathbb{P}^2$	$\mathcal{O}_{\mathbb{P}^2}(2)$
	$\mathbb{F}_e$	aE_0+bf,$a=2$ and $2+2e\leq b\leq e+8$
	$\mathbb{F}_0$	$3E_0+bf$, $b=3,4$
	elliptic $\mathbb{P}^1$-bundle, $e=2$	$2E_0+8f$
	elliptic $\mathbb{P}^1$-bundle, $e=1$	$2E_0+8f$, $b=6,7$
	elliptic $\mathbb{P}^1$-bundle, $e=0$	$2E_0+bf$, $b=4,5,6$
	elliptic $\mathbb{P}^1$-bundle, $e=-1$	$4E_0$; $3E_0+bf$, $b=1,2,3$ or $2E_0+bf$,$b=2,3,4,5$
	$\mathbb{P}^1$-bundle over a curve of genus 2, $e=-2$	$2E_0+2f$
	$Bl_s(\mathbb{F}_e)$, $e=0,1$	$4E_0+(2e+5)f-\sum_1^6 2E_i$, $s=6,7$
	$Bl_9(\mathbb{F}_0)$	$4E_0+6f-\sum_1^9 2E_i$
	$Bl_8(\mathbb{P}^2)$	$-(k+1)K_S+E_i$
	$Bl_8(\mathbb{P}^2)$	$\gamma^*(\mathcal{O}_{\mathbb{P}^2}(7))-\sum_1^8 2E_i$
	$Bl_7(\mathbb{P}^2)$	$-2K_S$
	$Bl_6(\mathbb{P}^2)$	$-2K_S$
	$Bl_5(\mathbb{P}^2)$	$-2K_S$
$k\geq 2$	$S\subset\mathbb{P}^g$ $K3$	$d(L)=2g-2$

1. BACKGROUND MATERIAL

NOTATION.

We will use the standard notation from Algebraic Geometry.

The ground field is always the field **C** of complex numbers.

S denotes a smooth projective surface, unless otherwise stated. $\mathcal{O}_S$ denotes the structure sheaf of S, K_S the canonical line bundle, i.e, the sheaf of holomorphic 2-forms on S and $k(S)$ denotes the Kodaira dimension of S. If $\mathcal{F}$ is a coherent sheaf on a variety V, $h^i(\mathcal{F})$ denotes the complex dimension of $H^i(V, \mathcal{F})$, for $i \geq 0$.

By $S^{[r]}$ we denote the Hilbert scheme of 0-dimensional subschemes $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ of S with $length(\mathcal{O}_{\mathcal{Z}}) = r$. Since we are working in characteristic zero, $length(\mathcal{O}_{\mathcal{Z}}) = h^0(\mathcal{O}_{\mathcal{Z}})$. Line bundles and divisors are used with no distinction as well as the multiplicative or additive notation.

if L is a line bundle on S, we use the following notation:
$|L|$ is the complete linear system associated to L;
$L^2 = deg(L)$ is the degree of the line bundle L (d stands for $deg(L)$, unless otherwise stated);
$g(L)$, the sectional genus of S, is the integer defined by the equality : $2g(L) - 2 = L^2 + K_S L$. If $L = \mathcal{O}_S(D)$ for a divisor D we also write $g(D)$ for $g(L)$ and if D is an irreducible reduced curve on a surface S , $g(D)$ is the arithmetic genus of D (g stands for $g(L)$ unless otherwise stated).

L is said to be *numerically effective* (nef) if $L \cdot C \geq 0$ for every curve C on S and in this case L is said to be *big* if $L^2 > 0$.
L is said to be Q-effective if, for some positive integer n, nD is effective.
L is said to be spanned if it is spanned by its global sections, i.e. $h^0(L \otimes \mathcal{I}_P) = h^0(L) - 1$ for every point on S.
L is said to be very ample if the map associated to $|L|$ embeds S in $P^{h^0(L)-1}$, i.e. L is spanned and $h^0(L \otimes \mathcal{I}_{\mathcal{Z}}) = h^0(L) - 2$ for any 0-dimensional subscheme $\mathcal{Z}$ of length 2 on S.

In this paper we always assume $k \geq 2$, unless otherwise stated.

SPECIAL SURFACES

For $n \geq 0$ $\mathbb{F}_n$ denotes the *Hirzebruch surface* of invariant n given by $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$.

A polarized surface (S, L) with L very ample is called a *scroll* (respectively a *conic-bundle*) over a non singular curve Y if there exists a morphism with connected fiber $p : S \to Y$ such that $L \cdot f = 1$ (resp. 2) for a general fiber $f \cong \mathbb{P}^1$. Let L be a k-very ample line bundle on S and let $\mathcal{L} = (k-1)K_S + L$ be ample. (S, L) is said to be a *k-conic bundle* if $(S, \mathcal{L})$ is a conic bundle over a curve Y. Equivalently there exists a morphism with connected fiber $p : S \to Y$ such that $kK_S + L = p^*(H)$ for some ample line bundle H on Y. It follows that $L|_f = \mathcal{O}_{\mathbb{P}^1}(k)$ and $L \cdot f = 2k$. Any reducible fiber of p consists of two distinct, irreducible components f_1, f_2 with $f_1 \cdot f_2 = 1$; it follows that $L \cdot f_1 = L \cdot f_2 = k$ and therefore $f_1 \cong f_2 \cong \mathbb{P}^1$. Notice that a scroll cannot be polarized by a k-very ample line bundle, since $L \cdot f < 2$(cf., 1.7).

The surface obtained blowing up S at n points is denoted by $Bl_n(S)$ or equivalently by $Bl_{P_1,\dots,P_n}(S)$, where $P_1, \dots, P_n$ are the points blown up .

Good references on the classification of minimal surfaces are [4, 15].

CLIFFORD INDEX AND GONALITY

The Clifford index and the gonality are very useful tools to study properties of special divisors on a smooth algebraic curve. A good survey on this subject can be found in [12].

Definition 1.1. *A curve C is called l-gonal if and only if l is the minimal degree of a covering map $C \to \mathbb{P}^1$.*

If $l = 2$ we say that C is hyperelliptic.

Definition 1.2. *Let L be a line bundle on C. The Clifford index of L is defined as:*

$$cl(L) = deg(L) - 2r(L)$$

where $r(L) = h^0(L) - 1$.

Definition 1.3. *The Clifford index of a curve C is defined as :*

$$cl(C) = min\{cl(L)|h^0(L) \geq 2, h^1(L) \geq 2\}$$

By the Clifford theorem if C is a curve of genus g and degree d in $\mathbb{P}^r$ and $d \leq 2g-1$ then $cl(C) \geq 0$ and equality holds if and only if C is hyperelliptic.

If L is a line bundle on C with $h^0(L) \geq 2$ and $h^1(L) \geq 2$ we say that L *contributes* to the Clifford index of C and if $cl(L) = cl(C)$ we say that L *computes* the Clifford index of C.

