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# Chern numbers of smooth varieties via homotopy continuation and intersection theory

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## ABSTRACT

Homotopy continuation provides a numerical tool for computing the equivalence of a smooth variety in an intersection product. Intersection theory provides a theoretical tool for relating the equivalence of a smooth variety in an intersection product to the degrees of the Chern classes of the variety. A combination of these tools leads to a numerical method for computing the degrees of Chern classes of smooth projective varieties in  $\mathbb{P}^n$ . We illustrate the approach through several worked examples.

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## 1. Introduction

In this paper, we describe a numerical method for computing the degrees of the Chern classes of a smooth projective variety. The main tools that will be used are drawn from intersection theory and numerical algebraic geometry. In particular, we will be using a refined version of Bézout's theorem (Flenner et al., 1999; Fulton, 1998; Vogel, 1984) together with ideas related to numerical polynomial algebra such as homotopy continuation, monodromy, and the numerical decomposition of zero sets (Li, 1997; Sommese et al., 2001; Sommese and Wampler, 2005; Stetter, 2004). In addition, we will

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need to numerically determine the degree of a certain residual zero-scheme (Bates et al., 2006; Dayton and Zeng, 2005; Lecerf, 2002; Leykin, 2008; Leykin et al., 2006). Implementations of many of the computational tools that will be used can be found in freely available software packages (e.g., Bates et al., 2010; CoCoA, 0000; Grayson and Stillman, 2009; Decker et al., 2010; Gunji et al., 2004; Lee et al., 2008; Stein and Joyner, 2010; Verschelde, 2010; Watson et al., 1997; Zeng, 2007). The algorithms developed in this paper primarily use the software package *Bertini* with help from the software package *Macaulay 2* (Bates et al., 2010; Grayson and Stillman, 2009).

From the generators of the ideal of a smooth projective variety, Chern numbers can be computed through a purely symbolic computation. This naturally leads one to ask if a numerical approach is useful or needed. A partial answer is that, in the numerical approach described in this paper, the algorithms work equally well whether the generators are sparse or dense and whether they have rational, algebraic or transcendental coefficients. This is not the case with purely symbolic methods. Furthermore, a surprising amount of information can be extracted from an ideal even in situations where the generators have inaccuracies in their coefficients (Hauenstein et al., 2009; Huber et al., 1998; Niyogi et al., 2008). Another important feature, which will likely play a dominant role in the future, is that homotopy continuation algorithms parallelize well. This will allow such algorithms to take advantage of the trend of more processors on a chip. These features lead us to believe that the introduction of a numerical approach is indeed useful and will allow experimentation in settings outside the domain of purely symbolic methods.

## 2. Intersection theory and homotopy continuation

In this section, we describe the main tools that will be used from intersection theory and from numerical algebraic geometry. Let  $X_1, \dots, X_r$  be a set of hypersurfaces in  $\mathbb{P}^r$ . Let  $Z$  be a smooth connected component of the scheme defined by  $X_1 \cap \dots \cap X_r$ . We present a well-known theorem from intersection theory that relates the equivalence of  $Z$  in the intersection product  $X_1 \dots X_r$  to a formula involving the total Chern classes of certain bundles related to  $X_1, \dots, X_r, Z$ , and  $\mathbb{P}^r$ . With mild assumptions on the scheme determined by the hypersurfaces, this theorem can be translated into a numerical condition involving the degrees of the Chern classes of  $Z$ , the degrees of the  $X_i$ , and the degree of a certain residual zero-scheme. Tools from numerical algebraic geometry can be used to compute the degree of the residual zero-scheme. By varying the degrees of the  $X_i$ , we obtain multiple numerical conditions that can be used to uniquely determine the degrees of the Chern classes of  $Z$ .

### 2.1. Intersection theory

Intersection theory has been primarily developed from the two different viewpoints found in Flenner et al. (1999), Fulton (1998) and Vogel (1984). Though each viewpoint seems to have its own advantages and disadvantages, van Gastel showed that the two are closely related (van Gastel, 1991). While either version would be adequate for our purposes, we follow the approach of Fulton and MacPherson, as described in Fulton (1998). We first need to fix the notation and definitions that will be used in this section.

**Definition 2.1.** Let  $Z$  and  $X$  be projective varieties in  $\mathbb{P}^r = \mathbb{P}_{\mathbb{C}}^r$  with  $Z \subseteq X$ .

- (1) Let  $A_*(X) = \bigoplus_k A_k(X)$ , where  $A_k(X)$  denotes the Chow group of  $k$ -dimensional cycles on  $X$  modulo rational equivalence.
- (2) For  $\alpha \in A_*(X)$ ,  $\{\alpha\}_0$  denotes the component of  $\alpha$  in  $A_0(X)$ . Let  $\alpha \in A_k(X)$ ,  $\alpha = \sum_{i=1}^m a_i V_i$ , where  $a_1, \dots, a_m \in \mathbb{Z}$  and  $V_1, \dots, V_m \subseteq X$  are irreducible subvarieties. The degree of  $\alpha$  is defined by  $\deg \alpha = \sum_i a_i \deg V_i$ .
- (3) If  $X$  is regularly embedded in  $\mathbb{P}^r$ ,  $N_X \mathbb{P}^r$  denotes the normal bundle of  $X$  in  $\mathbb{P}^r$ , and if  $X$  is smooth,  $T_X$  denotes the tangent bundle of  $X$ .
- (4) Given a vector bundle  $E$  of rank  $n$  on a smooth variety,  $c_i(E)$  denotes the  $i$ th Chern class of  $E$  and  $c(E) = 1 + c_1(E) + \dots + c_n(E)$  denotes the total Chern class of  $E$ .
- (5) The  $i$ th Chern class of a smooth variety  $X$  is defined as  $c_i(T_X)$ , and  $\deg(c_i(T_X))$  is called a Chern number of  $X$ .

