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PROJECTIVE DUALITY OF TORIC MANIFOLDS AND DEFECT POLYTOPES

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1. *Introduction*

In this paper we address two apparently disjoint questions, one of a purely complex geometrical nature and one of a purely combinatorial nature.

Let $i : X \hookrightarrow \mathbb{P}^n$ be an embedding of a non-singular complex variety. The associated dual variety is defined as:

$$X^* = \overline{\{H \in \mathbb{P}^{n*} \mid H \text{ is tangent to } X \text{ at some point } x \in X\}}.$$

Generally $\dim(X) + 1$ conditions are necessary for a hyperplane to be tangent at a point on X. Hence the dual variety is expected to be a hypersurface in $\mathbb{P}^{n*}$. The dual defect of the embedding i is $d(i) = n - 1 - \dim(X^*)$. An embedding has dual defect equal to zero when the dual variety has the expected dimension.

It is well known that there are no embeddings of curves with positive dual defect. In dimension 2 the only embedding having positive dual defect is $\mathrm{id} : \mathbb{P}^2 \hookrightarrow \mathbb{P}^2$. For higher dimensions the picture is not nearly as clear. The inequality $\dim(X^*) \geqslant \dim(X)$, [**23**, 2.5], implies that $d(i) \leqslant \mathrm{codim}(X) - 1$. In [**7**] Ein classified the maximal defect varieties, that is, embeddings for which $d(i) = \mathrm{codim}(X) - 1$, when $\dim(X) \leqslant \frac{2}{3}n$. In [**19**] Munõz gave a classification for $d(i) = \mathrm{codim}(X) - 2$ with the same bound on the dimension. In [**18**] and [**3**] non-singular embeddings of dimension up to 10 with positive dual defect are classified.

Most of the known examples of embeddings with positive dual defect are embeddings of rational varieties. Toric varieties represent an important class of examples of rational varieties. It is then natural to pose the following question.

QUESTION I. Which embeddings of non-singular toric varieties have positive dual defect?

EXAMPLE 1. Consider a projective bundle of the form

$$X = \mathbb{P}(\mathcal{L}_0 \oplus \ldots \oplus \mathcal{L}_k) \xrightarrow{\pi} Y,$$

where Y is a non-singular toric variety of dimension $m < k$ and $\mathcal{L}_i$ are equivariant line bundles on Y. Let ξ be the tautological line bundle associated to the vector bundle $\mathcal{L}_0 \oplus \ldots \oplus \mathcal{L}_k$ and let X be a non-singular toric variety. For every choice of an equivariant line bundle H on Y such that $H + \mathcal{L}_i$ is very ample, for $i = 0, \ldots, k$, the line bundle $\xi + \pi^*(H)$ is very ample; see Proposition 2. Hence it defines an

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equivariant embedding

$$I : X \hookrightarrow \mathbb{P}\Big(\bigoplus_{i=0}^{k} H^0(\mathcal{L}_i + H)\Big).$$

This embedding has defect $d(i) = k - m$; see [**17**, 5.2].

Using properties of the toric action and a convenient application of results in [**2**] we show that these are the only possible examples; see Theorem 2.

Answer to Question I. Let $i : X \hookrightarrow \mathbb{P}^n$ be an embedding of a non-singular r-dimensional complex toric variety. It has positive dual defect d if and only if there are invariant line bundles $H, \mathcal{L}_0, \ldots, \mathcal{L}_{(r+d)/2}$ over a non-singular toric variety Y of dimension $\frac{1}{2}(r-d)$ such that:

(1) $\mathcal{L}_i + H$ is very ample for all $i = 0, \ldots, \frac{1}{2}(r+d)$;
(2) $X = \mathbb{P}(\mathcal{L}_0 \oplus \ldots \oplus \mathcal{L}_{(r+d)/2})$;
(3) $i^*(\mathcal{O}_{\mathbb{P}^n}(1)) = \xi + \pi^*(H)$, where $\pi : X \to Y$ is the projection map and ξ is the tautological line bundle;
(4) $r + d \geqslant 4$.

In particular, the only embedding of an r-dimensional toric variety, with dual defect $d = r$ is $\mathrm{id} : \mathbb{P}^r \hookrightarrow \mathbb{P}^r$. All the necessary results on toric $\mathbb{P}^k$-bundles are proven in § 3.

Let us now turn to combinatorics. Let $P = \mathrm{Conv}(A)$ be a convex integral polytope of dimension r. Define the following combinatorial invariant:

$$c(P) := \sum_{F \in F(P)} (-1)^{\mathrm{codim}(F)} (\dim(F) + 1)! \,\mathrm{Vol}(F) \tag{1.1}$$

where $F(P)$ is the set of non-empty faces of P, including P itself, and the volume is the integral volume (see § 5) with the volume of a vertex set to be 1.

This combinatorial invariant was introduced in [**12**], where it is proven that $c(P)$ corresponds to the degree of a homogeneous function, called the regular A-determinant.

Example 2. The polytope $P = \Delta_2 \times I$, the product of the 2-dimensional standard simplex and the unit interval (see Figure 1), has

$$c(P) = 4! \tfrac{1}{2} - 3! 4 + 2 \cdot 9 - 6 = 0.$$

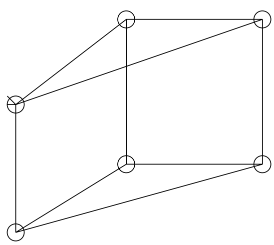

Figure 1. $P = \Delta_2 \times I$.

In order to use the results on dual varieties of toric manifolds we restrict our attention to Delzant polytopes; see Definition 3.

In Corollary 2 we prove that, for a convex integral Delzant polytope, the integer $c(P)$ is non-negative. This result has already been established in [**12**], where it is proven that, in the case when P is Delzant, the regular A-determinant is a polynomial. See § 6 for more details.

We define a *defect polytope* to be a Delzant convex integral polytope with $c(P) = 0$. It is natural to pose the following question.

QUESTION II. Which polytopes are defect polytopes?

Let $P_0, \dots, P_k$ be convex polytopes of the same combinatorial type; see Definition 2. In Definition 6 we define the *projective join* $\mathbb{P}(P_0, \dots, P_k)$ to be the convex polytope obtained by joining the vertices of the P_i by attaching standard k-dimensional simplices. Geometrically, projective joins correspond to the embeddings given in the answer to Question I. Because of this characterization, all projective joins have $c(P) = 0$. Theorem 3 proves that all defect polytopes are projective joins.

Answer to Question II. A polytope P is an r-dimensional defect polytope if and only if P is a projective join $\mathbb{P}(P_0, \dots, P_k)$, where P_i are Delzant polytopes having the same combinatorial type and

$$\max\left(2, \tfrac{1}{2}(r+1)\right) \leqslant k \leqslant r.$$

Note that P is the standard r-dimensional simplex when $r = k$.

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2. *Notation and set-up*

Throughout this paper X denotes a non-singular complex algebraic variety. We will use an additive structure on the group $\mathrm{Div}(X)$.

Embeddings and dual defect

Two embeddings $i : X \hookrightarrow \mathbb{P}^n$ and $j : X \hookrightarrow \mathbb{P}^m$ are said to be isomorphic if there is an isomorphism $\phi : \mathbb{P}^m \to \mathbb{P}^n$ such that $\phi j = i$. This implies that $n = m$.

NOTATION. The term *toric embedding* will denote an embedding of a non-singular toric variety.

The dual variety associated to an embedding i is defined as

$$X^*(i) = \overline{\{H \in \mathbb{P}^{n*} \mid H \text{ is tangent to } X \text{ at some point } x \in X\}}.$$

DEFINITION 1. The dual defect of an embedding $i : X \hookrightarrow \mathbb{P}^n$ is

$$d(i) = n - 1 - \dim(X^*).$$

We remark that isomorphic embeddings have the same dual defect.