Using Brill-Noether theory we have that: (cf., [2, Chapter V])

$$cl(C) \leq [(g-1)/2] \tag{1}$$

where $[x]$ means the greatest integer$\leq x$.

If C is a general curve, then $l = c+2$.

For the "exceptional curves" (conjectured to be very rare) we have that $l = c+3$.

Proposition 1.4. [12, 2.3] *If C is a l-gonal curve with Clifford index c then:*

$$c+2 \leq l \leq c+3$$

k-VERY AMPLENESS.

Definition 1.5. *A line bundle L on S is said to be* **k-very ample**, *for an integer $k \geq 0$ if, given any $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \in S^{[k+1]}$ the restriction map $H^0(L) \longrightarrow H^0(L_{\mathcal{Z}})$ is surjective.*

Note that L is 0-very ample if and only if it is spanned by its global sections and L is 1-very ample if and only if it is very ample.

Every k-very ample line bundle L gives an embedding $\psi_L^r : S^{[r]} \longrightarrow Grass(H^0(L), r)$ for $r \leq k$ (see [8], [11] for details) and in particular, the composition of ψ_L^k with the Plücker embedding in $\mathbb{P}^N$, $r_L : S^{[k]} \longrightarrow \mathbb{P}^N$ induces a natural correspondence between Pic(S) and Pic($S^{[k]}$) associating $L^{[k]} = r_L^*(\mathcal{O}_{\mathbb{P}^N}(1))$ to $L \in Pic(S)$.

If S is embedded in $\mathbb{P}^N$ via a very ample line bundle L, L being k-very ample is equivalent to there being no k-secant $(k-1)$-planes to S containing more then $k+1$ points. This interpretation relates closely the study of k-very ample line bundles on S to the search for a bound of the number of secant planes to S or to an hyperplane section.

In particular if S is embedded in $\mathbb{P}^N$ via a 2-very ample line bundle then S does not have trisecant lines. In this setting Le Barz formulas for trisecant lines, [18], are very useful. In the degree classification we use a formula by MacDonald for the number of $(k+1)$-secant $(k-1)$ planes to a curve $C \in \mathbb{P}^{2k}$, [2, pg 350-351].

In [5, 6] basic intersection properties of k-very ample line bundles on surfaces and curves are worked out. The following are fundamental tools in this paper:

Proposition 1.6. [5] *Let L be a k-very ample line bundle on S, then $LC \geq k$ for every effective irreducible curve C on S, with equality only if $C \cong \mathbb{P}^1$. Moreover $L \cdot C \geq k+2$ if $g(C) \geq 1$.*

Proposition 1.7. [6] *Let L be a k-very ample line bundle on an irreducible projective curve C. Then*

(1) *If $deg(L) = k+1$ and $k \geq 2$, then $C \cong \mathbb{P}^1$.*
(2) *If $deg(L) = k+2$, $k \geq 1$ then either $h^0(L) = k+3$ and $C \cong \mathbb{P}^1$, or C is isomorphic to a degree 3 plane curve.*

Proposition 1.8. [5, Theorem 1.2] *If L is a k-very ample line bundle on a curve C with $h^1(L_C) \neq 0$ then K_C is k-very ample and $g(C) \geq 2k+1$.*

ADJUNCTION PROPERTIES.

In the proof of Theorem 5.1 some results from classical adjunction theory are used. We refer to [9] for details.

It is very useful to study the adjunction properties of a k-very ample line bundle. The classical adjunction question would be whether the adjoint bundle $K_S + L$ is still k-very ample. The answer has been given by Beltrametti and Sommese in [6] and generalizes the classical adjunction theory for k=1.

Given a k-very ample line bundle L of degree $\geq 4k+5$ the k-adjoint bundle $kK_S + L$ is nef, unless S belongs to a short list of examples, and there exists a pair (S', L'), called k-reduction, with a morphism $r : S \longrightarrow S'$ such that:
- S is the blow up of S' at a finite number of point $B = \{P_1, ..., P_s\}$;
- $L' := (r_* L)^{**}$ is ample and $kK_S + L = r^*(kK_{S'} + L')$;
- $L \cong r^* L' - kr^{-1}(B)$ and there are no smooth rational curves E in S satisfying $E^2 = -1$ and $EL' \leq k$.

In this paper we use the nefness of the k-adjoint bundle according to the following theorem:

Theorem 1.9. [6, Theorem 3.1] *Let L be a k-very ample line bundle on a smooth connected surface S, $k \geq 2$. Then $tK_S + L$, $t = 0, ..., k-1$ is very ample and $kK_S + L$ is nef and big unless either:*

(1) *$S = \mathbb{P}^2$ and $L = \mathcal{O}_{\mathbb{P}^2}(a)$ for $k \leq a \leq 3k$;*
(2) *$(S, L) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b))$, with $k \leq min(a, b) \leq 2k$;*
(3) *S is a $\mathbb{P}^1$-bundle over a smooth curve and $Lf \leq 2k$, for every fiber ;*
(4) *S is a k-conic bundle , relative to L, over a smooth curve;*
(5) *S is a Del Pezzo surface with $L = -kK_S$.*

2. THE HYPERPLANE SECTION

Let $C \in |L|$ be the general hyperplane section. L_C is a k-very ample line bundle on C of degree d and genus g. The following results about k-very ample line bundles on curves turn out to be an important tool in the classification.

Lemma 2.1. *Let L be a k-very ample line bundle of degree d on an irreducible non-singular curve C of genus g. Then*

(a) $2g - 2 - d \geq k + 2h^1(L) - 2$ *if* $h^1(L) \geq 2$;
(b) $d \geq 2k + g$ *if* $h^1(L) = 1$ *and* $L \neq K_C$.

Proof. (a) is Theorem 2.3 in [3]. (b) is Theorem 2.5 in [3]. □

Lemma 2.2. [3, 2.10] *Let L be a k-very ample line bundle of degree d on an irreducible non-singular curve C of genus g. If $h^1(L) = 0$ then $d(L) \geq 2k + g + 1$ or $d(L) \geq 2g + k$.*

Lemma 2.3. [3, 2.8] *Let L be a k-very ample line bundle of degree d on a smooth connected surface S. Let $C \in |L|$ be a general hyperplane section. Then $h^0(L) \geq 2k+2$ and $h^0(L_C) \geq 2k+1$; if equality holds in either inequality, then $K_S \cong \mathcal{O}_S$ or $h^1(L_C) \geq 2$ and $H^0(L) \to H^0(L_C)$ is surjective. Moreover $h^0(L_C) \geq 2k+2$ if $h^1(L_C) = 0$.*

If L_C is a special k-very ample line bundle on a curve C then the gonality, l, and the Clifford index of C, c, are forced to be relatively big.