Let  $X_1, \dots, X_r$  be hypersurfaces in  $\mathbb{P}^r$  and let  $Z$  be a smooth connected component of  $X_1 \cap \dots \cap X_r$ . [Fulton \(1998\)](#) considers the intersection product  $X_1 \dots X_r \in A_0(X_1 \cap \dots \cap X_r)$  and defines  $(X_1 \dots X_r)^Z \in A_0(Z)$ , the equivalence of  $Z$  for  $X_1 \dots X_r$ , as the part of the intersection product supported on  $Z$ . The following is a simplified version of Proposition 9.1.1 of [Fulton \(1998\)](#) specialized to our setting.

**Proposition 2.2** (Fulton). *Let  $X_1, \dots, X_r$  be hypersurfaces in  $\mathbb{P}^r$  and let  $Z$  be a smooth connected component of  $X_1 \cap \dots \cap X_r$ . Let  $N_i$  be the restriction of  $N_{X_i \mathbb{P}^r}$  to  $Z$ . Then*

$$(X_1 \dots X_r)^Z = \left\{ \left( \prod_{i=1}^r c(N_i) \right) c(T_{\mathbb{P}^r|Z})^{-1} c(T_Z) \right\}_0. \tag{1}$$

We introduce notation for the purpose of stating a useful corollary of [Proposition 2.2](#). Let  $n_i = \deg X_i$ , let  $\dim Z = n$ , and let  $c_0, \dots, c_n$  be the Chern classes of  $Z$ . Here,  $c_0 \in A_*(Z)$  is represented by  $Z$  as a cycle on  $Z$  and thus  $\deg c_0 = \deg Z$ , the degree of the subscheme  $Z \subseteq \mathbb{P}^r$ . The  $k$ th elementary symmetric function in  $n_1, \dots, n_r$  will be denoted by  $\sigma_k$ . In other words,

$$\sigma_0 = 1, \quad \sigma_1 = \sum_i n_i, \quad \sigma_2 = \sum_{i < j} n_i n_j, \quad \sigma_3 = \sum_{i < j < k} n_i n_j n_k, \quad \dots$$

If we let

$$a_i = \sum_{j=0}^{n-i} (-1)^j \binom{r+j}{j} \sigma_{n-i-j},$$

then we obtain the following identity as a direct corollary of [Proposition 2.2](#).

**Corollary 2.3.**  $\deg (X_1 \dots X_r)^Z = \sum_{i=0}^n a_i \deg c_i$ .

**Proof.** Let  $H$  be the hyperplane class in  $\mathbb{P}^r$  and let  $H|_Z$  be its restriction to  $Z$ . Because  $c(T_{\mathbb{P}^r|Z}) = (1 + H|_Z)^{r+1}$ ,

$$c(T_{\mathbb{P}^r|Z})^{-1} = \sum_{j=0}^n (-1)^j \binom{r+j}{j} H|_Z^j.$$

As divisor classes on  $\mathbb{P}^r$ ,  $X_i = n_i H$ . The normal bundle  $N_{X_i \mathbb{P}^r}$  is the restriction to  $X_i$  of the line bundle on  $\mathbb{P}^r$  corresponding to the divisor  $X_i$ . Hence  $c(N_i) = 1 + n_i H|_Z$ , and we get that  $\prod_{i=1}^r (1 + c(N_i)) = \sum_{k=0}^n \sigma_k H|_Z^k$ . Thus formula (1) can be written as

$$(X_1 \dots X_r)^Z = \left\{ \left( \sum_{k=0}^n \sigma_k H|_Z^k \right) \left( \sum_{j=0}^n (-1)^j \binom{r+j}{j} H|_Z^j \right) \left( \sum_{i=0}^n c_i \right) \right\}_0.$$

The statement now follows, since  $\deg c_i = \deg (c_i H|_Z^{n-i})$  and the component of  $(X_1 \dots X_r)^Z$  in  $A_0(Z)$  is given by the sum of terms where  $k = n - i - j$ .  $\square$

The classical version of Bézout's theorem (Proposition 8.4 of [Fulton, 1998](#)) states that

$$\deg (X_1 \dots X_r) = \prod_{i=1}^r n_i.$$

A refined version of Bézout's theorem (Proposition 9.1.2 of [Fulton, 1998](#)) ties these results together with an extended Bézout formula.