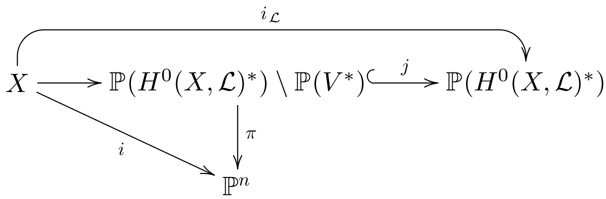
Consider an embedding $i : X \hookrightarrow \mathbb{P}^n$. Let $\mathcal{L} = i^*(\mathcal{O}_{\mathbb{P}^n}(1))$ be the pull back of the hyperplane line bundle. By definition $\mathcal{L}$ is very ample. A choice of a basis of $H^0(X, \mathcal{L})$ defines an embedding $i_{\mathcal{L}} : X \hookrightarrow \mathbb{P}(H^0(X, \mathcal{L})^*)$ whose isomorphism class does not depend on the basis. The dual defect of the embedding $i_{\mathcal{L}}$ is denoted by $d(X, \mathcal{L})$.

Proposition 1. *The dual defects $d(i)$ and $d(X, \mathcal{L})$ are equal.*

Proof. Let $r : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(X, \mathcal{L})$ be induced by i. The embedding i factors as

$$X \overset{j}{\hookrightarrow} \mathbb{P}^{n-k} \hookrightarrow \mathbb{P}^n,$$

where $\mathbb{P}^{n-k} \hookrightarrow \mathbb{P}^n$ is a linear embedding and k is the dimension of the kernel of r. The pull-back of $\mathcal{O}_{\mathbb{P}^n}(1)$ along this linear embedding $\mathbb{P}^{n-k} \hookrightarrow \mathbb{P}^n$ is $\mathcal{O}_{\mathbb{P}^{n-k}}(1)$ and hence $i_{\mathcal{L}}$ is isomorphic to $i_{j^*(\mathcal{O}_{\mathbb{P}^{n-k}}(1))}$. Moreover, the homomorphism $H^0(\mathbb{P}^{n-k}, \mathcal{O}_{\mathbb{P}^{n-k}}(1)) \to H^0(X, \mathcal{L})$ (induced by j) is a monomorphism. The difference between the dimensions of the dual variety of i and the dual variety of j is k which implies that $d(i) = d(j)$. This shows that r can be assumed to be injective. Let V be the cokernel of r. Note that there is the following commutative diagram:



where π is a linear projection. The dual of π induces an isomorphism between the dual projective space $\mathbb{P}^{n*}$ and $\{H \subset \mathbb{P}(H^0(X, \mathcal{L})^*) \mid \mathbb{P}(V^*) \subset H\}$. It follows that the dual variety of the embedding i, $X^*(i)$, is isomorphic to the intersection $X^*(i_{\mathcal{L}}) \cap \mathbb{P}^{n*}$, where $X^*(i_{\mathcal{L}})$ denotes the dual variety of the embedding $i_{\mathcal{L}}$. Because r is assumed to be injective, $\dim(X^*(i)) = \dim(X^*(i_{\mathcal{L}})) - \dim(V) - 1$ and thus $d(i) = d(X, \mathcal{L})$. $\square$

Proposition 1 shows that, when studying problems concerning the dual defect, without loss of generality, embeddings of X can be considered given by the complete linear system of the very ample line bundle $\mathcal{L} = i^*(\mathcal{O}_{\mathbb{P}^n}(1))$ on X.

Notation. We will denote the embedding $i : X \hookrightarrow \mathbb{P}^n$ by $(X, \mathcal{L})$ and the defect $d(i)$ by $d(X, \mathcal{L})$.

The vector bundle of principal parts

Let $(X, \mathcal{L})$ be an embedding of a non-singular r-dimensional complex variety. We shall recall briefly the definition and basic properties of the vector bundle of principal parts $P^1(\mathcal{L})$, associated to $\mathcal{L}$. For details we refer to [**16**, **3**].

Consider the projections $\pi_i : X \times X \to X$, for $i = 1, 2$, and the ideal sheaf $\mathcal{I}_\Delta$ of the diagonal $\Delta \subset X \times X$. This data defines a sheaf of principal parts, supported on the subscheme Δ:

$$\mathcal{P}^1(\mathcal{L}) = \pi_{2_*}\big(\pi_1^*(\mathcal{L}) \otimes \mathcal{O}_{X\times X}/\mathcal{I}_\Delta^2\big).$$

The sheaf $\mathcal{P}^1(\mathcal{L})$ is locally free of rank $r+1$.

There is a natural map of vector bundles

$$\mathcal{P}^1 : X \times H^0(X,\mathcal{L}) \to \mathcal{P}^1(\mathcal{L}). \tag{2.1}$$

After choosing local coordinates around a point $x \in X$, the first jet of a section $s \in H^0(X,\mathcal{L})$ at x is defined as $\mathcal{P}^1(x,s) = (s(x), ds(x)) \in \mathcal{P}^1(\mathcal{L})_x$.

The kernel of this map is the conormal bundle shifted by 1, $N^*_{X/\mathbb{P}^n}(1)$. The image of the projection map,

$$\mathbb{P}(N_{X/\mathbb{P}^n}(-1)) \to (\mathbb{P}^n)^*,$$

defines the dual variety X^*. For more details we refer to [**15**, **13**].

The dual defect is controlled by the vanishing of Chern classes of the vector bundle of principal parts.

Lemma 1 [**13**, 3.4; **2**, 0.16]. *The embedding $(X,\mathcal{L})$ has dual defect d if and only if $c_k(\mathcal{P}^1(\mathcal{L})) = 0$ for $k \geqslant r-d+1$ and $c_{r-d}(\mathcal{P}^1(\mathcal{L})) \neq 0$. In particular, $(X,\mathcal{L})$ has positive defect if and only if $c_r(\mathcal{P}^1(\mathcal{L})) = 0$.*

The nef-value morphism

Definition 2. The *nef value* of the embedding $(X,\mathcal{L})$ is

$$\tau_{(X,\mathcal{L})} = \min\{t \in \mathbb{R} \mid K_X + t\mathcal{L} \text{ is nef}\}.$$

In [**14**] Kawamata proved that $\tau_{(X,\mathcal{L})}$ is a rational number and the linear system $|m(K_X+\tau\mathcal{L})|$ is base point free for $m \gg 0$. Let $g : X \to \mathbb{P}^M$ be the morphism defined by the global sections of $|m(K_X+\tau\mathcal{L})|$. Consider the Remmert–Stein factorization $g = s \circ \psi_\tau$, where $\psi_\tau : X \to Y$ is a morphism with connected fibers and normal image, and s is a finite-to-one morphism onto $\mathrm{Im}(g)$. The morphism ψ_τ is called the *nef-value morphism* associated to the pair $(X,\mathcal{L})$.

Let $(X,\mathcal{L})$ be an embedding of a non-singular variety, $\tau = \tau_{(X,\mathcal{L})}$ and F be a general fiber of the nef-value morphism $\psi_\tau : X \to Y$. The restriction of $\mathcal{L}$ to F defines an embedding of F, $(F,\mathcal{L}|_F)$. The following lemma describes the relationship between the defect of the embedding $(X,\mathcal{L})$ and the defect of the embedding $(F,\mathcal{L}|_F)$.

Lemma 2 [**2**, 1.2]. *Let $m = \dim(Y)$. The following statements are equivalent:*

(i) $d(X,\mathcal{L}) > 0$;

(ii) $d(F,\mathcal{L}|_F) > 0$;

(iii) $d(X,\mathcal{L}) = d(F,\mathcal{L}|_F) - m$.

