ARTICLE IN PRESS

Journal of Pure and Applied Algebra ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at SciVerse ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

The variety of bad zero-schemes

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ARTICLE INFO

Article history:
Received 7 April 2011
Received in revised form 9 October 2011
Available online xxxx
Communicated by A.V. Geramita

MSC: 14C20; 14J25

ABSTRACT

The locus of reduced bad zero-schemes, $\mathfrak{B}_0 \subset X^{[b_0]}$, for a linear system $|V|$ on a non singular, n-dimensional, algebraic variety X is defined. The pairs (X, V) for which $\mathfrak{B}_0$ has the maximal dimension, $nb_0 - 1$, are characterized. For $n = 2$, non-reduced bad zero-schemes are also discussed.

1. Introduction

A bad zero-scheme for a linear system $\mathcal{S}$ on a smooth projective variety X is a zero-scheme whose scheme-theoretical containment forces all elements in $\mathcal{S}$ to be reducible or non reduced. Bad zero-schemes were introduced and studied in [3,5,4].

If $\mathcal{S}$ is somewhat positive, ample and free for example, and $\dim(X) \geq 2$, it is known that the locus of bad zero-schemes of length one is a finite set, when non-empty. This locus becomes an interesting object of study for higher lengths. For example, let $\mathcal{S}$ be a very ample linear system on a smooth projective variety X, and assume that there exist bad zero-schemes of length two. Then a classical result, see [1, Theorem 1.7.9], implies that X is a surface and the locus swept out by such zero-schemes is the union of the second symmetric products of lines contained in X. In particular, when X is a linear scroll, this locus has codimension one in the Hilbert scheme of zero-schemes of length two.

The aim of this paper is to properly define the locus of bad zero-schemes of minimal length for an ample and free linear system and study the case when it has maximal dimension, i.e. it is a codimension one subset of the Hilbert scheme of zero-schemes of the given length. As the Hilbert scheme of points on varieties of dimension $n \geq 3$ is a wild object, and far from being well understood, we first define $\mathfrak{B}_0$, the locus of reduced bad zero-schemes, in the closure of the open component of the Hilbert scheme parameterizing reduced zero-schemes, see Section 3. If $\mathfrak{B}_0$ has maximal dimension, Theorem 3.4 shows that $\dim(X) = 2$, bad zero-schemes of minimal length impose the maximum number of independent conditions on $\mathcal{S}$, and the locus of reducible elements of $\mathcal{S}$ has codimension 1 and thus it is a union of components of the discriminant. Moreover, the linear system of elements of $\mathcal{S}$ containing a general bad zero-scheme of minimal length has a fixed component.

When the linear system $\mathcal{S}$, is very ample, essentially because the discriminant variety is irreducible, the assumption that $\mathfrak{B}_0$ has maximal dimension is very restrictive. Corollary 3.6 gives a complete characterization. There are two possible classes of embeddings of such type, the 2-Veronese embedding and its isomorphic projections or scrolls, both in dimension 2.

On the other hand, when $\mathcal{S}$ is only ample and free, the situation turns out to be very different. Examples 4 and 5 provide a large class of surfaces with $\mathfrak{B}_0$ of maximal dimension, which indicates that a classification is out of reach.

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doi:10.1016/j.jpaa.2011.12.008

As the Hilbert scheme of points on surfaces is well understood, and given that surfaces play a relevant role in the above results, we devote Section 4 to the study of the more general locus $\mathfrak{B}$ of bad zero-schemes of minimal length (not necessarily reduced). The characterization for $\mathcal{S}$ very ample seen above still holds in the non reduced case, see Corollary 4.4.

In Section 5 a different invariant, s, is introduced as a measure of the degrees of freedom in constructing a bad zero-scheme. We also relate $\mathfrak{B}_0$ having maximal dimension with s reaching its allowed maximum.

A library of examples is provided in Section 6, which shows the complexity of the general situation. In particular, Example 4 provides a counterexample to [3, Conjecture 1].

2. Notation and background

Throughout this article X denotes a smooth, connected, projective variety of dimension $n \geq 2$, defined over the complex field $\mathbb{C}$. Its structure sheaf is denoted by $\mathcal{O}_X$. Cartier divisors, their associated line bundles and the invertible sheaves of their holomorphic sections are used with no distinction. Mostly additive notation is used for their group.

Let $X^{(t)}$ be the t-th symmetric power of X. The Hilbert scheme of zero-schemes of length t on X and its stratum of reduced zero-schemes will be denoted, respectively, by $X^{[t]}$ and $X_0^{[t]}$. We denote by $X^{[t]}_{1+r,1,1,\dots,1}$, for $0 \leq r \leq \min\{t-1, n\}$, the set of zero subschemes ξ of length t such that $\mathrm{Supp}(\xi) = \{x_1, x_2, \dots, x_{t-r}\}$, and $\mathcal{I}_\xi = \mathfrak{a} \cdot \mathfrak{m}_2 \cdot \ \dots \ \cdot \mathfrak{m}_{t-r}$ where $\mathfrak{m}_i = \mathfrak{m}_{x_i}$ is the maximal ideal of $\mathcal{O}_{X,x_i}$ and $\mathfrak{a} = (u_iu_j, u_{r+1}, \dots, u_n \mid 1 \leq i \leq j \leq r)$, $u_1, \dots u_n$ denoting local coordinates at x_1. Given a zero-scheme $\xi \in X_0^{[t]}$, we sometimes identify ξ and its support $\mathrm{Supp}(\xi)$; for example we write $x \in \xi$ to mean $x \in \mathrm{Supp}(\xi)$.

For any coherent sheaf $\mathcal{F}$ on X, $h^i(X, \mathcal{F})$ is the complex dimension of $H^i(X, \mathcal{F})$. Let L be an ample line bundle on X spanned by a vector subspace $V \subseteq H^0(X, L)$. We denote by $|V|$ the linear system associated with V; by $|V \otimes \mathcal{I}_Z|$, with a slight abuse of notation, the linear system of divisors in $|V|$ which contain, scheme-theoretically, the subscheme Z of X; and by φ_V the morphism defined by V. If $V = H^0(X, L)$ we write L instead of V in all of the above. By a general element of a possibly reducible subvariety we mean the general element of a maximal dimensional irreducible component.