**Proposition 2.4.** *Suppose that  $X_1 \cap \dots \cap X_r$  consists of a connected component  $Z$  and a finite set  $S$ . For  $p \in S$ , let  $m_p = i(p, X_1 \dots X_r; \mathbb{P}^r)$  denote the intersection multiplicity of  $p$  in  $X_1 \dots X_r$ . Then  $\deg (X_1 \dots X_r)^Z + \sum_{p \in S} m_p = \prod_{i=1}^r n_i$ .*

If  $n_1, \dots, n_r$  and  $\sum_{p \in S} m_p$  are known, then we can use [Proposition 2.4](#) to solve for  $\deg (X_1 \dots X_r)^Z$ . If  $Z$  is smooth, [Corollary 2.3](#) provides a linear relation among the degrees of the Chern classes of  $Z$ . We summarize these observations in the following theorem, implicit in [Fulton \(1998\)](#).

**Theorem 2.5.** Let  $F_1, \dots, F_r$  be homogeneous forms corresponding to hypersurfaces  $X_1, \dots, X_r \subset \mathbb{P}^r$ . Let  $n_i = \deg X_i$ , let  $\sigma_k$  be the  $k$ th elementary symmetric polynomial in  $n_1, \dots, n_r$  and let  $\vec{\mathbf{A}} = [a_0, \dots, a_n]$  with  $a_k = \sum_{i=0}^{n-k} (-1)^i \binom{r+i}{i} \sigma_{n-k-i}$ . Let  $Z$  be a smooth connected  $n$ -dimensional scheme with Chern classes  $c_0, \dots, c_n$  and let  $\vec{\mathbf{C}} = [\deg c_0, \dots, \deg c_n]$ . If the subscheme of  $\mathbb{P}^r$  defined by the ideal  $(F_1, \dots, F_r)$  is a disjoint union of  $Z$  and a (possibly empty) zero-scheme  $S$ , then  $\vec{\mathbf{A}} \cdot \vec{\mathbf{C}} = \sigma_r - \deg S$ .

In order to compute the degrees of the Chern classes of an  $n$ -dimensional smooth variety, it is enough to determine  $n + 1$  independent linear relations that they satisfy. Theorem 2.5 provides a mechanism for producing the linear relations provided we are able to find a sufficient number of  $r$ -tuples of homogeneous forms that satisfy the conditions of Theorem 2.5 and provided we have a method for computing  $\deg S$ . In the next two sections, we present a collection of tools that allow us to complete the latter task.

## 2.2. Homotopy continuation

In homotopy continuation, a polynomial ideal,  $I$ , is cast as a member of a parameterized family of polynomial ideals, one of which has known isolated solutions. Each of the known isolated solutions is tracked through a predictor/corrector method to a point which lies numerically close to the algebraic set  $V(I)$  determined by  $I$ . These points can then be refined to lie within a prescribed tolerance of  $V(I)$ . An introduction to general continuation methods can be found in Allgower and Georg (2003). Through the basic algorithms of numerical algebraic geometry (Sommese and Wampler, 2005), from  $I$  it is then possible to produce a collection of subsets of points such that the following hold.

- The subsets are in one to one correspondence with the irreducible components of the algebraic set  $V(I)$ .
- The points in a subset all lie within a prescribed tolerance of the irreducible component to which it corresponds.
- The number of points in the subset is the same as the degree of the irreducible component.
- The subset is a numerical approximation of the intersection of the irreducible component with a known linear space of complementary dimension.

Note that the decomposition above allows one to determine both the dimension and the degree of each algebraic variety appearing in the decomposition of an algebraic set.

## 2.3. Degree of a zero-dimensional scheme

To each irreducible component of an algebraic set, one can use the defining set of polynomials to attach a positive integer, called the *multiplicity*, that determines roughly how many times the component should be counted in a computation. In Dayton and Zeng (2005), Dayton and Zeng study the multiplicity inspired, to a large degree, by Macaulay's inverse systems approach. They provide an algorithm which yields as output the multiplicity of isolated solutions. This is essentially done by counting how many partial derivatives of the polynomials are forced to be zero. In Bates et al. (2006), an alternate approach is presented, inspired, to a large degree, by certain Gröbner basis calculations coupled with a fundamental result of Bayer and Stillman on regularity (Bayer and Stillman, 1987). Both of these algorithms have been implemented in Bertini (Bates et al., 2010). Thus, there is an implemented algorithm in place that allows one to determine the degree of a zero-dimensional scheme.

## 2.4. Discussion of accuracy

The certainty of the output of any implementation of the algorithms in this article depend on the certainty of the output of implementations of the numerical continuation algorithms for computation of the solution sets of polynomial systems. Numerical continuation algorithms are not certain. The remarkable speed and parallelism of these algorithms relies on the limited number of digits of floating-point arithmetic. The package we use, Bertini, was designed to adaptively use whatever number of

digits are needed to ensure the basic continuation algorithms are accurate. A realistic a priori analysis of a large software program such as *Bertini* based on adaptive precision algorithms, probabilistic algorithms, and many internal checks on accuracy, is for practical purposes impossible. On the other hand, though accuracy has never been quantified a posteriori, *Bertini* reaches correct conclusions on a growing number of complicated problems. So, though *Bertini* does not give mathematical certainty, it does give a tool to construct examples and explore mathematical theories involving algebraic geometry constructs.