Moreover, if one of the above conditions holds true then $\tau = \frac{1}{2}(r + d(X,\mathcal{L})) + 1$, $\mathrm{Pic}(F) \cong \mathbb{Z}$ and ψ_τ is a Mori-contraction of fiber type.

Toric varieties and polytopes

For basic definitions and properties of toric varieties we refer to [**20**, **9**]. Let N be a d-dimensional lattice and $V = N \otimes \mathbb{R}$. Throughout this paper the word ‘polytope’ will always mean an *integral convex polytope* $P \subset V$, that is, $P = \mathrm{Conv}(A)$ with $A \subset N$. We will denote by $P(k)$ the set of k-dimensional faces of P.

Let $\dim(P) = r$ and let $\mathcal{A}_P$ denote the affine span of P. There is an injective affine transformation $\phi : \mathcal{A}_P \to \mathbb{R}^d$ with $\phi(\mathcal{A}_P \cap V) = \mathbb{R}^r$. The volume of P is defined as

$$\mathrm{Vol}(P) = \mathrm{Leb}(\phi(P))$$

where Leb denotes the Lebesgue measure on $\mathbb{R}^r$. In particular:

(1) for every edge $\ell \in P(1)$ the length $\mathrm{Vol}(\ell) = |\ell \cap N| - 1$;

(2) if Δ_r denotes the r-dimensional standard simplex, then

$$|\Delta_r(k)| = \binom{r+1}{k+1} \quad \text{and} \quad \mathrm{Vol}(F) = \frac{1}{k!} \text{ for each face } F \in \Delta_r(k).$$

Recall that there is a one-to-one correspondence between polytopes and toric varieties embedded in projected space. We will denote by $(X(P), \mathcal{O}_P(1))$ the complete embedding $X(P) \hookrightarrow \mathbb{P}^{|P \cap N|-1}$ associated to P. The complete fan defining the toric variety $X(P)$ will be denoted by Δ_P. Geometry and combinatorics are related by the following one-to-one correspondences:

$$\begin{matrix} \text{codimension } t \text{ cones} \\ \sigma \in \Delta_P(r-t) \end{matrix} \longleftrightarrow \begin{matrix} t\text{-dimensional} \\ \text{invariant subvarieties} \\ V(\sigma) \subset X(P) \end{matrix} \longleftrightarrow \begin{matrix} t\text{-dimensional faces} \\ F_\sigma \in P(t) \end{matrix}$$

Notation. Two polytopes P_1 and P_2 have the *same combinatorial type* if

$$\Delta_{P_1} \cong \Delta_{P_2}.$$

Notice that if $P_1 = \mathrm{Conv}(A)$ and $P_2 = \mathrm{Conv}(B)$ have the same combinatorial type, then $|P_1(t)| = |P_2(t)|$, for $t \geqslant 0$, and $\dim(P_1) = \dim(P_2)$.

Definition 3. An r-dimensional polytope is *Delzant* if every vertex m is incident to exactly r edges $\{l_1, \dots, l_r\}$ and $\{m_1 - m, \dots, m_r - m\}$ forms a $\mathbb{Z}$-basis of N, where m_i are the lattice points on l_i next to m.

Remark 1. Note that the polytope P is Delzant if and only if $X(P)$ is a non-singular toric variety, [**20**, 2.22]. In other words, P is Delzant if and only if $(X(P), \mathcal{O}_{X(P)}(1))$ is a toric embedding.

3. *The geometry of a toric linear $\mathbb{P}^k$-bundle*

Definition 4. An embedding $(X, \mathcal{L})$ is called a *linear $\mathbb{P}^k$-bundle* over a variety Y if there is a surjective morphism $\pi : X \to Y$ such that for every fiber F of π, $(F, \mathcal{L}|_F) \cong (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$.

When X and Y are non-singular toric varieties we will use the term *toric linear $\mathbb{P}^k$-bundle*.

Note that $(X, \mathcal{L})$ is a linear $\mathbb{P}^k$-bundle over a variety Y if and only if

$$(X, \mathcal{L}) \cong (\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1)),$$

where E is a vector bundle of rank $k+1$ on Y.

The *divisor* group of a $\mathbb{P}^k$-bundle, $\mathbb{P}(E)$, over a variety Y, is generated by the pull-back of a basis of $\mathrm{Div}(Y)$ and the tautological line bundle ξ_E:

$$\mathrm{Div}(\mathbb{P}(E)) = \mathbb{Z}\cdot\xi_E \oplus \pi^*(\mathrm{Div}(Y)).$$

Moreover,

$$K_{\mathbb{P}(E)} = \pi^*(K_Y + \det(E)) - (k+1)\xi_E. \tag{3.1}$$

In this section we will prove the following.

THEOREM 1. *A pair $(X,\mathcal{L})$ is a toric linear $\mathbb{P}^k$-bundle if and only if*

$$(X,\mathcal{L}) \cong (\mathbb{P}(\mathcal{L}_0\oplus\ldots\oplus\mathcal{L}_k), \xi + \pi^*(H))$$

where ξ is the tautological line bundle associated to the vector bundle $\mathcal{L}_0\oplus\ldots\oplus\mathcal{L}_k$ and the line bundles $\mathcal{L}_i + H$ are very ample on Y.

Let $X \cong \mathbb{P}(E)$ be a $\mathbb{P}^k$-bundle over a non-singular toric variety Y of dimension m. Assume that X is a non-singular toric variety. The vector bundle E splits into the sum of line bundles; see [**6**, 1.1]. Hence,

$$X \cong \mathbb{P}(\mathcal{L}_0\oplus\ldots\oplus\mathcal{L}_k) \xrightarrow{\pi} Y$$

where the $\mathcal{L}_i$ are equivariant line bundles on Y. Let Δ and Δ' be the fans associated to X and Y respectively. The fan Δ is obtained from Δ' in the following way (see [**20**, 1.33]). Choose a $\mathbb{Z}$-basis for $\mathbb{R}^k$, $\{e_1,\ldots,e_k\}$, and set $e_0 = -e_1-\ldots-e_k$. Denote by σ^i the k-dimensional cone $\bigoplus_{j\neq i}(\mathbb{R}_+e_j)$. Recall that $\mathrm{Div}(Y) = \bigoplus\mathbb{Z}\cdot D_i$, where D_i are the invariant principal divisors associated to the edges $n_i\in\Delta'(1)$. Assume that

$$\mathcal{L}_j = \sum_i a_i^j D_i.$$

If $p:\mathbb{R}^m\to\mathbb{R}^m\oplus\mathbb{R}^k$ is the map given by by $p(n_i) = \left(n_i, \sum_{j=0}^k a_i^j e_j\right)$, then

$$\Delta = \{p(\sigma)+\sigma^i \mid \sigma\in\Delta', i=0,\ldots,k\}.$$

REMARK 2. Let $\mathbf{e}_j$ denote the vector $(0,e_j)\in\mathbb{R}^m\oplus\mathbb{R}^k$ and $w_i = p(n_i)$. Note that

$$\Delta(1) = \{\mathbf{e}_j \mid j=0,\ldots,k\}\cup\{w_i \mid n_i\in\Delta'(1)\}.$$

Hence the principal divisors on X are of the form $V(\mathbf{e}_j)$ for $j=0,\ldots,k$, or $V(w_i)$ for some $n_i\in\Delta'$.

LEMMA 3. *Let*

$$X = \mathbb{P}(\mathcal{L}_0\oplus\ldots\oplus\mathcal{L}_k) \xrightarrow{\pi} Y$$

be a toric $\mathbb{P}^k$-bundle. Then:

(i) $\{V(\mathbf{e}_0), V(w_i) \mid n_i\in\Delta'(1)\}$ *generate* $\mathrm{Div}(X)$ *up to linear equivalence;*

(ii) $\xi = V(\mathbf{e}_0) + \pi^*(\mathcal{L}_0)$.