Definition 2.1. (1) The *t-th bad locus* of (X, V), for $t \geq 1$ is:

$$\mathcal{B}_t(X, V) = \{\xi \in X^{[t]} \mid |V \otimes \mathcal{I}_\xi| \neq \emptyset \text{ and } \forall D \in |V \otimes \mathcal{I}_\xi|,\ D \text{ is reducible or non-reduced}\}.$$

(2) The *reduced t-th bad locus* of (X, V), for $t \geq 1$ is:

$$\mathcal{B}^0_t(X, V) = \mathcal{B}_t(X, V) \cap X_0^{[t]}.$$

(3) The *b-index* of the pair (X, V) is:

$$b(X, V) = \begin{cases} \infty & \text{if } \mathcal{B}_t(X, V) = \emptyset \text{ for every } t \geq 1 \\ \min\{t \mid \mathcal{B}_t(X, V) \neq \emptyset\} & \text{otherwise.} \end{cases}$$

(4) The *reduced b-index* of the pair (X, V) is:

$$b_0(X, V) = \begin{cases} \infty & \text{if } \mathcal{B}^0_t(X, V) = \emptyset \text{ for every } t \geq 1 \\ \min\{t \mid \mathcal{B}^0_t(X, V) \neq \emptyset\} & \text{otherwise.} \end{cases}$$

We write $\mathcal{B}_t(X, L)$ and $\mathcal{B}^0_t(X, L)$ if $V = H^0(X, L)$, and b and b_0, respectively, for $b(X, V)$ and $b_0(X, V)$, when the pair (X, V) is clear from the context. The above definitions imply immediately that $b \leq b_0$. Although $\mathcal{B}_t(X, V)$ is often closed, it is not always the case, as several examples will show.

3. The locus of reduced bad zero-schemes

Consider the following incidence diagram:

$$\mathfrak{I}_0 = \overline{\{(\xi, D) | \xi \in \mathcal{B}^0_{b_0}(X, V), D \in |V \otimes \mathcal{I}_\xi|\}} \subset \overline{X_0^{[b_0]}} \times |V|$$

where $\overline{X_0^{[b_0]}}$ is the closure of $X_0^{[b_0]}$ in the Hilbert scheme $X^{[b_0]}$.

$$\begin{array}{ccc} & \mathfrak{I}_0 & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \overline{X_0^{[b_0]}} & & |V| \end{array} \tag{1}$$

Definition 3.1. The image of π_1 in diagram (1) will be called the *locus of reduced bad zero-schemes* of (X, V) and will be denoted by $\mathfrak{B}_0$. Observe that $\mathfrak{B}_0 = \overline{\mathcal{B}^0_{b_0}(X, V)}$.

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Let $R(V)$ be the locus of reducible elements of the linear system $|V|$. Note that $R(V) \subseteq \mathcal{D}(V) \subset |V|$ where $\mathcal{D}(V)$ is the discriminant of (X, V), and note that $Im(\pi_2) \subseteq R(V)$. Recall that $\dim(\mathcal{D}(V)) \leq \dim(|V|) - 1$, see [7, Proposition 1.5], and that generically we have equality. One can then rewrite diagram (1) as follows:

$$\begin{array}{ccc} & \mathfrak{I}_0 & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathfrak{B}_0 & & R(V) \end{array} \tag{2}$$

Proposition 3.2. *Let $V \subseteq H^0(X, L)$ be a subspace spanning the ample line bundle L. Assume that for a general $\xi \in \mathfrak{B}_0$ it is $\dim(|V \otimes \mathcal{I}_\xi|) = \dim(|V|) - k$. Then*

(1) $\dim(\mathfrak{B}_0) \leq nb_0 - 1$.
(2) $\dim(R(V)) \geq (\dim(|V|) - 1) - \big((nb_0 - 1) - \dim(\mathfrak{B}_0)\big) + (b_0 - k)$.

Proof. As $\mathfrak{B}_0 \subseteq \overline{X_0^{[b_0]}}$, it is $\dim(\mathfrak{B}_0) \leq nb_0$. Consider the above diagram (2): let ξ be a general element in $\mathfrak{B}_0$ and let D be a general element in the image of π_2. It is

$$\dim(\mathfrak{B}_0) + \dim(\pi_1^{-1}(\xi)) = \dim(\mathfrak{I}_0) = \dim(Im(\pi_2)) + \dim(\pi_2^{-1}(D)).$$

Note that $\dim(\pi_1^{-1}(\xi)) = \dim(|V \otimes \mathcal{I}_\xi|) \geq \dim(|V|) - b_0$. On the other hand, $\dim(\pi_2^{-1}(D)) \leq b_0(n-1)$, as $\dim(D) = n-1$ and the general ξ such that $\pi_2(\xi, D) = D$, is reduced. Therefore

$$\dim(\mathfrak{B}_0) + \dim(|V|) - k \leq \dim(R(V)) + b_0(n-1),$$

which gives inequality (2). Moreover, if $\dim(\mathfrak{B}_0) = nb_0$, as $k \leq b_0$, it follows that

$$nb_0 + \dim(|V|) - b_0 \leq \dim(R(V)) + nb_0 - b_0.$$

This implies $R(V) = |V|$, which is impossible because the general element of $|V|$ is irreducible, being L ample and spanned by V. This proves (1). □

Remark 3.3. If $b_0 = 1$, the upper bound in Proposition 3.2, (1), is never achieved, as shown in [3, Corollary 2].

The inequality $\dim(\mathfrak{B}_0) \leq nb_0 - 1$ is sharp, as the following classical examples show.