Lemma 2.4. *If $h^1(L) \neq 0$ then the $l \geq k+2$ and $c \geq k-1$.*

Proof. Since L_C is special K_C is a k-very ample line bundle on C, by 1.8. It follows that for any 0-scheme $\mathcal{Z}$ on C of degree $\leq k+1$ the map $H^0(K_C) \to H^0(K_C|_{\mathcal{Z}})$ is surjective, i.e. $h^1(K_C - \mathcal{Z}) = h^0(\mathcal{Z}) = 1$. Then a pencil g_l^1 cannot have degree $\leq k+1$. Moreover $c \geq l-3 \geq k-1$. ☐

Proposition 2.5. *If $g(L) \leq 3k+1$ then $h^1(L) \leq 1$.*

Proof. If $h^1(L) \geq 2$ then the line bundle L would contribute to the Clifford index of C. Lemma 2.1 implies that $d \leq 2g - k - 2h^1(L) \leq 5k-2$ and therefore $cl(L_C) \leq 5k - 2 - 2h^0(L_C) + 2$. This is impossible because we would get $k-1 \leq k-2$, by Lemma 2.3 and Lemma 2.4. ☐

Corollary 2.6. *Let L be a k-very ample line bundle on a projective surface S with $g(L) \leq 3k+1$, then $K_S L \leq k-1$.*

Proof. If $C \in |L|$ then $h^1(L_C) \leq 1$ by 2.5. Using the adjunction formula $2g-2 = d + K_S L$ the bound is an easy consequence of Lemma 2.1, assuming $K_C \neq L_C$ in the case $h^1(L_C) = 1$. If $K_C = L_C$ we have that $K_S L = 0 \leq k-1$. ☐

3. SPECIAL CLASSES OF SURFACES

In this section we classify k-very ample line bundles L on $\mathbb{P}^2$, $\mathbb{F}_e$, $\mathbb{P}^1$-bundles and k-conic bundles, with $g(L) \leq 3k+1$. For more details on the characterization of k-very ample line bundles on those surfaces we refer to [3, 5, 6].

THE CASE $\mathbb{P}^2$.

Consider the generator of the Picard group, $H = \mathcal{O}_{\mathbb{P}^2}(1)$. If $L = \mathcal{O}_{\mathbb{P}^2}(a)$ is a k-very ample then $L \cdot H = a \geq k$. Using the fact that the tensor product of k very ample line bundles is k-very ample (cf., [9]) it is easy to see that the converse is also true, i.e. a line bundle $\mathcal{O}_{\mathbb{P}^2}(a)$ is k-very ample if and only if $a \geq k$.

Using the adjunction formula $a(a-3) = 2g-2$ it is easy to derive that if $g(L) \leq 3k+1$ then $a = k+n$ and $k \leq 9-2n$ for $n = 0, 1, 2, 3$. Therefore we have that:

Proposition 3.1. *Let L is a line bundle on $\mathbb{P}^2$ with $g(L) \leq 3k+1$. Then L is k-very ample if and only if $L = \mathcal{O}_{\mathbb{P}^2}(k)$ for $k \leq 9$, $L = \mathcal{O}_{\mathbb{P}^2}(k+1)$ for $k \leq 7$, $L = \mathcal{O}_{\mathbb{P}^2}(k+2)$ for $k \leq 5$ and $L = \mathcal{O}_{\mathbb{P}^2}(k+1)$ for $k \leq 3$*

THE CASE $\mathbb{F}_e$.

The Picard group of the Hirzebruch surface $\mathbb{F}_e$ is generated by a general fiber of the projection to $\mathbb{P}^1$, f, and E_0, the section of minimal self intersection. Recall that $f^2 = 0$, $E_0^2 = -e$ and $f \cdot E_0 = 1$.

Let $L = aE_0 + bf$ be a k-very ample line bundle on $\mathbb{F}_e$, then $L \cdot f = a \geq k$ and $L \cdot E_0 = b - ae \geq k$. The same decomposition argument as for $\mathbb{P}^2$, see [6], shows that the contrary is true i.e:

A line bundle $L = aE + bf$ on $\mathbb{F}_e$ is k-very ample if and only if $a \geq k$ and $b - ae \geq k$.

From the adjunction formula we have that

$$2g - 2 = (a-1)(2b - 2 - ae) - 2 \tag{2}$$

Assume that $g(L) \leq mk + 1$ for an integer $m \geq 3$.

This implies that $2mk \geq (k-1)(2k + ae - 2)$, i.e, $k^2 - (m+2)k + 1 \leq 0$, that gives

$$k \leq m + 2 \tag{3}$$

Plugging the possible values in 2 we obtain the following possibilities for a, b, k, e:

k	a	b	$\mathbb{F}_e$
$m+2$	$m+2$	$m+2$	$\mathbb{F}_0$
$m+1=4$	$m+1$	$m+1,\ldots,m+4$	$\mathbb{F}_0$
$m+1$	$m+1$	$m+1$	$\mathbb{F}_0$
m	m	$2m \leq b \leq \frac{3}{2}m+3$	$\mathbb{F}_1$
$m=3$	3	$3 \leq b \leq 6$	$\mathbb{F}_0$
$m=3$	4	4	$\mathbb{F}_0$
$m \geq 4$	m	m	$\mathbb{F}_0$
$m-1=2$	2	$2+2e \leq b \leq e+7$	$\mathbb{F}_e$, $e=0,\ldots,5$
$m-1$	m	$m, m+1$	$\mathbb{F}_0$
$m-1$	$m-1$	$m+5,\ldots,m-1$	$\mathbb{F}_0$
$\leq m-2$	$\leq m+1+\frac{m+1}{k-1}$	$\leq m+\frac{an+m+2}{k-1}$	$\mathbb{F}_e$

From the table above we have that:

Proposition 3.2. *Let $L = aE_0 + bf$ be a line bundle on $S = \mathbb{F}_n$ with $g(L) \leq 3k+1$. Then L is k-very ample if and only if $k \leq 5$ and (S, L) is one of the following:*

(1) $k=5$, $(S,L) \cong (\mathbb{F}_0, 5E_0+5f)$.
(2) $k=4$, $(S,L) \cong (\mathbb{F}_0, 4E_0+bf)$, *with* $b=4,\ldots,7$.
(3) $k=3$, $(S,L) \cong (\mathbb{F}_0, 4E_0+4f)$
$(S,L) \cong (\mathbb{F}_0, 3E_0+bf)$ *with* $b=3,\ldots,6$
$(S,L) \cong (\mathbb{F}_1, 3E_0+bf)$ *with* $b=6,7$
(4) $k=2$, $(S,L) \cong (\mathbb{F}_0, aE_0+bf)$ *with* $a=2$ *and* $b=2,\ldots,8$ *or* $a=3$ *and* $b=3,4$;
$(S,L) \cong (\mathbb{F}_e, 2E_0+bf)$ *with* $2+2e \leq b \leq e+7$ *for* $e=1,\ldots,5$

$\mathbb{P}^1$-BUNDLES OVER A CURVE OF GENUS $q \geq 1$.