### 3. Algorithms

In this section, we present the pseudocode for three algorithms. The first algorithm computes the equivalence of a connected scheme  $Z$  in an intersection product. The second algorithm determines a linear relation satisfied by the degrees of the Chern classes of a smooth connected scheme. The third algorithm computes the degrees of the Chern classes of a smooth connected scheme. For each of the algorithms, the input is a set of non-zero homogeneous generators for an ideal  $I$ . It is an assumption of the algorithms that the scheme determined by  $I$  is a disjoint union of  $Z$  and a zero-scheme  $S$ . The second and third algorithms have the additional assumption that  $Z$  is smooth.

**Algorithm 1.** *Equivalence\_of\_Z* ( $\{F_1, F_2, \dots, F_r\}; D$ )

**Input:** A set of  $r$  homogeneous polynomials  $\{F_1, F_2, \dots, F_r\} \subset \mathbb{C}[z_0, z_1, \dots, z_r]$ . The polynomials should generate an ideal  $I$  whose corresponding scheme is the disjoint union of a connected scheme  $Z$  and a possibly empty zero-scheme  $S$ .

**Output:**  $D = \deg(X_1 \dots X_r)^Z$ , where  $X_i$  is the hypersurface corresponding to  $F_i$ .

**Algorithm:**

Determine the support of  $S$ .

Determine the multiplicity of each point in the support of  $S$ .

Add up the multiplicities of the points in the support of  $S$  and store in  $\mu_S$ .

For each  $i$ , determine the degree of  $F_i$  and store in  $n_i$ .

Compute  $T = \prod_i n_i$ .

Compute  $T - \mu_S$  and store in  $D$ .

**Algorithm 2.** *Linear\_Relation\_On\_Chern\_Numbers* ( $\{F_1, F_2, \dots, F_r\}; \vec{\mathbf{A}}, D$ )

**Input:** A set of  $r$  homogeneous polynomials  $\{F_1, F_2, \dots, F_r\} \subset \mathbb{C}[z_0, z_1, \dots, z_r]$ . The polynomials should generate an ideal  $I$  whose corresponding scheme is the disjoint union of  $Z$  and  $S$ , where  $Z$  is a smooth connected scheme and  $S$  is a possibly empty zero-scheme.

**Output:**  $\vec{\mathbf{A}} = [a_0, \dots, a_n]$  and  $D \in \mathbb{Z}$ , where  $n$  denotes the dimension of  $Z$ . If  $c_i$  denotes the  $i$ th Chern class of  $Z$  and  $\vec{\mathbf{C}} = [\deg c_0, \dots, \deg c_n]$ , then the linear relation is  $\vec{\mathbf{A}} \cdot \vec{\mathbf{C}} = D$ .

**Algorithm:**

Determine the dimension of  $Z$  and store in  $n$ .

Compute *Equivalence\_of\_Z* ( $\{F_1, F_2, \dots, F_r\}; D$ ).

Compute the elementary symmetric functions  $\sigma_0, \sigma_1, \dots, \sigma_n$  of  $n_1, \dots, n_r$ .

Compute  $\vec{\mathbf{A}} = [a_0, a_1, \dots, a_n]$ , where  $a_k = \sum_{i=0}^{n-k} (-1)^i \binom{r+i}{i} \sigma_{n-k-i}$ .

**Algorithm 3.** *Chern\_Numbers* ( $\{F_1, F_2, \dots, F_t\}; \vec{\mathbf{C}}$ )

**Input:** A set of  $t$  homogeneous polynomials  $\{F_1, F_2, \dots, F_t\} \subset \mathbb{C}[z_0, z_1, \dots, z_r]$ . The polynomials should generate an ideal  $I$  whose corresponding scheme is the disjoint union of  $Z$  and  $S$ , where  $Z$  is a smooth connected scheme and  $S$  is a possibly empty zero-scheme.

**Output:**  $\vec{\mathbf{C}} = [\deg c_0, \dots, \deg c_n]$ , where  $n$  denotes the dimension of  $Z$  and  $c_i$  denotes the  $i$ th Chern class of  $Z$ .

**Algorithm:**

Determine the dimension of  $Z$  and store in  $n$ .  
 Determine the degrees of  $F_1, \dots, F_t$ , and store the maximal degree as  $b$ .  
 Set  $l_j = b$  for  $j = 1, \dots, n + 1$ .  
 FOR  $i = 1$  to  $n + 1$   
     Choose  $r$  random elements from  $I$  of degrees  $l_1, l_2, \dots, l_r$ .  
     Store the random elements as  $G_1, G_2, \dots, G_r$ .  
     Compute *Linear\_Relation\_On\_Chern\_Numbers* ( $\{G_1, G_2, \dots, G_r\}; \vec{A}_i, D_i$ ).  
     Let  $l_i = l_i + 1$ .  
 ENDFOR  
 Build the  $(n + 1) \times (n + 1)$  matrix  $M$  whose  $i$ th row is  $\vec{A}_i$ .  
 Build  $\vec{D} = [D_1, \dots, D_{n+1}]$ .  
 Set up the linear system  $M\vec{C} = \vec{D}$ , with  $\vec{C}$  and  $\vec{D}$  as column vectors.  
 Solve the linear system for  $\vec{C}$ .