Example 1. For $X = \mathbb{P}^n$ and $V = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$, $b_0 = \binom{n+1}{2}$, and every reduced bad zero-scheme is contained in a hyperplane. Let $R(V) \subset |V|$ be the subvariety of reducible hyperquadrics. It is $\dim(R(V)) = 2n$. Consider the diagram (2). The fiber of the map π_1 at the general point $\xi = (p_1, \ldots, p_{b_0})$ given by b_0 points generating a given hyperplane Π (and not lying on any of its quadrics) has dimension n. Indeed $|V \otimes \mathcal{I}_\xi| = \Pi + |\mathcal{O}_{\mathbb{P}^n}(1)|$. The general fiber of π_2 is the preimage of the union of two distinct hyperplanes and thus has dimension $b_0(n-1)$. It follows that

$$\dim(\mathfrak{B}_0) = 2n + b_0(n-1) - n = \frac{n}{2}(n^2 + 1).$$

Notice that $\dim(\mathfrak{B}_0)$ achieves the upper bound $nb_0 - 1$ if and only if $n = 2$.

Example 2. An n-dimensional scroll over a smooth curve has $b_0 = n$ and $\dim(\mathfrak{B}_0) = n^2 - n + 1$. Again $\dim(\mathfrak{B}_0)$ achieves the upper bound $nb_0 - 1$ if and only if $n = 2$.

The following result shows that the extreme case $\dim(\mathfrak{B}_0) = nb_0 - 1$, as the above examples suggest, is indeed achieved only in the surface case for $b_0 \geq 2$. Moreover, if the linear system is very ample, the surfaces achieving the upper bound are exactly the ones in Examples 1 and 2. It is also shown that $\mathfrak{B}_0$ having maximal dimension forces bad reduced zero-schemes to impose independent linear conditions on the linear system $|V|$ and a component of maximal dimension of $\mathcal{D}(V)$ to consist of reducible divisors.

Theorem 3.4. *If $\dim(\mathfrak{B}_0) = nb_0 - 1$ then:*

(1) $\dim(R(V)) = \dim(\mathcal{D}(V)) = \dim(|V|) - 1$;
(2) $n = 2$;
(3) *for a general $\xi \in \mathfrak{B}_0$ it is $\dim(V \otimes \mathcal{I}_\xi) = \dim(V) - b_0$;*
(4) *if $|V|$ is very ample then $R(V) = \mathcal{D}(V)$ and $b \leq 3$;*
(5) $\dim\,(\pi_2^{-1}(D)) = b_0$ *for a generic $D \in Im(\pi_2)$.*

Proof. (1) The hypothesis $\dim(\mathfrak{B}_0) = nb_0 - 1$ and Proposition 3.2, imply

$$\dim(R(V)) \geq (\dim(|V|) - 1) + (b_0 - k).$$

Because $k \leq b_0$ and $\dim(R(V)) \leq \dim(\mathcal{D}(V)) \leq \dim(|V|) - 1$, it follows that $b_0 = k$ and $\dim(R(V)) = \dim(\mathcal{D}(V)) = \dim(|V|) - 1$.

(2) The set of singularities of every singular element of a general pencil $\mathcal{P} \subset |V|$ is a finite set, according to [7, Corollary 2.5]. On the other hand, because of (1), $\mathcal{P}$ contains an element $D \in R(V)$. Hence, $\dim(\mathrm{Sing}(D)) \geq n - 2$. Therefore $n = 2$.

(3) It follows from the fact that $b_0 = k$ as shown in (1).

(4) If $|V|$ is very ample, note that $(X, L) \neq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, hence $\mathcal{D}(V)$ is irreducible and its general element has a single ordinary quadratic singularity. Thus (1) implies that $\mathcal{D}(V) = R(V)$. It follows that the presence of a singularity on a divisor $D \in |V|$ forces D to be reducible. As imposing an ordinary quadratic singularity requires $n + 1$ linear conditions it is $b \leq n + 1 = 3$.

(5) Our hypothesis, (1)–(3), and diagram (2) give

$$(\dim(|V|) - 1) + \dim(\pi_2^{-1}(D)) = 2b_0 - 1 + \dim(|V|) - b_0,$$

for a generic $D \in Im(\pi_2)$. □

The following Lemma is essentially due to Zak. We include it here for clarity and completeness.

Lemma 3.5. *Let $|V|$ be a very ample linear system on a smooth surface S such that $R(V) = \mathcal{D}(V) \neq \emptyset$. Then either $S = \mathbb{P}^2$ and $V \subseteq H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ or $|V|$ embeds S as a smooth scroll.*

Proof. Let H be the general element of a Lefschetz pencil of hyperplane sections of S. As $\mathcal{D}(V) = R(V)$, all singular elements of the pencil are reducible curves. It follows that all vanishing cycles on H are homologous to zero on H. Then the assertion follows from [8]. □

Corollary 3.6. *Let $|V|$ be a very ample linear system on X and assume $\dim(\mathfrak{B}_0) = nb_0 - 1$. Then $n = 2$ and either $b_0 = 3$, $X = \mathbb{P}^2$ and $V \subseteq H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)))$ or $b_0 = 2$ and $|V|$ embeds X as a scroll over a smooth curve.*

Proof. Theorem 3.4 gives $n = 2$ and $R(V) = \mathcal{D}(V)$. Then the assertion follows from Lemma 3.5. □

Notice that $b = b_0$ in both cases of Corollary 3.6, so that (4) in the statement of Theorem 3.4 holds with b_0 in place of b; moreover, for every $\xi \in \mathfrak{B}_0$, the linear system $|V \otimes \mathcal{I}_\xi|$ has a fixed component. This generalizes to ample and free linear systems as a simple consequence of $n = 2$.

Corollary 3.7. *If $\dim(\mathfrak{B}_0) = 2b_0 - 1$ then, for a general $\xi \in \mathfrak{B}_0$, $|V \otimes \mathcal{I}_\xi|$ has a fixed component.*

Proof. If $|V \otimes \mathcal{I}_\xi|$ has no fixed components then its base locus is finite and [4, Proposition 17, (iii)] implies that ξ cannot be reduced. □

4. The locus of bad zero-schemes on surfaces

Motivated by the results in Section 3, where it is shown that the maximum dimension of the locus of reduced bad zero-schemes is achieved only in dimension two, we will define and study the locus of bad zero-schemes, not necessarily reduced, in the case of surfaces. Another equally important reason for considering the case of dimension two is that the Hilbert scheme of points and its strata are well understood and enjoy nice properties such as irreducibility and smoothness. Let then X be a smooth, projective surface throughout this section.