If S is a $\mathbb{P}^1$-bundle over a smooth curve Y of genus $q \geq 1$, let f denote the fiber of the surjective morphism with connected fiber, $p : S \to Y$, and E_0 the section of minimal self-intersection $-e$. A theorem of Nagata gives $q \leq -e$. E_0 and f are the generators of the Picard group of S.

Certainly a k-very ample line bundle $L = aE_0 + bf$ on S has to satisfy: $L \cdot f = a \geq k$ and $L \cdot E_0 = b - ae \geq k+2$, since $g(E_0) = q \geq 1$, by 1.7.

If $e = 1$ we can use the following criterium:

Proposition 3.3. [6, 2.2] *A line bundle $L = aE + bf$ on a $\mathbb{P}^1$-bundle over a smooth elliptic curve , with $E^2 = 1$ is k-very ample* *if and only if* $a \geq k$; $a+b \geq k+2$; $a+2b \geq k+2$.

Lemma 3.4. *If* $1 \leq q \leq 4$ *then* $b-ae \geq k+2+q$, *unless* $q=1$ *and* $b-ae=k+2$.

Proof. L_{E_0} is a k-very ample line bundle on E_0 of degree $b-ae$ with $h^1(L_{E_0})=0$. In fact, if $h^1(L_{E_0}) \neq 0$, then we would have $q=g(L_{E_0}) \geq 2k+1 \geq 5$, by 1.8. Hence assuming $h^0(L_{E_0}) \geq k+3$, by Riemann-Roch we have that $k+3 \leq 1+b-ae-q$, i.e., $b-ae \geq k+2+q$. If $h^0(L_{E_0})=k+2$ then by [6, Propostion 1.5] (E_0, L_{E_0}) would be $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k+1))$ or E_0 would be isomorphic to a degree 3 plane curve, i.e. $q=1$ and $deg(L_{E_0})=k+2$. □

Lemma 3.5. *Let* L *be a line bundle on a* $\mathbb{P}^1$*-bundle of a curve of genus* $q \geq 2$ *with* $g(L) \leq 3k+1$. *Then* L *is* k*-very ample on* S *if and only if* $q=2$, $e=-2$ *and* $L=2E_0+2f$.

Proof. Assume L is k-very ample. From Hartshorne's formula [16] :

$$2g-2=2a(q-1)+(\frac{a-1}{a})d \tag{4}$$

using $d \geq 2k+g$ and $a \geq k$ we get $g \leq 4k-2k(q-1)$ that is impossible if $q \geq 3$. If $q=2$ then $b+2a \geq b-ae \geq k+4$ by Lemma 3.4. Moreover $a \leq k+1$, otherwise (4) would give $(k+3)g \geq 2(k+2)(k+3)+2k(k+1)$ that is impossible. Plugging the possible values of a and k in (4) , assuming $b \geq k+4-2a$ and $K_SL \leq k-1$ we get that the only possible values are $k=2$ and $a=b=2$. The converse is shown in [1, 3.1]. □

Proposition 3.6. *Let* S *be a* $\mathbb{P}^1$*-bundle over an elliptic curve and* L *a line bundle of genus* $\leq 3k+1$ *on* S. *Then* L *is* k*-very ample if and only if* (S, L) *is one of the following:*

1. $e=2$, $k=2$, $L=2E_o+8f$;
2. $e=1$, $k=2$, $L=2E_0+bf$, *where* $b=6,7$;
3. $e=0$, $k=2$, $L=2E_0+bf$, *where* $b=4,5,6$;
4. $e=-1$, $k=3$ $L=3E_0+2f$ *or* $k=2$ *and either* $L=4E_0$ $L=3E_0+bf$ *where* $b=1,2,3$ *or* $L=2E_0+bf$ *where* $b=2,3,4,5$.

Proof. Let $L = aE_0 + bf$ be a k-very ample line bundle on an elliptic $\mathbb{P}^1$-bundle. Since $b \geq k + 2 + ak$ from 4 we get:

$$6k \geq (k-1)(2k+4+ek)$$

Then $e \leq 2$ and the possible values are $a = k = 2$ if $e = 0, 1, 2$ and $k \leq 3$, $a \leq k+2$ if $e = -1$. Using $6k \geq (a-1)(2b - ae)$ and $b \geq k + 2 + ae$ we can easily determine b.

Assume now that L is as in cases (1)-(4). In the last case the criterium given by 3.3 shows the k-very ampleness.

For the remaining cases we use the "Reider type theorem", [5, 2.1]. We can apply the theorem since in all the cases the degree of the line bundle $L - K_S = 4E_0 + (b+e)f$ is bigger then 13. In order to show that L is 2-very ample it is enough to show that we cannot find an effective divisor, D, on S such that:

$$(L - K_S)\cdot D - 3 \leq D^2 < \frac{(L-K_S)\cdot D}{2} < 3 \tag{5}$$

Let $D = xE_0 + yf$ and assume $(L - K_S)\cdot D = x(b-e) + 4y \leq 5$. Then since $b - 3e \geq 2$ the only possibilities are $x = 0$ and $y = 1$ or $y = 0$ and $x \leq 2$. In the first case $D = f$ and $(L - K_S)\cdot D - 3 = 1 \geq f^2 = 0$ rules it out. In the second case we can have $D = E_0$ or $D = 2E_0$ when $e = 2$. In both cases we have that the first inequality in (5) implies $b \geq 2e + 3$ that is impossible. □

Remark.

In all the cases above E_0 is isomorphic to a plane curve of degree 3. If $b - ae = k + 2$ then E_0 is embedded in $\mathbb{P}^2$ by $L + \{[x_1]^{-1} + \ldots + [x_{k-1}]^{-1}\}$ (cf., [6] Lemma 1.1). If $b - ae > k + 2$ then we can just keep subtracting points until we get a line bundle, D of degree $3 = 2g + 1$ and therefore very ample. Moreover by Riemann-Roch and Clifford index theorem we get $h^0(D) = 3$.