Proof. The divisor group of a non-singular toric variety is generated by the principal divisors. Remark 2 implies that $\mathrm{Div}(X)$ is generated by

$$\{V(w_i) \mid n_i\in\Delta'(1)\}\cup\{V(\mathbf{e}_j) \mid j=0,\ldots,k\}.$$

The linear equivalences among the generators are

$$-V(\mathbf{e}_0) + V(\mathbf{e}_j) + \sum_{n_i \in \Delta'(1)} (a^i_j - a^0_j)V(w_i) = 0, \quad \text{for } j = 1, \ldots, k.$$

It follows that, for $j = 1, \ldots, k$,

$$V(\mathbf{e}_j) = V(\mathbf{e}_0) + \sum_{n_i \in \Delta'(1)} (a^0_j - a^i_j)V(w_i).$$

Let ξ be the tautological line bundle associated to the vector bundle $\mathcal{L}_0 \oplus \ldots \oplus \mathcal{L}_k$. The fact that $\pi^* V(n_i) = V(w_i)$ yields

$$V(\mathbf{e}_i) = V(\mathbf{e}_0) + \pi^*(\mathcal{L}_0) - \pi^*(\mathcal{L}_i). \tag{3.2}$$

The description of the canonical bundle in terms of the principal invariant divisors gives

$$K_X = -\sum_{n_i \in \Delta'(1)} V(w_i) - \sum_{j=0,\ldots,k} V(\mathbf{e}_j) \quad \text{and} \quad K_Y = -\sum_{n_i \in \Delta'(1)} V(n_i). \tag{3.3}$$

By comparing (3.1) and (3.3) we obtain

$$-\sum_{n_i \in \Delta'(1)} V(w_i) + \sum_{j=0,\ldots,k} \pi^*(\mathcal{L}_i) - (k+1)\cdot\xi = -\sum_{n_i \in \Delta'(1)} V(w_i) - \sum_{j=0,\ldots,k} V(\mathbf{e}_j).$$

The linear relations (3.2) yield

$$\sum_{j=0,\ldots,k} \pi^*(\mathcal{L}_i) - (k+1)\cdot\xi = -(k+1)V(\mathbf{e}_0) - (k+1)\pi^*(\mathcal{L}_0) + \sum_{j=0,\ldots,k} \pi^*(\mathcal{L}_i).$$

This proves that $\xi = V(\mathbf{e}_0) + \pi^*(\mathcal{L}_0)$. □

Lemma 3 implies that a line bundle $\mathcal{L}$ on X can be written as

$$\mathcal{L} = a\xi + \pi^*(H) = aV(\mathbf{e}_0) + \pi^*(a\mathcal{L}_0 + H)$$

where H is a line bundle on the non-singular toric variety Y and a is an integer. In order to understand the positivity of a line bundle on a toric variety one has to give an estimate of the intersection numbers of the line bundle with all the invariant rational curves. In particular, let $L = \sum a_i D_i$ be a line bundle on an r-dimensional non-singular toric variety X and let $V(\rho)$ be a rational invariant curve corresponding to the $(r-1)$-dimensional cone $\rho = \sigma_1 \cap \sigma_2$. Then, if $\sigma_1 = \langle v_1, v_2, \ldots, v_r\rangle$ and $\sigma_2 = \langle v_0, v_2, v_3, \ldots, v_r\rangle$, the intersection number $L \cdot V(\rho)$ is given by

$$L \cdot V(\rho) = a_0 + a_1 - \sum_{i=2}^{r} s_i a_i$$

where the s_i are the integers such that $v_0 + v_1 - \sum_2^r s_i v_i = 0$ (the existence of the integers s_i follows from the fact that the toric variety is non-singular).

Remark 3. The invariant rational curves on $X = \mathbb{P}(\mathcal{L}_0 \oplus \ldots \oplus \mathcal{L}_k)$ belong to two families.

Pull-backs $V(\rho^i)$. Let $\rho = \bigoplus_2^m \mathbb{R}_+ n_j$ be an $(m-1)$-dimensional cone in Δ'. It defines $(m-1)$-dimensional cones, $\rho^i \in \Delta$, for $i = 1, \ldots, k$:

$$\rho^i = \Big(\sum_{j=2}^{m} \mathbb{R}_+ p(n_j)\Big) + \Big(\sum_{l \neq i} \mathbb{R}_+ \mathbf{e}_l\Big).$$

Fiber-curves $V(\rho^\sigma_{ij})$. Let $\sigma = \bigoplus_l \mathbb{R}_+ n_l$ be an m-dimensional cone in Δ'. For every $i \neq j$ it determines an $(r-1)$-dimensional cone in Δ:

$$\rho_{ij} = \Big(\sum_l \mathbb{R}_+ p(n_l)\Big) + \Big(\sum_{t\neq i,j} \mathbb{R}_+ \mathbf{e}_t\Big).$$

Lemma 4. *Let*

$$X \cong \mathbb{P}(\mathcal{L}_0 \oplus \ldots \oplus \mathcal{L}_k) \xrightarrow{\pi} Y$$

be a toric linear $\mathbb{P}^k$-bundle, where X and Y are non-singular projective varieties. Then:

(i) $V(\mathbf{e}_0) \cdot V(\rho^i) = (\mathcal{L}_i - \mathcal{L}_0) \cdot \rho$ *if* $i \neq 0$*;*
(ii) $\pi^*(H) \cdot V(\rho^i) = H \cdot \rho$ *for every line bundle H on Y;*
(iii) $V(\mathbf{e}_0) \cdot V(\rho^\sigma_{ij}) = 1$ *for all i and j;*
(iv) $\pi^*(H) \cdot V(\rho^\sigma_{ij}) = 0$ *for every line bundle H on Y.*

Proof. Consider an $(r-1)$-dimensional cone of the form $V(\rho^i)$. Because X and Y are non-singular, there are integers s_i, $\overline{s}_i$ and h_j such that

$$n_0 + n_1 - s_2 n_2 - \ldots - s_m n_m = 0 \tag{3.4}$$

and

$$p(n_0) + p(n_1) - \overline{s}_2 n_2 - \ldots - \overline{s}_m n_m - \Big(\sum_{i\neq j} h_i \mathbf{e}_i\Big) = 0. \tag{3.5}$$

By definition, (3.4) and (3.5) give

$$n_0 + n_1 - \overline{s}_2 n_2 - \ldots - \overline{s}_m n_m + \mathbf{e}_0\Big(a^0_0 + a^0_1 - \sum \overline{s}_j a^0_j\Big) + \ldots + \mathbf{e}_k\Big(a^k_0 + a^k_1 - \sum \overline{s}_j a^k_j\Big) - \Big(\sum_{i\neq j} h_i \mathbf{e}_i\Big) = 0.$$

The relations $\mathbf{e}_0 + \ldots + \mathbf{e}_k = 0$ and $a^i_0 + a^i_1 - \sum \overline{s}_j a^i_j = \mathcal{L}_i \cdot \rho$ yield

$$n_0 + n_1 - \overline{s}_2 n_2 - \ldots - \overline{s}_m n_m - \sum_{j\neq i}(\mathbf{e}_j(\mathcal{L}_j \cdot \rho - \mathcal{L}_i \cdot \rho - h_j)) = 0.$$

It follows that

$$s_i = \overline{s}_i \quad \text{for } i = 2, \ldots, m \qquad \text{and} \qquad h_j = (\mathcal{L}_j - \mathcal{L}_i) \cdot \rho.$$