Remark 4.1. The Hilbert scheme $X^{[l]}$ of zero-subschemes of length l of X is non singular, for every l. Let

$$\varphi_l : X^{[l]} \to X^{(l)}$$

be the Hilbert–Chow morphism, which associates to the zero-scheme $\xi \in X^{[l]}$ the zero-cycle $\sum_{x \in \mathrm{Supp}(\xi)} h^0(\mathcal{O}_{\xi,x})\, x$ of length l. For every partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t > 0)$ of l, let $X_\lambda^{(l)}$ be the locally closed subset in $X^{(l)}$ consisting of zero-cycles of length l of the form $\sum_{1 \leq i \leq t} \lambda_i x_i$, and let $X_\lambda^{[l]} = \varphi_l^{-1}(X_\lambda^{(l)})$. It is $\dim(X_\lambda^{[l]}) = l + t$.

The stratum $X_{2,1,\ldots,1}^{[l]}$ is the only codimension 1 stratum and $X_{1,\ldots,1}^{[l]}$ is the open stratum of reduced zero-schemes. Moreover, the boundary

$$\partial X^{[l]} = X^{[l]} \setminus X_{1,\ldots,1}^{[l]} = \overline{X_{2,1,\ldots,1}},$$

the closure of $X_{2,1,\ldots,1}^{[l]}$, is irreducible, see [6] for details.

Let X be a non singular algebraic surface and L an ample line bundle spanned by V. Consider the following incidence diagram:

$$\mathfrak{I} = \overline{\{(\xi, D) | \xi \in \mathcal{B}_b(X, V), D \in |V \otimes \mathcal{I}_\xi|\}} \subset X^{[b]} \times |V|.$$

$$\begin{array}{ccc} & \mathfrak{I} & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ X^{[b]} \supset \overline{\mathcal{B}_b(X, V)} & & R(V) \subset |V| \end{array} \qquad (3)$$

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Definition 4.2. The image of π_1, $\overline{\mathcal{B}_b(X,V)}$, in diagram (3) will be called the *locus of bad zero-schemes* of (X,V) and will be denoted by $\mathfrak{B}$.

Remark 4.3. Notice that if $V \subseteq H^0(X,L)$ is a subspace spanning the ample line bundle L, Proposition 3.2 (1) implies $\dim(\mathfrak{B}) \le 2b-1$. Indeed if $\dim(\mathfrak{B}) = 2b$ then $\mathfrak{B}$ would have a non empty intersection with $X^{[b]}_{1,\dots,1}$, which is open in $X^{[b]}$. Hence $b = b_0$ and $\dim(\mathcal{B}_0) = 2b_0$, which contradicts Proposition 3.2 (1). Notice also that the inequality $\dim(\mathfrak{B}) \le 2b-1$ is sharp, as Example 4 shows.

As in Section 3 the very ample case can be characterized completely.

Corollary 4.4. *Let $|V|$ be a very ample linear system and assume $\dim(\mathfrak{B}) = 2b-1$. Then either $b = 3$, $X = \mathbb{P}^2$ and $V \subseteq H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ or $b = 2$ and $|V|$ embeds X as a scroll over a smooth curve.*

Proof. A dimension count in diagram (3) gives $R(V) = \mathcal{D}(V)$ and thus Lemma 3.5 implies the assertion. □

The hypothesis $\dim(\mathfrak{B}) = 2b-1$ imposes a rather strong positivity condition on the linear system. Observe that for a general $D \in Im(\pi_2)$ $\dim(\pi_2^{-1}(D)) \le b$ and thus a dimension count from the diagram (3) gives

$$2b-1+\dim(\pi_1^{-1}(\xi)) \le \dim(|V|)-1+b$$

which implies $\dim(\pi_1^{-1}(\xi)) = \dim(|V|)-b$. This means that the linear system $|V|$ *generically separates* the zero-schemes in $\mathfrak{B}$. More can be said if the open set where this happens covers all the open non-reduced locus.

Lemma 4.5. *Assume that $\dim(\mathfrak{B}) = 2b-1$ that $|V|$ is not very ample and that $|V|$ separates all zero-schemes in $X^{[b]}_{2,1,\dots,1}$. Then either the general $\xi \in \mathfrak{B}$ is reduced or $b = 2$.*

Proof. As above if the general $\xi \in \mathfrak{B}$ is not reduced then it must be $\mathfrak{B} \subseteq \partial X^{[b]}$. Because $\partial X^{[b]} = \overline{X^{[b]}_{2,1,\dots,1}}$ is irreducible of dimension $2b-1$ it is $\partial X^{[b]} = \mathfrak{B}$, i.e. every zero-scheme which is not reduced is in $\mathfrak{B}$ and the general one is in $X^{[b]}_{2,1,\dots,1}$.

Assume $b \ge 3$ and let $\eta \in X^{[2]}$. If η is not reduced, for any $\eta' \in X_0^{[b-2]}$ such that $\mathrm{Supp}(\eta') \cap \mathrm{Supp}(\eta) = \emptyset$, the zero-scheme $\xi = \eta \cup \eta' \in X^{[b]}_{2,1,\dots,1}$ and hence it is bad. On the other hand, if η is reduced, then for every zero-scheme $\eta' \in X^{[b-2]}_{2,1,\dots,1}$ such that $\mathrm{Supp}(\eta') \cap \mathrm{Supp}(\eta) = \emptyset$, the zero-scheme $\xi = \eta \cup \eta' \in X^{[b]}_{2,1,\dots,1}$ and hence it is bad. It follows that for any $\eta \in X^{[2]}$ there is an element $\xi \in X^{[b]}_{2,\dots,1}$ such that $\eta \subset \xi$. For such a ξ it is $\dim(\pi_1^{-1}(\xi)) = \dim(|V \otimes \mathcal{I}_\xi|) = \dim(|V|)-b$, by hypothesis. It follows that $\dim(|V \otimes \mathcal{I}_\eta|) = \dim(|V|)-2$, for every $\eta \in X^{[2]}$. This is a contradiction because $|V|$ is assumed not to be very ample. It follows that either $b = 2$ or the generic $\xi \in \mathfrak{B}$ is reduced. □