Corollary 3.7. *Let S be a $\mathbb{P}^1$-bundle over a curve of genus $q \geq 1$ and let L be a line bundle on S of genus $\leq 3k+1$. Then L is k-very ample if and only if (S, L) is as in Proposition* 3.6 *and Lemma* 3.5.

k-CONIC BUNDLES

Let (S, L) be a k-conic bundle and let $p : S \to Y$ the morphism with connected fibers, and H the ample line bundle on Y such that $kK_S + L = p^*(H)$. Since $Lf = 2k$, where f is a fiber of p then we have $kK_SL + d(L) = 2\delta k$, where $\delta = d(H)$. Moreover we can assume $d(kK_S + L) = k^2K_S^2 + 2kK_S + d(L) = 0$. Using the adjunction formula and assuming $g \leq 3k + 1$ we get:

$$(k-1)K_SL + 6k \geq 2\delta k \tag{6}$$

$$k^2K_S^2 + 2\delta k + kK_SL = 0 \tag{7}$$

Proposition 3.8. *Let L be a k-very ample line bundle on a smooth connected surface. Assume that (S, L) is a k-conic bundle and that $g(L) \leq 3k + 1$. Then (S, L) is one of the following:*

(a) $k = 3$, *S is the blow up $\gamma : S \to \mathbb{F}_e$ in 7 points and $L = 6E_0 + (3e + 7)f - 3\sum_1^7 E_i$, with $e = 0, 1$;*

(b) $k = 3$, *S is the blow up $\gamma : S \to \mathbb{F}_e$ in 9 points and $L = 6E_0 + (3e + 8f) - 3\sum_1^9 E_i$, $e = 0, 1, 2$;*

(c) $k = 2$, *S is the blow up $\gamma : S \to \mathbb{F}_0$ in 9 points and $L = 4E_0 + 6f - 2\sum_1^9 E_i$;*

(d) $k = 2$, *S is the blow up $\gamma : S \to \mathbb{F}_e$ in s, points, $s = 6, 7$, and $L = 4E_0 + (2e + 5)f - 2\sum_1^s E_i$, with $e = 0, 1$;*

(e) $k = 2$ *and S is an elliptic $\mathbb{P}^1$-bundle (cf., (4) in 3.7).*

Proof. If $\delta = 1$ then the fact that $kK_S + L$ is spanned implies $Y \cong \mathbb{P}^1$ and therefore $q = 0$. Assume $K_S^2 \geq 0$. Since $h^0(-K_S) \geq 0$ then $|-K_S|$ contains an effective divisor of genus 1 and thus $K_SL \leq -k - 2$ because L is k-very ample. It follows from (6) that $k \leq 3$ and $K_SL \geq -4 + \frac{-4}{k-1}$. If $k = 2$ then from (6) and (7) we get that $-4 \leq K_SL \leq -8$ and $K_S^2 \leq 3$. Let $L = 4\pi^*(E_0) + b\pi^*(f) - \sum a_iE_i$, where $\pi : S \to \mathbb{F}_e$ is the blowing up map in 7,6 or 5 points and E_i are the exceptional divisors. Again from (6) and (7) we see that the possible values for (K_SL, K_S^2) are $(-4, 1)$, $(-6, 2)$ and $(-8, 3)$. Using $d = -16 + 8b - \sum a_i^2$ and $K_SL = 4e - 2b - 8 + \sum a_i$ we have that $d + \sum a_i^2 = -4K_SL + 4\sum a_i - 32$, i.e,

$$\sum a_i(4 - a_i) = d + 4K_SL + 32 \tag{8}$$

Moreover since $h^0(\pi^*f - E_i) > 0$ for any i (consider a fiber passing through the point P_i blown up) $L\cdot(\pi^*(f)-E_i) = 4-a_i \geq 2$, i.e., $a_i \geq 2$ foe any i. At the same time $L \cdot E_i = a_i \leq 2$ and therefore $(a_1, a_2, ..., a_s) = (2, 2, ..., 2)$. Using again (6) and (7) it is easy to verify that $s = 6, 7$. Those are in fact the only values satisfying (8). The possible values for K_SL give $b = 2e + 5$ and $e = 0, 1$ and thus (d).

If $k = 3$ then $K_SL = -5, -6$ and thus from (6) and (7) we get that $K_S^2 = 1$, $K_SL = -6$ and $d = 24$. Writing $L = 6\pi^*(E_0) + b\pi^*(f) - \sum_1^7 a_iE_i$ and proceeding as before we get (a).

If $K_S^2 < 0$ it follows easily that $K_SL = k - 2$ and $d = 4k - k^2$ that is impossible since we can assume $d \geq 2k + 2$ (cf., [5, 1.4]). Assume now $\delta \geq 2$. If $q = 0$ and $K_S^2 \geq 0$ then:

$$\frac{2k(\delta-3)}{k-1} \leq K_SL \leq -k-2 \tag{9}$$

and therefore we get $K_S^2 = 0$, $\delta = 2$, $K_SL = -4$ and $k = 2$. Moreover K_S+L is a very ample line bundle (cf., [5] 4.1) of degree 8 and sectional genus 3 and since S cannot contain any line relative to L its first reduction coincides with itself. Checking on the list in [19] we find that $K_S + L = 2E_0 + (e+4)f$ and therefore $L = 4\pi^*(E_0) + (6+2e)\pi^*(f) - \sum_i E_i$ that is not 2-very ample. (7) and $g \leq 3k+1$ give $K_S^2 \geq -5$. Since we have already worked the case for elliptic $\mathbb{P}^1$-bundle out we can assume $K_S^2 < 0$ and $q \leq 1$. Let $L = 2k\pi^*(E_0) + \pi^*(b)f - \sum a_iE_i$, then $L(\pi^*(f) - E_i) \geq k$ implies $a_i = k$ for all i. Equations (6) and (7) give $K_SL \geq \frac{(k-3)}{2}$, and thus $d \leq 6k+2$ and $K_S^2 = -1$. Assume first $q = 1$ and thus $L = 2k\pi^*(E_0) + \pi^*(b)f - kE$. Using $d = 2k(b - 2ke) + 2kb - k^2$, $L \cdot \pi^*(E_0) = b - 2ke \geq k$ we get $e = -1$.Then $K_S \cdot L = -2k - 2b \geq \frac{(k-3)}{2}$ gives a contradiction.

We can then assume $q = 0$ and $L = 2k\pi^*(E_0) + b\pi^*(f) - \sum_1^9 kE_i$.

$$d = -4k^2e + 4kb - 9k^2$$

$$K_S \cdot L = 2ke - 2b + 5k$$

Then $d \leq 6k+2$ and $b-2ke \geq k$ give $e=0,1,2$. Consider the effective divisor $H = 2\pi^*(E_0)+(2+e)\pi^*(f)$ for $e=0,1,2$. Riemann-Roch gives $h^0(H) \geq 9$. Then it exists an effective divisor $\xi \in |H - E_1 - ... - E_8|$. L being k-very ample implies $L\cdot\xi = -4ke+2b+4k-8k \geq k$, i.e., $2b \geq 5k+4ke$. Using the inequality in the formula for K_SL we get $K_S \cdot L \leq 0$. Recalling that $K_SL \geq -2$ we can conclude that $K_S\cdot L = -1$ or -2.

If $K_S \cdot L = -2$, (6) and (7) give $k=\delta=2$.

If $K_S \cdot L = -1$ then same computations give $k=3$ and $\delta=2$.