There are two potential problems that should be addressed in the use of Algorithm 3. The first potential problem is whether the matrix  $M$  has full rank. The second potential problem is whether the random  $r$ -tuples  $G_1, G_2, \dots, G_r$ , produced in such a simple manner from  $I$ , satisfy the input requirements of Algorithm 2. These two questions are answered in Proposition 3.1 and Corollary 3.4, respectively.

**Proposition 3.1.** *The matrix  $M$  defined in Algorithm 3 satisfies  $\det(M) = \pm 1$ .*

**Proof.** View  $M$  as a function of the maximal degree  $b$ . Let  $\sigma_i$  be the  $i$ th elementary symmetric polynomial in  $r$  variables  $b_1, \dots, b_r$  and let

$$\sigma_i^j = \sigma_i(b_1 + 1, b_2 + 1, \dots, b_j + 1, b_{j+1}, \dots, b_r).$$

Define  $\tau_i^j \in \mathbb{Z}[b]$  by  $\tau_i^j = \sigma_i^j(b, b, \dots, b)$ ,  $\tau_i^0 = \sigma_i(b, b, \dots, b)$ . Define a  $(n + 1) \times (n + 1)$  matrix  $N(b)$  by

$$N = \begin{pmatrix} \tau_n^0 & \tau_{n-1}^0 & \dots & \tau_1^0 & 1 \\ \tau_n^1 & \tau_{n-1}^1 & \dots & \tau_1^1 & 1 \\ & & \vdots & & \\ \tau_n^n & \tau_{n-1}^n & \dots & \tau_1^n & 1 \end{pmatrix}.$$

Observe first that  $M$  can be factored as  $M = NR$ , where

$$R = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -(r + 1) & 1 & 0 & \dots & 0 & 0 \\ \binom{r+2}{2} & -(r + 1) & 1 & \dots & 0 & 0 \\ & & \vdots & & & \\ (-1)^n \binom{r+n}{n} & (-1)^{n-1} \binom{r+n-1}{n-1} & \dots & \binom{r+2}{2} & -(r + 1) & 1 \end{pmatrix}.$$

Since  $\det(R) = 1$ , we have  $\det(M) = \det(N)$ .

One checks that  $\det(N(0)) = \pm 1$ . We shall show that the determinant of  $N(b)$  does not depend on  $b$  by showing that

$$\frac{d(\det(N))}{db} = 0.$$

We shall use the following fact:

$$\frac{d\tau_i^j}{db} = (r - i + 1)\tau_{i-1}^j.$$

Let  $C_i$  be the  $i$ th column of  $N$ ,  $N = (C_0, C_1, \dots, C_n)$ . Then we have that

$$\frac{d(\det(N))}{db} = \sum_{i=0}^n \det \left( C_0, C_1, \dots, C_{i-1}, \frac{dC_i}{db}, C_{i+1}, \dots, C_n \right),$$

but each term in the above sum is zero since  $\frac{dC_i}{db} = (r - n + i + 1)C_{i+1}$  for  $0 \leq i \leq n - 1$  and  $\frac{dC_n}{db} = 0$ .  $\square$

Let  $I$  be an ideal defining a scheme  $X \subset \mathbb{P}^r$  and let  $I_X$  be the homogeneous ideal of  $X$ . For  $d \in \mathbb{N}$ , we say that  $X$  is cut out scheme theoretically in degree  $d$  if the saturation of the ideal generated by  $I(d)$  is equal to  $I_X$ .

**Proposition 3.2.** *Let  $I$  be an ideal defining a scheme in  $\mathbb{P}^r$  that is the disjoint union of a smooth connected scheme  $Z$  and a possibly empty zero-scheme  $S$ . Let  $G_1, \dots, G_k \in I$ ,  $1 \leq k \leq r$ , generate an ideal that defines a scheme  $X$ , such that the singular locus  $X_{\text{sing}}$  satisfies  $\text{codim}(X_{\text{sing}} \setminus S) > k$ , and  $\text{codim}(X \setminus Z) \geq k$ . Let  $d \in \mathbb{N}$  be such that  $Z \cup S$  is cut out scheme theoretically in degree  $d$  and fix integers  $n_{k+1}, \dots, n_r$  with  $n_i \geq d$  for all  $i$ . If  $G_{k+1}, \dots, G_r$  are general forms in  $I$  with  $\deg G_i = n_i$ , then the ideal  $J = (G_1, G_2, \dots, G_r)$  defines the disjoint union of  $Z$  and a possibly empty zero-scheme  $S'$ . If  $S$  is non-singular or empty, then  $S'$  is non-singular.*

**Proof.** Suppose that  $k \neq r$ . A generic form  $G_{k+1} \in I$  of degree  $n_{k+1}$  cuts down the dimension of every component of  $X \setminus (Z \cup S)$  as well as every component of  $X_{\text{sing}} \setminus S$ . By induction, the scheme  $Y$  defined by  $J$  is a disjoint union of  $Z$  and a zero-scheme  $S'$ , and  $Y$  is nonsingular away from  $S$ . In particular,  $Y$  is non-singular on  $Z$ .  $\square$