Corollary 4.6. *Assume that $\dim(\mathfrak{B}) = 2b-1$ and that $|V|$ separates all zero-schemes in $X^{[b]}_{2,1,\dots,1}$. Then for a general $\xi \in \mathfrak{B}$, $|V \otimes \mathcal{I}_\xi|$ has a fixed component.*

Proof. If $|V|$ is very ample, this follows from the characterization given in Corollary 4.4. Assume now $|V|$ is not very ample. Proceeding as in the proof of Corollary 3.7, one sees that for every $x \in \mathrm{Supp}(\xi)$, it is $\mathcal{I}_{\xi,x} \subset \mathfrak{m}_x^r$ for $r \ge 2$. It follows that $b = \mathrm{length}(\xi) \ge \binom{r+1}{2}|\mathrm{Supp}(\xi)| \ge 3$ and ξ not reduced at any point. This contradicts both possibilities allowed by Lemma 4.5. □

5. Spread

In Example 1, one can construct a bad zero-scheme ξ by arbitrarily choosing up to n distinct points which uniquely determine a component of the reducible divisor. The remaining points of $\mathrm{Supp}(\xi)$ have to lie on the same hyperplane. Similarly, in Example 2 one can choose arbitrarily only one point, while the remaining ones have to lie on the same fibre as the first one. The following definition is a way of encoding the degrees of freedom in choosing an element in $\mathfrak{B}_0$.

Definition 5.1. The *spread* of (X,V) is:

$$s = max\{0, r \in \mathbb{Z}_+ | \text{ for a general } \eta \in X_0^{[r]} \; \exists\, \xi \in \mathcal{B}^0_{b_0}(X,V) \text{ such that } \eta \subseteq \xi\}.$$

Remark 5.2. Notice that $0 \le s \le b_0-1$. In fact $s = b_0$ would imply $\overline{X_0^{[b_0]}} = \mathfrak{B}_0$. This is impossible as L is assumed to be ample and spanned by V. Moreover, the definition of s gives immediately that $\dim(\mathfrak{B}_0) \ge ns$.

Remark 5.3. If $s = 0$ then $\dim \mathfrak{B}_0 \le b_0(n-1)$. Let $\varphi : X^{[b_0]} \to X^{(b_0)}$ be the Hilbert–Chow morphism and let $Y = \overline{\cup_{\xi \in \mathfrak{B}_0} \mathrm{Supp}(\xi)}$. Because $s = 0$, then $\dim(Y) \le n-1$. Consider the restriction

$$\varphi_{|\mathfrak{B}_0} : \mathfrak{B}_0 \to Y^{(b_0)}.$$

The generic element of $\varphi(\mathfrak{B}_0)$ is a b_0-tuple of distinct points of Y and thus $\dim \mathfrak{B}_0 \le \dim(Y^{(b_0)}) \le b_0(n-1)$.

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The following incidence variety is used in the next two lemmata:

$$\begin{array}{ccc} & \mathfrak{I}_1 & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ \overline{X_0^{[b_0]}} & & \overline{X_0^{[s]}} \end{array} \tag{4}$$

where $s \geq 1$, $\mathfrak{I}_1 = \overline{\{(\xi, \eta) | \xi \in \mathcal{B}^0_{b_0}(X, V), \eta \in X_0^{[s]}, \eta \subset \xi\}} \subset \overline{X_0^{[b_0]}} \times \overline{X_0^{[s]}}$. Observe that $Im(p_1) = \overline{\mathcal{B}^0_{b_0}(X, V)} = \mathfrak{B}_0$.

Let $F_\eta = p_2^{-1}(\eta) \subset \{(\xi, \eta) | \eta \subset \xi\} \subset \{\eta\} \times X^{[b_0]}$. When $s = b_0 - 1$ the space $\{(\xi, \eta) | \eta \subset \xi\} \subset \{\eta\} \times X^{[b_0]}$ can be identified with the blow up of X at $\mathrm{Supp}(\eta)$, $Bl_\eta(X)$, and thus $F_\eta \subset Bl_\eta(X)$.

Lemma 5.4. *Let $s \geq 1$. Then the general fiber of p_2 has dimension:* $\dim(F_\eta) \leq (b_0 - s)(n - 1)$.

Proof. Let F_η be a generic fiber and let $(\xi, \eta) \in F_\eta$ be a generic element, so that ξ is reduced. Let η' be the residual zero-scheme, defined by the ideal $\mathcal{I}_{\xi/\eta}$. By definition of s, we can then assume that $\eta' \in X_0^{[b_0-s]}$, where each point of $\mathrm{Supp}(\eta')$ can vary in a subspace of codimension at least one. Because every element in F_η is uniquely determined by η and such an element η' we conclude that $\dim(F_\eta) \leq (b_0 - s)(n - 1)$. □

Lemma 5.5. *Let $s \geq 1$. Then $\dim(\mathfrak{B}_0) = nb_0 - 1$ if and only if $s = b_0 - 1$ and p_2 is a morphism with $\dim(F_\eta) = (n - 1)$ for a general fiber.*

Proof. Because $Im(p_1) = \mathfrak{B}_0$ and p_1 is finite, it is $\dim(\mathfrak{I}_1) = \dim(\mathfrak{B}_0) = nb_0 - 1$. By definition of s the map p_2 is dominant, hence

$$ns + \dim(F_\eta) = nb_0 - 1$$

where F_η is a general fiber. The above equality and Lemma 5.4 imply that $s \geq b_0 - 1$. Because $s \leq b_0 - 1$, as pointed out in Remark 5.2 it follows that $s = b_0 - 1$ and $\dim(F_\eta) = n - 1$ for a general η. The converse is immediate. □

Remark 5.6. Let $\eta \in X_0^{[b_0-1]}$ be a reduced zero-scheme. In the hypothesis of Lemma 5.4 the general fiber F_η can be identified with an effective divisor on $Bl_\eta(X)$. Notice that this divisor cannot be supported only on the exceptional divisor, because ξ is reduced for the general $(\xi, \eta) \in F_\eta$. Therefore F_η contains the strict transform of an effective divisor in X, D_η, passing possibly through some points of $\mathrm{Supp}(\eta)$.