We can conclude that

$$V(\mathbf{e}_0) \cdot V(\rho^i) = \begin{cases} 0 & \text{if } i = 0, \\ (\mathcal{L}_i - \mathcal{L}_0) \cdot \rho & \text{if } i \neq 0, \end{cases}$$

and

$$\pi^*(H) \cdot V(\rho^i) = H \cdot \rho \quad \text{if } H \in \mathrm{Div}(Y).$$

Let us now consider a fiber-curve $V(\rho^\sigma_{ij})$ and integers s_i and h_i for which the following equality holds:

$$\mathbf{e}_i + \mathbf{e}_j - \Big(\sum_{l\neq i,j} h_l \mathbf{e}_l\Big) - \Big(\sum_{t=1}^m s_t p(n_t)\Big) = 0.$$

Because the n_i can be chosen to form a $\mathbb{Z}$-basis for Δ' it is clear that

$$h_l = -1 \quad \text{for } l \neq i, j \qquad \text{and} \qquad s_t = 0 \quad \text{for } t = 1, \ldots, m.$$

This implies that

$$V(\mathbf{e}_0) \cdot V(\rho^\sigma_{ij}) = 1, \quad \text{for all } i, j,$$

and

$$\pi^*(H) \cdot V(\rho^\sigma_{ij}) = 0, \quad \text{for all } H \in \text{Div}(Y). \qquad \square$$

PROPOSITION 2. *The line bundle $\mathcal{L} = a\xi + \pi^*(H)$ is very ample if and only if $a \geqslant 1$ and $a\mathcal{L}_i + H$ is very ample on Y, for $i = 0, \ldots, k$.*

Proof. Recall that a line bundle $\mathcal{L}$ on a non-singular toric variety is very ample if and only if $\mathcal{L} \cdot V(\rho) \geqslant 1$ for all the invariant rational curves $V(\rho)$; see for example [**5**]. Lemma 4 implies that

$$\begin{aligned} \mathcal{L} \cdot V(\rho^i) &= (aV(\mathbf{e}_0) + \pi^*(a\mathcal{L}_0 + H)) \cdot V(\rho^i) \\ &= (a\mathcal{L}_i + H) \cdot \rho, \quad \text{for all } i = 0, \ldots, k, \end{aligned}$$

and

$$\mathcal{L} \cdot V(\rho^\sigma_{ij}) = (aV(\mathbf{e}_0) + \pi^*(a\mathcal{L}_0 + H)) \cdot V(\rho^\sigma_{ij}) = a, \quad \text{for every } i, j.$$

It follows that $\mathcal{L} = a\xi + \pi^*(H)$ is very ample on X if and only if the line bundles $a\mathcal{L}_i + H$ are very ample on Y, that is, $(a\mathcal{L}_i + H) \cdot \rho > 0$ for all $i = 0, \ldots, k$, and $a \geqslant 1$. $\square$

Proof of Theorem 1. Proposition 2 implies that

$$(X, \mathcal{L}) = (\mathbb{P}(\mathcal{L}_0 \oplus \ldots \oplus \mathcal{L}_k), a\xi + \pi^*(H)),$$

where H is a line bundle on the non-singular toric variety Y such that $a\mathcal{L}_i + H$ is very ample for $i = 0, \ldots, k$ and $a > 0$. The fact that $\mathcal{L}|_F \cong \mathcal{O}_{\mathbb{P}^{r-m}}(1)$ for every fiber of the morphism $\pi : X \to Y$ is equivalent to requiring that, for every fiber-curve $V(\rho^\sigma_{ij})$,

$$\begin{aligned} \mathcal{L} \cdot V(\rho^\sigma_{ij}) &= a(\xi \cdot V(\rho^\sigma_{ij})) + \pi^*(H) \cdot V(\rho^\sigma_{ij}) \\ &= a(V(\mathbf{e}_0) + \pi^*(\mathcal{L}_0 + H)) \cdot V(\rho^\sigma_{ij}) = a = 1. \end{aligned}$$

By Proposition 2 the line bundle $\xi + \pi^*(H)$ is very ample on X if and only if the line bundles $\mathcal{L}_i + H$ are very ample on Y. $\square$

REMARK 4. A linear $\mathbb{P}^k$-bundle over a variety of dimension $m \leqslant k$ has dual defect $d = k - m$ (see for example [**18**, 5.2]).

4. *Toric embeddings with degenerate dual variety*

In this section we will characterize toric embeddings $(X, \mathcal{L})$ with positive dual defect.

THEOREM 2. *Let $(X, \mathcal{L})$ be an r-dimensional toric embedding, with positive dual defect d. Then there is a non singular toric variety Y of dimension $\frac{1}{2}(r - d)$*

such that $(X,\mathcal{L})$ *is a toric linear* $\mathbb{P}^{(r+d)/2}$*-bundle over* Y*,*

$$(X,\mathcal{L}) = (\mathbb{P}(\mathcal{L}_0 \oplus \ldots \oplus \mathcal{L}_{(r+d)/2}), \xi + \pi^*(H))$$

where $\mathcal{L}_i$ *and* H *are line bundles over* Y*, such that* $\mathcal{L}_i + H$ *is very ample for every* i*. Moreover,* $r + d \geqslant 4$*.*

The following two lemmas collect some well-known results which will be essential in the proof of Theorem 2.

Lemma 5. *Let* $(X,\mathcal{L})$ *be a* k*-dimensional toric embedding, with positive dual defect* d*. If the Picard group of* X *has rank* 1*, then* $(X,\mathcal{L}) \cong (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$ *and* $d = k > 2$*.*

Proof. The only toric manifold with Picard group of rank 1 is $\mathbb{P}^k$. Hence $(X,\mathcal{L}) = (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(t))$. Because $d > 0$, it must be $t = 1$. In fact the Euler sequence gives $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^k}(t)) = \oplus^{k+1}\mathcal{O}_{\mathbb{P}^k}(t-1)$ and thus

$$c_k(\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^k}(t))) = (k+1)(t-1).$$

Lemma 1 implies that $t = 1$ and thus $d = k > 2$. □

Lemma 6. *Let* $(X,\mathcal{L})$ *be a toric embedding, with positive dual defect. Then there is a flat morphism with connected fibers* $\psi : X \to Y$*, where* Y *and a general fiber* F *are toric varieties.*

Proof. Consider the nef-value morphism ψ_τ, associated to $(X,\mathcal{L})$. Because every nef line bundle is globally generated on a toric manifold, the map ψ_τ is defined by the linear system $|K_X + \tau\mathcal{L}|$. By [**6**, 1.2] both Y and a general fiber are toric. Because $(X,\mathcal{L})$ has positive dual defect, Lemma 2 implies that ψ_τ is a Mori-contraction of fiber type. In [**22**, 2.6] a detailed description of such contractions is given. Let us just recall that in the above case the locus $A \subset X$, where ψ_τ is not an isomorphism, is a closed toric subvariety of dimension equal to the dimension of X. Hence $A = X$ and ψ_τ is a flat morphism. □