Using the formula for $K_S\cdot L$ we get the cases (b) and (c). □

The cases with $k=2$ in Proposition 3.8 are shown to be 2-very ample in [14, 17].

Let us examine the two cases for $k=3$.

Proposition 3.9. *Let S be the blow up of $\mathbb{F}_e$, $e=0,1$ in 7 points and $L = 6E+(3e+7)f-3\sum_1^7 E_i$. Then L is 3-very ample.*

Proof. Recalling that $\mathbb{F}_1$ is the blow up of $\mathbb{P}^2$ at one point and that $Bl_{P_1}(\mathbb{F}_0) = Bl_{P_1,P_2}(\mathbb{P}^2)$, we can think of S as a Del Pezzo surface of degree 1. Using the criteria proved in [13], in order to prove that L is 3-very ample we need to check that $L\cdot\xi \geq 3$ for all the (-1)-curves and 0-curves on S. If ξ is a (-1)-curve then $K_S\xi = -1$ and thus $L\xi = (-3K_S+2f)\cdot D \geq 3 + f\cdot D \geq 3$ since f is a nef line bundle on S. Similarly if ξ is a 0-curve then $L\cdot\xi \geq 6$. □

Proposition 3.10. *Let S be the blow up of $\mathbb{F}_e$, $e=0,1$ in 9 points and $L = L = 6E+(3e+8)f-3\sum_1^9 E_i$. Then L is not 3-very ample.*

Proof. First notice that $e \leq 1$. If $e=2$ then $L\cdot E_0 = -3e+8 = 2$ and therefore L cannot be 3-very ample. As above we can think of S as the blow up of a Del Pezzo surface X of degree 1 in two points. Consider the two blow ups:

$$S \to Y \to X$$

and let E_8, E_9 be the exceptional divisors. Since $h^0(-K_X)=2$, $-K_Y = -K_X - E_8$ is an effective irreducible divisor and from the adjunction formula we see that $p_a(C) = 1$ for a general curve $C \in |-K_Y|$. Then

$$L \cdot C = (-3K_S + 2f) \cdot C = (-3K_Y - E_9 + 2f) \cdot (-K_Y) = 4$$

that implies L not 3-very ample. $\square$

We can conclude that

Proposition 3.11. *Let L be a k-very ample line bundle on a smooth connected surface S. If (S, L) is a k-conic bundle with $g(L) \leq 3k+1$ then (S, L) is as in cases (a),(c),(d),(e) of Proposition 3.8. Moreover if L is as in cases (a),(c),(d),(e) of Proposition 3.8 then it is k-very ample.*

4. SURFACES OF POSITIVE KODAIRA DIMENSION

A bound for K_SL turns out to be very useful in the classification of surfaces with positive Kodaira dimension, because allows us to have a bound on the number of points blown up in the minimal resolution, according to the following Theorem proved in [3]. Let $S,'$ be the minimal model of S and set $\gamma := e(S) - e(S')$ where $e(Y)$ denotes the Euler characteristic of the space Y.

Theorem 4.1. *Let L be a k-very ample line bundle on S with $k(S) \geq 0$. Then $K_SL \geq \gamma k + k(S)(\frac{k+2}{2})$.*

Proposition 4.2. *Let L be a k-very ample line bundle on S, with $g(L) \leq 3k+1$. If $k(S) \neq \infty$ then S is a minimal $K3$ Surface.*

Proof. If $k(S) \geq 0$ then Theorem 4.1 and Proposition 2.6 imply that S is a minimal surface with $k(S) = 0$. Moreover $K_S \cong \mathcal{O}_S$, since $K_SL = 0$ and S cannot be a scroll (cf., [20, Proposition 0.9]) and therefore S is a minimal $K3$ surface embedded in $\mathbb{P}^g$ by a k-very ample line bundle of degree $2g-2$. $\square$

5. CLASSIFICATION

Using the classification for special surfaces that we have worked out in the previous two sections we have the following result as a simple corollary of Theorem 1.9.

Theorem 5.1. *Let L be a k-very ample line bundle on a smooth projective surface. Assume $g(L) \leq 3k+1$, then $tK_S + L$, $t = 0,...,k-1$ is very ample and $kK_S + L$ is nef and big unless (S, L) is :*

1. *as in* 3.1, 3.2, 3.7 , 3.11.
2. *S is a Del Pezzo surface of degree 2, $L = -kK_S$ and $k = 4, 3, 2$.*
3. *S is a Del Pezzo surface of degree 3, $L = -kK_S$ and $k = 3, 2$.*
4. *S is a Del Pezzo surface of degree 4, $L = -kK_S$ and $k = 2$.*

Proof. It follows directly from Theorem 1.9 and the criteria in [13]. □

Theorem 5.2. *Let S be a non minimal projective surface polarized by a k-very ample line bundle L with $g(L) \leq 3k+1$. Assume $tK_S + L$, $t = 0,...,k-1$ is very ample and $kK_S + L$ is nef and big. Then $K_S^2 \geq 0$.*

Proof. Since $kK_S + L$ is a nef and big line bundle we have that $g(kK_S + L) \geq 0$ that gives:

$$(k^2 + k)K_S^2 + 2kK_SL + 6k - 2 \geq 0 \tag{10}$$

It follows that $K_S^2 \geq -2$ and therefore $q = 0, 1$. Thus we can assume $\chi(\mathcal{O}_S) = 1, 0$ since $k(S) = -\infty$ from Theorem 10.

Case 1: q=0, $K_SL \geq 0$

Assume $K_S^2 = -1$, $K_SL \geq 0$ and let S be the blow up of $\mathbb{P}^2$ in 10 points or $\mathbb{F}_e$ in 9 points. Let $\gamma : S \dashrightarrow S'$ be the blow up in $P_1, ..., P_i$ and $L = \gamma^*(L') - \sum_i a_iE_i$. Since $h^0(-K_S) \geq i$ we can certainly find a curve $\xi \in |-K_S|$ passing through all the points blown up , but one, P_j, for any $j = 1, ..., i$. In fact if ξ passed through all the points $P_1, ..., P_i$ then $|-K_S|$ would have an effective divisor and then $K_SL \leq -k-2$, that is impossible. Let $\xi_j \in |-K_S + E_j|$, then since ξ is an elliptic curve $L\xi = -K_SL + a_j \geq k+2$ for $j = 1, ..., 10$. Let $K_SL = h$; $a_j \geq k+h+2$ implies $(k+2+h)K_S + L$ nef and big, in fact starting from the very ample line bundle $(k-1)K_S + L$ and considering the adjoint bundle $\mathcal{L} = K_S + (k-1)K_S + L$ we see that it is very ample since S

is not a Del Pezzo surface, nor a $\mathbb{P}^1$-bundle over an elliptic curve and there are no possible (-1)-curves relative to $\mathcal{L}$. We can keep going until we have $(k+1+h)K_S+L$ very ample. Then since S is not $\mathbb{P}^2$, nor a quadric, nor a scroll $\mathcal{L}=K_S+(k+1+h)K_S+L$ is in fact spanned and therefore $g(\mathcal{L})\geq 0$. using the adjunction formula and $d\leq 6k$ we get:

$$h^2\geq k^2-k+4>(k-1)^2$$

Since $a\geq 0$ this gives $K_SL>k-1$, that is impossible.