Assume $\dim(\mathfrak{B}_0) = nb_0 - 1$ and $|V|$ not very ample. Because the generic $\xi \in \mathfrak{B}_0$ is reduced, the system $|V \otimes \mathcal{I}_\xi|$ must have a fixed component, C_ξ, by [4, Prop. 17 (iii)]. It is natural to ask what relationship exists between the fixed component, C_ξ, for a generic ξ and D_η for $\eta \in X_0^{[b_0-1]}$ and $\eta \subset \xi$. The answer, as Examples 4 and 5 suggest, is provided by the following lemma.

Lemma 5.7. *Suppose that $\dim(\mathfrak{B}_0) = nb_0 - 1$. Let ξ be a general element of $\mathfrak{B}_0$. For every $x \in \mathrm{Supp}(\xi)$ let $\xi = \eta \cup x$. Then for every $x' \in C_\xi \setminus \mathrm{Supp}(\eta)$:*

(1) $\eta \cup x' \in \mathcal{B}^0_{b_0}(X, V)$;
(2) *the linear system $|V \otimes \mathcal{I}_{\eta \cup x'}|$ has the same fixed component as $|V \otimes \mathcal{I}_\xi|$;*
(3) *C_ξ is a component of D_η.*

Proof. Let $\xi = \eta \cup x$. Theorem 3.4 implies that $\dim(|V \otimes \mathcal{I}_\xi|) = \dim(|V|) - b_0$ and $\dim(|V \otimes \mathcal{I}_\eta|) = \dim(|V|) - b_0 + 1$. Notice that the evaluation map

$$ev : C_\xi \times (V \otimes \mathcal{I}_\eta) \to L$$

has generic maximal rank, i.e.

$$\dim(|V \otimes \mathcal{I}_{\eta \cup x'}|) = \dim(|V|) - b_0$$

for a generic $x' \in C_\xi$. In fact, if not, the map would be constantly zero and thus $|V \otimes \mathcal{I}_{\eta \cup x'}| = |V \otimes \mathcal{I}_\eta|$ for every $x' \in C_\xi$. But this would imply that C_ξ is a fixed component of $|V \otimes \mathcal{I}_\eta|$ and thus that η is a bad zero-scheme, which is impossible by definition.

Moreover, because $x' \in C_\xi$, it is $|V \otimes \mathcal{I}_\xi| \subseteq |V \otimes \mathcal{I}_{\eta \cup x'}|$ and thus, since they have the same dimension, they are equal. We can then conclude that for a generic $x' \in C_\xi \setminus \mathrm{Supp}(\eta)$, the zero-scheme $\xi' = \eta \cup x' \in \mathcal{B}^0_{b_0}(X, V)$ and that

$$C_{\xi'} = C_\xi.$$

Because a generic $x' \in C_\xi$ defines a $\xi' \in \mathcal{B}^0_{b_0}(X, V)$ such that $\eta \subset \xi'$, it is $x' \in D_\eta$ and thus

$$C_\xi \subseteq D_\eta$$

for every reduced ξ and every $\eta \subset \xi$ of length $b_0 - 1$. □

5.1. Non reduced spread on surfaces

As seen above, s achieving its maximum value is equivalent to $\mathfrak{B}_0$ reaching its maximum dimension, which implies that X is a surface, see Theorem 3.4. Therefore, analogously to what we did in Section 4, we introduce a non-reduced version s' of the spread for surfaces, and we show that a result analogous to Lemma 5.5 holds.

Definition 5.8. We set:

$$s' = \max\{0, r \in \mathbb{Z}_+ \mid \text{for a general } \eta \in X_0^{[r]}\ \exists\, \xi \in \mathcal{B}_b(X, V) \text{ such that } \eta \subseteq \xi\}.$$

Remark 5.9. Note that $s' \leq b - 1$. Otherwise, $s' = b = b_0 = s$, which contradicts Remark 5.2.

As above, consider the following incidence diagram:

$$\begin{array}{ccc} & \mathfrak{I}'_1 & \\ {\scriptstyle \phi_1} \swarrow & & \searrow {\scriptstyle \phi_2} \\ X^{[b]} & & \overline{X_0^{[s']}} \end{array} \tag{5}$$

where $s' \geq 1$, $\mathfrak{I}'_1 = \overline{\{(\xi, \eta) | \xi \in \mathcal{B}_b(X, V), \eta \in X_0^{[s']}, \eta \subset \xi\}}$. Observe that $Im(\phi_1) = \overline{\mathcal{B}_b(X, V)} = \mathfrak{B}$.

Let $F_\eta = \phi_2^{-1}(\eta) \subset \{(\xi, \eta) | \eta \subset \xi\} \subset \{\eta\} \times X^{[b]}$. When $s' = b - 1$ the space $\{(\xi, \eta) | \eta \subset \xi\} \subset \{\eta\} \times X^{[b]}$ can be identified with the blow up of X at $\mathrm{Supp}(\eta)$, $Bl_\eta(X)$, and thus $F_\eta \subset Bl_\eta(X)$.

Lemma 5.10. *Let* $\dim(X) = 2$, *and* $s' \geq 1$. *Then the general fiber of* ϕ_2 *has dimension:* $\dim(F_\eta) \leq (b - s')$.

Proof. One can argue exactly as in the proof of Lemma 5.4, noticing that if $(\xi, \eta) \in F_\eta$ is a generic element, then either ξ is reduced or $\xi \in X^{[b]}_{2,1,\dots,1}$. □

The proof of Lemma 5.5 can now be easily adapted to obtain the following result.