Proof of Theorem 2. Let $\psi : X \to Y$ be the flat morphism described in Lemma 6. Because $d > 0$, Lemma 2 implies that, for a general fiber F of the nef-value morphism ψ, the embedding $(F, \mathcal{L}|_F)$ has positive dual defect $d(F, \mathcal{L}|_F) > 0$ and $\mathrm{Pic}(F) \cong \mathbb{Z}$. Because F is toric and non-singular, it follows by Lemma 5 that $(F, \mathcal{L}|_F) \cong (\mathbb{P}^{r-m}, \mathcal{O}_{\mathbb{P}^{r-m}}(1))$. Note that for any fiber Z we have $\mathcal{L}^{r-m} \cdot Z = 1$, since ψ is flat and $\mathcal{L}^{r-m} \cdot F = 1$ for a general fiber F. It follows that Z is irreducible and reduced and that $(Z, \mathcal{L}|_Z) \cong (\mathbb{P}^{r-m}, \mathcal{O}_{\mathbb{P}^{r-m}}(1))$. This proves that $(X,\mathcal{L})$ has the structure of a toric linear $\mathbb{P}^{r-m}$-bundle. Because X is non-singular, the toric variety Y is also non-singular. Theorem 1 gives

$$(X,\mathcal{L}) \cong (\mathbb{P}(\mathcal{L}_0 \oplus \ldots \oplus \mathcal{L}_k), \xi + \pi^*(H))$$

with $\mathcal{L}_i + H$ very ample line bundles. Lemma 2 implies that $\tau = \frac{1}{2}(r+d) + 1$. Moreover, $K_F = K_X|_F = -\big(\frac{1}{2}(r+d) + 1\big)\mathcal{L}|_F$. This shows that $k = \dim(Y) = \frac{1}{2}(r-d)$ and $\dim(F) = \frac{1}{2}(r+d)$. Finally, since $d(F, \mathcal{L}|_F) > 0$, Lemma 5 implies that $\dim(F) = \frac{1}{2}(r+d) \geqslant 2$. Conversely, because $\dim(Y) = \frac{1}{2}(r-d) \leqslant \frac{1}{2}(r+d)$, Remark 4 implies that $d(X,\mathcal{L}) = \frac{1}{2}(r+d) - \frac{1}{2}(r-d) = d$. □

Remark 5. By the projection formula we have

$$\mathbb{P}(H^0(\mathbb{P}(\mathcal{L}_0 \oplus \ldots \oplus \mathcal{L}_{(r+d)/2}), \xi + \pi^*(H))) = \mathbb{P}(H^0(Y, \pi_*(\xi) + H)) = \mathbb{P}\Big(\bigoplus_{i=0}^{(r+d)/2} H^0(\mathcal{L}_i + H)\Big).$$

Corollary 1. *Let $(X, \mathcal{L})$ be an r-dimensional toric embedding, with positive dual defect d.*

(i) *If $r = 2$ then $d = 2$ and $(X, \mathcal{L}) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$.*
(ii) *If $d = r$ then $(X, \mathcal{L}) = (\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$.*
(iii) *If $d = r - 1$ then $(X, \mathcal{L})$ is a $\mathbb{P}^{r-1}$-bundle over $\mathbb{P}^1$.*

5. *Defect polytopes*

In this section we will use geometry to investigate combinatorial properties of polytopes. Consider the combinatorial invariant

$$c(P) = \sum_{F \subseteq P} (-1)^{\operatorname{codim}(F)} (\dim(F) + 1)! \operatorname{Vol}(F).$$

Example 3. Let Δ_r be the r-dimensional simplex, corresponding to $\mathbb{P}^r$ embedded in itself. The equality

$$\sum_{k=0}^{r} (-1)^{r-k} (k+1)! \binom{r+1}{k+1} \frac{1}{k!} = 0$$

yields

$$c(\Delta_r) := \sum_{k=0}^{r} (-1)^{r-k} (k+1)! \Big(\sum_{F \in \Delta_r(k)} \operatorname{Vol}(F) \Big) = 0.$$

Example 4. Let C_3 be the 3-dimensional hypercube, corresponding to the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7$. Then

$$c(C_3) = 4! - 6 \cdot 3! + 12 \cdot 2 - 8 = 4.$$

This invariant was introduced in [**12**], where it is proven that $c(P)$ is non-negative for every simple polytope. Proposition 3 provides an alternative geometrical proof for Delzant polytopes.

Proposition 3. *Let P be a Delzant polytope. Let $c_r(\mathcal{P}^1(\mathcal{O}_P(1)))$ be the top Chern class of the vector bundle $\mathcal{P}^1(\mathcal{O}_P(1))$. Then*

$$c(P) = c_r(\mathcal{P}^1(\mathcal{O}_P(1))).$$

Proof. In order to simplify notation we will write Ω^1_P for the cotangent bundle $\Omega^1_{X(P)}$, and $\mathcal{O}_P$ for the structural sheaf $\mathcal{O}_{X(P)}$. From the exact sequence

$$0 \to \Omega^1_P(1) \to \mathcal{P}^1(\mathcal{O}_P(1)) \to \mathcal{O}_P(1) \to 0,$$

one sees that (see [**13**, p. 150])

$$c_r(\mathcal{P}^1(\mathcal{O}_P(1))) = \sum_{k=0}^{r}(r+1-k)c_k(\Omega_P^1)\cdot\mathcal{O}_P(1)^{r-k}.$$

Consider the generalized Euler sequence, [**1**, 12.1],

$$0 \to \Omega_P^1 \to \bigoplus_{\xi\in\Delta_P(1)} \mathcal{O}_P(-V(\xi)) \to {\mathcal{O}_P}^{|M\cap P(r-1)|-r} \to 0.$$

The Chow groups $A_k(X(P))$ are generated by the classes $[V(\sigma)]$, for $\sigma\in\Delta(r-k)$, [**9**, 5.1]. The total Chern class of Ω_P^1 is then (see [**4**, 11.4])

$$c(\Omega_P^1) = \prod_{\xi\in\Delta_P(1)}(1-[V(\xi)])$$

and

$$\begin{aligned} c_k(\Omega_P^1) &= (-1)^k \sum_{\xi_1\neq\ldots\neq\xi_k\in\Delta(1)} [V(\xi_1)]\cdot\ldots\cdot[V(\xi_k)] \\ &= (-1)^k \sum_{\sigma\in\Delta_P(k)}[V(F_\sigma)] = \sum_{F_\sigma\in P(r-k)}[V(F_\sigma)]. \end{aligned}$$

Moreover [**9**, 5.3],

$$\mathrm{Vol}(F_\sigma) = \frac{\deg(\mathcal{O}_P(1)^k\cap[V(F_\sigma)])}{k!}. \tag{5.1}$$

It follows that

$$c_r(\mathcal{P}^1(\mathcal{O}_P(1))) = \sum_{k=0}^{r}(r+1-k)(-1)^k\Bigg(\sum_{F_\sigma\in P(r-k)}[V(F_\sigma)]\Bigg)\cdot\mathcal{O}_P(1)^{r-k},$$

and thus

$$\begin{aligned} c_r(\mathcal{P}^1(\mathcal{O}_P(1))) &= \sum_{k=0}^{r}(r+1-k)(-1)^k(r-k)!\Bigg(\sum_{F_\sigma\in P(r-k)}\mathrm{Vol}(F_\sigma)\Bigg) \\ &= \sum_{k=0}^{r}(-1)^{r-k}(k+1)!\Bigg(\sum_{F\in P(k)}\mathrm{Vol}(F)\Bigg) = c(P). \end{aligned}$$

□

The non-negativity of the integer $c(P)$ is a simple consequence of this identification.

Corollary 2. *If P is a* (*integral convex*) *Delzant polytope, then* $c(P)\geqslant 0$.

Proof. The fact that the line bundle $\mathcal{O}_P(1)$ defines an embedding implies that the map

$$H^0(X(P),\mathcal{O}_P(1)) \to H^0(x,\mathcal{O}_P(1)\otimes\mathcal{O}_{X(P)}/m_x^2)$$

is surjective at every point x. It follows that the vector bundle map (2.1) in § 2 is surjective on stalks and hence it is a surjection of vector bundles. This means that the global sections of the first jet bundle are generated by $H^0(X(P),\mathcal{O}_P(1))$. Any globally generated vector bundle has non-negative top Chern class, [**10**, 12.1.7], which proves that $c(P)\geqslant 0$. □

The classification of defect polytopes

We will classify Delzant polytopes minimal with respect to $c(P)$.