**Corollary 3.3.** *Suppose that  $I$  and  $d$  are as in Proposition 3.2. If  $G_1, \dots, G_k \in I$  is a regular sequence in  $I$  such that the scheme  $X$  defined by the ideal  $(G_1, \dots, G_k)$  is generically smooth, then, for general forms  $G_{k+1}, \dots, G_r$  with  $\deg G_i = n_i \geq d$ , the ideal  $J = (G_1, G_2, \dots, G_r)$  defines the disjoint union of  $Z$  and a possibly empty zero-scheme  $S'$ .*

**Proof.** Every component of  $X$  has codimension  $k$  and is of multiplicity 1. We may therefore apply Proposition 3.2.  $\square$

The following corollary implies that, for generic choices of the forms  $G_1, \dots, G_r$  in Algorithm 3, the assumptions on the input of Algorithm 2 are satisfied.

**Corollary 3.4.** *Let  $I \neq 0$  be as in Proposition 3.2 and let  $\{F_1, \dots, F_t\}$  be a set of non-zero generators of  $I$ . Put  $d = \max\{\deg F_1, \dots, \deg F_t\}$  and fix integers  $n_1, \dots, n_r$  with  $n_i \geq d$  for all  $i$ . If  $G_1, \dots, G_r$  are general forms in  $I$  with  $\deg G_i = n_i$ , then the ideal  $J = (G_1, G_2, \dots, G_r)$  defines the disjoint union of  $Z$  and a possibly empty zero-scheme  $S'$ .*

**Proof.** Observe that the ideal generated by  $I(d)$  is equal to  $\bigoplus_{e \geq d} I(e)$ . Since the ideal  $\bigoplus_{e \geq d} I(e)$  and  $I$  have the same saturation, namely the homogeneous ideal of  $Z \cup S$ , the scheme  $Z \cup S$  is cut out scheme theoretically in degree  $d$ .  $\square$

For reasons of efficiency, it is desirable to keep the degrees of the forms  $G_1, \dots, G_r$  in Algorithm 3 as low as possible. We give two examples that have a bearing on the question of how much Corollary 3.4 can be strengthened.

**Example 3.5.** Consider a conic  $Z \subset \mathbb{P}^3$ . The conic is cut out by a plane  $P$  and a quadric  $Q$ . The corresponding forms of degree 1 and 2 generate the homogeneous ideal  $I$  of  $Z$ . In this case we cannot choose the generic forms  $G_1, G_2, G_3 \in I$  of degrees (3, 1, 1). A cubic and two planes, all three containing the conic, will intersect in the conic union a line, since the planes are necessarily both equal to  $P$ . The residue in this case is thus a line and not a finite scheme as required.

**Example 3.6.** This example is along the same lines as Example 3.5. Let  $A$  be a  $2 \times 3$  matrix of general linear forms in  $\mathbb{C}[x_0, \dots, x_4]$ . Let  $F_2, F_3, F_4$  be the  $2 \times 2$  minors of  $A$  and let  $F_1$  be a general form of degree 3. The ideal  $I = (F_1, F_2, F_3, F_4)$  is the homogeneous ideal of a curve  $Z$  in  $\mathbb{P}^4$ , and the minors  $F_2, F_3, F_4$  define a surface. We thus have a minimal generating set of  $I$  of degrees (3, 2, 2, 2), but four general forms of degrees (4, 2, 2, 2) will define a union of the curve  $Z$  and another curve, violating the input requirements of Algorithm 2.

#### 4. Examples

In this section, we present several examples of computations of Chern numbers which illustrate the algorithms of the previous section.

##### 4.1. Curves

We consider the case of a smooth connected curve in  $\mathbb{P}^r$  (Fulton, 1998 Example 9.1.1). If  $Z$  is a smooth curve of genus  $g$  in  $\mathbb{P}^r$ , then the first Chern class of  $Z$  is  $-K_Z$  and  $\deg(-K_Z) = 2 - 2g$ . If  $(F_1, F_2, \dots, F_r) \subset \mathbb{C}[z_0, z_1, \dots, z_r]$  is a homogeneous ideal  $I$  whose corresponding scheme is the disjoint union of the curve  $Z$  and a zero-scheme  $S$ , then Corollary 2.3 leads to

$$\deg(X_1 \dots X_r)^Z = (n_1 + \dots + n_r - (r + 1)) \deg Z + 2 - 2g,$$

where  $X_i$  denotes the hypersurface corresponding to  $F_i$  and  $n_i = \deg X_i$ . From Theorem 2.5, we can also write the equation as

$$\prod_i n_i - \deg S = (n_1 + \dots + n_r - (r + 1)) \deg Z + 2 - 2g. \quad (2)$$

**Example 4.1.** The homogeneous ideal of the twisted cubic curve in  $\mathbb{P}^3$  is  $I = (x^2 - wy, y^2 - xz, wz - xy) \subset \mathbb{C}[w, x, y, z]$ . It is well known that this curve has degree 3 and genus 0. If we choose  $F_1, F_2, F_3 \in I$  of degrees  $(2, 2, 2)$ , then the numerical irreducible decomposition implemented in Bates et al. (2010) determines that the corresponding scheme consists of a degree 3 curve and no additional points ( $S$  is empty). Eq. (2) gives the relation

$$2 \cdot 2 \cdot 2 - 0 = (2 + 2 + 2 - (3 + 1)) \cdot 3 + 2 - 2g,$$

which we can solve to get  $g = 0$ .