Lemma 5.11. *Let* $\dim(X) = 2$ *and* $s' \geq 1$. *Then* $\dim(\mathfrak{B}) = 2b - 1$ *if and only if* $s' = b - 1$ *and* ϕ_2 *is a morphism with* $\dim(F_\eta) = 1$ *for a general fiber.*

6. Examples

Example 3. Let $(S = C^{(2)}, L)$ be as in [3, Example 9], where C is a smooth hyperelliptic curve of genus ≥ 2. We recall that $L^2 = 4$, $h^0(L) = 3$, and the base locus $\mathrm{Bs}|K_S + L|$ of the adjoint linear system contains a smooth rational curve Γ as a component. Moreover, for all $x \in \Gamma$, the pencil $|L \otimes \mathcal{I}_x|$ contains exactly one element singular at x, which is reducible, while every other member of the pencil is smooth at x with the same tangent direction τ_x, by [2, Section 5.2 and Proposition 6.3]. Let ξ be the zero-scheme of length 2 supported at x, defined by any tangent direction $\tau \neq \tau_x$. Then $|L \otimes \mathcal{I}_\xi|$ consists only of the reducible element of $|L \otimes \mathcal{I}_x|$. This shows that ξ is a bad zero-scheme, hence $b \leq 2$. Taking into account [5, Proposition 7.4] it thus follows that $b = 2$. Letting $P := \mathbb{P}_\Gamma(T_{S|_\Gamma})$ we see that the family of bad zero-schemes ξ as x varies on Γ is exactly the complement of the section defined by τ_x in the $\mathbb{P}^1$-bundle P. Note that P is a component (perhaps not the only one) of $\mathfrak{B}$, which turns out to have dimension 2. In particular we can note that $\mathcal{B}_2$ is not closed. Finally, note that if $x \in S$ is general, then $\varphi_L(x)$ is in general position with respect to the branch locus of the 4-tuple cover $\varphi_L : S \to \mathbb{P}^2$, hence there are no bad zero-schemes whose support contains x. Therefore $s = 0$.

Example 4. Let $(S, \mathcal{L})$ be a scroll over a smooth curve C, with projection $p' : S \to C$, where $\mathcal{L}$ is an ample and spanned line bundle. Let $B \in \mathrm{Pic}(S)$ be a line bundle numerically equivalent to $[\alpha C_0 + \beta f]$, where C_0 and f denote a section of minimal self-intersection and a fiber respectively and α, β are integers. Suppose that $\alpha \geq 2$. Under suitable conditions on α and β we can suppose that $|2B|$ contains a smooth divisor Δ. Let $\pi : X \to S$ be the double cover branched along Δ. Then X is a smooth surface and $L := \pi^*\mathcal{L}$ is an ample and spanned line bundle. The scroll projection p' induces a fibration $p = p' \circ \pi : X \to C$ whose general fiber $F = \pi^* f$ is a smooth curve of genus $g(F) = \alpha - 1$. As $\pi_*\mathcal{O}_X = \mathcal{O}_S \oplus [-B]$, the projection formula gives

$$h^0(L) = h^0(\pi_* L) = h^0(\mathcal{L}) + h^0(\mathcal{L} - B).$$

On the other hand $(\mathcal{L} - B)f = 1 - \alpha < 0$, since $\alpha \geq 2$. Then $\mathcal{L} - B$ cannot be effective and so $h^0(L) = h^0(\mathcal{L})$. This means that $|L| = \pi^*|\mathcal{L}|$; equivalently, for every element $D \in |L|$ we have $D = \pi^* h$ with $h \in |\mathcal{L}|$. Let ξ be the zero-subscheme of X of length 2 supported at two points x_1, x_2 lying on the same fibre $\overline{F}$ of p. Note that if $\pi(x_1) = \pi(x_2)$ then $|L \otimes \mathcal{I}_\xi| = \pi^*|\mathcal{L} \otimes \mathcal{I}_{\pi(x_1)}|$ and the general element of this linear system is a smooth curve provided that $\pi(x_1)$ is not in $\mathcal{J}_2(S, \mathcal{L})$ (the second jumping set, see [7]). Now suppose that $\pi(x_1) \neq \pi(x_2)$ and denote by ξ' the zero-subscheme of S consisting of $\pi(x_1)$ and $\pi(x_2)$. As ξ' is contained in the fiber $\overline{f} = \pi(\overline{F})$ of S and $\mathcal{L}f = 1$, we have that

$$|\mathcal{L} \otimes \mathcal{I}_{\xi'}| = \overline{f} + |M|,$$

$|M|$ standing for the moving part of the linear system. Consequently,

$$|L \otimes \mathcal{I}_\xi| = \pi^*|\mathcal{L} \otimes \mathcal{I}_{\xi'}| = \overline{F} + \pi^*|M|.$$

This shows that ξ is a bad zero-scheme for (X, L). It follows that $b = 2$. Moreover, $s = 1$, as we can produce a bad zero-scheme like ξ choosing any point of X as x_1. Finally, note that $\mathcal{B}_2$ consists of the set of pairs of points of X lying on a same fiber of p, not exchanged by the involution defined by π. Thus, even in this case we can note that $\mathcal{B}_2$ is not closed. Moreover $\mathfrak{B}$, which is given by the fiber product of $p : X \to C$ with itself, has dimension 3 (the maximum). Note that the same construction can be done by replacing the double cover with a cyclic cover of any degree.

It deserves to note that the pair (X, L) above provides a counterexample to [3, Conjecture 1]. Actually, for the general point $x \in X$ we have that $|L \otimes \mathcal{I}_x^2| = \pi^*|\mathcal{L} \otimes \mathcal{I}_{\pi(x)}^2|$. The fact that $(S, \mathcal{L})$ is a scroll implies that $|\mathcal{L} \otimes \mathcal{I}_{\pi(x)}^2| = \overline{f} + |M|$, where $|M|$ stands for the moving part of the linear system and $\overline{f}$ is the fiber of S through $\pi(x)$. Therefore $|L \otimes \mathcal{I}_x^2| = \overline{F} + \pi^*|M|$, where $\overline{F}$ is the fiber of X through x. This shows that x belongs to the rude locus of (X, L), as defined in [3, Section 5]. Another example suggested by the construction above is the following.