Assume now $K_S^2=-2$, then from (10) $K_SL=k-1$. Following the setting as above with one more point blown up, the linear system $|-K_S-E_i-E_j|$ has an effective divisor ξ, such that $L\xi=-K_SL-a_i-a_j\geq k+2$. Thus $a_i+a_j\geq 2k+1$ for any i,j; this implies that either $a_i\geq k+1$ for any i or there exist a unique $a_j=k$ and $a_i\geq k+1$ for any $i\neq j$. The first case is impossible because $(k+1)K_S+L$ would be a spanned line bundle of genus 0. In the second case the k-reduction of (S,L) would be $(S',L'=L+kE_j)$, where S' is the surface obtained contracting the (-1)-curve E_j, with $K_{S'}+L'$ k-very ample and $(k+1)K_{S'}+L'$ spanned. Then, since $d(L')=d+k^2$ and $K_{S'}L'=K_SL-k=-1$, we have $0\leq 2g((k+1)K_{S'}+L')=d-5k-3$, that would imply $d\geq 5k+3$, impossible because the adjunction formula gives $d=2g-2-K_SL\leq 6k-k+1=5k-1$.

Case 2: q=1, $K_SL\geq 0$.

Assume $K_S^2=-1$ and Let S be the blow up of an elliptic $\mathbb{P}^1$-bundle in one point and $L=\gamma^*(aE_0+bf)-cE$. It is known that $\xi=2E-f$ is numerically equivalent to an elliptic curve and then $L\xi=b+2a\geq k+2$. It follows that $K_SL=-2b-a+c\leq -k-2+c$, i.e. $c\geq K_SL+k+2$. As above we conclude that $(K_SL+c+2)K_S+L$ is spanned and therefore we reach an absurd. If $K_S^2=-2$ and $L=\gamma^*(aE_0+bf)-c_1E_1-c_2E_2$ we can repeat the exact same argument as for $q=0$ imposing that either $c_1,c_2\geq k+1$ or the k-reduction consists in only one curve blown down and then conclude that this is impossible.

Case 3: q=0,1 and $K_SL < 0$.

From $\frac{k}{2} - 2 < 0$ we have $k = 2$, $K_SL = -1$ and $d = 2g - 1$, i.e. $2g - 1 \geq g + 2k$. Therefore we only have to consider the cases $g = 5, 6, 7$. If $g = 6$, and $d = 11$, checking on the list in [19], with $t_0 = 0$ the only case is $S = \mathbb{P}^2$ and $L = \sigma^*(\mathcal{O}_{\mathbb{P}^2}(9)) - \sum_1^6 3E_i - \sum_1^4 2E_1$, where $\sigma : S \to \mathbb{P}^2$ is the blow up along 10 points $P_1, ..., P_{10}$. Note that there are no plane cubics passing through all the 10 points otherwise $-K_S$ would be an effective divisor and $K_SL \leq -3$. Consider the plane cubics, ξ_i passing through all points P_j, for all $j \neq i$ but P_i, this gives an effective elliptic curve in the linear system $-K_S + E_i$ and $L\xi_i = 3 < k + 2$ that is impossible .

If $g = 5$ we still have only one case, a blow up of $\mathbb{P}^2$ in 10 points that we can rule out as above (cf., [5]).

If $g = 7$ and $d = 13$ then $h^0(L) = 8$, since $h^0(L_C) \geq 7$ and thus we can use the Le Barz formulas as stated in [1, pg 89] to rule this case out. ☐

Theorem 5.3. *Let S be a non minimal projective surface polarized by a k-very ample line bundle L with $g(L) \leq 3k+1$. Assume $tK_S + L$, $t = 0, ..., k-1$ is very ample and $kK_S + L$ is nef and big and $K_S^2 \geq 0$ then (S, L) is one of the following:*

(1) *$k = 3, 2$, S is a Del Pezzo surface of degree 1 and $L = -(k+1)K_S + E_i$, where E_1 is an exceptional divisor of the blow-up $\gamma : S \to \mathbb{P}^2$.*

(2) *$k = 2$, S is a Del Pezzo surface of degree 1 and $L = \gamma^*(\mathcal{O}_{\mathbb{P}^2}(7)) - \sum_1^8 2E_i$.*

(3) *$k = 2$, S is a Del Pezzo surface of degree 2 and $L = -3K_S$.*

Proof. Since S has negative Kodaira dimension, by 4.2, then S is a rational surface with $K_S^2 > 0$ and therefore the linear system $|-K_S|$ contains an effective divisor and thus $K_SL \leq -k - 2$. Moreover since $kK_S + L$ is nef and big $(kK_S + L)K_S \geq k + 4$, by [3, Corollary 5.2] which gives $d \geq k(k+2) + k + 4$, i.e., $k = 2, 3$.

If $k = 3$ the only possible case is $K_SL = -5$ and $d = 23$ that gives $K_S^2 = 1$, by Hodge index Theorem. Checking the existence of $2K_S + L$, that is a very ample line bundle of genus 3 and degree 7, on the lists in [19] we get (1).

If $k = 2$ than it is easy to check that the only possibilities are:

(a) $K_SL = -4$, $d = 14$ and $K_S^2 = 1$;
(b) $K_SL = -5$, $d = 17$ and $K_S^2 = 1$;
(c) $K_SL = -6$, $d = 18$ and $K_S^2 = 1$;
(d) $K_SL = -6$, $d = 18$ and $K_S^2 = 2$;

Checking for the existence of the very ample line bundle $K_S + L$ we easily see that (a) and (b) give cases (1) and (2) respectively and that (c) cannot occur. In (d) S can only be a Del Pezzo surface of degree 2 and $K_S(3K_S + L) = 0$ implies $L = -3K_S$ and thus (3). $\square$

The following proposition complete the proof of the sufficiency of the final list, except for the $K3$ case.

Proposition 5.4. *let* (S, L) *be a polarized Del Pezzo surface as in* 5.1 *or* 5.3. *Then* L *is* k*-very ample.*

Proof. According to the criterium proved in [13] we have to check that $L \cdot \xi \geq k$ for all the (-1)-curves and 0-curves on S. In 5.1 $L = -kK_S$, which is k-very ample since $-kK_S\xi = 1$ if ξ is a (-1)-curve and $-kK_S\xi = 2$ if ξ is a 0-curve.