Definition 5. A polytope is said to be a *defect polytope* if it is Delzant and $c(P) = 0$.

Example 5. In dimension 1 there is no defect polytope. In dimension 2 there are no defect polytopes other than the 2-dimensional simplex. This can be checked using Pick's formula:

$$|P \cap N| = \text{Area}(P) + \tfrac{1}{2} \cdot \text{Perimeter}(P) + 1. \tag{5.2}$$

If $r = 2$ then

$$\begin{aligned} c(P) &= 6 \cdot \text{Area}(P) - 2 \cdot \text{Perimeter}(P) + |P(0)| \\ &= 6|P \cap N| - 5 \cdot \text{Perimeter}(P) + |P(0)| - 6 \geqslant 0. \end{aligned}$$

Because $|P \cap N| \geqslant \text{Perimeter}(P)$ and $|P \cap N| \geqslant 4$, we have $|P(0)| \geqslant 4$ unless P is a simplex in which case $|P \cap N| = |P(0)| = 3$ and thus $c(P) = 0$.

In higher dimensions this is no longer the case. See, for example, the polytope in Figure 1 or in Example 3.

Definition 6. Consider m-dimensional polytopes, $P_0, \ldots, P_k$, having the same combinatorial type, $P_i = \text{Conv}\{m_1^i, \ldots, m_s^i\}$. Let $(e_1, \ldots, e_k)$ be a basis for $\mathbb{Z}^k$ and $e_0 = 0$. The *projective join* of $P_0, \ldots, P_k$ is the $(m+k)$-dimensional polytope defined as

$$\mathbb{P}(P_0, \ldots, P_k) = \text{Conv}\{(m_j^i, e_i)\}_{j=1,\ldots,s, i=0,\ldots,k}.$$

Example 6. Figure 2 shows the projective join of the three 1-dimensional polytopes P, Q and S of length 2, 2 and 3 respectively. The corresponding toric embedding is the toric linear $\mathbb{P}^2$-bundle

$$(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)), \xi + \pi^*(\mathcal{O}_{\mathbb{P}^1}(1)).$$

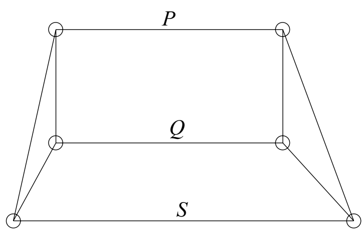


Figure 2. $\mathbb{P}(P, Q, S)$.

Example 7. A standard r-dimensional simplex is given by $\mathbb{P}(\overbrace{m, \ldots, m}^{r})$, for any 0-dimensional polytope m.

Example 8. The 2-dimensional projective joins are the standard simplex and the 4-polygons with two parallel edges and two edges of length 1.

EXAMPLE 9. The product of a k-dimensional standard simplex and an m-dimensional polytope Q is given by the projective join $\mathbb{P}(\underbrace{Q,\dots,Q}_{k+1})$.

THEOREM 3. *An r-dimensional Delzant polytope P is a defect polytope if and only if there exist a positive integer k and $(r-k)$-dimensional Delzant polytopes, $P_0,\dots,P_k$ having the same combinatorial type, such that:*
(i) $\max(2, \frac{1}{2}(r+1)) \leqslant k \leqslant r$, *and*
(ii) $P = \mathbb{P}(P_0,\dots,P_k)$.

Proof. Note that P is a defect polytope if and only if $c(P) = 0$. By Lemma 1 and Proposition 3, P is a defect polytope if and only if the toric embedding $(X(P), \mathcal{O}_P(1))$ has a positive dual defect. Theorem 2 implies that

$$(X(P), \mathcal{O}_P(1)) = (\mathbb{P}(\mathcal{L}_0 \oplus \dots \oplus \mathcal{L}_k), \xi + \pi^*(H)).$$

The $\mathcal{L}_i$ are line bundles on a non-singular toric variety Y, defined by a fan Δ' of dimension $r-k$. The polytope P is defined by the line bundle

$$\mathcal{O}_P(1) = \xi + \pi^*(H).$$

By Theorem 1, the line bundles $\mathcal{L}_i + H$ are very ample and hence define Delzant polytopes $P_0,\dots,P_k$. The polytope P is constructed dually to the fan of $\mathbb{P}(\mathcal{L}_0 \oplus \dots \oplus \mathcal{L}_k)$, described in the previous section. In particular, the lengths of the edges are given by the intersection numbers of the line bundle $\mathcal{O}_P(1)$ with the invariant curves, as described in Theorem 1. If $P_i = \mathrm{Conv}\{m_1^i,\dots,m_s^i\}$ then we have the following correspondences.

Dimension 0. For every $\sigma \in \Delta'\big(\frac{1}{2}(r-d)\big)$ the r-dimensional cone $p(\sigma) + \sigma^i$ corresponds to a vertex m_σ^i of P.

Dimension 1. Every $(r-1)$-dimensional cone of the form

$$\rho^i = (p(\sigma_1) + \sigma^i) \cap (p(\sigma_2) + \sigma^i)$$

corresponds to the edge connecting $m_{\sigma_1}^i$ and $m_{\sigma_2}^i$. Every $(r-1)$-dimensional cone of the form ρ_{ij}^σ corresponds to the edge connecting m_σ^i and m_σ^j.

Dimension $r-k$. The k-dimensional cones $p(0) + \sigma^i$ correspond to the $(r-k)$-dimensional polytope P_i.

Dimension k. Every $(r-k)$-dimensional cone of the form $p(\sigma)$ corresponds to a standard simplex Δ_k.

It is straightforward to check that

$$P = \mathbb{P}(P_0,\dots,P_k) = \mathrm{Conv}\{m_\sigma^i\}_{\sigma\in\Delta'(r-k), i=0,\dots,k}. \qquad \square$$

COROLLARY 3. *Let P be a defect polytope. Then the following hold:*
(i) *if* $\dim(P) = 2$ *then P is the standard simplex* $\mathbb{P}(m,m,m)$, *as observed earlier;*
(ii) *if* $\dim(P) = 3$ *then P is the standard simplex* $\mathbb{P}(m,m,m,m)$ *or* $\mathbb{P}(P_0,P_1,P_2)$, *where P_0, P_1 and P_2 are* 1-*dimensional polytopes;*
(iii) *if* $\dim(P) = 4$ *then P is the standard simplex* $\mathbb{P}(m,m,m,m,m)$ *or* $\mathbb{P}(P_0,P_1,P_2,P_3)$, *where P_0, P_1, P_2 and P_3 are* 1-*dimensional.*

One could give a similar explicit classification for any dimension.

6. *Final remarks and conjectures*

The degree of the dual variety

For any polytope (not necessarily Delzant), $P = \mathrm{Conv}(A)$, $c(P)$ corresponds to the degree of a rational homogeneous function, D_P, called the *regular A-determinant*. In [**12**, 11,1.6] it is proven that $c(P)$ is non-negative for any simple polytope. Recall that an r-dimensional polytope P is simple if every vertex is incident to exactly r edges. Note that when P is simple, but not Delzant, the invariant $c(P)$ no longer coincides with $c_r(\mathcal{P}^1(\mathcal{O}_P(1)))$ and thus our methods fail to show the non-negativity.

Several numerical experiments though suggest that the invariant $c(P)$ should be non-negative for any polytope. The following example shows that even when the function D_P is not polynomial, the invariant $c(P)$ can be non-negative.