If we choose  $F_1, F_2, F_3 \in I$  of degrees  $(2, 2, 3)$ , then we obtain a degree 3 curve and one additional point of multiplicity 1. Eq. (2) gives the relation

$$2 \cdot 2 \cdot 3 - 1 = (2 + 2 + 3 - (3 + 1)) \cdot 3 + 2 - 2g,$$

again leading to  $g = 0$ .

If we did not know the degree of  $Z$ , the two computations in this example would yield

$$2 \cdot 2 \cdot 2 - 0 = (2 + 2 + 2 - (3 + 1)) \deg Z + 2 - 2g,$$

$$2 \cdot 2 \cdot 3 - 1 = (2 + 2 + 3 - (3 + 1)) \deg Z + 2 - 2g.$$

The unique solution to these two equations is  $\deg Z = 3$  and  $g = 0$ .

##### 4.2. Surfaces

Now consider the case where  $Z$  is a smooth connected surface in  $\mathbb{P}^r$  (Fulton, 1998 Example 9.1.5). Let  $I = (F_1, F_2, \dots, F_r) \subset \mathbb{C}[z_0, z_1, \dots, z_r]$  be a homogeneous ideal. If the scheme corresponding to  $I$  is the disjoint union of the surface  $Z$  and a zero-scheme  $S$ , then Corollary 2.3 and Theorem 2.5 lead to the equation

$$\prod_i n_i - \deg S = \deg(X_1 \dots X_r)^Z = a_2 \deg c_2 + a_1 \deg c_1 + a_0 \deg Z, \quad (3)$$

where

$$a_2 = 1, \quad a_1 = \sum_{i=1}^r n_i - (r + 1) \quad \text{and} \quad a_0 = \sum_{i < j} n_i n_j - (r + 1) \sum_{i=1}^r n_i + \binom{r + 2}{2}.$$

**Example 4.2.** In characteristic zero and up to standard modifications, the Horrocks–Mumford bundle  $E$  is the only known indecomposable rank 2 vector bundle on  $\mathbb{P}^4$  (Horrocks and Mumford, 1973). For a general section  $s$  of  $E$ , the zero set  $V(s)$  is a smooth surface in  $\mathbb{P}^4$  called a Horrocks–Mumford

surface. We shall compute the degree of a smooth Horrocks–Mumford surface  $Z$  as well as the degrees of its Chern classes  $c_1$  and  $c_2$ . It is known that  $Z$  is an Abelian surface of degree 10 (Horrocks and Mumford, 1973). Hence  $\deg c_0 = 10$  and  $c_1 = c_2 = 0$ . The homogeneous ideal  $I$  of  $Z$  is generated by three quintics and 15 sextics, and  $Z$  is cut out scheme theoretically by three quintics and one sextic. Generators for  $I$  can be found by a variety of methods, e.g., by a Beilinson monad (Decker et al., 1993). In this example, we do not follow Algorithm 3 in detail, but the example is covered by Proposition 3.2.

The following table shows the number of solutions resulting from zero-dimensional, adaptive precision runs in Bates et al. (2010) on four random elements of  $I$  for various choices of degrees. We let  $(n_1, n_2, n_3, n_4)$  denote the degrees of these elements.

$(n_1, n_2, n_3, n_4)$	Isolated points	Non-isolated points
(5, 5, 5, 6)	0	750
(5, 5, 6, 6)	40	860
(5, 6, 6, 6)	100	980

In each of these computations, the zero set consists of the Abelian surface together with a finite set  $S$  of points of multiplicity 1. In each row of the table the entry in the middle column is the cardinality of  $S$  and the entry in the rightmost column is the Bézout number  $\prod_{i=1}^4 n_i$  minus the entry in the middle column, i.e.,

$$n_1 n_2 n_3 n_4 - \sum_{p \in S} m_p.$$

By Proposition 2.4, this is also the degree of the equivalence of  $Z$ . Evaluating (3) in each case gives

$$\begin{aligned} \deg c_2 + 16 \deg c_1 + 75 \deg Z &= 750 \\ \deg c_2 + 17 \deg c_1 + 86 \deg Z &= 860 \\ \deg c_2 + 18 \deg c_1 + 98 \deg Z &= 980. \end{aligned}$$

This system has the unique solution  $\deg Z = 10, \deg c_1 = \deg c_2 = 0$ .

### 4.3. Higher-dimensional varieties

If  $Z$  is a smooth  $n$ -dimensional variety in  $\mathbb{P}^r$ , then the algorithm proceeds in a similar manner to the curve and surface cases. We use homotopy continuation to determine the equivalence of  $Z$  in various intersection products. This combines with the formulas appearing in Corollary 2.3 and Theorem 2.5 to produce a linear system involving the degrees of the Chern classes of  $Z$ . Finally, we solve the linear system to determine these Chern numbers.