Example 5. Let Δ be a smooth plane curve of degree $2a$, and let $\pi : X \to \mathbb{P}^2$ be the double cover branched along Δ. Then X is a smooth surface and $L := \pi^*\mathcal{O}_{\mathbb{P}^2}(2)$ is an ample and spanned line bundle. The projection formula gives

$$h^0(L) = h^0(\pi_*L) = h^0(\mathcal{O}_{\mathbb{P}^2}(2)) + h^0(\mathcal{O}_{\mathbb{P}^2}(2 - a)).$$

So $h^0(L) = h^0(\mathcal{O}_{\mathbb{P}^2}(2))$ provided that $a \geq 3$. Equivalently, for every element $D \in |L|$ we have $D = \pi^*h$ with $h \in |\mathcal{O}_{\mathbb{P}^2}(2)|$. Let ξ be the zero-subscheme of X of length 3 supported at three points x_1, x_2, x_3 whose images in $\mathbb{P}^2$ are distinct and lying on a line ℓ. Denote by ξ' the zero-subscheme of $\mathbb{P}^2$ consisting of $\pi(x_1), \pi(x_2)$ and $\pi(x_3)$. As ξ' is contained in ℓ, we have

$$|\mathcal{O}_{\mathbb{P}^2}(2) \otimes \mathcal{I}_{\xi'}| = \ell + |M|,$$

$|M|$ standing for the moving part of the linear system. Consequently,

$$|L \otimes \mathcal{I}_\xi| = \pi^*\ell + \pi^*|M|.$$

This shows that ξ is a bad zero-scheme for (X, L). It follows that $b = 3$. Moreover, $s = 2$, as we can produce a bad zero-scheme like ξ choosing any two points of X not exchanged by the involution defined by π as x_1 and x_2. Finally, note that $\mathcal{B}_3$ consists of the set of triplets of points of X mapped to distinct collinear points by π. Thus, even in this case we can note that $\mathcal{B}_3$ is not closed. Moreover $\mathfrak{B}$ has dimension 5 (the maximum). Note that X is a surface of general type as soon as $a \geq 4$. A similar construction can be done by replacing the double cover with a cyclic cover of any degree.

Example 6. Let $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k))$ with $k \geq 3$. We have $b = k + 1$, the general bad zero-scheme ξ of minimal length consisting of $k + 1$ collinear points, so that the line containing them is the fixed component of $|\mathcal{O}_{\mathbb{P}^2}(k) \otimes \mathcal{I}_\xi|$. Clearly, $s = 2$ and $\dim(\mathfrak{B}) = 4 + k - 1 = k + 3$.

Example 7. Let X be a del Pezzo surface with $K_X^2 = 2$ and $L = -K_X$. Then L is ample and spanned and $\varphi_L : X \to \mathbb{P}^2$ is a double cover branched along a smooth quartic curve Δ. Here $b = 2$ and the general bad zero-scheme of length 2 consists of two points, not on the same fiber of φ_L, lying on the divisor $D = \varphi_L^*(\ell)$, where $\ell \subset \mathbb{P}^2$ is a bitangent line to Δ [4, Example 21]. Clearly, $s = 0$ and $\dim(\mathfrak{B}) = 2$.

Example 8. Let X be a del Pezzo surface with $K_X^2 = 1$ and $L = -3K_X$. Here L is very ample and $b = 3$ as shown in [4, Example 29]. The general bad zero-scheme of length 3 consists of 3 points lying on a singular curve $\Gamma \in |-K_X|$, one of them being the singular point. Recall that if X is general in moduli, then $|-K_X|$ contains exactly 12 (irreducible) singular curves Γ. Then $s = 0$ and $\dim(\mathfrak{B}) = 2$.

Example 9. Let E_i $(i = 1, 2)$ be a smooth curve of genus one and let $\mathcal{L}_i$ be a line bundle of degree 2 on E_i. Set $S = E_1 \times E_2$ and $L = p_1^*\mathcal{L}_1 \otimes p_2^*\mathcal{L}_2$, where $p_i : S \to E_i$ is the projection onto the i-th factor. Note that L is ample and spanned, but not very ample, $L \equiv 2(E_1 + E_2)$, and $h^0(L) = 4$. As $LE_i = 2$, for any reduced zero-scheme ξ' consisting of three points lying on a same fibre, say $\overline{E}$, of p_i the linear system $|L \otimes \mathcal{I}_{\xi'}|$ has to contain $\overline{E}$ as a fixed component. Hence $b \leq b_0 \leq 3$. We claim that $b = b_0 = 2$. As $b \neq 1$, by [5, Corollary 1.3], and as $b \leq b_0$, it is enough to produce a reduced zero-scheme of length two. The following argument relies on the notion of jumping sets, for which we refer the reader to [7]. As $|\mathcal{L}_i|$ is a g_2^1 on E_i, the jumping set $\mathcal{J}_1(\mathcal{L}_i)$ consists of the 4 ramification points of the morphism $\varphi_{\mathcal{L}_i} : E_i \to \mathbb{P}^1$. Consequently, as in [7, Example 1.8(2)], the jumping sets of L are the following:

$$\mathcal{J}_1(L) = (\mathcal{J}_1(\mathcal{L}_1) \times E_2) \cup (E_1 \times \mathcal{J}_1(\mathcal{L}_2)),$$
$$\mathcal{J}_2(L) = \mathcal{J}_1(\mathcal{L}_1) \times \mathcal{J}_1(\mathcal{L}_2).$$

Relying on basic properties of jumping sets one can then see that bad reduced zero-schemes $\xi = x + y$ of length two on S can be constructed in two different ways

1. by choosing $x \in \mathcal{J}_2(L)$ and choosing $y \in p_i^{-1}(p_i(x)) \setminus \{x\}$ for any $i = 1, 2$;

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2. by choosing $x \in \mathcal{J}_1(L) \setminus \mathcal{J}_2(L)$ and, assuming up to renaming the factors that $p_1(x) \in \mathcal{J}_1(\mathcal{L}_1)$, choosing $y \in p_2^{-1}(p_2(x)) \setminus \{x\}$.

From both construction approaches one sees that $s = 0$. One can also conclude, from 2. above, that $\dim \mathfrak{B}_0 = 2$, the maximum allowed by Remark 5.3.

Acknowledgements

The authors acknowledge financial support from: DePaul University, Vetenskapsrådet's grant NT:2006-3539, the Gustafsson Foundation, Università degli Studi di Milano (FIRST 2006 and 2007), and MiUR of the Italian Government in the framework of PRIN "Algebraic Varieties etc." (Cofin 2006) and "Algebraic Geometry etc." (Cofin 2008). The authors are grateful to the anonymous referee for several useful remarks.

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