Example 10. Let $e_1, \ldots, e_6$ be a $\mathbb{Z}$-basis for $\mathbb{R}^6$. Consider the 5-dimensional hypersimplex

$$\Delta(3,6) = \mathrm{Conv}\{(e_i + e_j + e_k) \mid 1 \leqslant i < j < k \leqslant k\}.$$

The rational function $D_{\Delta(3,6)}$ is not polynomial, [**12**, 11, 2.5]. Let us compute the invariant $c(\Delta(3,6))$. The number of faces of $\Delta(3,6)$ are described in [**11**, 2.5]:

k	0	1	2	3	4	5
$\lvert\Delta(3,6)(k)\rvert$	20	90	120	60	12	1

Every face is a hypersimplex given by disjoint partitions of the set $\{1, \ldots, n\}$. Using the formula, [**8**],

$$\mathrm{Vol}(\Delta(k,n)) = \frac{A_{n,k}}{(n-1)!}$$

where the numbers $A_{n,k}$ are the Eulerian numbers, we get

$$c(\Delta(3,6)) = 352.$$

We make the following conjecture.

Conjecture 4. The invariant $c(P)$ is non-negative for any integral polytope.

When P is simple the multiplicities of the faces have to be taken into account leading to considering a modified combinatorial invariant. Moreover, since $X(P)$ is no longer smooth, the arguments are more involved and have to deal with orbi-bundles on orbifolds. This line of results will be presented in a sequel paper.

If $X(P)$ is non-singular and has dual defect equal to zero, then the dual variety is a hypersurface defined by a polynomial of degree equal to the top Chern class of the first jet bundle $\mathcal{P}^1(\mathcal{O}_{X(P)}(1))$. Hence

$$\deg(X(P)^*) = c(P).$$

This is the way this integer was defined in [**12**].

In order to avoid confusion Zak introduced the term *codegree* and defined

$$\operatorname{codeg} X = \deg(X^*).$$

A classification of non-singular algebraic varieties of a given codegree is a very classical problem, which is still open for codegree greater than or equal to 4.

Example 11. Consider $\Delta_2(2)$, the dilated simplex with edge-length equal to 2. One computes that $c(\Delta_2(2)) = 3$. The corresponding embedding is the Veronese embedding of degree 2 of $\mathbb{P}^2$ in $\mathbb{P}^5$. It is well known that the dual variety is a hypersurface of degree 3.

Example 12. Let P_1 be the 2-dimensional polytope with two parallel edges of lengths 1 and 2 and the other two edges of length 1. One has $c(P_1) = 3$. The corresponding embedding, which is the embedding of the blow-up of $\mathbb{P}^2$ in one point into $\mathbb{P}^4$, has codegree 3.

Using the classification up to codegree 3 (see [**23**]), we can conclude that the only Delzant polytopes having positive $c(P) \leqslant 3$ have dimension 2 and correspond to the two examples given above.

Lemma 7. *Let P be a Delzant polytope. Then $c(P) \geqslant 3$ unless P is as in Theorem 3, $P = \Delta_2(2)$ or $P = P_1$.*

Proof. These are the only linearly normal toric embeddings listed by Zak in [**23**, 5.2]. □

Other invariants

The invariant $c(P)$ may not seem to be the most natural object to consider. For $t \in \mathbb{Z}$ define

$$c_t(P) = \sum_{F \subseteq P} (-1)^{\operatorname{codim}(F)} (\dim(F) + t)! \operatorname{Vol}(F).$$

The sum runs over all non-empty faces, including the polytope P. The invariant $c_0(P)$ is certainly combinatorially more attractive, but unfortunately negative in several examples. Even for standard simplices Δ_r one can easily see that

$$\begin{aligned} c_0(\Delta_r) &= (-1)^r, \\ c_i(\Delta_r) &= 0 \quad \text{for } i = 1, \dots, r, \\ c_i(\Delta_r) &> 0 \quad \text{for } i \geqslant r+1. \end{aligned}$$

The invariant $c(P) = c_1(P)$ is in fact the first for which one could have hoped to prove a non-negativity result. Moreover, it is a corollary of Corollary 2 that every $c_t(P)$ is non-negative for $t \geqslant 1$.

Corollary 4. *If P is Delzant and $t \geqslant 1$ then the invariant $c_t(P) \geqslant 0$.*

Proof. Proceed by induction on t. The assertion is true for $t = 1$, by Corollary 2. Consider the polytope $P' = P \times [0, m]$. We have

$$c_t(P') = \sum_{F \subset P} [(-1)^{\operatorname{codim}(F)-1}(\dim(F) + t)! \operatorname{Vol}(F)] \\ + m[(-1)^{\operatorname{codim}(F)}(\dim(F) + t + 1)! \operatorname{Vol}(F).$$

In the limit, as m goes to infinity, the dominant term is $mc_{t+1}(P)$. It follows that $c_{t+1}(P)$ is non-negative if $c_t(P)$ is non-negative. □

If F is a face of an r-dimensional polytope P, let

$$\mathbb{Z}^F = (\text{Affine span of } F) \cap \mathbb{Z}^r.$$

Consider the following function of n:

$$f(P, n) = \sum_0^r (-n)^{r-k}(k + 1)! \sum_{F \in P(k)} |\mathbb{Z}^F \cap nF| = \sum_0^r d_i(P)n^i.$$

If $\dim(F) = k$ then the coefficient of the n^k term in $|\mathbb{Z}^F \cap nF|$ is $\operatorname{Vol}(F)$. This implies that

$$c(P) = d_r(P).$$

Because of the equality (5.1) in § 5 the polynomial $f(P, n)$ has integer coefficients. Some numerical experiments provide evidence that every coefficient of this polynomial is non-negative.

Conjecture 5. The function $f(P, n)$ is a polynomial in n with non-negative integer coefficients.

Equivalently, we conjecture that, for a toric embedding $(X(P), \mathcal{O}_P(1))$,

$$f(P, n) = \sum_0^r (-n)^{r-k}(k + 1)! \sum_{F \in P(k)} \dim(H^0(X(P), \mathcal{O}_F(n)))$$

is a polynomial in n with non-negative integer coefficients. Because the line bundle $\mathcal{O}_F(1)$ is very ample for each face F this would mean that

$$f(P, n) = \sum_0^r (-n)^{r-k}(k + 1)! \sum_{F \in P(k)} \chi(\mathcal{O}_F(n)) \tag{6.1}$$

is a polynomial in n with non-negative integer coefficients.

One could give one additional interpretation of (6.1). Let C_n^m denote the m-dimensional cube with edges of length n. Geometrically, it corresponds to the Segre embedding $(1, \dots, 1)$ of m copies of the nth Veronese embedding of $\mathbb{P}^1$:

$$V_n^m := \mathbb{P}^1 \times \dots \times \mathbb{P}^1 \to \mathbb{P}^{(n+1)^m - 1}.$$

For any face F consider the polytope $F \times C_n^m$ associated to the variety $V(F) \times V_n^m$. Let π_i be the projection onto the ith factor, for $i = 1, 2$. The polytope $F \times C_n^m$ is defined by the invertible sheaf

$$\pi_1^*(\mathcal{O}_F(1)) + \pi_2^*(\mathcal{O}_{V_n^m}(1)) = \mathcal{O}_{V(F) \times V_n^m}(1).$$

We conjecture that

$$f(P,n) = \sum_F (-1)^{\operatorname{codim}(F)} (\dim(F))! \chi(\mathcal{O}_{V(F)\times V_n^m}(1))$$

is a polynomial in n with non-negative integer coefficients.

For standard simplices one can show the non-negativity of the constant term because

$$d_0(\Delta_r) = c_{r+1}(\Delta_r).$$

This identity is easily shown by using the Chu–Vandermonde identity [**21**].

We were not able to find examples of polytopes with $c(P) = 0$ and which do not have the structure of a projective join.

The use of duality and intersection estimates relies strongly on the fact that the variety is non-singular. A purely combinatorial proof of the results contained in this paper could possibly open the door to a generalization.

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