**Example 4.3.** Let  $I$  be the ideal defined by the  $4 \times 4$  minors of a  $4 \times 5$  matrix of general linear forms in  $\mathbb{C}[x_0, x_1, \dots, x_5]$ , and let  $Z$  be the corresponding threefold in  $\mathbb{P}^5$ . The following table shows the number of solutions resulting from zero-dimensional, adaptive precision runs in Bates et al. (2010) on five random elements of  $I$  for various choices of degrees. We let  $(n_1, n_2, n_3, n_4, n_5)$  denote the degrees of these elements.

$(n_1, n_2, n_3, n_4, n_5)$	Isolated points	Non-isolated points
(4, 4, 4, 4, 4)	0	1024
(4, 4, 4, 4, 5)	1	1279
(4, 4, 4, 5, 5)	6	1594
(4, 4, 5, 5, 5)	21	1979

In each of these computations, the zero set consists of the threefold together with a finite set  $S$  of points of multiplicity 1. The formulas from Corollary 2.3 and Theorem 2.5 lead to

$$\begin{aligned} \deg c_3 + 14 \deg c_2 + 61 \deg c_1 + 44 \deg Z &= 1024 \\ \deg c_3 + 15 \deg c_2 + 71 \deg c_1 + 65 \deg Z &= 1279 \\ \deg c_3 + 16 \deg c_2 + 82 \deg c_1 + 92 \deg Z &= 1594 \\ \deg c_3 + 17 \deg c_2 + 94 \deg c_1 + 126 \deg Z &= 1979. \end{aligned}$$

The unique solution is  $\deg Z = 10, \deg c_1 = 0, \deg c_2 = 45, \deg c_3 = -46$ .

**Example 4.4.** Let  $V$  be the image of the Segre embedding  $i : \mathbb{P}^1 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^7$ . Let  $Z$  be the intersection of  $V$  with a general hyperplane and let  $I$  be the homogeneous ideal of  $Z$ . The following table shows the number of solutions resulting from zero-dimensional, adaptive precision runs in Bates et al. (2010) on six random elements of  $I$  for various choices of the degrees  $(n_1, \dots, n_6)$  of these elements.

$(n_1, n_2, n_3, n_4, n_5, n_6)$	Isolated points	Non-isolated points
(2, 2, 2, 2, 2, 2)	0	64
(2, 2, 2, 2, 2, 3)	0	96
(2, 2, 2, 2, 3, 3)	2	142
(2, 2, 2, 3, 3, 3)	10	206

In each of these computations the zero set of the six random elements consists of the threefold together with a finite set of points of multiplicity 1. The formulas from Corollary 2.3 and Theorem 2.5 lead to

$$\begin{aligned} \deg c_3 + 5 \deg c_2 + 4 \deg c_1 - 8 \deg Z &= 64 \\ \deg c_3 + 6 \deg c_2 + 7 \deg c_1 - 10 \deg Z &= 96 \\ \deg c_3 + 7 \deg c_2 + 11 \deg c_1 - 11 \deg Z &= 142 \\ \deg c_3 + 8 \deg c_2 + 16 \deg c_1 - 10 \deg Z &= 206. \end{aligned}$$

The unique solution is  $\deg Z = 4, \deg c_1 = 10, \deg c_2 = 10, \deg c_3 = 6$ .

**Remark 4.5.** The code used to compute the examples given in this section may be found at <http://www.math.kth.se/~daek/>. As discussed in Section 2.4, numerically based implementations of the algorithms presented in this paper may fail. However, when setting *Bertini* in adaptive precision mode, the examples above seem to run smoothly. That the outputs given above are truly the correct numbers have been checked by computing the Hilbert polynomials of the respective projective varieties and utilizing double-point formulae.

The run times for the examples are roughly as follows (in single-processor mode on a AMD Athlon 64 processor 3500+, 2211 MHz):

Example	4.1	4.2	4.3	4.4
Run time	0.25 s	19 m	7 h 38 m	57 s

## 5. Conclusion

The results of this paper demonstrate the viability of computing Chern numbers of smooth varieties through numerical homotopy continuation. Homotopy continuation via square systems is a natural venue through which to compute the equivalence of a scheme in the context of an important generalized version of Bézout's theorem from intersection theory. It should be noted that the algorithms of this paper could be implemented in a purely symbolic setting as well. The advantage of the numerical approach is that the algorithms work equally well whether the generators are sparse or dense and whether they have rational, algebraic or transcendental coefficients. In addition, meaning can often be attached to the computations in situations where the generators have inaccuracies in their coefficients. Finally, homotopy continuation algorithms parallelize well allowing such algorithms to take full advantage of multi-processor machines. These features suggest the complementary nature of the approach to purely symbolic methods.

In future work, the authors intend to extend the approach to take further advantage of the ideas of intersection theory. In particular, intersection theory for non-square systems leads to a method for computing intersection numbers of Chern classes.

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