Let us examine the cases in 5.3. The last case follows from what showed above. Let L be as in the case (1), $L = -(k+1)K_S + E_i$ and let ξ be a (-1)-curve of the form $\gamma^*(\mathcal{O}_{\mathbb{P}^2}(a)) - \sum a_iE_i$, where $\gamma : S \to \mathbb{P}^2$ is the blowing up map. Then $L \cdot \xi = (k+1) + a_i \geq k$. Similarly for a 0-curve.

Let $L = \gamma^*(\mathcal{O}_{\mathbb{P}^2}(7)) - \sum_1^8 2E_i$ be as in the case (2), then $L = -2K_S + \gamma^*(\mathcal{O}_{\mathbb{P}^2}(1))$. It follows that $L \cdot \xi \geq k + 1 + a$ if ξ is expressed as above and therefore k-very ample. $\square$

At this stage we can just collect the results of this section and have a final classification :

Corollary 5.5. *Let* L *be a* k*-very ample line bundle on a smooth connected surface* S. *Assume that* $k \geq 2$ *and that* $g(L) \leq 3k+1$, *then* (S, L) *is as in* 5.1 *or* 5.3 *(cfr. TABLE 1). Moreover If*

(S, L) is as in 5.1 or 5.3 then L is k-very ample except possibly for S being a $K3$ surface.

Corollary 5.6. *Let L be a k-very ample line bundle on S with $g \leq 3k+1$. Then*

(1) *if $k(S) = -\infty$ then $k \leq 9$;*
(2) *if $k(S) \neq -\infty$ then S is a minimal $K3$ surface of degree $\geq 4k$.*

6. THE DEGREE CLASSIFICATION

Using the classification for k-very ample line bundles of genus $\leq 3k+1$ it is possible to complete and extend the classification by degree done in [3] . The value of $h^1(L_C)$ plays a key role in the relation between the degree and the genus of a k-very ample line bundle, as shown in 2.1.

Proposition 6.1. *Let L be a k-very ample line bundle on a smooth connected surface S and let $C \in |L|$ be a general hyperplane section. Assume $h^1(L_C) \geq 2$, then $d(L_C) \geq 5k+2$ or $k \geq 14$.*

Proof. If $h^1(L_C) \geq 2$ then $2g-2-d \geq k+2h^1(L_C)-2$, by 2.1. This implies $d(K_S|_C) \geq k+2h^0(K_S|_C)-2$ and thus $cl(K_S|_C) \geq k$. Then $cl(K_S|_C) = cl(L_C) \geq k$. If $h^0(L_C) \geq 2k+2$ than $k \leq d-4k-2$ gives $d \geq 5k+2$. Assume $h^0(L_C) = 2k+1$ and $d < 5k+2$. Then since $cl(L_C) \geq k$, $d = 5k, 5k+1$. If $d = 5k$ and $h^0(L_C) = 2k+1$ the Castelnuovo bound (cf., [2]) gives $g \leq 4k+1$ and therefore $h^1(L_C) \leq k+1$. If $d = 5k+1$ then the Castelnuovo bound gives $g \leq 4k+2$ and thus $h^1(L_C) \leq k+2$. Since L_C is k-very ample the number, ν_k, of $(k+1)$-secant $(k-1)$-planes of $C \in \mathbb{P}^{2k}$ has to be zero. Using all the possible values and the formula for ν_k (see [2, pg 351]):

$$\nu_k = \sum_0^{k+1} (-1)^i \binom{g+2k-d}{i} \binom{g}{k+1-i} \tag{11}$$

numerical computations show that $\nu_k \neq 0$ for $k \leq 13$. □

Proposition 6.2. *let L be a k-very ample line bundle on a smooth projective surface S, assume $k \geq 2$ and $d \leq 5k+1$ then (S, L) is as in 5.5 or $k \geq 14$ and $d = 5k, 5k+1$.*

Proof. Let $C \in |L|$. If $h^1(L_C) = 0, 1$ then $d \geq g + 2k$ by 2.1 and therefore $g \leq 3k + 1$ that gives (S, L) as in 5.3. If $h^1(L_C) \geq 2$ then we are done by 6.1. ☐

Corollary 6.3. *let L be a k-very ample line bundle on a smooth projective surface S, assume $k \geq 2$ and $d \leq 4k + 4$ then (S, L) is as in* 5.5.

Proof. $4k + 4 \leq 5k + 1$ if $k \geq 3$, and $4k + 4 \geq 5k - 1$ if $k \leq 5$. Then we are done by 6.2 unless $k = 2$ and $d = 12$. In this last case $g \leq 3k + 1$ unless $h^0(L_C) = 2k + 1, 2k + 2$, for a general hyperplane section $C \in |L|$. The first case does not occur since 11 is not zero. If $h^0(L_C) = 6$ then $S \in \mathbb{P}^6$ (cf., [3, Lemma 2.8]) and the Le Barz formula for 3-secant lines, as stated in [1, pg 89], rules this case out. ☐

Let S be a surface polarized by a k-very ample line bundle of degree $\leq 4k + 4$ then it is one of the cases in the following table:

TABLE 2

k	S	L
4	$\mathbb{P}^2$	$\mathcal{O}_{\mathbb{P}^2}(4)$
3	$\mathbb{P}^2$	$\mathcal{O}_{\mathbb{P}^2}(3)$
	elliptic $\mathbb{P}^1$-bundle, $e = -1$	$3E_0 + 2f$
	$Bl_8(\mathbb{P}^2)$	$-4K_S + E_i$
2	$\mathbb{P}^2$	$\mathcal{O}_{\mathbb{P}^2}(2)$, $\mathcal{O}_{\mathbb{P}^2}(3)$
	$\mathbb{F}_0$	$aE_0 + bf$,$a = 2$ and $2 \leq b \leq 3$
	elliptic $\mathbb{P}^1$-bundle, $e = 1$	$2E_0 + 8f$, $b = 6, 7$
	elliptic $\mathbb{P}^1$-bundle, $e = 0$	$2E_0 + 4f$
	$\mathbb{P}^1$-bundle over $Bl_7(\mathbb{F}_e)$, $e = 0, 1$	$4E_0 + (2e + 5)f - \sum_1^7 2E_i$
	$Bl_9(\mathbb{F}_0)$	$4E_0 + 6f - \sum_1^9 2E_i$
	$Bl_8(\mathbb{P}^2)$	$-3K_S + E_i$
	$Bl_7(\mathbb{P}^2)$	$-2K_S$
	$Bl_6(\mathbb{P}^2)$	$-2K_S$
$k \geq 2$	$S \subset \mathbb{P}^g$ K3	$d(L) = 4k, 4k + 2, 4k + 4$

Corollary 6.4. *Let L be a k-very ample line bundle on S with $d \leq 4k + 4$. Then*

(1) *if $k(S) = -\infty$ then $k \leq 4$;*
(2) *if $k(S) \neq -\infty$ then S is a minimal $K3$ surface of degree* $= 4k, 4k + 2, 4k + 4$

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